WEIGHTED NORM INEQUALITIES
FOR SINGULAR INTEGRAL
OPERATORS

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Abstract

For a Calderón–Zygmund singular integral operator $T$, we show that the following weighted inequality holds

$$
\int_{\mathbb{R}^n} |Tf(y)|^p \, w(y) \, dy \leq C \int_{\mathbb{R}^n} |f(y)|^p \, M^{[p]+1} w(y) \, dy,
$$

where $M^k$ is the Hardy–Littlewood maximal operator $M$ iterated $k$ times, and $[p]$ is the integer part of $p$. Moreover, the result is sharp since it does not hold for $M^{[p]}$.

We also give the following endpoint result:

$$
w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, M^2 w(y) \, dy.
$$
1 Introduction and statements of the results

A classical result due to C. Fefferman and E. Stein [4] states that the Hardy–Littlewood maximal operator $M$ satisfies the following inequality for arbitrary $1 < p < \infty$, and weight $w$

$$\int_{\mathbb{R}^n} |Mf(y)|^p w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p Mw(y)dy,$$

(1)

where $C$ is independent of $f$. A weight $w$ in $\mathbb{R}^n$ will always be a nonnegative locally integrable function.

The study of weighted inequalities like the above, for other operators has played a central rôlle in modern Harmonic Analysis since they appear in duality arguments. We refer the reader to [5] Chapters 5 and 6 for a very nice exposition.

Although we could work with any Calderón–Zygmund operator (cf. §3), we shall only consider singular integral operators of convolution type defined by:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x - y)f(y)dy,$$

where the kernel $k$ is $C^1$ away from the origin, has mean value on the unit sphere centered at the origin and satisfies for $y \neq 0$

$$|k(y)| \leq \frac{C}{|y|^n} \quad \text{and} \quad |\nabla k(y)| \leq \frac{C}{|y|^{n+1}}.$$

It is well known that the analogous version of inequality (1) fails for the Hilbert transform for all $p$. In [3] A. Córdoba and C. Fefferman have shown that there is a similar inequality for any $T$, but with $Mw$ replaced by the pointwise larger operator $M_r w = M(w^r)^{1/r}$, $r > 1$, that is, for $1 < p < \infty$

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y)dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M_r w(y)dy,$$

(2)

with $C$ independent of $f$.

The purpose of this paper is to prove weighted norm inequalities of the form (2), where $M_r w$, $r > 1$, will be replaced by appropriate smaller maximal–type operators $w \rightarrow Nw$ satisfying

$$Mw(x) \leq Nw(x) \leq C M_r w(x),$$

(3)
for each $x \in \mathbb{R}^n$. We shall also be concern with corresponding endpoints results such as weak type $(1, 1)$ and $H^1-L^1$ estimates.

Before stating our main results, we shall make the following observation. Let $M^k$ be the Hardy–Littlewood maximal operator $M$ iterated $k$ times, where $k = 1, 2, \cdots$. We claim that for $k = 2, \cdots$, and $r > 1$, there exists a positive constant $C$ independent of $w$ such that

$$Mw(x) \leq M^k w(x) \leq CM_r w(x),$$

for each $x \in \mathbb{R}^n$. The left inequality follows from the Lebesgue differentiation theorem; for the other, we let $B$ be the best constant in Coifman’s estimate $M(M_r w) \leq BM_r w$, where $B$ is independent of $w$. Then, it follows easily that $M^k w \leq B^{k-1} M_r w$, $k = 1, 2, \cdots$.

In view of this observation, it is natural to consider whether or not (2) holds for some $M^k$, with $k = 2, 3, \cdots$. In a very interesting paper [8], M. Wilson has recently obtained the following partial answer to this question: Let $1 < p < 2$, then

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M^2 w(y) dy.$$  \hfill(5)

Moreover, he shows that this estimate does not hold for $p \geq 2$, and also that when $p = 2$, $M^2 w$ can be replaced by $M^3 w$. However, his method does not yield corresponding estimates for $p > 2$ (cf. §3 of that paper), and $M^2 w$ must be replaced by a much more complicated expression.

M. Wilson’s approach to this problem is based on certain (difficult) estimates for square functions that he obtained in the same paper, together with a couple of related estimates for the area function, obtained essentially by S. Chanillo and R. Wheeden in [1].

In this paper we give a complete answer to Wilson’s problem by means of a different method. Our main result is the following.

**Theorem 1.1:** Let $1 < p < \infty$, and let $T$ be a singular integral operator. Then, there exists a constant $C$ such that for each weight $w$

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y) dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M^{[p]+1} w(y) dy,$$  \hfill(6)

where $[p]$ is the integer part of $p$. Furthermore, the result is sharp since it does not hold for $M^{|p|}$.

The corresponding weak–type $(1, 1)$ version of this result is the following.
Theorem 1.2: Let $T$ be a singular integral operator. Then, there exists a constant $C$ such that for each weight $w$ and for all $\lambda > 0$

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M^2 w(y) dy. \quad (7)$$

Remark 1.3: Let $1 < p < \infty$, a natural question is whether (7) can be extended to the case $(p,p)$, that is whether

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p M^p w(y) dy,$$

holds for some constant $C$ and for all $\lambda > 0$. At the end of section 2 we give an example showing that this inequality is false when $p$ is not an integer; however, we do not know what happens when $p$ is an integer.

Although we do not know whether (7) holds for $Mw$ (cf. remark 1.7) we can give the following estimate. For a measure $\mu$ we shall denote by $H^1(\mu)$ the subspace of $L^1(\mu)$ of functions $f$ which can be written as $f = \sum_j \lambda_j a_j$, where $a_j$ are $\mu$–atoms and $\lambda_j$ are complex numbers with $\sum_j |\lambda_j| < \infty$. A function $a$ is a $\mu$–atom if there is a cube $Q$ for which $supp(a) \subset Q$, so that

$$|a(x)| \leq \frac{1}{\mu(Q)},$$

and

$$\int_Q a(y) dy = 0.$$

Theorem 1.4: Let $T$ be a singular integral operator. Then, there exists a constant $C$ such that for each weight $w$

$$\int_{\mathbb{R}^n} |Tf(y)| w(y) dy \leq C \|f\|_{H^1(Mw)}.$$

(8)
for each $x \in \mathbb{R}^n$, where $r > 1$. $A$ stands for a Young function; i.e. $A : [0, \infty) \to [0, \infty)$ is continuous, convex and increasing satisfying $A(0) = 0$. To define $M_A$ we introduce for each cube $Q$ the $A$–average of a function $f$ over $Q$ by means of the following Luxemburg norm

$$\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f(y)|}{\lambda}\right) \, dy \leq 1\}.$$ 

We define the maximal operator $M_A$ by

$$M_A f(x) = \sup_{x \in Q} \|f\|_{A,Q},$$

where $f$ is a locally integrable functions, and where the supremum is taken over all the cubes containing $x$. When $A(t) = t^r$ we get $M_A = M_r$, but more interesting examples are provided by Young functions like $A(t) = t \log(1 + t)$, $\epsilon > 0$.

The optimal class of Young functions $A$ is characterized by the following theorem.

**Theorem 1.5:** Let $1 < p < \infty$, and let $T$ be a singular integral operator. Suppose that $A$ is a Young function satisfying the condition

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p' - 1} \frac{dt}{t} < \infty,$$  \hspace{1cm} (9)

for some $c > 0$. Then, there exists a constant $C$ such that for each weight $w$

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y) \, dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M_A w(y) \, dy.$$  \hspace{1cm} (10)

Furthermore, condition (9) is also necessary for (10) to hold for all the Riesz transforms: $T = R_1, R_2, \cdots, R_n$.

We recall that the $j$–th Riesz transform $R_j$, $j = 1, 2, \cdots, n$, is the singular integral operator defined by

$$R_j f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) \, dy.$$ 

The proof of this theorem is given in §2, and it is based on the following inequality of E.M. Stein [7]

$$\int_Q w(y) \log^k(1 + w(y)) \, dy \leq C \int_Q Mw(y) \log^{k-1}(1 + Mw(y)) \, dy,$$  \hspace{1cm} (11)
with \( k = 1, 2, 3, \ldots \).

As for the strong case, there is an estimate sharper than (7).

**Theorem 1.6:** Let \( T \) be a singular integral operator. For arbitrary \( \epsilon > 0 \), consider the Young function

\[
A_{\epsilon}(t) = t \log^\epsilon(1 + t).
\]

(12)

Then, there exists a constant \( C \) such that for each weight \( w \) and for all \( \lambda > 0 \)

\[
w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_{A_{\epsilon}}w(y) dy.
\]

(13)

**Remark 1.7:** For \( 1 < p < \infty \) let us denote by \( B_p \) the collection of all Young functions \( A \) satisfying condition (9):

\[
\int_c^\infty \left( \frac{t}{A(t)} \right)^{p'} dt < \infty,
\]

for some \( c > 0 \). Observe that \( B_p \subset B_q \), \( 1 < p < q < \infty \). Then it follows easily from the proof of last theorem that we may replace \( A_{\epsilon} \) by any Young function belonging to the smallest class \( \cap_{p>1} B_p \). We could consider for instance

\[
A_{\epsilon}(t) = t \log(1 + t) [\log \log(1 + t)]^\epsilon.
\]

(14)

If we let \( \epsilon = 0 \) in (12) \( M_{A_0} = M \) is the Hardy–Littlewood maximal operator. Since \( A_0 \) does not belong to \( \cap_{p>1} B_p \) we think that the estimate:

\[
w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M w(y) dy,
\]

(15)

for some constant \( C \), and for all \( \lambda > 0 \), does not hold.

2 Proof of the Theorems

**Proof of Theorem 1.5:**

We prove first that condition (9) is sufficient for (10) to hold for any singular integral operator \( T \).

We may assume that \( M_{A_{\epsilon}}w \) is finite almost everywhere, and we let \( T^* \) be the adjoint operator of \( T \). \( T^* \) is also a singular integral operator with kernel \( k^*(x) = k(-x) \). Then, by duality (10) is equivalent to
\[
\int_{\mathbb{R}^n} |T^*f(y)|^{p'} M_A w(y)^{1-p'} dy \leq C \int_{\mathbb{R}^n} |f(y)|^{p'} w(y)^{1-p'} dy. \tag{16}
\]

We shall be using some well known facts about the $A_p$ theory of weights for which we remit the reader to [5] Chapter 4.

To prove (16) we shall use the following fundamental estimate due to Coifman ([2]):

Let $T$ be any singular integral operator; then for each $0 < p < \infty$, and each $u \in A_\infty$, there exists $C = C_{u,p} > 0$ such that for each $f \in C^\infty_0(\mathbb{R}^n)$

\[
\int_{\mathbb{R}^n} |Tf(y)|^p u(y) dy \leq C \int_{\mathbb{R}^n} Mf(y)^p u(y) dy. \tag{17}
\]

Therefore, to apply this estimate to $T^*$ we need to show that $(M_A w)^{1-p'}$ satisfies the $A_\infty$ condition.

To check this, we claim first that $(M_A w)^{\delta}$ satisfies the $A_1$ condition for $0 < \delta < 1$. However, this is a straightforward generalization of the well known fact that $(M w)^{\delta} \in A_1$, $0 < \delta < 1$, also due to Coifman (cf. [5] p. 158), and we shall omit its proof.

Now, since $w^{1-r} \in A_r$, for any $w \in A_1$ and $r > 1$, we have that

\[
(M_A w)^{1-p'} = \left[(M_A w)^{\frac{p'}{p'-1}}\right]^{1-r} \in \cap_{r>p} A_r \subset A_\infty.
\]

After these observations, we have reduced the problem to showing that

\[
\int_{\mathbb{R}^n} Mf(y)^{p'} M_A w(y)^{1-p'} dy \leq C \int_{\mathbb{R}^n} |f(y)|^{p'} w(y)^{1-p'} dy. \tag{18}
\]

But this is a particular instance of the following characterization which can be found in [6] Theorem 4.4.

**Theorem 2.1:** Let $1 < p < \infty$. Let $A$ be a Young function, and denote $B = \overline{A(t^{p'})}$. Then the following are equivalent.

i) \[
\int_c^\infty \left(\frac{t}{A(t)}\right)^{p-1} \frac{dt}{t} < \infty; \tag{19}
\]

ii) there is a constant $c$ such that

\[
\int_{\mathbb{R}^n} M_B f(y)^p dy \leq c \int_{\mathbb{R}^n} f(y)^p dy \tag{20}
\]
for all nonnegative, locally integrable functions \( f \);

iii) there is a constant \( c \) such that

\[
\int_{\mathbb{R}^n} M_B f(y)^p \ u(y) \, dy \leq c \int_{\mathbb{R}^n} f(y)^p \ M u(y) \, dy \tag{21}
\]

for all nonnegative, locally integrable functions \( f \) and \( u \);

iv) there is a constant \( c \) such that

\[
\int_{\mathbb{R}^n} M f(y)^p \ \frac{u(y)}{[M_A(w)(y)]^{p-1}} \, dy \leq c \int_{\mathbb{R}^n} f(y)^p \frac{M u(y)}{w(y)^{p-1}} \, dy, \tag{22}
\]

for all nonnegative, locally integrable functions \( f, w \) and \( u \).

Observe that (18) follows from (22) by taking \( u = 1 \), and by replacing \( p \) by \( p' \).

Now we shall prove that condition (9) is also necessary for (10) to hold for all the Riesz transforms. That is, suppose that the Young function \( A \) is fixed, and that

\[
\int_{\mathbb{R}^n} |T f(x)|^p \ w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \ M A w(x) dx, \tag{23}
\]

is verified for each Riesz transform \( T = R_j, \ j = 1, 2, \ldots, n \).

Fix one of these \( j \). As above, by duality (23) is equivalent to

\[
\int_{\mathbb{R}^n} |R_j f(x)|^{p'} \ M A w(x)^{1-p'} dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{1-p'} dx, \tag{24}
\]

We shall adapt an argument from [5] p. 561. We define the cone

\[ E_j = \{ x \in \mathbb{R}^n : \max\{|x_1|, |x_2|, \ldots, |x_n|\} = x_j \}, \]

so that \( \mathbb{R}^n = \bigcup_{j=1}^n (E_j \cup (-E_j)) \). Let \( B \) be the unit ball, and consider the function \( f = w = \chi_{B \cap (-E_j)} \). Then, (24) implies

\[
\infty > C \int_{\mathbb{R}^n} |f(x)|^{p'} w(x)^{1-p'} dx = C \ |B \cap (-E_j)| \geq \int_{E_j \cap \{|x| > 2\}} |R_j f(x)|^{p'} M_A f(x)^{1-p'} dx.
\]
Observe that for $|x| > 2$, $M_A f(x) \approx A^{-1}(|x|^n)^{-1}$. Also, for every $x \in E_j$

$$R_j f(x) = C \int_{B \cap (-E_j)} \frac{x_j - y_j}{|x - y|^{n+1}} \, dy \geq C \int_{B \cap (-E_j)} \frac{1}{|x - y|^n} \, dy \geq \frac{C}{|x|^n}.$$ 

Therefore

$$\int_{E_j \cap \{|x| > 2\}} \frac{1}{|x|^{np'}} A^{-1}(|x|^n)^{p'-1} \, dx \leq C |B \cap (-E_j)|.$$ 

A corresponding estimate can be proved for $E_j$, and for each $j = 1, 2, \ldots, n$, by using in each case the corresponding Riesz transform. Since the family of cones $\{\pm E_j\}_{j=1,2,\ldots,n}$ is disjoint, we finally have that

$$\int_{|x| > 2} \frac{1}{|x|^{np'}} A^{-1}(|x|^n)^{p'-1} \, dx \approx \int_{c}^{\infty} \frac{1}{B A(t)^{p'-1}} \frac{dt}{t} \approx \int_{c}^{\infty} \left( \frac{t}{A(t)} \right)^{p'-1} \frac{dt}{t} < \infty,$$

since $t A'(t) \approx A(t)$. This concludes the proof of the theorem.

\[\square\]

**Proof of Theorem 1.6:**

We shall assume that $M_A w$ is finite almost everywhere, since otherwise there is nothing to be proved.

For $f \in C_0^\infty(R^n)$ we consider the standard Calderón–Zygmund decomposition of $f$ at level $\lambda$ (cf. [5] p. 414).

Let $\{Q_j\}$ be the Calderón–Zygmund nonoverlapping dyadic cubes satisfying

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)| \, dx \leq 2^n \lambda. \quad (25)$$

If we let $\Omega = \bigcup_j Q_j$, we also have that $|f(x)| \leq \lambda$ a.e. $x \in R^n \setminus \Omega$.

Using the notation $f_{Q_j} = \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx$, we write $f = g + b$ where $g$, the “good part”, is given by

$$g(x) = \begin{cases} f(x) & x \in R^n \setminus \Omega \\ f_{Q_j} & x \in Q_j \end{cases}$$

Observe that $|g(x)| \leq 2^n \lambda$ a.e.
The “bad part” can be split as $b = \sum_j b_j$, where $b_j(x) = (f(x) - f_{Q_j})\chi_{Q_j}(x)$.

Let $\tilde{Q}_j = 2Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$.

We have

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda/2\}) \leq w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tf(y)| > \lambda/2\}) + 2w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tb(y)| > \lambda/2\}).$$

Pick any $p > 1$ such that $1 < p < 1 + \epsilon$. Then, it follows that $A_\epsilon = t \log^\epsilon(1 + t)$ satisfies condition

$$\int_{c}^{\infty} \left( \frac{t}{A_\epsilon(t)} \right)^{p'-1} \frac{dt}{t} < \infty,$$

for some $c > 0$. Thus, we can apply Theorem 1.5 with this $p$ to the first term, together with the fact that $|g(x)| \leq 2^n \lambda$ a.e. Then, using an idea from [1] p. 282

$$w(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tf(y)| > \lambda/2\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |Tg(y)|^p w(y)dy \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_A(w_{\mathbb{R}^n \setminus \tilde{\Omega}})(y)dy =$$

$$\frac{C}{\lambda} \left( \int_{\mathbb{R}^n \setminus \Omega} |f(y)| M_A(w(y)dy + \int_{\Omega} |g(y)| M_A(w_{\mathbb{R}^n \setminus \tilde{\Omega}})(y)dy \right) =$$

$$\frac{C}{\lambda} (I + II)$$

Since $I \leq \int_{\mathbb{R}^n} |f(y)| M_A(w(y)dy$ we only need to estimate II:

$$II \leq \sum_j \int_{Q_j} |f_{Q_j}| M_A(w_{\mathbb{R}^n \setminus \tilde{\Omega}})(y)dy \leq$$

$$\sum_j \int_{Q_j} |f(x)| dx \frac{1}{|Q_j|} \int_{Q_j} M_A(w_{\mathbb{R}^n \setminus \tilde{\Omega}})(y)dy.$$

We shall make use of the following fact: for arbitrary Young function $A$, non-negative function $w$ with $M_A w(x) < \infty$ a.e., cube $Q$, and $R > 1$ we have

$$M_A(\chi_{\mathbb{R}^n \setminus RQ}w)(y) \approx M_A(\chi_{\mathbb{R}^n \setminus RQ}w)(z)$$

for each $y, z \in Q$. This is an observation whose proof follows exactly as for the case of the Hardy–Littlewood maximal operator $M$, cf. for instance [5] p. 159.
Then,

\[ II \leq C \sum_j \int_{Q_j} |f(x)| \, dx \inf_{Q_j} M_{A_w}(w_{x_{\mathbb{R}^n \setminus 2Q_j}}) \leq C \sum_j \int_{Q_j} |f(x)| M_{A_w}(w(x)) \, dx \]

\[ \leq C \int_{\mathbb{R}^n} |f(x)| M_{A_w}(w(x)) \, dx. \]

The second term is estimated as follows:

\[ w(\tilde{\Omega}) \leq C \sum_j \frac{w(\tilde{Q}_j)}{|Q_j|} \leq \]

\[ \frac{C}{\lambda} \sum_j \frac{w(\tilde{Q}_j)}{|Q_j|} \int_{Q_j} |f(x)| \, dx \leq \frac{C}{\lambda} \sum_j \int_{Q_j} |f(x)| M_{w}(x) \, dx \leq \]

\[ \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_{w}(x) \, dx. \]

To estimate the last term we use the inequality

\[ \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |Tb_j(y)| \, w(y) \, dy \leq C \int_{\mathbb{R}^n} b_j(y) M_{w}(y) \, dy, \]

with \( C \) independent of \( b_j \), which can be found in Lemma 3.3, p. 413, of [5]. Now, using this estimate with \( w \) replaced by \( w_{x_{\mathbb{R}^n \setminus \tilde{Q}_j}} \) we have

\[ w(\{ y \in \mathbb{R}^n \setminus \tilde{\Omega} : |Tb(y)| > \lambda/2 \}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tb(y)| \, w(y) \, dy \leq \]

\[ \frac{C}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |Tb_j(y)| w(y) \, dy \leq \frac{C}{\lambda} \sum_j \int_{\mathbb{R}^n} |b_j(y)| M(\chi_{x_{\mathbb{R}^n \setminus \tilde{Q}_j}})(y) \, dy \leq \]

\[ \frac{C}{\lambda} \sum_j \int_{Q_j} |b(y)| M(\chi_{x_{\mathbb{R}^n \setminus \tilde{Q}_j}})(y) \, dy. \]

Since \( b = f - g \) this is at most

\[ \frac{C}{\lambda} \sum_j \left( \int_{Q_j} |f(y)| M_{w}(y) \, dy + \int_{Q_j} |g(y)| M(\chi_{x_{\mathbb{R}^n \setminus \tilde{Q}_j}})(y) \, dy \right) = \frac{C}{\lambda} (A + B) \]
To conclude the proof of the theorem is clear that we only need to estimate $B$. However
\[
B = \sum_j \int_{Q_j} |f_{Q_j}| M(w \chi_{R^n \setminus \tilde{Q}_j})(y) dy \leq 
\]
\[
\sum_j \int_{Q_j} |f(x)| dx \frac{1}{|Q_j|} \int_{Q_j} M(w \chi_{R^n \setminus \tilde{Q}_j})(x) dx \leq 
\]
\[
\sum_j \int_{Q_j} |f(x)| dx \inf_{Q_j} M(w \chi_{R^n \setminus \tilde{Q}_j}) \leq \sum_j \int_{Q_j} |f(x)| M(w \chi_{R^n \setminus 2Q_j})(x) dx \leq 
\]
\[
C \int_{R^n} |f(y)| Mw(y) dy
\]
Here we have used again that $M(w \chi_{R^n \setminus 2Q})(y) \approx M(w \chi_{R^n \setminus 2Q})(z)$ for each $y, z \in Q$.

This concludes the proof of the theorem since we always have that $Mw(x) \leq M_A w(x)$ for each Young function $A$ and for each $x$.

\[\square\]

**Proof of Theorem 1.1:**

Let us assume that $M^{[p]+1}w$ is finite almost everywhere, since otherwise (6) is trivial. Let $A$ be the Young function
\[
A(t) = t \log^{[p]}(1 + t).
\]

A simple computation shows that $A$ satisfies condition (9), which is the hypothesis of Theorem 1.5. Then, Theorem 1.1 will follow if we prove the pointwise inequality
\[
M_A w(x) \leq C M^{[p]+1}w(x).
\]
(27)

Recall that $M_A$ is defined by $M_A f(x) = \sup_{x \in Q} \|f\|_{A,Q}$, where
\[
\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_{Q} A\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\}.
\]

Then, it is enough to prove that there is constant $C$ such that for each cube $Q$
\[
\|f\|_{A,Q} \leq \frac{C}{|Q|} \int_{Q} M^{[p]}w(x) dx.
\]
2 Proof of the Theorems

By assumption, the right hand side average is finite, and by homogeneity we can assume that is equal to one. Then, by the definition of Luxemburg norm we need to prove

$$\frac{1}{|Q|} \int_Q A(w(y)) \, dy = \frac{1}{|Q|} \int_Q w(y) \log^{[p]}(1 + w(y)) \, dy \leq C.$$  

But this is a consequence of iterating the following inequality of E.M. Stein [7]

$$\int_Q w(y) \log^k(1 + w(y)) \, dy \leq C \int_Q Mw(y) \log^{k-1}(1 + Mw(y)) \, dy,$$  

with $k = 1, 2, 3, \cdots$.

To conclude the proof of the theorem, we are left with showing that for arbitrary $1 < p < \infty$, the inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M^{[p]} w(x) \, dx,$$  

is false in general. To prove this assertion we consider the Hilbert transform

$$Hf(x) = pv \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy.$$

Then, by duality (29) is equivalent to

$$\int_{\mathbb{R}} \left| Hf(x) \right|^p M^{[p]} w(x)^{1-p'} \, dx \leq C \int_{\mathbb{R}} \left| f(x) \right|^p M^{[p]} w(x)^{1-p'} \, dx.$$  

(30)

Let $f = w = \chi_{(-1,1)}$. A standard computation shows that

$$M^k f(x) \approx \frac{\log^{k-1}(1 + |x|)}{|x|}, \quad |x| \geq e$$

for each $k = 1, 2, 3, \cdots$. Then, we have

$$\int_{\mathbb{R}} \left| Hf(x) \right|^p M^{[p]} w(x)^{1-p'} \, dx \geq C \int_{x>e} \left( \frac{1}{x} \right)^{p'} \left( \frac{\log^{[p]-1}(x)}{x} \right)^{1-p'} \, dx \approx \int_{x>e} \log^{(p-1)(1-p')}(x) \, dx = \infty,$$
since \((p - 1)(1 - p') + 1 \geq 0\). However, the right hand side of (30) equals \(\int_{\mathbb{R}} f(y)dy = 2 < \infty\).

\[\square\]

**Proof of Theorem 1.2:**

As above, we shall assume that \(M^2w\) is finite almost everywhere. For \(0 < \epsilon < 1\) set as before \(A_\epsilon(t) = t \log^\epsilon(1 + t)\). Then, the inequality

\[\int_Q w(y) \log^\epsilon(1 + w(y)) dy \leq C \int_Q Mw(y) dy,\]

whose proof is analogue to that of (28) using that the derivative of \(A_\epsilon(t)\) is less than \(1/t\), implies exactly as in the proof of Theorem 1.1 that

\[M_{A_\epsilon}w(x) \leq C M^2w(x).\]

This concludes the proof of Theorem 1.2.

\[\square\]

**Proof of Theorem 1.4:** By an standard argument, it is enough to show that there is a constant \(C\) such that

\[\int_{\mathbb{R}^n} |Ta(y)| w(y)dy \leq C\]

for each \(Mw\)-atom \(a\). To prove this, suppose that \(\text{supp}(a) \subset Q\) for some cube \(Q\). Then

\[\int_{\mathbb{R}^n} |Ta(y)| w(y)dy = \int_{3Q} |Ta(y)| w(y)dy + \int_{\mathbb{R}^n \setminus 3Q} |Ta(y)| w(y)dy = I + II.\]

Now, \(II\) is majorized, as in the proof of Theorem 1.6, by using Lemma 3.3, p. 413 of [5]

\[II \leq C \int_{\mathbb{R}^n} |a(y)| Mw(y)dy \leq \frac{C}{Mw(Q)} \int_Q Mw(y)dy = C,\]

where \(C\) is independent of \(a\).

For \(I\) we use the fact that any singular integral operator \(T : L^{\infty}(Q, \frac{dx}{|Q|}) \rightarrow L_{Lexp}(Q, \frac{dx}{|Q|})\). Then

\[I = |3Q| \frac{1}{|3Q|} \int_{3Q} |Ta(y)| w(y)dy \leq C|Q| \|Ta\|_{L_{Lexp,3Q}} \|w\|_{L_{Lexp,3Q}} \leq \]
\[ \leq C |Q| \|a\|_{\infty, 3Q} \frac{1}{|3Q|} \int_{3Q} Mw(y)dy \leq C, \]

by (28) and by the definition of $Mw$–atom. This finishes the proof of Theorem 1.4.

\[ \square \]

We shall end this section by disproving inequality

\[ w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda \}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p M^{|p|}w(y)dy \quad (31) \]

from remark 1.3, whenever $p$ is greater than one but not an integer.

Consider $T = H$ the Hilbert transform as above. For $\lambda > 0$, we let $f = \chi_{(1, e^{\lambda})}$, and $w = \chi_{(0, 1)}$. Then for $y \neq 1, e^{\lambda}$

\[ Hf(y) = \log \left| \frac{y-1}{y-e^{\lambda}} \right|. \]

When $y \in (0, 1)$ we have

\[ |Hf(y)| = |\log \left| \frac{y-1}{y-e^{\lambda}} \right|| = \log \frac{e^{\lambda} - y}{1 - y} > \log e^{\lambda} = \lambda. \]

Then, assuming that (31) holds for all $\lambda$ we had

\[ 1 = \int_0^1 w(y)dy \leq w(\{y \in (0, 1) : |Hf(y)| > \lambda \}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(y)|^p M^{|p|}w(y)dy = \frac{C}{\lambda^p} \int_1^{e^{\lambda}} M^{|p|}w(y)dy \approx \frac{1}{\lambda^p} \int_1^{e^{\lambda}} \log^{p-1} w(y)dy \approx \lambda^{p-\epsilon}. \]

By letting $\lambda \to \infty$ we see that this a contradiction when $p$ is not an integer.

There is another argument due to S. Hofmann, and is as follows. Since $p$ is not an integer we can find an small $\epsilon > 0$ such that $[p] < p - \epsilon < p < p + \epsilon < [p] + 1$. Then, (31) implies that $M$ is at once of weak type $(p - \epsilon, p - \epsilon)$ and $(p + \epsilon, p + \epsilon)$ with respect to the weights $(w, M^{|p|}w)$. Then, by the Marcinkiewicz interpolation theorem $M$ is of strong type $(p, p)$ with respect to the weights $(w, M^{|p|}w)$. But this is a contradiction as shown in Theorem 1.1.
3 Calderón–Zygmund operators

In this section we shall state our main results for the more general Calderón–Zygmund operators.

We recall the definition of a Calderón–Zygmund operator in $\mathbb{R}^n$. A kernel on $\mathbb{R}^n \times \mathbb{R}^n$ will be a locally integrable complex-valued function $K$, defined on $\Omega = \mathbb{R}^n \times \mathbb{R}^n \setminus$ diagonal. A kernel $K$ on $\mathbb{R}^n$ satisfies the standard estimates, if there exist $\delta > 0$ and $C < \infty$ such that for all distinct $x, y \in \mathbb{R}^n$ and all $z$ such that $|x - z| < |x - y|/2$:

(i) $|K(x, y)| \leq C |x - y|^{-n};$

(ii) $|K(x, y) - K(z, y)| \leq C \left( \frac{|x - z|}{|x - y|} \right)^\delta |x - y|^{-n};$

(iii) $|K(y, x) - K(y, z)| \leq C \left( \frac{|x - z|}{|x - y|} \right)^\delta |x - y|^{-n}.$

We say that a linear and continuous operator $T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is associated with a kernel $K$, if

$$\langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) g(x) f(y) \, dx \, dy,$$

whenever $f, g \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$.

We say that $T$ is a Calderón–Zygmund operator if the associated kernel $K$ satisfies the standard estimates, and if it extends to a bounded linear operator in $L^2(\mathbb{R}^n)$.

**Theorem 3.1:** Let $1 < p < \infty$, and let $T$ be a Calderón–Zygmund operator. Then, there exists a constant $C$ such that for each weight $w$

$$\int_{\mathbb{R}^n} |Tf(y)|^p w(y) \, dy \leq C \int_{\mathbb{R}^n} |f(y)|^p M^{|p|+1} w(y) \, dy,$$

and there exists another constant $C$ such that for all $\lambda > 0$

$$w(\{y \in \mathbb{R}^n : |Tf(y)| > \lambda \}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|^2 M^2 \, w(y) \, dy.$$

The proof of Theorem 3.1 is essentially the same as Theorems 1.1 and 1.2, after observing that the adjoint $T^*$ of any Calderón–Zygmund operator $T$ is also a Calderón–Zygmund operator with kernel $K^*(x, y) = K(y, x)$. 
There are corresponding results to Theorems 1.2, 1.4, 1.5, and for 1.6 for any Calderón–Zygmund operator. We shall omit the obvious statements.

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**References**


