EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR DELAY EVOLUTION EQUATIONS OF SECOND ORDER IN TIME

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Abstract

We prove some results on the existence and uniqueness of solutions for a class of evolution equations of second order in time, containing some hereditary characteristics. Our theory is developed from a variational point of view, and in a general functional setting which permits us to deal with several kinds of delay terms. In particular, we can consider terms which contain spatial partial derivatives with deviating arguments.

1. Introduction

In the modelling of many evolution phenomena arising in physics, biology, engineering, etc., some hereditary characteristics such as time-delay can appear in the variables. Typical examples can be found in the researches of materials with termal memory, biochemical reactions, population models, etc. (see, for instance, Wu [12] and references cited therein). Thus these problems are better modeled by considering a functional differential equation which takes into account the history of the system.

From the pioneering works of Artola [1,2], Baiocchi [3] and Travis and Webb [10, 11], a wide literature has appeared on the existence of different kind of solutions (strong, mild, integral, etc.) to functional differential equations of first order in time, even in the more general context of differential inclusions (see, for instance, Ruess [7, 8] and references cited therein). However, to our knowledge, in the case of functional differential equations of second order in time, there is only partial results.

Recently, in Kartsatos and Markov [5], some questions on existence of solutions for functional differential inclusions of second order in time, and in particular, for equations of the form

$$\begin{cases} u''(t) + A(t)u(t) = F(t, u_t), \quad t \geq 0, \\ u(t) = \psi(t), \quad t \in [-h, 0], \end{cases} \tag{1.1}$$

have been analized.

Our aim is to obtain some results of existence and uniqueness of solution for some problems that are related to (1.1) in the case in which a damping term is added. Then, we can consider right hand terms of the form $F(t, u_t, u'_t)$ with F eventually depending of spatial derivatives of u and/or u'.

Our analysis will be made from a variational point of view, in the spirit of Artola [1], and makes use of the results of Lions and Strauss [6] and Strauss [9].

In Section 2, we prove some results for the case in which $F(t, u_t, u'_t)$ does not depend on the spatial derivatives of u and/or u'. These are extensions of previous results of Artola [1]. In Section 3, we study, in a hilbertian framework, the case in which $F(t, u_t, u'_t)$ depends of the spatial derivatives of u and/or u', and we obtain some new results of existence of solutions under a coercivity condition (cf. condition (H) in Theorem 3.1 below). Finally, two examples are given in Section 4 to illustrate our theory.

2. The case without spatial derivatives in the delayed terms.

To start off, let us state the abstract framework in which our analysis will be carried out. Let V a real Hilbert space, H a real separable Hilbert space and W a reflexive real Banach space, such that $V \cup W \subset H$, V and W are dense in H and the injections of V and W in H are continuous.

We assume also that $V \cap W$ is separable and a dense subset of V and W.

We identify H with its dual space H^*, and we have

$$V \cap W \subset V \subset H \subset V^* \subset (V \cap W)^*,$$

$$V \cap W \subset W \subset H \subset W^* \subset (V \cap W)^*,$$

where the injections are continuous and dense.

We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V^* respectively; by $\|\cdot\|_W$ the norm in W, and by $\|\cdot\|_{W^*}$ the norm in W^*. We also denote by $((\cdot,\cdot))$ and $(\cdot,\cdot)$ the scalar products in V and H respectively; and by $\langle \cdot, \cdot \rangle$ the duality product between V^* and V or the duality product between W^* and W.

Let us fix a real number $T > 0$, and consider given $\{A(t); t \in [0,T]\}$ a family of linear operators satisfying:

$(A.1)$ $\{A(t); t \in [0,T]\} \subset \mathcal{L}(V, V^*)$ and $A(t)$ is selfadjoint for each $t \in [0,T]$.

$(A.2)$ there exist $\alpha > 0$ such that $\langle A(t)u, u \rangle \geq \alpha \|u\|^2, \quad \forall u \in V, \ \forall t \in [0,T]$.

$(A.3)$ $\langle A(\cdot)u, \widetilde{u} \rangle \in C^1([0,T]) \ \forall u, \widetilde{u} \in V$, and, if we denote for $t \in [0,T]$ by $\langle A'(t)u, \widetilde{u} \rangle$ the value of $\dfrac{d}{dt} \langle A(t)u, \widetilde{u} \rangle$, the operator $A'(t)$ so defined belongs to $\mathcal{L}(V; V^*)$.

$(A.4)$ $\quad \langle A'(t)u, u \rangle \leq 0 \quad \forall u \in V, \ \forall t \in [0,T]$.

Observe that $A(\cdot)$ and $A'(\cdot)$ belong to $L^\infty(0,T; \mathcal{L}(V, V^*))$.

Given real numbers $a < b$, and a Banach space $\mathcal{V}$, we will denote by $C(a,b;\mathcal{V})$ the Banach space of all continuous functions from $[a,b]$ into $\mathcal{V}$ equipped with *sup* norm.

We will denote $u' = \dfrac{du}{dt}$, and $u'' = \dfrac{d^2u}{dt^2}$, the first and second derivatives of u as a vectorial distribution, and by p' the conjugate exponent of p. We have the following result due to Strauss [9].

Theorem 2.1 *Assume that hypothesis* $(A.1)$-$(A.4)$ *hold. For* $p \in (1,\infty)$ *given, let* $u \in L^\infty(0,T;V)$ *such that* $u' \in L^p(0,T;W) \cap L^\infty(0,T;H)$*, and* $u'' + Au = G \in L^{p'}(0,T;W^*) + L^1(0,T;H)$*. Then,* $u \in C(0,T;V)$*,* $u' \in C(0,T;H)$*,* $\langle Au, u \rangle \in C([0,T])$*, and for each* $t \in [0,T]$*,*

$$|u'(t)|^2 + \langle A(t)u(t), u(t) \rangle = |u'(0)|^2 + \langle A(0)u(0), u(0) \rangle + \int_0^t \langle A'(s)u(s), u(s) \rangle \, ds + 2 \int_0^t \langle G(s), u'(s) \rangle \, ds. \tag{2.1}$$

Let $B(t,\cdot) : W \longrightarrow W^*$ be a family of nonlinear operators defined a.e. $t \in (0,T)$ and satisfying:

$(B.1)$ $\forall v \in W$, the map $t \in (0,T) \longmapsto B(t,v) \in W^*$ is Lebesgue measurable.

$(B.2)$ the map $\theta \in \mathbb{R} \longmapsto \langle B(t, v + \theta w), z \rangle \in \mathbb{R}$ is continuous $\forall v, w, z \in W$, a.e. $t \in (0,T)$.

For some $p \in (1,\infty)$,

$(B.3)$ there exists $c > 0$ such that $\|B(t,v)\|_{W^*} \leq c \|v\|_W^{p-1} \quad \forall v \in W$, a.e. $t \in (0,T)$.

$(B.4)$ there exists $\beta > 0$ such that $\langle B(t,v), v\rangle \geq \beta \|v\|_W^p \quad \forall\, v \in W$, a.e. $t \in (0,T)$.

$(B.5)$ $\langle B(t,v) - B(t,\widetilde{v}), v - \widetilde{v}\rangle \geq 0 \quad \forall v, \widetilde{v} \in W, \quad$ a.e. $t \in (0,T)$.

Using Theorem 2.1 and the results in Lions and Strauss [6], we have:

Theorem 2.2 *Assume that hypothesis $(A.1)$-$(A.4)$ and $(B.1)$-$(B.5)$ hold. Then, for every $u_0 \in V$, $v_0 \in H$ and $f \in L^{p'}(0,T;W^*) + L^1(0,T;H)$, there exists a unique solution to the problem*

$$\begin{cases} u \in L^\infty(0,T;V), \quad u' \in L^\infty(0,T;H) \cap L^p(0,T;W), \\ u''(t) + A(t)u(t) + B(s, u'(t)) = f(t), \quad t \in (0,T), \\ u(0) = u_0, \; u'(0) = v_0. \end{cases}$$

Moreover, the solution u satisfies $u \in C(0,T;V)$, $u' \in C(0,T;H)$, and for each $t \in [0,T]$,

$$|u'(t)|^2 + \langle A(t)u(t), u(t)\rangle + 2\int_0^t \langle B(s,u'(s)), u'(s)\rangle \, ds =$$
$$|v_0|^2 + \langle A(0)u_0, u_0\rangle + \int_0^t \langle A'(s)u(s), u(s)\rangle \, ds + 2\int_0^t \langle f(s), u'(s)\rangle \, ds.$$

Consider also fixed a real number $h > 0$. For a given Banach space $\mathcal{V}$, if we consider a function $x : [-h,T] \mapsto \mathcal{V}$, for each $t \in [0,T]$ we will denote by x_t the function defined by $x_t(s) = x(t+s)$, $s \in [-h,0]$.

Let now $F_0 : (0,T) \times C(-h,0;V) \times C(-h,0;H) \longrightarrow H$ be a family of nonlinear operators defined a.e. $t \in (0,T)$ such that:

$(F_0.1)$ $\forall (\xi,\eta) \in C(-h,0;V) \times C(-h,0;H)$, the map $t \in (0,T) \longmapsto F_0(t,\xi,\eta) \in H$ is Lebesgue measurable,

$(F_0.2)$ $F_0(t,0,0) = 0$, a.e. $t \in (0,T)$,

$(F_0.3)$ $\exists C_{F_0} > 0$ such that $\forall \xi, \widetilde{\xi} \in C(-h,0;V)$, $\forall \eta, \widetilde{\eta} \in C(-h,0;H)$ and a.e. $t \in (0,T)$,

$$|F_0(t,\xi,\eta) - F_0(t,\widetilde{\xi},\widetilde{\eta})|^2 \leq C_{F_0}\left(\|\xi - \widetilde{\xi}\|^2_{C(-h,0;V)} + |\eta - \widetilde{\eta}|^2_{C(-h,0;H)} \right).$$

Remark 2.1 If $(u,v) \in C(-h,T;V) \times C(-h,T;H)$, it is not difficult to deduce from $(F_0.1)$-$(F_0.3)$ that the mapping $t \in (0,T) \mapsto F_0(t,u_t,v_t) \in H$ is measurable, and consequently the function $F_0(t,u_t,v_t)$ belongs to $L^\infty(0,T;H)$.

We consider the problem

$$\begin{cases} u \in C(-h,T;V), \quad u' \in C(-h,T;H) \cap L^p(0,T;W), \\ u''(t) + A(t)u(t) + B(t,u'(t)) = F_0(t,u_t,u_t') + f(t),\ t \in (0,T), \\ u(t) = \psi(t),\ t \in [-h,0], \end{cases} \tag{P_0}$$

where $f \in L^{p'}(0,T;W^*) + L^1(0,T;H)$, and $\psi \in C(-h,0;V)$ such that $\psi' \in C(-h,0;H)$, are given.

We can prove the following result:

Theorem 2.3 *Assume that $(A.1)$-$(A.4)$, $(B.1)$-$(B.5)$ and $(F_0.1)$-$(F_0.3)$ hold. Then, for each $f \in L^{p'}(0,T;W^*) + L^1(0,T;H)$, and $\psi \in C(-h,0;V)$ such that $\psi' \in C(-h,0;H)$, there exists a unique solution u to the problem (P_0).*

Proof.

Uniqueness of solutions. Assume that $u, \widetilde{u}$ are two solutions of problem (P_0). Denote $v(t) = u'(t)$ and $\widetilde{v}(t) = \widetilde{u}'(t)$, $t \in [-h,T]$. Then, by Theorem 2.1, for each $t \in [0,T]$, we obtain

$$\begin{aligned} &|v(t) - \widetilde{v}(t)|^2 + \langle A(t)(u(t) - \widetilde{u}(t)), u(t) - \widetilde{u}(t)\rangle = \\ &\int_0^t \langle A'(s)(u(s) - \widetilde{u}(s)), u(s) - \widetilde{u}(s)\rangle \, ds \\ &- 2\int_0^t \langle B(s,v(s)) - B(s,\widetilde{v}(s)), v(s) - \widetilde{v}(s))\rangle \, ds \\ &+ 2\int_0^t (F_0(s,u_s,v_s) - F_0(s,\widetilde{u}_s,\widetilde{v}_s), v(s) - \widetilde{v}(s))\, ds. \end{aligned}$$

¿From this equality, and $(A.2), (A.4)$ and $(B.5)$, we have for each $t \in [0,T]$,

$$\begin{aligned} &\sup_{0\le\theta\le t} |v(\theta) - \widetilde{v}(\theta)|^2 + \alpha \sup_{0\le\theta\le t} \|u(\theta) - \widetilde{u}(\theta)\|^2 \\ &\le 4\int_0^t |(F_0(s,u_s,v_s) - F_0(s,\widetilde{u}_s,\widetilde{v}_s), v(s) - \widetilde{v}(s)))| \, ds. \end{aligned} \tag{2.2}$$

By $(F_0.3)$,

$$\begin{aligned} &4\int_0^t |(F_0(s,u_s,v_s) - F_0(s,\widetilde{u}_s,\widetilde{v}_s), v(s) - \widetilde{v}(s))| \, ds \\ &\le \frac{1}{2}\left[\sup_{0\le s\le t} |v(s) - \widetilde{v}(s)|^2\right] + 8T\int_0^t |F_0(s,u_s,v_s) - F_0(s,\widetilde{u}_s,\widetilde{v}_s)|^2 \, ds \\ &\le \frac{1}{2}\left[\sup_{0\le s\le t} |v(s) - \widetilde{v}(s)|^2\right] \\ &+ 8TC_{F_0}\int_0^t \left(\sup_{0\le\theta\le s} \|u(\theta) - \widetilde{u}(\theta)\|^2 + \sup_{0\le\theta\le s} |v(\theta) - \widetilde{v}(\theta)|^2\right) ds. \end{aligned}$$

Thus, from (2.2), we can assure that there exists a constant $k > 0$ such that

$$\sup_{0\le\theta\le t} |v(\theta) - \widetilde{v}(\theta)|^2 + \sup_{0\le\theta\le t} \|u(\theta) - \widetilde{u}(\theta)\|^2$$
$$\le k \int_0^t \left(\sup_{0\le\theta\le s} \|u(\theta) - \widetilde{u}(\theta)\|^2 + \sup_{0\le\theta\le s} |v(\theta) - \widetilde{v}(\theta)|^2 \right) ds,$$

for all $t \in [0, T]$. Uniqueness follows immediately from Gronwall's lemma.

Existence of solutions: We denote $u^0 \equiv v^0 \equiv 0 \in V \cap W$, and define by recurrence a sequence $\{u^n, v^n\}_{n\ge1}$ of pairs of functions as solutions to the problem

$$\begin{cases} u^n \in C(-h,T;V), \quad v^n \in C(-h,T;H) \cap L^p(0,T;W), \\ v^n(t) = (u^n)'(t), \quad t \in [-h,T], \\ v^n(t) + \int_0^t A(s)u^n(s)\,ds + \int_0^t B(s, v^n(s))\,ds = \psi'(0) \\ + \int_0^t (F_0(s, u_s^{n-1}, v_s^{n-1}) + f(s))\,ds, \ t \in [0,T], \\ u^n(t) = \psi(t), \quad t \in [-h, 0]. \end{cases} \tag{P_{0_n}}$$

The existence and uniqueness of solution to the problem (P_{0_n}) is guaranteed by Remark 1.1. and Theorem 2.2. Now, we want to prove that $\{u^n\}_{n\ge1}$ converges to the solution of (P_0).

Applying Theorem 2.1 to $u^{n+1} - u^n$, and using $(A.2)$, $(A.4)$ and $(B.5)$, we obtain

$$\sup_{0\le\theta\le t} \left|v^{n+1}(\theta) - v^n(\theta)\right|^2 + \alpha \sup_{0\le\theta\le t} \left\|u^{n+1}(\theta) - u^n(\theta)\right\|^2$$
$$\le 4 \int_0^t \left|(F_0(s, u_s^n, v_s^n) - F_0(s, u_s^{n-1}, v_s^{n-1}), v^{n+1}(s) - v^n(s))\right| ds. \tag{2.3}$$

Thanks to condition $(F_0.3)$, we have

$$4 \int_0^t \left|(F_0(s, u_s^n, v_s^n) - F_0(s, u_s^{n-1}, v_s^{n-1}), v^{n+1}(s) - v^n(s))\right| ds$$
$$\le \frac{1}{2} \left[\sup_{0\le s\le t} \left|v^{n+1}(s) - v^n(s)\right|^2 \right]$$
$$+ 8TC_{F_0} \int_0^t \left(\sup_{0\le\theta\le s} \left\|u^n(\theta) - u^{n-1}(\theta)\right\|^2 + \sup_{0\le\theta\le s} \left|v^n(\theta) - v^{n-1}(\theta)\right|^2 \right) ds. \tag{2.4}$$

Thus, if we define for each $t \in [0,T]$

$$\rho^{n+1}(t) = \sup_{0\le\theta\le t} \left|v^{n+1}(\theta) - v^n(\theta)\right|^2 + \sup_{0\le\theta\le t} \left\|u^{n+1}(\theta) - u^n(\theta)\right\|^2, \quad \forall n \ge 1,$$

from (2.3)-(2.4) we can assure that there exists a constant $k > 0$ such that

$$\rho^{n+1}(t) \leq k \int_0^t \rho^n(s)\, ds, \quad \forall n \geq 1, \quad \forall t \in [0,T]. \tag{2.5}$$

¿From (2.5), it is easy to deduce that

$$\rho^{n+1}(t) \leq \frac{k^n T^n}{n!} \rho^1(T), \quad \forall n \geq 1, \ \forall t \in [0,T]. \tag{2.6}$$

¿From (2.6), and the fact that $u^{n+1}(t) = u^n(t)$ and $v^{n+1}(t) = v^n(t)$, $\forall t \in [-h, 0]$ and $\forall n \geq 1$, we obtain that $\{u^n\}_{n\geq 1}$ is a Cauchy sequence in $C(-h,T;V)$, and that $\{v^n\}_{n\geq 1}$ is a Cauchy sequence in $C(-h,T;H)$, being $(u^n)'(t) = v^n(t)$ in $[-h,T]$. Consequently, there exists $u \in C(-h,T;V)$, with $u' \in C(-h,T;H)$ such that

$$u^n \to u \text{ in } C(-h,T;V), \quad v^n \to u' \text{ in } C(-h,T;H). \tag{2.7}$$

¿From (2.7), the linearity and uniform boundedness of $A(t)$ and $A'(t)$, and $(F_0.3)$, we obtain

$$\begin{aligned} &A(t)u^n(t) \to A(t)u(t) \quad \text{in } C(0,T;V^*), \\ &A'(t)u^n(t) \to A'(t)u(t) \quad \text{in } L^\infty(0,T;V^*), \\ &F_0(t,u^n_t,v^n_t) \to F_0(t,u_t,u'_t) \quad \text{in } L^\infty(0,T;H). \end{aligned} \tag{2.8}$$

On the other hand, applying (2.1) to u^n, we obtain

$$\begin{aligned} &|v^n(t)|^2 + \langle A(t)u^n(t), u^n(t)\rangle + 2\int_0^t \langle B(s,v^n(s)), v^n(s)\rangle\, ds = \\ &|\psi'(0)|^2 + \langle A(0)\psi(0), \psi(0)\rangle + 2\int_0^t \big(F_0(s,u^{n-1}_s, v^{n-1}_s), v^n(s)\big)\, ds \\ &+ 2\int_0^t \langle f(s), v^n(s)\rangle\, ds + \int_0^t \langle A'(s)u^n(s), u^n(s)\rangle\, ds. \end{aligned} \tag{2.9}$$

But, if we write $f = f_1 + f_2$ with $f_1 \in L^{p'}(0,T;W^*)$ and $f_2 \in L^1(0,T;H)$, then by Young's inequality, for all $t \in [0,T]$,

$$2\int_0^t \langle f_1(s), v^n(s)\rangle\, ds \leq \beta \int_0^t \|v^n(s)\|_W^p\, ds + \frac{2^{p'}}{p'(\beta p)^{p'/p}} \int_0^T \|f_1(s)\|_{W^*}^{p'}\, ds.$$

Hence, using this last inequality, $(A.2)$, $(A.4)$, $(F_0.3)$, $(B.4)$, $(F_0.4)$ and the boundedness of $\{u^n\}_{n\geq 1}$ in $C(-h,T;V)$ and $\{v^n\}_{n\geq 1}$ in $C(-h,T;H)$, one obtain from (2.9) that $\{v^n\}_{n\geq 1}$ is bounded in $L^p(0,T;W)$. Hence, by $(B.3)$, the sequence $\{B(t,v^n(t))\}_{n\geq 1}$ is bounded in

$L^{p'}(0,T;W^*)$. Consequently, there exist a subsequence $\{v^{n_k}\}_{n_k\geq 1}\subset\{v^n\}_{n\geq 1}$, and two functions $v\in L^p(0,T;W)$ and $\mathcal{B}(t)\in L^{p'}(0,T;W^*)$, such that

$$v^{n_k}\rightharpoonup v \text{ in } L^p(0,T;W),\quad B(t,v^{n_k}(t))\rightharpoonup\mathcal{B}(t) \text{ in } L^{p'}(0,T;W^*),$$

where $\rightharpoonup$ denotes weak convergence. Obviously, as $v^n\to u'$ in $C(-h,T;H)$, $v=u'$ in $(0,T)$. Thus, we can take limits in (P_{0_n}), and obtain that u is solution of

$$\begin{cases} u\in C(-h,T;V),\quad u'\in C(-h,T;H),\\ u'(t)+\int_0^t A(s)u(s)\,ds+\int_0^t \mathcal{B}(s)\,ds=\psi'(0)\\ +\int_0^t (F_0(s,u_s,u_s')+f(s))\,ds,\ t\in[0,T],\\ u(t)=\psi(t),\ t\in[-h,0]. \end{cases}\tag{$\widehat{P_0}$}$$

To simplify the notation, observe that from $(\widehat{P_0})$, $\mathcal{B}$ is uniquely determined by u, and thus, the whole sequence $\{B(t,v^n(t))\}_{n\geq 1}$ converges weakly to $\mathcal{B}$ in $L^{p'}(0,T;W^*)$.

In order to prove that u is in fact a solution of problem (P_0), we only need to prove that $\mathcal{B}(t)=B(t,u'(t))$ in $(0,T)$. Firstly, from (2.7)-(2.9), we have

$$\begin{aligned} &|u'(T)|^2+\langle A(T)u(T),u(T)\rangle+2\limsup_{n\to\infty}\int_0^T\langle B(s,v^n(s)),v^n(s)\rangle\,ds\\ \leq&|\psi'(0)|^2+\langle A(0)\psi(0),\psi(0)\rangle+2\int_0^T(F_0(s,u_s,u_s'),u'(s))\,ds\\ &+2\int_0^T\langle f(s),u'(s)\rangle\,ds+\int_0^T\langle A'(s)u(s),u(s)\rangle\,ds, \end{aligned}$$

and consequently, using identity (2.1) applied to $(\widehat{P}_0)$, we obtain

$$\limsup_{n\to\infty}\int_0^T\langle B(s,v^n(s)),v^n(s)\rangle\,ds\leq\int_0^T\langle\mathcal{B}(s),u'(s)\rangle\,ds.$$

¿From $(B.5)$,

$$\int_0^T\langle B(s,v^n(s))-B(s,X(s)),v^n(s)-X(s)\rangle\,ds\geq 0\quad\forall X\in L^p(0,T;W),\quad\forall n\geq 1.\tag{2.10}$$

Taking limits in (2.10), we get

$$\int_0^T\langle\mathcal{B}(s)-B(s,X(s)),u'(s)-X(s)\rangle\,ds\geq 0\quad\forall X\in L^p(0,T;W).\tag{2.11}$$

Now, if we set $X = u' - \delta Z$, with $Z \in L^p(0,T;W)$ and $\delta > 0$, we obtain from (2.11)

$$\int_0^T \langle \mathcal{B}(s) - B(s, u'(s) - \delta Z(s)), \delta Z(s)\rangle \, ds \geq 0 \quad \forall Z \in L^p(0,T;W), \quad \forall \delta > 0. \tag{2.12}$$

If we divide by δ in (2.12), and take limits as $\delta \to 0$, we obtain from $(B.2)$ and $(B.3)$

$$\int_0^T \langle \mathcal{B}(s) - B(s, u'(s)), Z(s)\rangle \, ds \geq 0 \quad \forall\, Z \in L^p(0,T;W),$$

and consequently, $\mathcal{B}(s) = B(s, u'(s))$ in $(0,T)$. ■

Remark 2.2 Suppose now that F_0 satisfy $(F_0.1)$-$(F_0.3)$, and

$(F_0.4)$ $\exists K_{F_0} > 0$ such that $\forall x, \tilde{x} \in C(-h,T;V)$, $\forall y, \tilde{y} \in C(-h,T;H)$, and $\forall t \in [0,T]$,

$$\int_0^t |F_0(s, x_s, y_s) - F_0(s, \widetilde{x}_s, \widetilde{y}_s)|^2 \, ds \leq K_{F_0} \int_{-h}^t \left(\|x(s) - \widetilde{x}(s)\|^2 + |y(s) - \widetilde{y}(s)|^2 \right) ds. \tag{2.13}$$

We know that, by $(F_0.1)$-$(F_0.3)$, if $(x,y) \in C(-h,T;V) \times C(-h,T;H)$, the function $F_0^{(x,y)}$: $(0,T) \mapsto H$ defined by $F_0^{(x,y)}(t) = F_0(t, x_t, y_t)$ a.e.$t \in (0,T)$, belongs to $L^\infty(0,T;H)$. But, thanks to $(F_0.4)$, the mapping

$$\Xi_0 : (x,y) \in C(-h,T;V) \times C(-h,T;H) \longmapsto F_0^{(x,y)} \in L^2(0,T;H)$$

has a unique extension to a mapping $\widetilde{\Xi}_0$ which is uniformly continuous from the product space $L^2(-h,T;V) \times L^2(-h,T;H)$ into $L^2(0,T;H)$. From now on, we will also write $F_0(t, x_t, y_t) = \widetilde{\Xi}_0(x,y)(t)$ for each $(x,y) \in L^2(-h,T;V) \times L^2(-h,T;H)$. Consequently, for every $x, \widetilde{x} \in L^2(-h,T;V)$, $y, \widetilde{y} \in L^2(-h,T;H)$ and $\forall\, t \in [0,T]$, inequality (2.13) will be satisfied.

Now, we are in a position to prove the following result:

Theorem 2.4 *Assume that hypothesis $(A.1)$-$(A.4)$, $(B.1)$-$(B.5)$ and $(F_0.1)$-$(F_0.4)$ hold. Then, for each $u_0 \in V$, $v_0 \in H$, $f \in L^{p'}(0,T;W^*) + L^1(0,T;H)$, $\psi \in L^2(-h,0;V)$ and $\phi \in L^2(-h,0;H)$ given, there exists a unique solution (u,v) to the problem*

$$\begin{cases} u \in L^2(-h,T;V) \cap C(0,T;V), \quad v \in L^2(-h,T;H) \cap C(0,T;H) \cap L^p(0,T;W), \\ u'(t) = v(t), \ t \in [0,T], \\ v'(t) + A(t)u(t) + B(t, v(t))\, ds = F_0(t, u_t, v_t) + f(t), \ t \in (0,T), \\ u(0) = u_0, \ u'(0) = v_0, \\ u(t) = \psi(t), \ v(t) = \phi(t), \quad \text{a.e.}\, t \in (-h, 0), \end{cases} \tag{Q_0}$$

Proof. The proof is similar to that of Theorem 2.3, and so, we omit it.

Remark 2.3 Observe that in Theorems 2.3 and 2.4 we can add to $A(t)u(t)$ terms of the form $\widetilde{A}_0(t, u(t))$, and to $B(t, u'(t))$ terms of the form $\widetilde{B}_0(t, u'(t))$, with $\widetilde{A}_0$ and $\widetilde{B}_0$ satisfying adequate conditions. More exactly, consider given $\widetilde{A}_0(t, \cdot) : V \mapsto H$ and $\widetilde{B}_0(t, \cdot) : H \mapsto H$, two families of nonlinear operators defined a.e. $t \in (0, T)$ and satisfying:

($\widetilde{A}_0$.1) $\forall u \in V$, the map $t \in (0, T) \mapsto \widetilde{A}_0(t, u) \in H$ is Lebesgue measurable,

($\widetilde{A}_0$.2) $\widetilde{A}_0(t, 0) = 0$, a.e. $t \in (0, T)$.

($\widetilde{A}_0$.3) $\exists L_{\widetilde{A}_0} > 0$ such that $|\widetilde{A}_0(t, u) - \widetilde{A}_0(t, \widetilde{u})| \leq L_{\widetilde{A}_0} \|u - \widetilde{u}\| \quad \forall\, u, \widetilde{u} \in V$, a.e. $t \in (0, T)$,

($\widetilde{B}_0$.1) $\forall v \in H$, the map $t \in (0, T) \mapsto \widetilde{B}_0(t, v) \in H$ is Lebesgue measurable,

($\widetilde{B}_0$.2) $\widetilde{B}_0(t, 0) = 0$, a.e. $t \in (0, T)$.

($\widetilde{B}_0$.3) $\exists L_{\widetilde{B}_0} > 0$ such that $|\widetilde{B}_0(t, v) - \widetilde{B}_0(t, \widetilde{v})| \leq L_{\widetilde{B}_0} |v - \widetilde{v}| \quad \forall\, v, \widetilde{v} \in H$, a.e. $t \in (0, T)$.

Then, under the conditions of Theorem 2.3, we can assert existence and uniqueness of solution to the problem

$$\begin{cases} u \in C(-h, T; V), \quad u' \in C(-h, T; H) \cap L^p(0, T; W), \\ u''(t) + A(t)u(t) + \widetilde{A}_0(t, u(t)) + B(t, u'(t)) + \widetilde{B}_0(t, u'(t)) = \\ F_0(t, u_t, u_t') + f(t), \; t \in (0, T), \\ u(t) = \psi(t), \; t \in [-h, 0], \end{cases} \tag{$\widetilde{P}_0$}$$

and, under the conditions of Theorem 2.4, we can assert existence and uniqueness of solution to the problem

$$\begin{cases} u \in L^2(-h, T; V) \cap C(0, T; V), \quad v \in L^2(-h, T; H) \cap C(0, T; H) \cap L^p(0, T; W), \\ u'(t) = v(t), \; t \in [0, T], \\ v'(t) + A(t)u(t) + \widetilde{A}_0(s, u(t)) + B(t, v(t)) + \widetilde{B}_0(t, v(t)) = \\ F_0(t, u_t, v_t) + f(t), \; t \in (0, T), \\ u(0) = u_0, \; v(0) = v_0, \\ u(t) = \psi(t), \quad v(t) = \phi(t), \; \text{a.e. } t \in (-h, 0). \end{cases} \tag{$\widetilde{Q}_0$}$$

It is enough to observe that we can substitute in problem (P_0), or in problem (Q_0), the term $F_0(t, u_t, u_t')$ by the term $\widetilde{F}_0(t, u_t, u_t')$, with $\widetilde{F}_0$ defined by

$$\widetilde{F}_0(t, \xi, \eta) = F_0(t, \xi, \eta) - \widetilde{A}_0(t, \xi(0)) - \widetilde{B}_0(t, \eta(0)),$$

$\forall\, \xi \in C(-h, 0; V)$, $\forall\, \eta \in C(-h, 0; H)$, a.e. $t \in (0, T)$.

3. The case with delays depending on the spatial derivatives

We consider the hypothesis of Section 2, in the particular case $W = V$ and $p = 2$.

Let $F_1 : (0,T) \times C(-h,0;V) \times C(-h,0;H) \longrightarrow V^*$ and $F_2 : (0,T) \times C(-h,0;V) \times C(-h,0;V) \longrightarrow V^*$ be two families of nonlinear operators defined a.e. $t \in (0,T)$ such that:

$(F_1.1)$ $\forall (\xi,\eta) \in C(-h,0;V) \times C(-h,0;H)$, the map $t \in (0,T) \longmapsto F_1(t,\xi,\eta) \in V^*$ is Lebesgue measurable,

$(F_1.2)$ $F_1(t,0,0) = 0$, a.e. $t \in (0,T)$,

$(F_1.3)$ $\exists C_{F_1} > 0$ such that $\forall \xi, \widetilde{\xi} \in C(-h,0;V)$, $\forall \eta, \widetilde{\eta} \in C(-h,0;H)$ and a.e. $t \in (0,T)$,

$$\|F_1(t,\xi,\eta) - F_1(t,\widetilde{\xi},\ \widetilde{\eta})\|_*^2 \le C_{F_1}\left(\|\xi - \widetilde{\xi}\|_{C(-h,0;V)}^2 + |\eta - \widetilde{\eta}|_{C(-h,0;H)}^2\right),$$

$(F_2.1)$ $\forall (\xi,\eta) \in C(-h,0;V) \times C(-h,0;V)$, the map $t \in (0,T) \longmapsto F_2(t,\xi,\eta) \in V^*$ is Lebesgue measurable,

$(F_2.2)$ $F_2(t,0,0) = 0$, a.e. $t \in (0,T)$,

$(F_2.3)$ $\exists C_{F_2} > 0$ such that $\forall \xi, \widetilde{\xi}, \eta, \widetilde{\eta} \in C(-h,0;V)$, and a.e. $t \in (0,T)$,

$$\|F_2(t,\xi,\eta) - F_2(t,\widetilde{\xi},\ \widetilde{\eta})\|_*^2 \le C_{F_2}\left(\|\xi - \widetilde{\xi}\|_{C(-h,0;V)}^2 + \|\eta - \widetilde{\eta}\|_{C(-h,0;V)}^2\right),$$

$(F_2.4)$ $\exists K_{F_2} > 0$ such that $\forall x, \widetilde{x}, y, \widetilde{y} \in C(-h,T;V)$, and $\forall t \in [0,T]$,

$$\int_0^t \|F_2(s,x_s,y_s) - F_2(s,\widetilde{x}_s,\widetilde{y}_s)\|_*^2\, ds \le K_{F_2}\int_{-h}^t \left(\|x(s) - \widetilde{x}(s)\|^2 + \|y(s) - \widetilde{y}(s)\|^2\right) ds. \quad (3.1)$$

We consider the problem

$$\begin{cases} u \in C(-h,T;V), \quad u' \in L^2(-h,T;V) \cap C(-h,T;H), \\ u''(t) + A(t)u(t) + B(t,u'(t)) = F_1(t,u_t,u_t') + F_2(t,u_t,u_t') + f(t),\ t \in (0,T), \\ u(t) = \psi(t),\ t \in [-h,0], \end{cases} \qquad (P)$$

where $f \in L^2(0,T;V^*) + L^1(0,T;H)$, and $\psi \in C(-h,0;V)$ such that $\psi' \in L^2(-h,0;V) \cap C(-h,0;H)$ are given.

Remark 3.1 If $(x,y) \in C(-h,T;V) \times C(-h,T;H)$, we deduce from $(F_1.1)$-$(F_1.3)$ that the function $F_1(t,x_t,y_t)$ belongs to $L^\infty(0,T;V^*)$. Also, by $(F_2.1)$-$(F_2.3)$, if $(x,y) \in C(-h,T;V) \times C(-h,T;V)$, the function $F_2^{(x,y)} : (0,T) \longrightarrow V^*$ defined by $F_2^{(x,y)}(t) = F_2(t,x_t,y_t)$ a.e. $t \in (0,T)$, belongs to $L^\infty(0,T;V^*)$. Then, thanks to $(F_2.4)$, the mapping

$$\Xi_2 : (x,y) \in C(-h,T;V) \times C(-h,T;V) \longmapsto F_2^{(x,y)} \in L^2(0,T;V^*)$$

has a unique extension to a mapping $\widetilde{\Xi}_2$ which is uniformly continuous from the product space $L^2(-h,T;V)\times L^2(-h,T;V)$ into $L^2(0,T;V^*)$. From now on, we will also write $F_2(t,x_t,y_t) = \widetilde{\Xi}_2(x,y)(t)$ for each $(x,y)\in L^2(-h,T;V)\times L^2(-h,T;V)$, and thus, for every $x,y,\widetilde{x},\widetilde{y}\in L^2(-h,T;V)$ and $\forall\, t\in[0,T]$ the inequality (3.1) will continue to hold.

As a consequence of the preceding remark, the terms appearing in problem (P) make sense. Now, we are interested in establishing some results on the existence and uniqueness of solution to (P) under some additional assumptions. Firstly, we can prove the following result:

Theorem 3.1 *Assume that hypothesis* $(A.1)$-$(A.4)$, $(B.1)$-$(B.5)$ *with* $W=V$ *and* $p=2$, $(F_1.1)$-$(F_1.3)$ *and* $(F_2.1)$-$(F_2.4)$ *hold. Suppose also the following condition:*

(H) $\exists\gamma>0$, $\lambda,\widehat{\lambda}\geq 0$ *such that* $\forall x,\widetilde{x},y,\widetilde{y}\in L^2(-h,T;V)$, *and* $\forall t\in[0,T]$,

$$\begin{aligned}
&2\int_0^t \mathrm{e}^{-\lambda s}\left\langle B(s,y(s))-B(s,\widetilde{y}(s)),y(s)-\widetilde{y}(s)\right\rangle ds\\
&\quad+\lambda\int_0^t \mathrm{e}^{-\lambda s}\left|y(s)-\widetilde{y}(s)\right|^2 ds+\lambda\int_0^t \mathrm{e}^{-\lambda s}\left\langle A(s)(x(s)-\widetilde{x}(s)),x(s)-\widetilde{x}(s)\right\rangle ds\\
&\quad+\widehat{\lambda}\int_{-h}^0 \mathrm{e}^{-\lambda s}\left(\|x(s)-\widetilde{x}(s)\|^2+\|y(s)-\widetilde{y}(s)\|^2\right) ds\\
&\geq\gamma\int_0^t \mathrm{e}^{-\lambda s}\|y(s)-\widetilde{y}(s)\|^2 ds\\
&\quad+2\int_0^t \mathrm{e}^{-\lambda s}\left\langle F_2(s,x_s,y_s)-F_2(s,\widetilde{x}_s,\widetilde{y}_s),y(s)-\widetilde{y}(s)\right\rangle ds. \qquad (3.2)
\end{aligned}$$

Then, for each $f\in L^2(0,T;V^*)+L^1(0,T;H)$, $\psi\in C(-h,0;V)$ *such that* $\psi'\in C(-h,0;H)\cap L^2(-h,0;V)$ *given, there exists a unique solution to the problem* (P).

Proof.

Uniqueness of solutions. Assume that u and $\widetilde{u}$ are two solutions of problem (P). Denote $v(t)=u'(t)$ and $\widetilde{v}(t)=\widetilde{u}'(t)$, $t\in[-h,T]$. Then, by Theorem 2.1 we obtain for each $t\in[0,T]$,

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\left|v(t)-\widetilde{v}(t)\right|^2+\mathrm{e}^{-\lambda t}\left\langle A(t)(u(t)-\widetilde{u}(t)),u(t)-\widetilde{u}(t)\right\rangle\\
&\quad+\lambda\int_0^t \mathrm{e}^{-\lambda s}\left|v(s)-\widetilde{v}(s)\right|^2 ds+\lambda\int_0^t \mathrm{e}^{-\lambda s}\left\langle A(s)(u(s)-\widetilde{u}(s)),u(s)-\widetilde{u}(s)\right\rangle ds\\
&=\int_0^t \mathrm{e}^{-\lambda s}\left\langle A'(s)(u(s)-\widetilde{u}(s)),u(s)-\widetilde{u}(s)\right\rangle ds\\
&\quad-2\int_0^t \mathrm{e}^{-\lambda s}\left\langle B(s,v(s))-B(s,\widetilde{v}(s)),v(s)-\widetilde{v}(s)\right\rangle ds
\end{aligned}$$

$$+\,2\int_0^t \mathrm{e}^{-\lambda s}\left\langle F_1(s,u_s,v_s)-F_1(s,\widetilde{u}_s,\widetilde{v}_s),v(s)-\widetilde{v}(s)\right\rangle\,ds$$
$$+\,2\int_0^t \mathrm{e}^{-\lambda s}\left\langle F_2(s,u_s,v_s)-F_2(s,\widetilde{u}_s,\widetilde{v}_s),v(s)-\widetilde{v}(s)\right\rangle\,ds.$$

By $(A.2)$, $(A.4)$, (H), the fact that $\mathrm{e}^{-\lambda T}\le \mathrm{e}^{-\lambda t}\le 1$, $\forall\, t\in[0,T]$, and $(F_1.3)$, we obtain

$$\begin{aligned}
&|v(t)-\widetilde{v}(t)|^2+\alpha\,\|u(t)-\widetilde{u}(t)\|^2+\gamma\int_0^t \|v(s)-\widetilde{v}(s)\|^2\,ds\\
&\le 2\mathrm{e}^{\lambda T}\int_0^t \left\langle F_1(s,u_s,v_s)-F_1(s,\widetilde{u}_s,\widetilde{v}_s),v(s)-\widetilde{v}(s)\right\rangle\,ds\\
&\le \frac{\gamma}{2}\int_0^t \|v(s)-\widetilde{v}(s)\|^2\,ds\\
&\quad+\frac{2\mathrm{e}^{2\lambda T}C^{F_1}}{\gamma}\int_0^t\left(\sup_{0\le\theta\le s}\|u(\theta)-\widetilde{u}(\theta)\|^2+\sup_{0\le\theta\le s}|v(\theta)-\widetilde{v}(\theta)|^2\right)ds.
\end{aligned}\tag{3.3}$$

¿From (3.3) we obtain that there exists $k>0$ such that for all $t\in[0,T]$

$$\begin{aligned}
&\sup_{0\le\theta\le t}|v(\theta)-\widetilde{v}(\theta)|^2+\sup_{0\le\theta\le t}\|u(\theta)-\widetilde{u}(\theta)\|^2\\
&\le k\int_0^t\left(\sup_{0\le\theta\le s}\|u(\theta)-\widetilde{u}(\theta)\|^2+\sup_{0\le\theta\le s}|v(\theta)-\widetilde{v}(\theta)|^2\right)ds,
\end{aligned}$$

and thus, uniqueness follows from Gronwall's lemma.

Existence of solutions. We will proceed in two steps.

Step 1. Firstly, we consider that $F_1\equiv 0$. We have to prove existence of solution to

$$\begin{cases}
u\in C(-h,T;V),\quad v\in C(-h,T;H)\cap L^2(-h,T;V),\\
u'(t)=v(t),\ t\in[-h,T],\\
v(t)+\displaystyle\int_0^t A(s)u(s)\,ds+\int_0^t B(s,v(s))\,ds=\psi'(0)\\
+\displaystyle\int_0^t\left(F_2(s,u_s,v_s)+f(s)\right)ds,\ t\in[0,T],\\
u(t)=\psi(t),\ t\in[-h,0].
\end{cases}\tag{$\widetilde{P}$}$$

We will use a Galerkin scheme. Let $\{w_i\}_{i\ge1}$ be a Hilbert basis of H such that $\{w_i\}_{i\ge1}\subset V$ and the subspace of V spanned by $\{w_i\}_{i\ge1}$ is dense in V.

We will denote by V_m the subspace of V spanned by $\{w_1,...,w_m\}$, by $P_m\in\mathcal{L}(H;V_m)$, the orthogonal proyection from H onto V_m, and by $\widetilde{P}_m\in\mathcal{L}(V;V_m)$, the orthogonal proyection from V onto V_m.

We consider the problem

$$\begin{cases} u^m \in L^2(-h,T;V_m)\cap C(0,T;V_m),\ v^m \in L^2(-h,T;V_m)\cap C(0,T;V_m), \\ (u^m)'(t) = v^m(t),\ t\in[0,T], \\ (v^m(t),w) + \int_0^t \langle A(s)u^m(s),w\rangle\, ds + \int_0^t \langle B(s,v^m(s)),w\rangle\, ds \\ = (P_m\psi'(0),w) + \int_0^t \langle F_2(s,u^m_s,v^m_s)+f(s),w\rangle\, ds,\ t\in[0,T],\ \forall w\in V_m, \\ u^m(t) = \widetilde{P}_m\psi(t),\ t\in[-h,0], \\ v^m(t) = \widetilde{P}_m\psi'(t),\ t\in(-h,0). \end{cases} \qquad (\widetilde{P}_m)$$

The existence and uniqueness of solution to problem $(\widetilde{P}_m)$ is guaranteed by Theorem 2.4 (notice that in this case $V = W = H = V^* = W^* = V_m$).

It is easy to obtain for $t\in[0,T]$,

$$\begin{aligned} &\mathrm{e}^{-\lambda t}\left|v^m(t)\right|^2 + \mathrm{e}^{-\lambda t}\langle A(t)u^m(t),u^m(t)\rangle + \lambda\int_0^t \mathrm{e}^{-\lambda s}\left|v^m(s)\right|^2 ds + \lambda\int_0^t \mathrm{e}^{-\lambda s}\langle A(s)u^m(s),u^m(s)\rangle\, ds \\ =&\left|P_m\psi'(0)\right|^2 + \left\langle A(0)\widetilde{P}_m\psi(0),\widetilde{P}_m\psi(0)\right\rangle + \int_0^t \mathrm{e}^{-\lambda s}\langle A'(s)u^m(s),u^m(s)\rangle\, ds \\ &- 2\int_0^t \mathrm{e}^{-\lambda s}\langle B(s,v^m(s)),v^m(s)\rangle\, ds + 2\int_0^t \mathrm{e}^{-\lambda s}\langle F_2(s,u^m_s,v^m_s)+f(s),v^m(s)\rangle\, ds. \end{aligned} \qquad (3.4)$$

By (H), $(A.2)$, $(A.4)$ and thanks to the fact that $|P_m\psi'(0)|\le|\psi'(0)|$ and $\|\widetilde{P}_m\psi(0)\|\le\|\psi(0)\|$, if we denote $a_0 = \|A(0)\|_{\mathcal{L}(V,V^*)}$, we obtain from (3.4), for all $t\in[0,T]$,

$$\begin{aligned} &\left|v^m(t)\right|^2 + \alpha\left\|u^m(t)\right\|^2 + \gamma\int_0^t \left\|v^m(s)\right\|^2\, ds \\ \le&\mathrm{e}^{\lambda T}\left|\psi'(0)\right|^2 + a_0 e^{\lambda T}\|\psi(0)\|^2 + \widehat{\lambda}\mathrm{e}^{\lambda T}\int_{-h}^0 (\|u^m(s)\|^2 + \|v^m(s)\|^2)\, ds \\ &+ 2\mathrm{e}^{\lambda T}\int_0^t |\langle f(s),v^m(s)\rangle|\, ds. \end{aligned} \qquad (3.5)$$

Observe that, if $f = f_1 + f_2$ with $f_1\in L^2(0,T;V^*)$ and $f_2\in L^1(0,T;H)$, then

$$\begin{aligned} &2\mathrm{e}^{\lambda T}\int_0^t |\langle f(s),v^m(s)\rangle|\, ds \\ \le&\frac{\gamma}{2}\int_0^t \left\|v^m(s)\right\|^2\, ds + \frac{2\mathrm{e}^{2\lambda T}}{\gamma}\int_0^T \left\|f_1(s)\right\|_*^2\, ds \\ &+ \frac{1}{6}\sup_{0\le s\le T}\left|v^m(s)\right|^2 + 6\mathrm{e}^{2\lambda T}\left(\int_0^T |f_2(s)|\, ds\right)^2. \end{aligned} \qquad (3.6)$$

As

$$\int_{-h}^0 \left\|u^m(s)\right\|^2\, ds = \int_{-h}^0 \left\|\widetilde{P}_m\psi(s)\right\|^2\, ds \le \int_{-h}^0 \left\|\psi(s)\right\|^2\, ds$$

and

$$\int_{-h}^{0} \|v^m(s)\|^2 \, ds = \int_{-h}^{0} \left\| \widetilde{P}_m \psi'(s) \right\|^2 \, ds \le \int_{-h}^{0} \|\psi'(s)\|^2 \, ds,$$

we obtain from (3.5)-(3.6) that there exists a constant $C > 0$ such that

$$|v^m(t)|^2 + \alpha \|u^m(t)\|^2 + \frac{\gamma}{2} \int_0^t \|v^m(s)\|^2 \, ds \le C, \tag{3.7}$$

for all $m \ge 1$ and all $t \in [0, T]$. Consequently, the sequence $\{u^m\}_{m\ge1}$ is bounded in $C(0, T; V)$, and $\{v^m\}_{m\ge1}$ is bounded in $C(0, T; H)$. Moreover, $\{u^m\}_{m\ge1}$ and $\{v^m\}_{m\ge1}$ are bounded in $L^2(-h, T; V)$ (observe that $\widetilde{P}_m\psi \to \psi$ and $\widetilde{P}_m\psi' \to \psi'$ in $L^2(-h, 0; V)$).

Thanks to $(B.3)$ (with $p = 2$), the sequence $\{B(\cdot, v^m(\cdot))\}$ is bounded in $L^2(0, T; V^*)$. Also, by $(F_2.2)$ and $(F_2.4)$ the sequence $\{F_2(\cdot, u^m_\cdot, v^m_\cdot)\}_{m\ge1}$ is bounded in $L^2(0, T; V^*)$.

Thus, there exist $\{u^{m_k}(\cdot)\}_{m_k\ge0} \subset \{u^m(\cdot)\}_{m\ge1}$, $\xi \in V$, $\eta \in H$, and four functions $u \in L^2(-h, T; V)$, $v \in L^2(-h, T; V)$, $\mathcal{B} \in L^2(0, T; V^*)$ and $\mathcal{F}_2 \in L^2(0, T; V^*)$ such that

$$\begin{aligned}
&u^{m_k} \rightharpoonup u \quad \text{in } L^2(-h, T; V), \text{ and in } L^\infty(0, T; V) \text{ weak star},\\
&u^{m_k}(T) \rightharpoonup \xi \quad \text{in } V,\\
&v^{m_k} \rightharpoonup v \quad \text{in } L^2(-h, T; V), \text{ and in } L^\infty(0, T; H) \text{ weak star},\\
&v^{m_k}(T) \rightharpoonup \eta \quad \text{in } H,\\
&B(\cdot, v^{m_k}(\cdot)) \rightharpoonup \mathcal{B}(\cdot) \quad \text{in } L^2(0, T; V^*),\\
&F_2(\cdot, u^{m_k}_\cdot, v^{m_k}_\cdot) \rightharpoonup \mathcal{F}_2(\cdot) \quad \text{in } L^2(0, T; V^*).
\end{aligned}$$

Observe that $A(\cdot)u^{m_k}(\cdot) \rightharpoonup A(\cdot)u(\cdot)$ and $A'(\cdot)u^{m_k}(\cdot) \rightharpoonup A'(\cdot)u(\cdot)$ in $L^2(0, T; V^*)$. Observe also that the sequence u^m converges to ψ in $L^2(-h, 0; V)$, that v^m converges to ψ' in $L^2(-h, 0; V)$, and that $v^m(0)$ converges to $\psi'(0)$ in H. Also, for each $t \in [-h, 0]$, $u^m(t)$ converges to $\psi(t)$ in V. Consequently, $u = \psi$ in $[-h, 0]$ and $v = \psi'$ in $(-h, 0)$.

Now, we prove that $u' = v$ in $(0, T)$. As $(u^{m_k})' = v^{m_k}$ in $[0, T]$, if χ is an absolutely continuous function on $[0, T]$ such that $\chi' \in L^2(0, T)$ and $\chi(T) = 0$, and we fix $w \in H$, then

$$-(\widetilde{P}_{m_k}\psi(0), w)\chi(0) = \int_0^T (v^{m_k}(s), w)\chi(s) \, ds + \int_0^T (u^{m_k}(s), w)\chi'(s) \, ds,$$

and consequently, making $k \to \infty$,

$$-(\psi(0), w)\chi(0) = \int_0^T (v(s), w)\chi(s) \, ds + \int_0^T (u(s), w)\chi'(s) \, ds \quad \forall w \in H. \tag{3.8}$$

Fix $t \in (0,T)$, and for each $n \geq 1$ such that $t + \dfrac{1}{2n} \leq T$, define

$$\chi^n(s) = \begin{cases} 1 & \text{if } \ 0 \leq s \leq t - \dfrac{1}{2n}, \\ \dfrac{1}{2} + n(t-s) & \text{if } \ t - \dfrac{1}{2n} \leq s \leq t + \dfrac{1}{2n}, \\ 0 & \text{if } \ t + \dfrac{1}{2n} \leq s \leq T. \end{cases} \tag{3.9}$$

Then, by (3.8),

$$-(\psi(0), w) = \int_0^T (v(s), w)\chi^n(s)\, ds - n \int_{t-\frac{1}{2n}}^{t+\frac{1}{2n}} (u(s), w)\, ds \quad \forall w \in H,$$

and thus, by the separability of H, making $n \to \infty$ we obtain that a.e.$t \in (0,T)$,

$$-(\psi(0), w) = \int_0^t (v(s), w)\, ds - (u(t), w) \quad \forall w \in H,$$

and consequently, a.e.$t \in (0,T)$, then $u(t) = \psi(0) + \int_0^t v(s)\, ds$. If we define

$$\widehat{u}(t) = \begin{cases} \psi(0) + \displaystyle\int_0^t v(s)\, ds, & \text{if } t \in [0,T], \\ \psi(t), & \text{if } t \in [-h,0], \end{cases}$$

then $\widehat{u} \in C(-h,T;V)$, $\widehat{u} = u$ a.e.$t \in [-h,T]$ and $\widehat{u}' = v$ in $[-h,T]$. We can thus redefine $u \equiv \widehat{u}$ and we obtain that $u \in C(-h,T;V)$, $u = \psi$ in $[-h,0]$ and $u' = v$ in $[-h,T]$. Observe also that

$$u^{m_k}(T) = \widetilde{P}_{m_k}\psi(0) + \int_0^T v^{m_k}(s)\, ds,$$

and thus, taking weak limits in V as $k \to \infty$, we obtain $\xi = \psi(0) + \int_0^T v(s)\, ds$, and consequently, $u(T) = \xi$.

On the other hand, if we continue to denote by χ an absolutely continuous real function on $[0,T]$ such that $\chi' \in L^2(0,T)$ and $\chi(T) = 0$, and fix m_j and $w \in V_{m_j}$, differentiating $(v^{m_k}(t), w)\chi(t)$ with $1 \leq m_j \leq m_k$, we get

$$\begin{aligned} -(P_{m_k}\psi'(0), w)\chi(0) = &- \int_0^T \langle A(s)u^{m_k}(s), w\rangle\, \chi(s)\, ds - \int_0^T \langle B(s, v^{m_k}(s)), w\rangle\, \chi(s)\, ds \\ &+ \int_0^T \langle F_2(s, u_s^{m_k}, v_s^{m_k}) + f(s), w\rangle\, \chi(s)\, ds + \int_0^T \langle v^{m_k}(s), w\rangle\, \chi'(s)\, ds. \end{aligned}$$

We can take limits in the last equality as $m_k \to \infty$, and observing that m_j is arbitrary and that $\cup_{m \geq 1} V_m$ is dense in V, we can ensure that

$$\begin{aligned} -(\psi'(0), w)\chi(0) = &- \int_0^T \langle A(s)u(s), w\rangle\, \chi(s)\, ds - \int_0^T \langle \mathcal{B}(s), w\rangle\, \chi(s)\, ds \\ &+ \int_0^T \langle \mathcal{F}_2(s) + f(s), w\rangle\, \chi(s)\, ds + \int_0^t \langle v, w\rangle\, \chi'(s) ds,\ \forall w \in V. \end{aligned} \tag{3.10}$$

If we fix $t \in (0,T)$ and use the functions χ^n defined by (3.9), we obtain from (3.10)

$$-(\psi'(0), w) = -\int_0^T \langle A(s)u(s), w\rangle \chi^n(s)\, ds - \int_0^T \langle \mathcal{B}(s), w\rangle \chi^n(s)\, ds + \int_0^T \langle \mathcal{F}_2(s) + f(s), w\rangle \chi^n(s)\, ds - n\int_{t-\frac{1}{2n}}^{t+\frac{1}{2n}} \langle v(s), w\rangle \, ds, \tag{3.11}$$

$\forall\, w \in V$. We can take limits in (3.11) and obtain

$$-(\psi'(0), w) = -\int_0^t \langle A(s)u(s), w\rangle \, ds - \int_0^t \langle \mathcal{B}(s), w\rangle \, ds + \int_0^t \langle \mathcal{F}_2(s) + f(s), w\rangle \, ds - \langle v(t), w\rangle, \tag{3.12}$$

a.e.$t \in (0,T)$, $\forall w \in V$. By the separability of V, we obtain from (3.12)

$$v(t) = \psi'(0) - \int_0^t A(s)u(s)\, ds - \int_0^t \mathcal{B}(s)\, ds + \int_0^t (\mathcal{F}_2(s) + f(s))\, ds, \tag{3.13}$$

(equality in V^*), a.e.$t \in (0,T)$. Thus, if we define

$$\widehat{v}(t) = \begin{cases} \psi'(0) - \int_0^t A(s)u(s)\, ds - \int_0^t \mathcal{B}(s)\, ds + \int_0^t (\mathcal{F}_2(s) + f(s))\, ds, & \text{if } t \in [0,T], \\ \psi'(t), & \text{if } t \in [-h, 0], \end{cases}$$

we have that $\widehat{v} = v$ a.e.$t \in (-h,T)$, and consequently $\widehat{v} \in L^2(-h,T;V) \cap L^\infty(0,T;H)$. Moreover, as $\psi'(0) \in H$, $A(\cdot)u(\cdot)$, $\mathcal{B}(\cdot)$, $\mathcal{F}_2(\cdot)$ and f belong to $L^2(0,T;V^*) + L^1(0,T;H)$, and $\widehat{v} = u$, with $u \in L^\infty(0,T;V)$ and $u' \in L^\infty(0,T;H)$, according to Theorem 2.1, we can assert that $\widehat{v} \in C(-h,T;H)$. Thus, we choose $\widehat{v}$ as being v, and we obtain that $v \in C(-h,T;H) \cap L^2(-h,T;V)$, and satisfies (3.13) for all $t \in [0,T]$. Finally, if $w \in V_{m_j}$ and $m_j \leq m_k$, then

$$(v^{m_k}(T), w) = (P_m\psi'(0), w) - \int_0^T \langle A(s)u^m(s), w\rangle \, ds - \int_0^T \langle B(s, v^m(s)), w\rangle \, ds + \int_0^T \langle F_2(s, u_s^m, v_s^m) + f(s), w\rangle \, ds,$$

and thus, taking weak limits in H as $k \to \infty$, and using that the vector space spanned by $\{w_j\}_{j\geq 1}$ is dense in V, we obtain

$$(\eta, w) = (\psi'(0), w) - \int_0^T \langle A(s)u(s), w\rangle \, ds - \int_0^T \langle \mathcal{B}(s), w\rangle \, ds + \int_0^T \langle \mathcal{F}_2(s) + f(s), w\rangle \, ds, \ \forall\, w \in V,$$

and consequently $v(T) = \eta$.

We have thus proved that

$$\begin{cases} u \in C(-h,T;V), \quad v \in C(-h,T;H)\cap L^2(-h,T;V),\\ v(t)=u'(t), \quad t\in[-h,T],\\ v(t)+\displaystyle\int_0^t A(s)u(s)\,ds+\int_0^t \mathcal{B}(s)\,ds=\psi'(0)+\int_0^t(\mathcal{F}_2(s)+f(s))\,ds,\ \forall\, t\in[0,T],\\ u(t)=\psi(t), \quad t\in[-h,0],\\ u(T)=\xi, \quad v(T)=\eta. \end{cases}$$

To finish with the step 1, it is enough to prove that $\mathcal{B}(t)-\mathcal{F}_2(t)=B(t,v(t))-F_2(t,u_t,v_t)$, $t\in(0,T)$. Consider X, $Y\in L^2(-h,T;V)$, such that $X=\psi$, $Y=\psi'$ a.e.$t\in(-h,0)$. Define

$$\begin{aligned} a^{m_k}=&2\int_0^T \mathrm{e}^{-\lambda t}\left\langle B(t,v^{m_k}(t))-B(t,Y(t)),v^{m_k}(t)-Y(t)\right\rangle\,dt\\ &+\lambda\int_0^T \mathrm{e}^{-\lambda t}\left\langle A(t)(u^{m_k}(t)-X(t)),u^{m_k}(t)-X(t)\right\rangle\,dt\\ &+\widehat{\lambda}\int_{-h}^0 \mathrm{e}^{-\lambda t}\left(\left\|\widetilde{P}_{m_k}\psi(t)-\psi(t)\right\|^2+\left\|\widetilde{P}_{m_k}\psi'(t)-\psi'(t)\right\|^2\right)\,dt\\ &+\lambda\int_0^T \mathrm{e}^{-\lambda t}\left|v^{m_k}(t)-Y(t)\right|^2\,dt-\gamma\int_0^T \mathrm{e}^{-\lambda t}\left\|v^{m_k}(t)-Y(t)\right\|^2\,dt\\ &-2\int_0^T \mathrm{e}^{-\lambda t}\left\langle F_2(t,u_t^{m_k},v_t^{m_k})-F_2(t,X_t,Y_t),v^{m_k}(t)-Y(t)\right\rangle\,dt\\ &-\int_0^T \mathrm{e}^{-\lambda t}\left\langle A'(t)(u^{m_k}(t)-X(t)),u^{m_k}(t)-X(t)\right\rangle\,dt. \end{aligned}$$

Then, thanks to hypothesis (H) and $(A.3)$, we can assert that $a^{m_k}\geq 0$.

Now, define

$$\begin{aligned} b^{m_k}=&2\int_0^T \mathrm{e}^{-\lambda t}\left\langle B(t,v^{m_k}(t)),v^{m_k}(t)\right\rangle\,dt+\lambda\int_0^T \mathrm{e}^{-\lambda t}\left\langle A(t)u^{m_k}(t),u^{m_k}(t)\right\rangle\,dt\\ &+\lambda\int_0^T \mathrm{e}^{-\lambda t}\left|v^{m_k}(t)\right|^2\,dt-\gamma\int_0^T \mathrm{e}^{-\lambda t}\left\|v^{m_k}(t)\right\|^2\,dt\\ &-2\int_0^T \mathrm{e}^{-\lambda s}\left\langle F_2(t,u_t^{m_k},v_t^{m_k}),v^{m_k}(t)\right\rangle-\int_0^T \mathrm{e}^{-\lambda t}\left\langle A'(t)u^{m_k}(t),u^{m_k}(t)\right\rangle\,dt. \end{aligned}$$

Then

$$\begin{aligned} \lim_{k\to\infty}(a^{m_k}-b^{m_k})=&-2\int_0^T \mathrm{e}^{-\lambda t}\left\langle \mathcal{B}(t),Y(t)\right\rangle\,dt-2\int_0^T \mathrm{e}^{-\lambda t}\left\langle B(t,Y(t)),v(t)-Y(t)\right\rangle\,dt\\ &-\lambda\int_0^T \mathrm{e}^{-\lambda t}\left\langle A(t)X(t),u(t)-X(t)\right\rangle\,dt+\int_0^T \mathrm{e}^{-\lambda t}\left\langle A'(t)u(t),X(t)\right\rangle\,dt\\ &-\lambda\int_0^T \mathrm{e}^{-\lambda t}\left\langle A(t)u(t),X(t)\right\rangle\,dt+\lambda\int_0^T \mathrm{e}^{-\lambda t}\left|Y(t)\right|^2\,dt \end{aligned}$$

$$
\begin{aligned}
&- 2\lambda \int_0^T \mathrm{e}^{-\lambda t}((v(t), Y(t)))\, dt - \gamma \int_0^T \mathrm{e}^{-\lambda t} \left\|Y(t)\right\|^2 \, dt \\
&+ 2\gamma \int_0^T \mathrm{e}^{-\lambda t}((v(t), Y(t)))\, dt + 2 \int_0^t \mathrm{e}^{-\lambda s} \left\langle \mathcal{F}_2(t), Y(t) \right\rangle \, dt \\
&+ 2 \int_0^t \mathrm{e}^{-\lambda s} \left\langle F_2(t, X_t, Y_t), v(t) - Y(t) \right\rangle \, dt \\
&+ \int_0^T \mathrm{e}^{-\lambda t} \left\langle A'(t)X(t), u(t) - X(t) \right\rangle \, dt. \qquad (3.14)
\end{aligned}
$$

On the other hand, by (3.4) written in $t = T$,

$$
\begin{aligned}
b^{m_k} =& - \mathrm{e}^{-\lambda T} \left|v^{m_k}(T)\right|^2 - \mathrm{e}^{-\lambda T} \left\langle A(T)u^{m_k}(T), u^{m_k}(T) \right\rangle \\
&+ \left|P_{m_k}\psi'(0)\right|^2 + \left\langle A(0)\widetilde{P}_{m_k}\psi(0), \widetilde{P}_{m_k}\psi(0) \right\rangle \\
&- \gamma \int_0^T \mathrm{e}^{-\lambda t} \left\|v^{m_k}(t)\right\|^2 \, dt + 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle f(t), v^{m_k}(t) \right\rangle \, dt,
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
\limsup_{k\to\infty} b^{m_k} \leq& - \mathrm{e}^{-\lambda T} \left|v(T)\right|^2 - \mathrm{e}^{-\lambda T} \left\langle A(T)u(T), u(T) \right\rangle + \left\langle A(0)\psi(0), \psi(0) \right\rangle \\
&+ \left|\psi'(0)\right|^2 - \gamma \int_0^T \mathrm{e}^{-\lambda t} \left\|v(t)\right\|^2 \, dt + 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle f(t), v(t) \right\rangle \, dt. \qquad (3.15)
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&\mathrm{e}^{-\lambda T} \left|v(T)\right|^2 + \mathrm{e}^{-\lambda T} \left\langle A(T)u(T), u(T) \right\rangle - \left|\psi'(0)\right|^2 - \left\langle A(0)\psi(0), \psi(0) \right\rangle \\
&+ \lambda \int_0^T \mathrm{e}^{-\lambda t} \left|v(t)\right|^2 \, dt + \lambda \int_0^T \mathrm{e}^{-\lambda t} \left\langle A(t)u(t), u(t) \right\rangle \, dt \\
=& \int_0^T \mathrm{e}^{-\lambda t} \left\langle A'(t)u(t), u(t) \right\rangle \, dt - 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{B}(t), v(t) \right\rangle \, dt \\
&+ 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{F}_2(t) + f(t), v(t) \right\rangle \, dt,
\end{aligned}
$$

we obtain from (3.15),

$$
\begin{aligned}
\limsup_{k\to\infty} b^{m_k} \leq& \lambda \int_0^T \mathrm{e}^{-\lambda t} \left|v(t)\right|^2 \, dt + \lambda \int_0^T \mathrm{e}^{-\lambda t} \left\langle A(t)u(t), u(t) \right\rangle \, dt \\
&- \int_0^T \mathrm{e}^{-\lambda t} \left\langle A'(t)u(t), u(t) \right\rangle \, dt - \gamma \int_0^T \mathrm{e}^{-\lambda t} \left\|v(t)\right\|^2 \, dt \\
&+ 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{B}(t), v(t) \right\rangle \, dt - 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{F}_2(t), v(t) \right\rangle \, dt. \qquad (3.16)
\end{aligned}
$$

¿From (3.14) and (3.16), we have

$$
0 \leq \limsup_{k\to\infty} a^{m_k} \leq 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{B}(t) - B(t, Y(t)), v(t) - Y(t) \right\rangle \, dt
$$

$$
\begin{aligned}
&- \int_0^T \mathrm{e}^{-\lambda t} \langle A'(t)(u(t) - X(t)), u(t) - X(t) \rangle \, dt + \lambda \int_0^T e^{-\lambda t} |v(t) - Y(t)|^2 \, dt \\
&- \gamma \int_0^T \mathrm{e}^{-\lambda t} \|v(t) - Y(t)\|^2 \, dt - 2 \int_0^T \mathrm{e}^{-\lambda t} \langle \mathcal{F}_2(t) - F_2(t, X_t, Y_t), v(t) - Y(t) \rangle \, dt \\
&+ \lambda \int_0^T \mathrm{e}^{-\lambda t} \langle A(t)(u(t) - X(t)), u(t) - X(t) \rangle \, dt. \qquad (3.17)
\end{aligned}
$$

If we take in (3.17) $X = u - \delta\widetilde{X}$, and $Y = v - \delta\widetilde{Y}$, with $\delta > 0$, $\widetilde{X}$ and $\widetilde{Y}$ in $L^2(-h, T; V)$ such that $\widetilde{X}(t) = \widetilde{Y}(t) = 0$ a.e.$t \in (-h, 0)$, we obtain

$$
\begin{aligned}
0 \leq & 2\delta \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{B}(t) - B(t, v(t) - \delta\widetilde{Y}(t)), \widetilde{Y}(t) \right\rangle dt \\
&- \delta^2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle A'(t)\widetilde{X}(t), \widetilde{X}(t) \right\rangle dt \\
&- 2\delta \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{F}_2(t) - F_2(t, u_t - \delta\widetilde{X}_t, v_t - \delta\widetilde{Y}_t), \widetilde{Y}(t) \right\rangle dt \\
&+ \lambda\delta^2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle A(t)\widetilde{X}(t), \widetilde{X}(t) \right\rangle dt + \lambda\delta^2 \int_0^T \mathrm{e}^{-\lambda t} \left|\widetilde{X}(t)\right|^2 dt. \qquad (3.18)
\end{aligned}
$$

Dividing by δ in (3.18), and letting $\delta \to 0$, we get by $(B.2)$, $(B.3)$, $(F_2.4)$ and Remark 3.1,

$$
0 \leq 2 \int_0^T \mathrm{e}^{-\lambda t} \left\langle \mathcal{B}(t) - \mathcal{F}_2(t) - B(t, v(t)) + F_2(t, u_t, v_t), \widetilde{Y}(t) \right\rangle dt. \qquad (3.19)
$$

As $\widetilde{Y}$ is an arbitrary element of $L^2(0, T; V)$, from (3.19) we obtain clearly that $\mathcal{B}(t) - \mathcal{F}_2(t) = B(t, v(t)) - F_2(t, u_t, v_t)$ as elements of $L^2(0, T; V^*)$.

Step 2. Now, we consider the problem (P) under the conditions in the theorem. We denote $u^0 \equiv v^0 \equiv 0 \in V$, and consider the sequence $\{u^n, v^n\}_{n\geq 1}$ of pairs of functions defined recursively by

$$
\begin{cases}
u^n \in C(-h, T; V), \quad v^n \in C(-h, T; H) \cap L^2(-h, T; V), \\
(u^n)'(t) = v^n(t) \; t \in [-h, T], \\
v^n(t) + \displaystyle\int_0^t A(s)u^n(s)\, ds + \int_0^t B(s, v^n(s))\, ds = \psi'(0) \\
+ \displaystyle\int_0^t (F_1(s, u_s^{n-1}, v_s^{n-1}) + F_2(s, u_s^n, v_s^n) + f(s))\, ds, \; t \in [0, T], \\
u^n(t) = \psi(t), \; t \in [-h, 0].
\end{cases} \qquad (P_n)
$$

Remember that if $u^{n-1} \in C(-h, T; V)$ and $v^{n-1} \in C(-h, T; H)$, then, $F_1(t, u_t^{n-1}, v_t^{n-1}) \in L^2(0, T; V^*)$. Consequently, by Step 1, we can ensure that problem (P_n) has a unique solution pair.

Now, we can argue as in the proof of Theorem 2.3, and prove first that $\{u^n\}_{n\geq 1}$ is a Cauchy sequence in $C(-h,T;H)$, and that $\{v^n\}_{n\geq 1}$ is a Cauchy sequence in $L^2(-h,T;V)\cap C(-h,T;H)$. Then, $\{u^n\}_{n\geq 1}$ converges in $C(-h,T;H)$ to a function u that is the solution to (P). We easily obtain by $(F_1.3)$, that there exists $k>0$ such that

$$\begin{aligned}
&\sup_{0\leq s\leq t}\left|v^{n+1}(s)-v^n(s)\right|^2+\sup_{0\leq s\leq t}\left\|u^{n+1}(s)-u^n(s)\right\|^2+\int_0^t\left\|v^{n+1}(s)-v^n(s)\right\|^2\,ds\\
\leq& k\int_0^t\sup_{0\leq\theta\leq s}\left\|u^{n+1}(s)-u^n(s)\right\|^2\,ds\\
&+k\int_0^t\left(\sup_{0\leq\theta\leq s}\left\|u^n(\theta)-u^{n-1}(\theta)\right\|^2+\sup_{0\leq\theta\leq s}\left|v^n(\theta)-v^{n-1}(\theta)\right|^2\right)\,ds,
\end{aligned}\tag{3.20}$$

for all $n\geq 1$ and all $t\in[0,T]$. In particular, if we denote

$$\chi^n(t)=\sup_{0\leq s\leq t}\left|v^n(s)-v^{n-1}(s)\right|^2+\sup_{0\leq s\leq t}\left\|u^n(s)-u^{n-1}(s)\right\|^2,$$

then

$$\chi^{n+1}(t)\leq k\int_0^t\chi^n(s)\,ds+k\int_0^t\chi^{n+1}(s)\,ds,\quad\forall\, t\in[0,T],\ \forall n\geq 1.$$

Consequently, if we fix $t\in(0,T]$, then

$$\chi^{n+1}(\theta)\leq k\int_0^t\chi^n(s)\,ds+k\int_0^\theta\chi^{n+1}(s)\,ds,\quad\forall\,\theta\in[0,t],$$

for each $n\geq 1$, and thus, by Gronwall lemma,

$$\chi^{n+1}(\theta)\leq\left(k\int_0^t\chi^n(s)\,ds\right)\mathrm{e}^{kT},\forall\theta\in[0,t].$$

In particular,

$$\chi^{n+1}(t)\leq\left(k\int_0^t\chi^n(s)\,ds\right)\mathrm{e}^{kT},\forall\, t\in[0,T],$$

for each $n\geq 1$, and from this last inequality, one easily deduces that

$$\chi^{n+1}(T)\leq\frac{(k\mathrm{e}^{kT})^n}{n!}\chi^1(T)\quad\forall n\geq 1.\tag{3.21}$$

¿From (3.20) and (3.21), and the fact that $u^n(t)=\psi(t)$ and$v^n(t)=\psi'(t)$ for all $t\in[-h,0]$ and all $n\geq 1$, we deduce that $\{u^n\}_{n\geq 1}$ is a Cauchy sequence in $C(-h,T;V)$, and $\{v^n\}_{n\geq 1}$ is a Cauchy sequence in $L^2(-h,T;V)\cap C(-h,T;H)$. Thus, there exist u and v such that $u^n\to u$ in $C(-h,T;V)$, and $v^n\to v$ in $L^2(-h,T;V)\cap C(-h,T;H)$. Now, by a similar argument to the one in the proof of Theorem 2.3, we can deduce that u is the solution of problem (P). ■

With a similar proof to that in Theorem 3.1, we can obtain the following result:

Theorem 3.2 *Assume that hypothesis* $(A.1)$-$(A.4)$, $(B.1)$-$(B.5)$ *with* $W = V$ *and* $p = 2$, $(F_1.1)$-$(F_1.3)$, $(F_2.1)$-$(F_2.4)$, *and* (H) *hold. Suppose also given a family of operators* $\widetilde{F}_1 : (0,T) \times C(-h,0;V) \times C(-h,0;V) \longrightarrow H$, *such that:*

$(\widetilde{F}_1.1)$ $\forall(\xi,\eta) \in C(-h,0;V) \times C(-h,0;V)$, *the map* $t \in (0,T) \longmapsto \widetilde{F}_1(t,\xi,\eta) \in H$ *is Lebesgue measurable,*

$(\widetilde{F}_1.2)$ *a.e.* $t \in (0,T)$, *the map* $(\xi,\eta) \in C(-h,0;V) \times C(-h,0;V) \longmapsto \widetilde{F}_1(t,\xi,\eta) \in H$ *is linear,*

$(\widetilde{F}_1.3)$ *there exists* $C_{\widetilde{F}_1} > 0$ *such that* $\forall\, \xi,\eta \in C(-h,0;V)$ *and a.e.*$t \in (0,T)$,

$$|\widetilde{F}_1(t,\xi,\eta)|^2 \le C_{\widetilde{F}_1}\left(\|\xi\|^2_{C(-h,0;V)} + \|\eta\|^2_{C(-h,0;V)}\right),$$

$(\widetilde{F}_1.4)$ *there exists* $K_{\widetilde{F}_1} > 0$ *such that* $\forall x,y \in C(-h,T;V)$, *and* $\forall t \in [0,T]$,

$$\int_0^t \left|\widetilde{F}_1(s,x_s,y_s)\right|^2 ds \le K_{\widetilde{F}_1}\int_{-h}^t \left(\|x(s)\|^2 + \|y(s)\|^2\right) ds.$$

Then, for each $f \in L^2(0,T;V^*) + L^1(0,T;H)$, $u_0 \in V$, $v_0 \in H$, *and* $\psi \in C(-h,0;V)$ *such that* $\psi' \in C(-h,0;H) \cap L^2(-h,0;V)$ *given, there exists a unique solution to the problem*

$$\begin{cases} u \in C(-h,T;V), \quad u' \in L^2(-h,T;V) \cap C(-h,T;H), \\ u''(t) + A(t)u(t) + B(t,u'(t)) = \psi'(0) + F_1(t,u_t,u'_t) \\ +\widetilde{F}_1(t,u_t,u'_t) + F_2(t,u_t,u'_t) + f(t),\ t \in (0,T), \\ u(t) = \psi(t),\ t \in [-h,0]. \end{cases}$$

Remark 3.2 Suppose now that F_1 satisfies

$(F_1.4)$ $\exists K_{F_1} > 0$ such that $\forall x,\tilde{x} \in C(-h,T;V)$, $\forall y,\tilde{y} \in C(-h,T;H)$, and $\forall t \in [0,T]$,

$$\int_0^t \|F_1(s,x_s,y_s) - F_1(s,\widetilde{x}_s,\widetilde{y}_s)\|_*^2\, ds \le K_{F_1}\int_{-h}^t \left(\|x(s)-\widetilde{x}(s)\|^2 + |y(s)-\widetilde{y}(s)|^2\right) ds.$$

Then, reasoning as in the proof of theorem 3.3 above, one can obtain the following result:

Theorem 3.3 *Assume that hypothesis* $(A.1)$-$(A.4)$, $(B.1)$-$(B.5)$ *with* $W = V$ *and* $p = 2$, $(F_1.1)$-$(F_1.4)$, $(\widetilde{F}_1.1)$-$(\widetilde{F}_1.4)$, $(F_2.1)$-$(F_2.4)$ *and* (H) *hold. Then, for* $f \in L^2(0,T;V^*) +$

$L^1(0,T;H)$, $u_0 \in V$, $v_0 \in H$, $\psi \in L^2(-h,0;V)$ *and* $\phi \in L^2(-h,0;V)$ *given, there exists a unique solution to the problem*

$$\begin{cases} u \in L^2(-h,T;V) \cap C(0,T;V), \quad v \in L^2(-h,T;V) \cap C(0,T;H), \\ u'(t) = v(t),\ t \in [0,T], \\ v'(t) + A(t)u(t) + B(t,v(t)) = v_0 + F_1(t,u_t,v_t) \\ +\widetilde{F}_1(t,u_t,v_t) + F_2(t,u_t,v_t) + f(t),\ t \in (0,T), \\ u(0) = u_0, \\ u(t) = \psi(t), \quad v(t) = \phi(t),\ \text{a.e. } t \in (-h,0). \end{cases}$$

Remark 3.3 Consider given $\widetilde{A}(t,\cdot) : V \longrightarrow V^*$ and $\widetilde{B}(t,\cdot) : H \longrightarrow V^*$, two families of nonlinear operators defined a.e. $t \in (0,T)$ and satisfying:

($\widetilde{A}$.1) $\forall u \in V$, the map $t \in (0,T) \longmapsto \widetilde{A}(t,u) \in V^*$ is Lebesgue measurable,

($\widetilde{A}$.2) $\widetilde{A}(t,0) = 0$, a.e. $t \in (0,T)$.

($\widetilde{A}$.3) $\exists L_{\widetilde{A}} > 0$ such that $\|\widetilde{A}(t,u) - \widetilde{A}(t,\widetilde{u})\|_* \le L_{\widetilde{A}} \|u - \widetilde{u}\| \quad \forall\, u, \widetilde{u} \in V$, a.e. $t \in (0,T)$,

($\widetilde{B}$.1) $\forall v \in H$, the map $t \in (0,T) \longmapsto \widetilde{B}(t,v) \in V^*$ is Lebesgue measurable,

($\widetilde{B}$.2) $\widetilde{B}(t,0) = 0$, a.e. $t \in (0,T)$.

($\widetilde{B}$.3) $\exists L_{\widetilde{B}} > 0$ such that $\|\widetilde{B}(t,v) - \widetilde{B}(t,\widetilde{v})\|_* \le L_{\widetilde{B}} |v - \widetilde{v}| \quad \forall\, v, \widetilde{v} \in H$, a.e. $t \in (0,T)$.

Suppose also given $\widehat{B}(\cdot) \in L^\infty(0,T;\mathcal{L}(V;H))$.

Then, as in Remark 2.3, we can assert, under the conditions of Theorem 3.2, existence and uniqueness of solution to the problem

$$\begin{cases} u \in C(-h,T;V), \quad u' \in L^2(-h,T;V) \cap C(-h,T;H), \\ u''(t) + A(t)u(t) + \widetilde{A}(t,u(t)) + B(s,u'(t)) + \widetilde{B}(t,u'(t)) + \widehat{B}(t)u'(t) \\ = \psi'(0) + F_1(t,u_t,u'_t) + \widetilde{F}_1(t,u_t,u'_t) + F_2(t,u_t,u'_t) + f(t),\ t \in (0,T), \\ u(t) = \psi(t),\ t \in [-h,0], \end{cases} \tag{R}$$

Also, under the conditions of Theorem 3.3, we can assert existence and uniqueness of solution to the problem

$$\begin{cases} u \in L^2(-h,T;V) \cap C(0,T;V), \quad v \in L^2(-h,T;V) \cap C(0,T;H), \\ u'(t) = v(t),\ t \in [0,T], \\ v'(t) + A(t)u(t) + \widetilde{A}(t,u(t)) + B(t,v(t)) + \widetilde{B}(t,v(t)) + \widehat{B}(t)v(t) = \\ v_0 + F_1(t,u_t,v_t) + \widetilde{F}_1(t,u_t,v_t) + F_2(t,u_t,v_t) + f(t),\ t \in (0,T), \\ u(0) = u_0, \\ u(t) = \psi(t), \quad v(t) = \phi(t),\ \text{a.e. } t \in (-h,0). \end{cases}$$

4. Examples

To illustrate our theory, we shall consider two examples.

Example 1. Assume $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set with smooth boundary $\partial\mathcal{O}$. Let us set $H = L^2(\mathcal{O})$ and $V = H^1(\mathcal{O})$. Let $A(t) = -\Delta$ for all $t \in (0, T)$.

Let $\widetilde{h} : [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $\widetilde{h}(t, 0, 0) = 0$ for all $t \in [0, T]$, and there exists $L_{\widetilde{h}} > 0$ such that

$$|\widetilde{h}(t, a, y) - \widetilde{h}(t, \widetilde{a}, \widetilde{y})| \le L_{\widetilde{h}}(|a - \widetilde{a}| + |y - \widetilde{y}|),$$

for all $(a, y), (\widetilde{a}, \widetilde{y}) \in \mathbb{R} \times \mathbb{R}^n$, and all $t \in [0, T]$.

For each $w \in H^1(\mathcal{O})$, and $t \in [0, T]$, denote by $\widetilde{A}_0(t, w)$ the element of $L^2(\mathcal{O})$ defined by $\widetilde{A}_0(t, w)(x) = \widetilde{h}(t, w(x), \nabla w(x))$, a.e. $x \in \mathcal{O}$.

Let $k : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $(k(t, a) - k(t, \widetilde{a}))(a - \widetilde{a}) \ge 0$ $\forall\, a, \widetilde{a} \in \mathbb{R}$, $\forall\, t \in [0, T]$, and such that there exist $p > 1$, $c_k > 0$ and $\beta > 0$ satisfying

$$|k(t, a)| \le c_k |a|^{p-1}, \quad k(t, a)a \ge \beta |a|^p \quad \forall\, a \in \mathbb{R}, \quad \forall\, t \in [0, T].$$

A classical example of such a function is $k(t, a) = \beta a^3$, for $p = 4$.

Given $w \in L^p(\mathcal{O})$, denote, for $t \in [0, T]$, by $B(t, w)$ the function of $L^{p/p-1}(\mathcal{O})$ defined by

$$B(t, w)(x) = k(t, w(x)), \quad \text{a.e. } x \in \mathcal{O}.$$

Consider given $\widetilde{k} : [0, T] \times \mathbb{R} \to \mathbb{R}$, a measurable function such that $\widetilde{k}(t, 0) = 0$ for all $t \in [0, T]$, and there exists $L_{\widetilde{k}} > 0$ such that

$$|\widetilde{k}(t, a) - \widetilde{k}(t, \widetilde{a})| \le L_{\widetilde{k}} |a - \widetilde{a}|,$$

for all $a, \widetilde{a} \in \mathbb{R}$ and $t \in [0, T]$, and denote, for $w \in L^2(\mathcal{O})$, $t \in [0, T]$, by $\widetilde{B}_0(t, w)$ the functions of $L^2(\mathcal{O})$ defined by $\widetilde{B}_0(t, w)(x) = \widetilde{k}(t, w(x))$, a.e. $x \in \mathcal{O}$.

Let us consider now a measurable map, $f_0 : [0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and three measurable functions $\omega_i : [0, T] \to \mathbb{R}$, $i = 1, 2, 3$, such that $0 \le \omega_i(t) \le h$ for all $t \in [0, T]$. Suppose that $f_0(t, 0, 0, 0) = 0$, $\forall\, t \in [0, T]$, and that there exists $L > 0$ such that

$$|f_0(t, a, y, b) - f_0(t, \widetilde{a}, \widetilde{y}, \widetilde{b})| \le L(|a - \widetilde{a}| + |y - \widetilde{y}| + |b - \widetilde{b}|),$$

$\forall\, t \in [0,T],\ \forall\, a, \widetilde{a}, b, \widetilde{b} \in \mathbb{R},\ \forall\, y, \widetilde{y} \in \mathbb{R}^n$.

For each $(t,\xi,\eta) \in [0,T] \times C(-h,0;V) \times C(-h,0;H)$, denote by $F_0(t,\xi,\eta)$ the function of $L^2(\mathcal{O})$ defined by

$$F_0(t,\xi,\eta)(x) = f_0(t, \xi(-\omega_1(t))(x), \nabla\xi(-\omega_2(t))(x), \eta(-\omega_3(t))(x)).$$

Then, all the conditions in Theorem 2.3 and Remark 2.3 are satisfied, and consequently, for each $f \in L^{p/p-1}(\mathcal{O} \times (0,T)) + L^1(0,T;L^2(\mathcal{O}))$, and $\psi \in C(-h,0;H^1(\mathcal{O}))$ such that $\psi' \in C(-h,0;L^2(\mathcal{O}))$ we can assert the existence and uniqueness of a solution $u \in C(-h,T;H^1(\mathcal{O}))$, such that $\dfrac{\partial u}{\partial t} \in C(-h,T;L^2(\mathcal{O})) \cap L^p(\mathcal{O} \times (0,T))$, of the corresponding problem $(\widetilde{P}_0)$. This solution can be seen as generalized solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u(t) + \widetilde{h}(t,u(t),\nabla u(t)) + k(t,\dfrac{\partial u}{\partial t}(t)) + \widetilde{k}(t,\dfrac{\partial u}{\partial t}(t)) = \\ f_0(t, u(t-\omega_1(t)), \nabla u(t-\omega_2(t)), \dfrac{\partial u}{\partial t}(t-\omega_3(t))) + f(t), \quad \text{in } \mathcal{O} \times (0,T), \\ \dfrac{\partial u}{\partial \vec{n}} = 0, \quad \text{in } \partial\mathcal{O} \times (0,T), \\ u(t) = \psi(t), \quad \text{in } \mathcal{O} \times [-h,0], \end{cases}$$

where we denote by $\vec{n}$ the outward unit normal to $\partial\mathcal{O}$.

If the functions ω_i are such that for $i = 1,2,3$, $\omega_i \in C^1([0,T])$ and $\max_{t\in[0,T]} \omega_i'(t) < 1$, the F_0 satisfies the condition $(F_0.4)$, and consequently, for each $u_0 \in H^1(\mathcal{O})$, $v_0 \in L^2(\mathcal{O})$, $f \in L^{p/p-1}(\mathcal{O}\times(0,T)) + L^1(0,T;L^2(\mathcal{O}))$, $\psi \in L^2(-h,0;H^1(\mathcal{O}))$ and $\phi \in L^2((-h,0)\times\mathcal{O})$ we can also assert the existence and uniqueness of a solution $u \in L^2(-h,T;H^1(\mathcal{O})) \cap C(0,T;H^1(\mathcal{O}))$, such that $\dfrac{\partial u}{\partial t} \in L^2((-h,T)\times\mathcal{O}) \cap L^p(\mathcal{O}\times(0,T)) \cap C(0,T;L^2(\mathcal{O}))$, of the corresponding problem $(\widetilde{Q}_0)$. Now, this solution can be seen as generalized solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u(t) + \widetilde{h}(t,u(t),\nabla u(t)) + k(t,\dfrac{\partial u}{\partial t}(t)) + \widetilde{k}(t,\dfrac{\partial u}{\partial t}(t)) = \\ f_0(t, u(t-\omega_1(t)), \nabla u(t-\omega_2(t)), \dfrac{\partial u}{\partial t}(t-\omega_3(t))) + f(t), \quad \text{in } \mathcal{O} \times (0,T), \\ \dfrac{\partial u}{\partial \vec{n}} = 0, \quad \text{in } \partial\mathcal{O} \times (0,T), \\ u(0) = u_0, \quad u'(0) = v_0, \quad \text{in } \mathcal{O}, \\ u(t) = \psi(t), \quad u'(t) = \phi(t), \quad \text{in } \mathcal{O} \times (-h,0). \end{cases}$$

Example 2. Assume $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set. Let us set $H = L^2(\mathcal{O})$, $V = H_0^1(\mathcal{O})$ and $V^* = H^{-1}(\mathcal{O})$. Let $A(t)w = -\Delta w + w$ for all $w \in H_0^1(\mathcal{O})$ and all $t \in [0,T]$.

Let $\widetilde{h} : [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function such that $\widetilde{h}(t,0,0) = 0$ for all $t \in [0,T]$, and there exists $L_{\widetilde{h}} > 0$ such that

$$|\widetilde{h}(t,a,y) - \widetilde{h}(t,\widetilde{a},\widetilde{y})| \leq L_{\widetilde{h}}(|a-\widetilde{a}| + |y-\widetilde{y}|),$$

for all $(a,y), (\widetilde{a},\widetilde{y}) \in \mathbb{R} \times \mathbb{R}^n$, and all $t \in [0,T]$.

For each $w \in H_0^1(\mathcal{O})$, and $t \in [0,T]$, denote by $\widetilde{A}(t,w)$ the element of $H^{-1}(\mathcal{O})$ defined by

$$\left\langle \widetilde{A}(t,w), v \right\rangle = \int_{\mathcal{O}} \widetilde{h}(t, w(x), \nabla w(x)) \cdot \nabla v(x)\, dx - \int_{\mathcal{O}} w(x)v(x)\, dx \quad \forall\, v \in H_0^1(\mathcal{O}),$$

where we denote by $\cdot$ the escalar product in $\mathbb{R}^n$.

For each $w \in L^2(\mathcal{O})$, and $t \in [0,T]$, denote by $\widehat{B}(t,w)$ the element of $L^2(\mathcal{O})$ defined by $\widehat{B}(t,w)(x) = -w(x)$, a.e. $x \in \mathcal{O}$.

Consider also given $k : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, a continuous function such that $k(t,0) = 0$ for all $t \in [0,T]$, there exists $c > 0$ such that $|k(t,y)| \leq c|y|$ for all $y \in \mathbb{R}^n$ and all $t \in [0,T]$, and

$$(k(t,y) - k(t,\widetilde{y})) \cdot (y - \widetilde{y}) \geq 0, \quad \forall\, y, \widetilde{y} \in \mathbb{R}^n,\ \forall\, t \in [0,T].$$

For each $w \in H_0^1(\mathcal{O})$, and $t \in [0,T]$, denote by $B(t,w)$ the element of $H^{-1}(\mathcal{O})$ defined by

$$\langle B(t,w), v \rangle = \beta \int_{\mathcal{O}} \nabla w(x) \cdot \nabla v(x)\, dx + \int_{\mathcal{O}} w(x)v(x)\, dx + \int_{\mathcal{O}} k(t, \nabla w(x)) \cdot \nabla v(x)\, dx,$$

$\forall\, v \in H_0^1(\mathcal{O})$, with $\beta > 0$ fixed.

Let now n measurable maps $f_{1_j} : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $3n$ measurable functions $\rho_{i_j} : [0,T] \to \mathbb{R}$, $i = 1,2,3$, $j = 1, ..., n$, such that for each (i,j), $0 \leq \rho_{i_j}(t) \leq h$ for all $t \in [0,T]$. Suppose that $f_{1_j}(t,0,0,0) = 0$, $\forall\, t \in [0,T]$ for all $j = 1, ..., n$, and that for each j, there exists $L_{f_{1_j}} > 0$ such that, $\forall\, a, \widetilde{a}, b, \widetilde{b} \in \mathbb{R}$, $\forall\, y, \widetilde{y} \in \mathbb{R}^n$, $\forall\, t \in [0,T]$,

$$|f_{1_j}(t,a,y,b) - f_{1_j}(t,\widetilde{a},\widetilde{y},\widetilde{b})| \leq L_{f_{1_j}}(|a-\widetilde{a}| + |y-\widetilde{y}| + |b-\widetilde{b}|).$$

Denote by $F_1(t,\cdot,\cdot)$ the family of operators defined by

$$\langle F_1(t,\xi,\eta), v \rangle = -\sum_{j=1}^{n} \int_{\mathcal{O}} f_{1_j}(t, \xi(-\rho_{1_j}(t))(x), \nabla\xi(-\rho_{2_j}(t))(x), \eta(-\rho_{3_j}(t))(x)) \frac{\partial v}{\partial x_j}(x)\, dx,$$

$\forall\, (\xi,\eta) \in C(-h,0;V) \times C(-h,0;H)$, $\forall\, v \in V$, for each $t \in [0,T]$.

Consider now n measurable maps $f_{2_j} : [0,T]\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, and $4n$ functions $\tau_{i_j} : [0,T] \to \mathbb{R}$, $i = 1,2,3,4$, $j = 1,...,n$, such that for each (i,j), $\tau_{i_j} \in C^1([0,T])$, $0 \le \tau_{i_j}(t) \le h$ for all $t \in [0,T]$, and $\tau_j^* = \max_{1\le i\le 4}(\max_{t\in[0,T]} \tau_{i_j}'(t)) < 1$. Suppose that $f_{2_j}(t,0,0,0,0) = 0,\ \forall\, t \in [0,T]$ for all $j = 1,...,n$, and that for each j, there exists $L_{f_{2_j}} > 0$ such that

$$|f_{2_j}(t,a,y,b) - f_{2_j}(t,\widetilde{a},\widetilde{y},\widetilde{b})|^2 \le L_{f_{2_j}}(|a-\widetilde{a}|^2 + |y-\widetilde{y}|^2 + |b-\widetilde{b}|^2 + |z-\widetilde{z}|^2),$$

$\forall\, a,\widetilde{a},b,\widetilde{b} \in \mathbb{R},\ \forall\, y,\widetilde{y},z,\widetilde{z} \in \mathbb{R}^n,\ \forall\, t \in [0,T]$.

Denote by $F_2(t,\cdot,\cdot)$ the family of operators defined by

$$\langle F_2(t,\xi,\eta),v\rangle =$$

$$-\sum_{j=1}^n \int_{\mathcal{O}} f_{2_j}(t,\xi(-\tau_{1_j}(t))(x),\nabla\xi(-\tau_{2_j}(t))(x),\eta(-\tau_{3_j}(t))(x),\nabla\eta(-\tau_{4_j}(t))(x))\frac{\partial v}{\partial x_j}(x)\,dx,$$

$\forall\,(\xi,\eta) \in C(-h,0;V)\times C(-h,0;V),\ \forall\, v \in V$, for each $t \in [0,T]$.

Then, if it is true one of the following conditions

$$2\sum_{j=1}^n \left(\frac{L_{f_{2_j}}}{1-\tau_j^*}\right)^{\frac{1}{2}} \le \beta \mathrm{e}^{-\beta T}, \quad 2\sum_{j=1}^n \left(\frac{L_{f_{2_j}}}{1-\tau_j^*}\right)^{\frac{1}{2}} < \mathrm{e}^{-1}\max(2\beta,T^{-1}),$$

all the hypothesis in Theorem 3.3 and Remark 3.3 are satisfied, and consequently, for each $f \in L^2(0,T;H^{-1}(\mathcal{O}))$, and $\psi \in C(-h,0;H_0^1(\mathcal{O}))$ such that $\psi' \in C(-h,0;L^2(\mathcal{O}))\cap L^2(0,T;H_0^1(\mathcal{O}))$, we can assert the existence and uniqueness of a solution $u \in C(-h,T;H_0^1(\mathcal{O}))$, such that $\dfrac{\partial u}{\partial t} \in C(-h,T;L^2(\mathcal{O}))\cap L^2(-h,T;H_0^1(\mathcal{O}))$, of the corresponding problem (R). This solution can be seen as generalized solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u(t) - \nabla\cdot\widetilde{h}(t,u(t),\nabla u(t)) - \beta\Delta\left(\dfrac{\partial u}{\partial t}(t)\right) - \nabla\cdot k(t,\nabla(\dfrac{\partial u}{\partial t}(t))) = \\ \displaystyle\sum_{j=1}^n \frac{\partial f_{2_j}}{\partial x_j}(t,u(t-\tau_{1_j}(t)),\nabla u(t-\tau_{2_j}(t)),\frac{\partial u}{\partial t}(t-\tau_{3_j}(t)),\nabla(\frac{\partial u}{\partial t}(t-\tau_{4_j}(t)))) + f(t) \\ \displaystyle+\sum_{j=1}^n \frac{\partial f_{1_j}}{\partial x_j}(t,u(t-\rho_{1_j}(t)),\nabla u(t-\rho_{2_j}(t)),\frac{\partial u}{\partial t}(t-\rho_{3_j}(t))), \quad \text{in } \mathcal{O}\times(0,T), \\ u = 0, \quad \text{in } \partial\mathcal{O}\times(0,T), \\ u(t) = \psi(t), \quad \text{in } \mathcal{O}\times[-h,0]. \end{cases}$$

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References

1 M. Artola, Sur les pertubations des équations d'évolution, application à des problèmes de retard, *Ann. Sci. Ecole Norm. Sup.* 4 **2** (1969) 137-253.

2 M. Artola, Équations et inéquations variationnelles à retard, *Ann. Sc. Math. Québec* vol. I, no. 2 (1977), 131-152

3 C. Baiocchi, Sulle equazioni differenziali astratte lineari del primo et del secondo ordine negli spazi di Hilbert, *Ann. Mat. Pura Appl.* (4), **76** (1967), 233-304.

4 A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, "Representation and Control of Infinite Dimensional Systems," Vol I, "Systems and Control: Foundations and Applications," Birkhäuser, Boston, 1992.

5 A. G. Kartsatos and L. P. Markov, An L_2-approach to second-order nonlinear functional evolutions involving m-accretive operators in Banach spaces, *Diff. Integral Eqns.* **14** No. 7 (2001), 833-866.

6 J. L. Lions and W. A. Strauss, Some non-linear evolution equations, *Bull. Soc. Math. France* **93** (1965), 43-96.

7 W. Ruess, Existence of solutions to partial functional differential equations with delay, in "Theory and Applications of Nonlinear Operators of Accretive and Monotone Type," (A.G. Kartsatos, Eds.), pp 259-288, Dekker, New York (1996).

8 W. Ruess, Existence and stability of solutions to partial functional differential equations with delay, *Adv. Differential Equations* 4, No. 6 (1999), 843-876.

9 W. A. Strauss, On continuity of functions with values in various Banach spaces, *Pacific J. Math.* **19** No. 3 (1966), 543-551.

10 C. C. Travis and G. F. Webb, Existence and stability of solutions to partial functional differential equations, *Trans. Amer. Math. Soc.* **200** (1974), 395-418.

11 C. C. Travis and G. F. Webb, Existence, stability and compactness in the α-norm for partial functional differential equations, *Trans. Amer. Math. Soc.* **240** (1978), 129-143.

12 J. Wu, "Theory and applications of partial functional differential equations", Springer-Verlag, New York, 1996.