The Product Formula

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ABSTRACT

A useful property of the Brouwer degree relates the degree of a composition of maps to the degree of each map. This property, which can be generalized for the Leray Schauder degree and in some cases for the A-proper maps [see 4], is called the Product Formula.

In [3] we develop a generalized degree theory for a class of mappings, this class contains the class of A-proper mappings and compact mappings. In this paper we prove a generalization of the Product Formula when one factor is of the Identity+Compact type and the other is an A-compact mapping.

1. Introduction

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In [3] we develop a generalized degree theory for a class of mappings, this class contains the class of A-proper mappings and compact mappings. In this paper we prove a generalization of the Product Formula when one factor is of the Identity+Compact type and the other is an A-compact mapping.
2. Notation and Previous Results

Throughout this paper $\Gamma_n = \{X, P_n, X_n\}$ will be a projectionally complete Banach space (see definition in [2]). For any subset $G$ of $X$, $\text{Cl}(G)$ will denote the closure of $G$, $\text{bdry}(G)$ the boundary of $G$ and $G_n = G \cap X_n$. We consider the following classes of mappings:

1. $A(G) = \{F: \text{Cl}(G) \hookrightarrow X \mid F \text{ is } A-\text{proper and continuous}\}$ (see definition in [1])

2. $K(G) = \{C: \text{Cl}(G) \hookrightarrow X \mid C \text{ is } \text{Compact}\}$

3. $\Delta(G) = \{T: \text{Cl}(G) \hookrightarrow X \mid \lim_{n \rightarrow \infty} \sup_{x \in G_n} d(T(x), X_n) = 0\}$

By $\mathbb{Z}$ we denote the set of integer numbers and $\mathbb{Z}^* = \mathbb{Z} \cup \{+\infty\} \cup \{-\infty\}$.

If $A, B \subset \mathbb{Z}^*$, $A + B = \{a \in \mathbb{Z}^* \mid a = a_1 + a_2, a_1 \in A, a_2 \in B\}$. By definition $\{+\infty\} + \{-\infty\} = \mathbb{Z}^*$.

**Definition 1.** Let $\Gamma_n = \{X, P_n, X_n\}$ be a projectionally complete Banach space, $G$ an open bounded subset of $X$, $T$ a mapping from $\text{Cl}(G)$ into $X$ such that $P_n T$ is continuous. We say that $T$ is A-compact on $\text{Cl}(G)$ with respect to the approximation scheme $\Gamma_n$ if for any $y \in X$, for any sequence $\{n_j\}$ of positive integers with $n_j \rightarrow \infty$ and for any sequence $\{x_{n_j}\}$, $x_{n_j} \in G \cap X_{n_j}$, if:

$$\lim_{n_j \rightarrow \infty} P_{n_j} T x_{n_j} = y$$

then there exists a subsequence $\{x_{n_{j_k}}\}$ such that:

$$\lim_{n_{j_k} \rightarrow \infty} T x_{n_{j_k}} = y$$

We will denote by $A - K(G)$ the class of A-compact mappings.

**Lemma 1.** ([3])

Let $T$ be an A-compact mapping from $\text{Cl}(G)$ into $X$ with respect to $\Gamma_n$. Suppose $y \notin \text{Cl}(T(\text{bdry}(G)))$, then there exist $\delta > 0$ and $n_o \in \mathbb{N}$ such that for every $n > n_o$ we obtain:

$$\|P_n T(x) - P_n(y)\| \geq \delta$$

for every $x \in \text{bdry}(G) \cap X_n$. 
**Definition 2.** Let $\Gamma_n$ be a projectionally complete Banach space, $G$ an open bounded subset of $X$, $T$ a mapping in $A - K(G)$ and let us assume that $y \notin Cl(T \text{bdry}(G))$. We define:

$$D(T, G, y) = \left\{ z \in Z : z = \lim_{n_j \to \infty} \{d(P_{n_j} T, G_{n_j}, P_{n_j} y)\} \text{ and } \{n_j\} \text{ an increasing sequence of positive integers} \right\}$$

where $G_{n_j} = G \cap X_{n_j}$ and $d(P_{n_j} T, G_{n_j}, P_{n_j} y)$ is the classical Brouwer degree.

**Theorem 1.** ([3])

Let $T$ be an $A$-compact mapping from $Cl(G)$ into $X$ with respect to $\Gamma_n$. Suppose that $y \notin Cl(T(\text{bdry}(G)))$. Then:

1. $D(G, T, y) \neq \emptyset$
2. $D(I, G, y) = \{1\}$
3. If $G_1$ and $G_2$ are open subsets of $G$, $G_1 \cap G_2 = \emptyset$ and $y \notin Cl(T(Cl(G - (G_1 \cup G_2))))$ we obtain:

$$D(T, G, y) \subseteq D(T, G_1, y) + D(T, G_2, y)$$

where equality holds if either $D(T, G_1, y)$ or $D(T, G_2, y)$ is a singleton.

4. Let $H$ be a mapping from $[0, 1] \times Cl(G)$ into $X$. Suppose that the following hypotheses are all satisfied:

$$H(t, .) \in A - K(G) \text{ for every } t \in [0, 1]$$

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $\|H(t_1, .) - H(t_2, .)\| < \epsilon$ for every $x \in Cl(G)$. Then $D(H(t, .), G, y(t))$ is independent of $t$, where $y : [0, 1] \to X$ is continuous and $y(t) \notin Cl(H(t, \text{bdry}(G)))$

5. If $T(Cl(G))$ is closed and $D(T, G, y) \neq \{0\}$ there exists a point $x \in G$ such that:

$$T(x) = y$$

6. $K(G) \cup \Delta(G) \cup A(G) \subset A - K(G)$

7. If $T \in A - K(G)$ and $C \in K(G)$ we obtain $T + C \in A - K(G)$
3. Results

Lemma 2.

Let \( \Gamma_n = \{X, P_n, X_n\} \) be a projectionally complete Banach space, \( G \) and \( U \) open bounded subsets of \( X, C \in K(G), I + C(Fr(G) \subset Cl(U) \) and \( T \) continuous and \( A \)-compact, if:

(1) There exists \( n_0 \) such that if \( n > n_0 \) \( P_n(I + C)(Cl(G)) \subset Cl(U) \).

(2) \( y \notin Cl(T(I + C)bdry(G)) \).

Then there exists \( n_1 \) such that if \( n > n_1 \)

\[
d(P_nTP_n(I + C), G_n, P_ny) = d(P_nT, G_n, P_ny)
\]

Proof. By the homotopy property for the Brouwer degree it is sufficient to prove that there exists \( n_1 \) such that for all \( n > n_1 \):

\( P_ny \notin (tP_nT(I + C) + (1-t)P_nTP_n(I + C))(bdry(G_n)) \) for all \( t \in [0,1] \).

If we suppose this assertion is false there exist sequences \( \{x_n\}, \{t_n\} \) with \( x_n, \in X_n \) and \( t_n, \in [0,1] \) such that:

\[
t_nP_nT(I + C)x_n + (1 - t_n)P_nTP_n(I + C)x_n = P_ny
\]

We can suppose \( t_n, \longrightarrow t_0 \) \((0,1]\) and \( P_ny \longrightarrow y, \) thus:

\[
\lim_{n, \rightarrow \infty} t_0P_nT(I + C)x_n + (1 - t_0)P_nTP_n(I + C)x_n = y
\]

As \( C \) is a compact mapping there exists a convergent subsequence of \( \{Cx_n\} \) and so there exists a subsequence of \( \{P_nT(I + C)x_n\} \) convergent to \( y \) which is not possible by lemma 1 and hypothesis (2). \( \square \)

Lemma 3.

If in addition to the hypothesis of lemma 2 we have:

(1) \( T \) is proper

(2) \( U - (I + C)bdry(G)) = V \) is connected.

There exists \( n_0 \) such that for all \( n > n_0 \):

\[
d(P_nTP_n(I + C), G_n, P_ny) = D(I + C, G, V)d(P_nT, V_n, P_ny)
\]

Where \( D(I + C, G, V) \) is the Leray-Schauder degree for any \( x \in V \)
Proof. By the Product Formula for the Brouwer degree whenever $n$ is sufficiently large

$$d(P_n T P_n (I + C), G_n, P_n y) = \sum_{i_n=1}^{k} d(P_n (I + C), G_n, V_n^{i_n}) d(P_n T, V_n^{i_n}, P_n y)$$

where $V_n^{i_n}$ are the finite connected components of $V_n - P_n (I + C)(\bdry(G_n))$ which satisfy $d(P_n T, V_n^{i_n}, P_n y) \neq 0$.

The lemma will be proved if when $h \neq k$ whatever the subsequences, whenever $d(P_n T, V_n^k, P_n y)$ and $d(P_n T, V_n^h, P_n y)$ are not zero:

$$d(P_n (I + C), G_n^k, V_n^h) = d(P_n (I + C), G_n^h, V_n^k) = D(I + C, G, V)$$

is satisfied.

In order to prove the equality it suffices to show that if $\{x_n\}$ is a sequence of points with $x_n \in V_n - P_n (I + C)(\bdry(G_n))$ and $P_n T x_n = P_n y$, there exists $n_1$ such that for all $n > n_1$:

(1) $$d(P_n (I + C), G_n, x_n) = D((I + C), G, V)$$

At first we prove that there exists $\delta > 0$ satisfying

$$d(x_n, (I + C)(\bdry(G))) > \delta > 0$$

Suppose that this assertion is false. If we consider the sets

$$V^j = \{x \in V \mid d(x, (I + C)(\bdry(G))) < 1/j\}$$

for all $j$ there would exist a subsequence $\{x_{n_j}\} \subset V^j$. As $T$ is $A$-compact and proper it must be closed, so there exists $\{x_j\}$ with $x_j \in V^j$ such that $T(x_j) = y$. As $T$ is proper and $I + C$ is closed then $T(x) = y$ with $x \in I + C(\bdry(G))$, which is not possible from the hypothesis.

Let $x \in V$ and $d(x, (I + C)(\bdry(G))) > \delta > 0$. If we prove that there exists $n_0$ such that for $n > n_0$

$$d(P_n (I + C), G_n, x_n) = d(P_n (I + C), G_n, P_n x)$$

is verified, the equality (1) would be proved since $d(P_n (I + C), G_n, P_n x) = D((I + C), G, x) = D((I + C), G, V)$ whenever $n$ is sufficiently large.
By the homotopy property we only need to show that:

$$tP_n x + (1 - t) x_n \notin P_n(I + C)(\text{bdry}(G))$$

If this assertion were false there would exist sequences \( \{z_{n_i}\}, \{t_{n_i}\} \) with \( z_{n_i} \in \text{bdry}(G_{n_i}) \) and \( t_{n_i} \in [0, 1] \) such that:

$$t_{n_i} P_{n_i} x + (1 - t_{n_i}) x_{n_i} = P_{n_i}(I + C)z_{n_i}$$

Hence there would exist \( t_0 \in [0, 1] \) and a subsequence of \( \{n_i\} \), which we denote \( \{n_i\} \) again, such that there exists \( n_0 \) satisfying for all \( n > n_1 \)

$$\|t_0 x + (1 - t_0) x_{n_i} - P_{n_i}(I + C)z_{n_i}\| < \frac{1}{j}$$

which is not possible because \( d(x_n, (I + C)(\text{bdry}(G))) > \delta > 0 \) for all \( n \) and \( d(x, (I + C)(\text{bdry}(G))) > \delta > 0 \). \( \square \)

**Theorem 2.**

*With the hypotheses of lemmas 2 and 3*

$$D(T(I + C), G, y) = D(I + C, G, V)D(T, V, y)$$

The proof is deduced from lemmas 1 and 2.

**References**