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RATE OF CONVERGENCE UNDER WEAK CONTRACTIVENESS CONDITIONS

DAVID ARIZA-RUIZ* EYVIND MARTOL BRISEID**, ANTONIO JIMÉNEZ-MELADO*** AND GENARO LÓPEZ-ACEDO*

* Dept. Análisis Matemático, Fac. Matemáticas,
Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain.
E-mail: dariza@us.es, glopez@us.es

** Dept. of Mathematics,
The University of Oslo, Postboks 1053, Blindern, 0316 Oslo, Norway.
E-mail address: eyvindmb@math.uio.no

*** Dept. Análisis Matemático, Fac. Ciencias,
Universidad de Málaga, Apdo. 29071 Málaga, Spain
E-mail address: melado@uma.es

Abstract. We introduce a new class of selfmaps T of metric spaces, which generalizes the weakly Zamfirescu maps (and therefore weakly contraction maps, weakly Kannan maps, weakly Chatterjea maps and quasi-contraction maps with constant $h < \frac{1}{2}$). We give an explicit Cauchy rate for the Picard iteration sequences $\{T^n x_0\}_{n\in\mathbb{N}}$ for this type of maps, and show that if the space is complete, then all Picard iteration sequences converge to the unique fixed point of T. Our Cauchy rate depends on the space (X, d), the map T, and the starting point $x_0 \in X$ only through an upper bound $b \geq d(x_0, Tx_0)$ and certain moduli θ, μ for the map, but is otherwise fully uniform. As a step on the way to proving our fixed point result we also calculate a modulus of uniqueness for this type of maps.
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1. Introduction

Let (X, d) be a metric space and $T : X \to X$ a map. The well-known Banach contraction principle states that if T is contractive (that is, there exists $\alpha \in [0,1)$ such that $d(Tx,Ty) \leq \alpha\, d(x,y)$ for all $x,y \in X$) and (X,d) is complete then T has a unique fixed point p in X, and for any $x_0 \in X$ the Picard iteration sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to p. Moreover, we have the following estimate of the rate of convergence

$$d(T^n x_0, p) \leq \frac{\alpha^n}{1-\alpha}\, d(x_0, Tx_0) \quad \text{for all } n \in \mathbb{N}.$$

The rate of convergence for the Picard iteration sequences $\{T^n x_0\}_{n\in\mathbb{N}}$ which we obtain from this is highly uniform – it depends on the space (X,d), the map T, and the starting point $x_0 \in X$ only through a bound $b \geq d(x_0, Tx_0)$ and the constant α. It is of interest to obtain similar rates of convergence or Cauchy rates for other classes of mappings – where instead of the constant α the rates then depend on other constants or moduli governing the behavior of the mappings in question, e.g. a modulus of uniform continuity. In [3, 4, 5] such explicit rates of convergence were calculated for very general classes of maps.[1] These rates of convergence are given in terms of appropriate moduli for the classes of mappings considered, and so one possible route to obtaining new rates of convergence for other classes of maps is to show that these are e.g. asymptotic contractions in the sense of Kirk, for which a rate of convergence is given in [3], and then to calculate the moduli for an asymptotic contraction from the moduli of the mapping at hand. However, since it often in this case is far from obvious how to turn one set of moduli into another, it is worthwhile to calculate explicit rates of convergence for more specific classes of maps directly in terms of the moduli through which these are given. This approach can also lead to better rates of convergence which can be expressed more simply than the rates of convergence for the more general classes of maps.

An example of a class of maps for which it would be interesting to give a simple rate of convergence are the weakly contractive maps. This concept was introduced by Dugundji and Granas [8] by replacing the contraction constant α by a function $\alpha = \alpha(x,y)$ compactly less than 1. In this paper we introduce a new type of maps that includes both the class of weakly contractive maps and several other well-known types of maps, such as the class of quasi-contraction

[1] In particular, Proposition 3.7 in [3] and Theorem 30 below together imply that in the setting of complete, bounded metric spaces the mappings for which we in this paper calculate a rate of convergence are included in the asymptotic contractions considered in [3].

maps with constant $h < \frac{1}{2}$. The main result of the paper is an explicit and highly uniform Cauchy rate (which makes sense without assuming the completeness of the space) for this class of mappings, expressed in terms of certain moduli θ and μ; and as a followup the result that if the space is complete, then all Picard iteration sequences converge to the unique fixed point. The structure of the paper is as follows:

In Section 2 we introduce the condition $(wE_{\alpha,\mu})$, and construct our Cauchy rate for mappings which satisfy this in addition to some extra condition.

In Section 3 we show that the condition introduced in this paper includes several well-known types of maps such as weakly contractive maps, quasi-contraction maps with constant $h < \frac{1}{2}$, and others.

In Section 4 we prove that the maps satisfying this new condition have the property of existence of a modulus of uniqueness. This is used to establish our fixed point result, which we give in Section 5, along with a local version of the theorem.

To a certain extent our methodology of focussing on the moduli through which the mappings T are given, rather than on the mappings themselves, as well as in particular some of the arguments involving a modulus of uniqueness, are inspired by so-called *proof mining* techniques (see [13]), which were the explicit background for the work on rates of convergence in [3, 4, 5].

Finally a notational point: We will throughout this paper let $\mathbb{N}$ denote the set of nonnegative integer numbers.

2. Cauchy rate

We recall the two basic definitions that are the matter of interest of this paper.

Definition 1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d). We say that $\gamma : (0,\infty) \to \mathbb{N}$ is a Cauchy rate for $\{x_n\}_{n\in\mathbb{N}}$ if for all $\varepsilon > 0$ we have that $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \leq \varepsilon$ for each $k \in \mathbb{N}$.

Definition 2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d). We say that $\gamma : (0,\infty) \to \mathbb{N}$ is a rate of convergence for $\{x_n\}_{n\in\mathbb{N}}$ to $z \in X$ if for all $\varepsilon > 0$ we have that $n \geq \gamma(\varepsilon)$ gives $d(z, x_n) \leq \varepsilon$.

In [10], J. Garcia-Falset et al. introduced a condition which is weaker than nonexpansiveness. This condition reads as follows: let (X, d) be a metric space, D a nonempty subset of X and $\mu \geq 1$ a real number. A map $T : D \to X$ is said to satisfy the condition (E_μ) on D if

$$d(x, Ty) \leq d(x, y) + \mu\, d(x, Tx), \tag{E_μ}$$

for all $x, y \in D$. In this paper, we shall consider a version of (E_μ) for weakly contractive mappings obtained by replacing in (E_μ) the nonexpansiveness condition by the weakly contractive condition. In order to do this, we need the following concept.

Definition 3. Let D be a nonempty subset of a metric space (X, d). A map $\alpha : D \times D \to [0, 1]$ is called compactly less than 1 if $\theta(a, b) < 1$ for all $0 < a \leq b$, where θ is given by

$$\theta(a, b) := \begin{cases} \sup\limits_{a \leq d(x,y) \leq b} \alpha(x, y) & \text{if } \{(x, y) \in D \times D : a \leq d(x, y) \leq b\} \neq \emptyset, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Remark 4. It is easy to check that $\alpha : D \times D \to [0, 1]$ is compactly less than 1 if, and only if, there exists a function $\theta : (0, \infty) \times (0, \infty) \to [0, 1)$ such that $\alpha(x, y) \leq \theta(a, b) < 1$ for all $0 < a \leq b$ and all $x, y \in D$ with $a \leq d(x, y) \leq b$. This equivalent formulation is more convenient for our needs and will be used throughout this paper. So in a context where $\alpha : D \times D \to [0, 1]$ is a map which is compactly less than 1 we will let θ be some function as given in this remark rather than necessarily the one appearing in Definition 3.

Definition 5. Let D be a nonempty subset of a metric space (X, d). We say that $T : D \to X$ satisfies condition $(wE_{\alpha,\mu})$ if

$$d(x, Ty) \leq \alpha(x, y)\, d(x, y) + \mu\, d(x, Tx) \tag{$wE_{\alpha,\mu}$}$$

for all $x, y \in D$, where $\mu \geq 1$ and $\alpha : D \times D \to [0, 1]$ is compactly less than 1.

Remark 6. Suppose that $T : D \to X$ satisfies condition $(wE_{\alpha,\mu})$. If for any fixed $\sigma \in (0, 1)$ we define $\alpha_\sigma : D \times D \to [0, 1]$ by

$$\alpha_\sigma(x, y) := \max\{\alpha(x, y), \sigma\},$$

then T also satisfies the above condition with μ, α_σ, and, moreover, we have that

$$\theta_\sigma(a,b) := \sup\{\alpha_\sigma(x,y) : a \le d(x,y) \le b\} > 0,$$

for all $0 < a \le b$. Thus, from now on, when considering an α as in Definition 5 we will assume that for the corresponding θ as in Definition 3 we have $\theta(a,b) > 0$ for all $0 < a \le b$.

Theorem 7. Let (X,d) be a metric space and $T : X \to X$ be a map which satisfies condition $(wE_{\alpha,\mu})$ and the following condition

$$d(Tx, T^2x) \le \alpha(x, Tx)\, d(x, Tx), \tag{1}$$

for all $x \in X$. Let $x_0 \in X$ be the starting point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} := Tx_n$. Let $b > 0$ satisfy $d(x_0, x_1) \le b$, and define $\rho : (0,\infty) \to (0,\infty)$ by

$$\rho(\varepsilon) := \min\left\{ \frac{\varepsilon}{2\mu}\,\left(1 - \theta(\tfrac{\varepsilon}{2}, \varepsilon)\right), b \right\}.$$

Define $\gamma : (0,\infty) \to \mathbb{N}$ by

$$\gamma(\varepsilon) := \left\lceil \frac{\log \rho(\varepsilon) - \log b}{\log \theta(\rho(\varepsilon), b)} \right\rceil + 1, \tag{2}$$

where $\lceil\cdot\rceil$ denotes the ceiling function given by $\lceil x \rceil := \min\{n \in \mathbb{Z} : x \le n\}$. Then γ is a Cauchy rate for $\{x_n\}_{n\in\mathbb{N}}$.

Remark 8. Note that γ depends only on θ, μ, b and ε. Thus, we should consider

$$\gamma : (0,1)^{(0,\infty)\times(0,\infty)} \times [1,\infty) \times (0,\infty) \times (0,\infty) \to \mathbb{N}$$

and $\gamma(\theta,\mu,b,\varepsilon)$ instead of $\gamma(\varepsilon)$. We shall use both notations interchangeably. Observe also that if (X,d) is complete, then $\gamma(\theta,\mu,b,\,\cdot\,)$ is a rate of convergence for $\{x_n\}_{n\in\mathbb{N}}$ to $z := \lim_{n\to\infty} x_n$.

Proof. Let $x_0 \in X$ and define the Picard iteration sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} := Tx_n$ for $n \in \mathbb{N}$. Note that (1) gives

$$d(x_{n+1}, x_{n+2}) \le \alpha(x_n, x_{n+1})\, d(x_n, x_{n+1}) \tag{3}$$

for all $n \in \mathbb{N}$, so since $0 \le \alpha(x_n, x_{n+1}) \le 1$ it follows that the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is nonincreasing. Let now $\varepsilon > 0$. We will first prove that $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) < \rho(\varepsilon)$. In order to do this, we assume $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) \ge \rho(\varepsilon)$

and we shall get a contradiction. Note that $\rho(\varepsilon) \leq d(x_i, x_{i+1}) \leq b$ for all $0 \leq i \leq \gamma(\varepsilon)$. Thus, $\alpha(x_i, x_{i+1}) \leq \theta(\rho(\varepsilon), b)$ for $0 \leq i \leq \gamma(\varepsilon)$. Then,

$$\begin{aligned} \rho(\varepsilon) &\leq d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) \\ &\leq \prod_{i=1}^{\gamma(\varepsilon)} \alpha(x_{\gamma(\varepsilon)-i}, x_{\gamma(\varepsilon)+1-i}) \cdot d(x_0, x_1) \\ &\leq \theta(\rho(\varepsilon), b)^{\gamma(\varepsilon)} \cdot b \\ &< \rho(\varepsilon), \end{aligned}$$

which is a contradiction. Here the last step

$$\theta(\rho(\varepsilon), b)^{\gamma(\varepsilon)} \cdot b < \rho(\varepsilon)$$

follows from

$$\begin{aligned} \theta(\rho(\varepsilon), b)^{\gamma(\varepsilon)} \cdot b < \rho(\varepsilon) &\iff \\ \gamma(\varepsilon) \cdot \log \theta(\rho(\varepsilon), b) + \log b < \log \rho(\varepsilon) &\iff \\ \gamma(\varepsilon) > \frac{\log \rho(\varepsilon) - \log b}{\log \theta(\rho(\varepsilon), b)} & \end{aligned}$$

and the definition of γ. Therefore, we have

$$d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}) < \rho(\varepsilon). \tag{4}$$

Next, we will prove by induction that $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \leq \varepsilon$ for $k \in \mathbb{N}$. This inequality is true for $k = 1$ by (4), since $\rho(\varepsilon) < \varepsilon$. Assuming $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) < \varepsilon$, let us see $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k+1}) < \varepsilon$. Taking $x = x_{\gamma(\varepsilon)}$ and $y = x_{\gamma(\varepsilon)+k}$, from condition $(wE_{\alpha,\mu})$ we obtain

$$d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k+1}) \leq \alpha(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k})\, d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) + \mu\, d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+1}). \tag{5}$$

If $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) < \varepsilon/2$, using that $0 \leq \alpha(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \leq 1$ and (4), we have

$$d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k+1}) < \frac{\varepsilon}{2} + \mu\, \rho(\varepsilon) \leq \frac{\varepsilon}{2} + \mu\, \frac{\varepsilon}{2\mu} = \varepsilon.$$

And if $d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \geq \varepsilon/2$, applying the induction hypothesis, we have

$$\frac{\varepsilon}{2} \leq d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) < \varepsilon.$$

Then, $\alpha(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k}) \leq \theta(\frac{\varepsilon}{2}, \varepsilon)$. Thus, from (4) and (5), we conclude that

$$\begin{aligned} d(x_{\gamma(\varepsilon)}, x_{\gamma(\varepsilon)+k+1}) &\leq \theta(\tfrac{\varepsilon}{2}, \varepsilon)\, \varepsilon + \mu\, \rho(\varepsilon) \\ &\leq \theta(\tfrac{\varepsilon}{2}, \varepsilon)\, \varepsilon + \mu\, \frac{\varepsilon}{2\mu}\, (1 - \theta(\tfrac{\varepsilon}{2}, \varepsilon)) \\ &= \frac{\varepsilon}{2}\, (1 + \theta(\tfrac{\varepsilon}{2}, \varepsilon)) \\ &< \varepsilon. \end{aligned}$$

□

3. Some examples

The aim of this section is to show that some well-known types of maps satisfy condition $(wE_{\alpha,\mu})$.

3.1. **Weakly Zamfirescu maps.** In 1972 Zamfirescu [16], combining the contractive condition, Kannan's condition [11] and Chatterjea's condition [6], defined the following type of maps.

Definition 9. Let D be a nonempty subset of a metric space (X, d). A map $T : D \to X$ is a Zamfirescu map if there exists a constant $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha\, M_T(x, y) \tag{Z}$$

for all $x, y \in D$, where

$$M_T(x, y) := \max\left\{d(x, y), \tfrac{1}{2}\,[d(x, Tx) + d(y, Ty)], \tfrac{1}{2}\,[d(x, Ty) + d(y, Tx)]\right\}.$$

In [2], the authors defined a new class of map which generalizes the class of Zamfirescu maps and the class of weakly contraction maps [8, 9] and weakly Kannan maps [1].

Definition 10. Let (X, d) be a metric space, $D \subseteq X$ and $T : D \to X$. We say that T is a weakly Zamfirescu map if there exists $\alpha : D \times D \to [0, 1]$ compactly less than 1 such that

$$d(Tx, Ty) \leq \alpha(x, y)\, M_T(x, y) \tag{wZ}$$

for all $x, y \in D$, where

$$M_T(x, y) := \max\left\{d(x, y), \frac{1}{2}\big[d(x, Tx) + d(y, Ty)\big], \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big]\right\}.$$

It is clear that every Zamfirescu map is a weakly Zamfirescu map. Next we shall show with Example 21 of [2] that the converse is not true.

Example 11. Consider the metric space (X, d), where $X = [0, 1]$ and d is the usual metric. The map $T : [0, 1] \to [0, 1]$ given as

$$Tx = \begin{cases} \frac{2}{3}x & \text{if } 0 \leq x < 1, \\ \\ 0 & \text{if } x = 1, \end{cases}$$

is a weakly Zamfirescu map, since T is a weakly Chatterjea map (for more details, see [2]). Moreover, T is not a Zamfirescu map, since

$$\lim_{x \to 1^-} \frac{d(Tx, T1)}{M_T(x, 1)} = \lim_{x \to 1^-} \frac{\frac{2}{3}x}{\max\{1 - x, \frac{1}{2}(\frac{x}{3} + 1), \frac{1}{2}(x + 1 - \frac{2}{3}x)\}} = 1.$$

The following result assures that every weakly Zamfirescu map satisfies condition $(wE_{\alpha,\mu})$ and inequality (1).

Proposition 12. Let (X, d) be a metric space. If $T : X \to X$ is a weakly Zamfirescu map, then T satisfies condition $(wE_{\alpha,\mu})$ and inequality (1) with $\mu = 3$.

Proof. First we will show that T satisfies condition $(wE_{\alpha,\mu})$. Let $x, y \in X$.

Case 1: If $M_T(x, y) = d(x, y)$, then

$$d(x, Ty) \leq d(x, Tx) + d(Tx, Ty) \leq d(x, Tx) + \alpha(x, y)\, d(x, y).$$

Case 2: If $M_T(x, y) = \frac{1}{2}\big[d(x, Tx) + d(y, Ty)\big]$, then

$$\begin{aligned} d(x, Ty) &\leq d(x, Tx) + d(Tx, Ty) \\ &\leq d(x, Tx) + \frac{\alpha(x, y)}{2}\big[d(x, Tx) + d(y, Ty)\big] \\ &\leq \left[\frac{\alpha(x, y)}{2} + 1\right] d(x, Tx) + \frac{\alpha(x, y)}{2}\,[d(x, y) + d(x, Ty)]\ . \end{aligned}$$

Hence,

$$d(x, Ty) \leq \frac{\alpha(x, y)}{2 - \alpha(x, y)}\, d(x, y) + \frac{2 + \alpha(x, y)}{2 - \alpha(x, y)}\, d(x, Tx).$$

Case 3: If $M_T(x,y) = \frac{1}{2}\big[d(x,Ty)+d(y,Tx)\big]$, then

$$\begin{aligned} d(x,Ty) &\le d(x,Tx) + d(Tx,Ty) \\ &\le d(x,Tx) + \frac{\alpha(x,y)}{2}\big[d(x,Ty)+d(y,Tx)\big] \\ &\le \left[\frac{\alpha(x,y)}{2}+1\right] d(x,Tx) + \frac{\alpha(x,y)}{2}\left[d(x,y)+d(x,Ty)\right] . \end{aligned}$$

Hence,

$$d(x,Ty) \le \frac{\alpha(x,y)}{2-\alpha(x,y)}\, d(x,y) + \frac{2+\alpha(x,y)}{2-\alpha(x,y)}\, d(x,Tx).$$

Since $\frac{t}{2-t} \le t$ and $1 \le \frac{2+t}{2-t} \le 3$ for all $t \in [0,1]$, we get

$$d(x,Ty) \le \alpha(x,y)\, d(x,y) + 3\, d(x,Tx).$$

Next, we will prove that T satisfies inequality (1). Let $x \in X$. We note that

$$\begin{aligned} M_T(x,Tx) &= \max\left\{ d(x,Tx), \frac{1}{2}\big[d(x,Tx)+d(Tx,T^2x)\big], \frac{1}{2}d(x,T^2x)\right\} \\ &\le \max\left\{ d(x,Tx), \frac{1}{2}\big[d(x,Tx)+d(Tx,T^2x)\big], \right. \\ &\qquad\qquad \left. \frac{1}{2}\big[d(x,Tx)+d(Tx,T^2x)\big]\right\} \\ &= \max\left\{ d(x,Tx), \frac{1}{2}\big[d(x,Tx)+d(Tx,T^2x)\big]\right\} . \end{aligned}$$

So, since T is a weakly Zamfirescu map,

$$d(Tx,T^2x) \le \alpha(x,Tx)\, \max\left\{ d(x,Tx), \frac{1}{2}\big[d(x,Tx)+d(Tx,T^2x)\big]\right\}$$

If the maximum is equal to $d(x,Tx)$, we have finished. Otherwise, we have

$$d(Tx,T^2x) \le \frac{\alpha(x,Tx)}{2}\,\big[d(x,Tx)+d(Tx,T^2x)\big].$$

Hence,

$$d(Tx,T^2x) \le \frac{\alpha(x,Tx)}{2-\alpha(x,Tx)}\, d(x,Tx) \le \alpha(x,Tx)\, d(x,Tx).$$

□

As a consequence of the above result and Theorem 7 we get a new result concerning a Cauchy rate for weakly Zamfirescu maps.

Corollary 13. Let (X, d) be a metric space and $T : X \to X$ be a weakly Zamfirescu map. Let $x_0 \in X$ be the starting point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} := Tx_n$. Let $b > 0$ satisfy $d(x_0, x_1) \leq b$, and define $\rho : (0, \infty) \to (0, \infty)$ by

$$\rho(\varepsilon) := \min\left\{\frac{\varepsilon}{6}\left(1 - \theta(\tfrac{\varepsilon}{2}, \varepsilon)\right), b\right\}.$$

If $\gamma : (0, \infty) \to \mathbb{N}$ is defined by (2), then γ is a Cauchy rate for $\{x_n\}_{n\in\mathbb{N}}$.

3.2. **Quasi-contraction maps.** In order to prove a generalization of the Banach contraction principle, in 1974 Ćirić [7] introduced the concept of quasi-contraction maps.

Definition 14. Let D be a nonempty subset of a metric space (X, d). A map $T : D \to X$ is said to be a quasi-contraction if there exists $h \in [0, 1)$ such that

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for any $x, y \in D$.

Clearly, the concept of a quasi-contraction map is more general than the concept of a Zamfirescu map and, therefore, of a contraction, Kannan, and Chatterjea map. However, the class of quasi-contractions with $h < \frac{1}{2}$ coincides with the class of Zamfirescu maps, since one can take $\alpha = 2h$ to show that a quasi-contraction with $h < \frac{1}{2}$ is a Zamfirescu map.

In the following result we prove that quasi-contractions with constant h less than 1/2 satisfy condition $(wE_{\alpha,\mu})$ and inequality (1) with $\alpha = \frac{h}{1-h}$ and $\mu = \frac{1}{1-h}$, improving the constants for the case of weakly Zamfirescu maps in Proposition 12.

Proposition 15. Every quasi-contraction map with $h < \frac{1}{2}$ satisfies condition $(wE_{\alpha,\mu})$ and inequality (1), with $\alpha = \frac{h}{1-h}$ and $\mu = \frac{1}{1-h}$.

Proof. We note that putting $y = Tx$ for any $x \in X$, we have that

$$\begin{aligned} d(Tx, T^2x) &\leq h \max\{d(x, Tx), d(x, Tx), d(Tx, T^2x), d(x, T^2x), d(Tx, Tx)\} \\ &= h \max\{d(x, Tx), d(Tx, T^2x), d(x, T^2x)\} \\ &\leq h \max\{d(x, Tx), d(Tx, T^2x), d(x, Tx) + d(Tx, T^2x)\} \\ &= h\,[d(x, Tx) + d(Tx, T^2x)], \end{aligned}$$

i.e.,

$$d(Tx, T^2x) \leq \frac{h}{1-h}\, d(x, Tx).$$

Thus, T satisfies inequality (1), since $h < \frac{1}{2}$ implies that $\frac{h}{1-h} < 1$.

Next we will prove that T satisfies condition $(wE_{\alpha,\mu})$ with $\alpha = \frac{h}{1-h}$, and $\mu = \frac{1}{1-h}$. Let $x, y \in X$. From the quasi-contraction condition, we get

$$\begin{aligned} d(x, Ty) &\leq d(x, Tx) + d(Tx, Ty) \\ &\leq d(x, Tx) + h\, \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &\leq d(x, Tx) + h\, \max\{d(x, y), d(x, Tx), d(x, y) + d(x, Ty), d(x, Ty), \\ &\qquad\qquad d(x, y) + d(x, Tx)\} \\ &\leq d(x, Tx) + h\, \max\{d(x, y) + d(x, Ty), d(x, y) + d(x, Tx)\}. \end{aligned}$$

Thus,

$$\begin{aligned} d(x, Ty) &\leq \max\left\{\frac{h}{1-h}, h\right\} d(x, y) + \max\left\{\frac{1}{1-h}, 1+h\right\} d(x, Tx) \\ &= \frac{h}{1-h}\, d(x, y) + \frac{1}{1-h}\, d(x, Tx). \end{aligned}$$

□

The following example shows that, in the above result, the constant 1/2 is sharp.

Example 16. Consider the metric space (X, d), where $X = \{1, 2, 3\}$ and $d : X \times X \to \mathbb{R}$ is defined as follows: $d(1,1) = d(2,2) = d(3,3) = 0$, $d(1,2) = d(2,1) = d(2,3) = d(3,2) = 1$ and $d(1,3) = d(3,1) = 2$. Define $T : X \to X$ by $T1 = 2$, $T2 = 2$ and $T3 = 1$. Then, T is a quasi-contraction map with constant $h = \frac{1}{2}$, but T does not satisfy condition $(wE_{\alpha,\mu})$ because, for all $\mu \geq 1$ and any $\alpha : X \times X \to [0, 1]$ compactly less than 1, we have that

$$\alpha(2, 3)\, d(2, 3) + \mu\, d(2, T2) = \alpha(2, 3) \leq \theta(\tfrac{1}{4}, \tfrac{4}{3}) < 1 = d(2, T3).$$

3.3. **More examples.** We now give two examples of how to build a family of maps which satisfy condition $(wE_{\alpha,\mu})$. The first one is motivated by Example 3 in [10].

Example 17. Let a_1, a_2, c, h be four real numbers such that $a_1 < 0 < a_2$, $0 \leq c < 1$ and $|h| \leq c\, a_2$. If we define $T : [a_1, a_2] \to \mathbb{R}$ as

$$Tx = \begin{cases} c\ |x| & \text{if } x \in [a_1, a_2), \\ \\ h & \text{if } x = a_2, \end{cases}$$

then T satisfies condition $(wE_{\alpha,\mu})$, with $\alpha = c$ and $\mu = \frac{1+c}{1-c}$. We will consider the following non-trivial cases.

Case 1.: $x \in [a_1, 0]$ and $y \in [a_1, a_2]$. In this case, we note that

$$|x - Tx| = |x + cx| = (1 + c)\ |x|\,.$$

Thus,

$$\begin{aligned} |x - Ty| &\leq |x| + |Ty| \\ &\leq |x| + c\ |y| \\ &\leq |x| + c\,[|x| + |x - y|] \\ &\leq (1 + c)\ |x| + c\ |x - y| \\ &= |x - Tx| + c\ |x - y|\,. \end{aligned}$$

Case 2.: $x \in [0, a_2)$ and $y \in [a_1, a_2]$. Since

$$|x - Tx| = |x - cx| = (1 - c)\ |x|\ ,$$

then

$$\begin{aligned} |x - Ty| &\leq |x| + |Ty| \\ &\leq |x| + c\ |y| \\ &\leq |x| + c\,[|x| + |x - y|] \\ &\leq (1 + c)\ |x| + c\ |x - y| \\ &= \frac{1 + c}{1 - c}\ |x - Tx| + c\ |x - y|\,. \end{aligned}$$

Case 3.: $x = a_2$ and $y \in [a_1, a_2)$. In this case,

$$|x - Tx| = |a_2 - h| = a_2 - h \geq (1 - c)\, a_2,$$

since $|h| \leq c\, a_2$. Thus,

$$\begin{aligned} |x - Ty| &= |a_2 - c\,|y|| \\ &\leq (1-c)\, a_2 + c\, |a_2 - |y|| \\ &\leq (1-c)\, a_2 + c\, |a_2 - y| \\ &\leq |x - Tx| + c\, |x - y|\,. \end{aligned}$$

Remark 18. Note that there exist maps belonging to the family given in Example 17 which are not weakly Zamfirescu maps. Indeed, if we take $a_1 = -1$, $a_2 = 1$, $c = \frac{4}{5}$ and $h = -\frac{4}{5}$, then T is not a weakly Zamfirescu map, since

$$\frac{\left|T\frac{1}{2} - T1\right|}{M_T(\frac{1}{2}, 1)} = \frac{\frac{6}{5}}{\frac{19}{20}} = \frac{24}{19} > 1.$$

Example 19. Let X be the set of nonnegative real numbers with its usual metric. We assume that $g : [0, \infty) \to \mathbb{R}$ is a function such that for some fixed $n \in \mathbb{N}$ there exists $x_n \in [0, \infty)$ with $g(x_n) = 2n + \frac{3}{2}$. For fixed $k \in (0, 1)$ and $\tau > x_n$, we consider $T : [0, \infty) \to \mathbb{R}$ defined as

$$Tx = \begin{cases} k\, x\, \sin(\pi\, g(x)) & \text{if } 0 \leq x \leq x_n, \\[2ex] -k\, x_n & \text{if } x_n < x \leq x_{n,\tau}, \\[2ex] \frac{k\, x\, (\tau - x)}{x - x_n} & \text{if } x > x_{n,\tau}, \end{cases}$$

where $x_{n,\tau}$ is a fixed real number with $x_{n,\tau} > \tau$. It is easy to check that T is continuous on $[0, \infty)$ if and only if

$$x_{n,\tau} = \frac{x_n + \tau}{2} \left(1 + \sqrt{1 - \left(\frac{2\, x_n}{x_n + \tau}\right)^2}\right).$$

Before proving that T satisfies condition $(wE_{\alpha,\,\mu})$ we will show that $x \leq \frac{1}{1-k}\, |x - Tx|$ for all $x \geq 0$. In order to do this, we consider three cases.

Case 1: Suppose that $x \in [0, x_n]$. Then,

$$|x - Tx| = |x - k\, x\, \sin(\pi\, g(x))| = x\, (1 - k\, \sin(\pi\, g(x))) \geq x\, (1 - k).$$

Case 2: If $x \in (x_n, x_{n,\tau}]$,

$$|x - Tx| = x + k\, x_n \geq x.$$

Case 3: Suppose that $x > x_{n,\tau}$.

$$|x - Tx| = \left| x - \frac{k\, x\, (\tau - x)}{x - x_n} \right| = x \left(1 + \frac{k\, (x - \tau)}{x - x_n} \right) \geq x.$$

Therefore,

$$x \leq \frac{1}{1-k}\ |x - Tx|$$

for all $x \geq 0$. Bearing in mind that $|Tx| \leq k\, x$ for all $x \geq 0$, we get

$$\begin{aligned} |x - Ty| &\leq |x| + |Ty| \\ &\leq x + k\, y \leq (1+k)\, x + k\ |x - y| \\ &\leq \frac{1+k}{1-k}\ |x - Tx| + k\ |x - y|, \end{aligned}$$

for all $x, y \geq 0$. Hence, T satisfies condition $(wE_{\alpha,\mu})$ with $\alpha = k$, and $\mu = \frac{1+k}{1-k}$.

4. Modulus of uniqueness

In this section we will study the existence of a modulus of uniqueness for the class of maps satisfying condition $(wE_{\alpha,\mu})$. We will use the results of this section to improve Theorem 7, and to establish the fixed point results that appear in the next section. In the literature on approximation theory moduli of uniqueness often show up as rates of strong uniqueness or strong unicity, whereas in its full generality the concept of a modulus of uniqueness was introduced by Kohlenbach [12].

Definition 20. Let (X, d) be a metric space, and let $T : X \to X$. We say that $\phi : (0, \infty) \to (0, \infty)$ is a modulus of uniqueness (for fixed points of T) if for all $\varepsilon > 0$ and $x, y \in X$ we have

$$\left. \begin{array}{l} d(x, Tx) \leq \phi(\varepsilon) \\ \\ d(y, Ty) \leq \phi(\varepsilon) \end{array} \right\} \Rightarrow d(x, y) \leq \varepsilon.$$

Theorem 21. Let (X, d) be a metric space. Suppose that $T : X \to X$ satisfies condition $(wE_{\alpha,\mu})$. Define $\psi(\theta, \mu, \cdot\,, \cdot\,) : (0,\infty) \times (0,\infty) \to (0,\infty)$ by

$$\psi(\theta, \mu, b, t) := \frac{t\,(1 - \theta(t, b))}{\mu + 1},$$

where θ and μ are as in Definition 3 and Definition 5. Then for all $\varepsilon \in (0,\infty)$ and for all $x, y \in X$ with $d(x,y) \le b$ we have the following implication:

$$\left.\begin{aligned} d(x,Tx) &\le \psi(\theta,\mu,b,\varepsilon) \\ d(y,Ty) &\le \psi(\theta,\mu,b,\varepsilon) \end{aligned}\right\} \Rightarrow d(x,y) \le \varepsilon.$$

Proof. Let $\varepsilon, b \in (0,\infty)$ and $x, y \in X$ with $d(x,y) \le b$. Assume that $d(x,Tx) \le \psi(\theta,\mu,b,\varepsilon)$ and $d(y,Ty) \le \psi(\theta,\mu,b,\varepsilon)$. We will prove that $d(x,y) \le \varepsilon$. To do this, we will argue by contradiction. Suppose that $d(x,y) > \varepsilon$. We note

$$\begin{aligned} d(x,y) &\le d(x,Ty) + d(y,Ty) \\ &\le \alpha(x,y)\, d(x,y) + \mu\, d(x,Tx) + d(y,Ty) \\ &\le \alpha(x,y)\, d(x,y) + (\mu+1)\, \psi(\theta,\mu,b,\varepsilon). \end{aligned}$$

So, we have that $\alpha(x,y) \le \theta(\varepsilon,b) < 1$. Thus,

$$d(x,y) \le \theta(\varepsilon,b)\, d(x,y) + (\mu+1)\, \psi(\theta,\mu,b,\varepsilon),$$

i.e.,

$$d(x,y) \le \frac{\mu+1}{1-\theta(\varepsilon,b)}\, \psi(\theta,\mu,b,\varepsilon).$$

Therefore,

$$\varepsilon < d(x,y) \le \frac{\mu+1}{1-\theta(\varepsilon,b)} \cdot \psi(\theta,\mu,b,\varepsilon) = \varepsilon,$$

which is a contradiction. □

Corollary 22. Let $\mu \ge 1$ and let $\theta : (0,\infty) \times (0,\infty) \to [0,1)$. Let $b > 0$ and define $\psi(\theta,\mu,b,\,\cdot\,)$ as in Theorem 21. Then $\psi(\theta,\mu,b,\,\cdot\,)$ is a modulus of uniqueness for any map $T : X \to X$ on a b-bounded metric space (X,d) which satisfies $(wE_{\alpha,\mu})$, with μ and θ satisfying the requirements in Definition 5 and Definition 3.

Proof. By Theorem 21. □

The following proposition shows that we cannot remove the dependence on $b > 0$ in the theorem above: Given (X,d) and $T : X \to X$ which satisfy $(wE_{\alpha,\mu})$, there does not exist a modulus of uniqueness for T which depends on (X,d) and T only through μ and θ.

Proposition 23. There does not exist a function $\phi : [0,1)^{(0,\infty)\times(0,\infty)} \times [1,\infty) \times (0,\infty) \to (0,\infty)$ such that $\phi(\theta,\mu,\cdot\,) : (0,\infty) \to (0,\infty)$ works as a modulus of uniqueness for all metric spaces (X,d) and all maps $T : X \to X$ satisfying $(wE_{\alpha,\mu})$ with μ and θ satisfying the requirements in Definition 5 and Definition 3.

Proof. Assume for a contradiction that $\phi(\theta,\mu,\cdot\,) : (0,\infty) \to (0,\infty)$ satisfies the condition

$$\forall\varepsilon > 0 \forall x,y \in X\Big(\big(d(x,Tx) \le \phi(\theta,\mu,\varepsilon) \wedge d(y,Ty) \le \phi(\theta,\mu,\varepsilon)\big) \Rightarrow d(x,y) \le \varepsilon\Big)$$

for all metric spaces (X,d) and maps $T : X \to X$ which satisfy $(wE_{\alpha,\mu})$ with μ and θ satisfying the requirements in Definition 5 and Definition 3.

Let $\{x_n^1\}_{n\in\mathbb{N}}$ and $\{x_n^2\}_{n\in\mathbb{N}}$ be two sequences of real numbers, with $x_n^i \neq x_m^j$ if $n \neq m$ or $i \neq j$. Let $\delta \in (0,1)$ and $k_0 \in \mathbb{N}$ with $k_0 \ge \frac{1}{4\delta}$. We equip $X = \{x_n^1\}_{n\ge k_0} \cup \{x_n^2\}_{n\ge k_0}$ with the metric

$$d(x_n^i, x_m^j) = \begin{cases} 0 & \text{if } n = m \text{ and } i = j, \\ \delta & \text{if } n = m \text{ and } i \neq j, \\ n+m & \text{if } n \neq m. \end{cases}$$

Let $T : X \to X$ be a map given by $Tx_n^1 = x_n^2$ and $Tx_n^2 = x_n^1$ for each $n \ge k_0$. Next we will check that T satisfies condition $(wE_{\alpha,\mu})$ with $\mu := 2$, $\alpha : X \times X \to [0,1]$ given by

$$\alpha(x,y) := \begin{cases} \max\left\{\frac{1}{2}, 1 - \frac{1}{d(x,y)^2}\right\} & \text{if } x \neq y, \\ \\ 0 & \text{if } x = y, \end{cases}$$

and with θ (as in Remark 4) given by

$$\theta(a,b) := \max\left\{\frac{1}{2}, 1 - \frac{1}{b^2}\right\}.$$

We consider the non-trivial case $x = x_n^i$ and $y = x_m^j$ with $n,m \ge k_0$, $n \neq m$ and $i,j \in \{1,2\}$. Since

$$\frac{1}{n+m} \le \frac{1}{2k_0} \le 2\delta$$

we get

$$\begin{aligned} d(x, Ty) &= n + m \\ &\leq n + m - \frac{1}{n+m} + 2\,\delta \\ &= d(x,y) - \frac{1}{d(x,y)} + 2\,d(x,Tx) \\ &\leq \alpha(x,y)\,d(x,y) + 2\,d(x,Tx). \end{aligned}$$

Note that μ and also θ are the same for all choices of $\delta \in (0,1)$. Let now $\delta := \min\{1/2, \phi(\theta,\mu,1)\}$. By taking $n = k_0 + 1$ and $m = k_0 + 2$ we get

$$d(x_n^1, Tx_n^1) = d(x_n^1, x_n^2) = \delta \leq \phi(\theta,\mu,1)$$

and

$$d(x_m^1, Tx_m^1) = d(x_m^1, x_m^2) = \delta \leq \phi(\theta,\mu,1).$$

So,

$$d(x_n^1, x_m^1) \leq 1,$$

which is a contradiction, since $d(x_n^1, x_m^1) = n + m \geq 2$. □

We can use Theorem 21 to improve Theorem 7 in the following way:

Theorem 24. Let (X,d) be a metric space and $T : X \to X$ be a map which satisfies conditions $(wE_{\alpha,\mu})$ and (1), with μ and θ as in Definition 5 and Definition 3. Let $x_0 \in X$ be the starting point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} := Tx_n$. If in addition to the starting point $x_0 \in X$ we have another starting point $y_0 \in X$, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0, \tag{6}$$

where $\{y_n\}_{n\in\mathbb{N}}$ is defined by $y_n = T^n y_0$, for every $n \in \mathbb{N}$.

Proof. In order to show (6), suppose this equality does not hold. Then there exist $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for every $n \geq k_0$ we have

$$d(x_{k_0}, x_n) < \frac{\varepsilon}{4}\,,$$

$$d(y_{k_0}, y_n) < \frac{\varepsilon}{4}$$

and

$$d(x_{k_0}, y_{k_0}) = \varepsilon.$$

So,

$$\frac{\varepsilon}{2} < d(x_n, y_n) < \frac{3\,\varepsilon}{2}$$

for all $n \geq k_0$. Now, we can take $b = \frac{3\varepsilon}{2}$ in Theorem 21 and $n \geq k_0$ so big that

$$d(x_n, x_{n+1}) \leq \psi(\theta, \mu, b, \tfrac{\varepsilon}{2})$$

and

$$d(y_n, y_{n+1}) \leq \psi(\theta, \mu, b, \tfrac{\varepsilon}{2})$$

to get

$$d(x_n, y_n) \leq \frac{\varepsilon}{2}\,,$$

which is a contradiction. □

Using the above theorem, Theorem 7, and Theorem 21 we can show that for maps satisfying conditions $(wE_{\alpha,\mu})$ and (1) there exists a modulus of uniqueness which does not depend on a bound $b > 0$ on the diameter of the space. Proposition 23 shows that this is in sharp contrast with the situation for maps satisfying only condition $(wE_{\alpha,\mu})$. Before proving this result, we need two lemmas.

Lemma 25. Let (X, d) be a metric space, $\varepsilon > 0$, $x_0 \in X$, and $T : X \to X$ a map satisfying conditions $(wE_{\alpha,\mu})$ and (1), with μ and θ as in Definition 5 and Definition 3. Then, we have the following implication:

$$d(y_0, x_{k_0}) > b \cdot \gamma(\theta, \mu, b, \varepsilon) + 3\,\varepsilon \;\Rightarrow\; d(y_0, Ty_0) > b$$

for all $y_0 \in X$ and $b \in (0, \infty)$, where $k_0 = \gamma(\theta, \mu, d(x_0, x_1), \varepsilon)$ and γ is the Cauchy rate from Theorem 7.

Proof. Let $y_0 \in X$ and $b \in (0, \infty)$. Suppose that $d(y_0, Ty_0) \leq b$. Using the triangle inequality, we have

$$d(y_0, y_{k_1}) \leq k_1\, b,$$

where $k_1 = \gamma(\theta, \mu, b, \varepsilon)$. By Theorem 7, $d(y_{k_1}, y_n) \leq \varepsilon$ for all $n \geq k_1$. So, we have that

$$d(y_0, y_n) \leq d(y_0, y_{k_1}) + d(y_{k_1}, y_n) \leq k_1\, b + \varepsilon$$

for all $n \geq k_1$. Again, using Theorem 7, we get $d(x_{k_0}, x_n) \leq \varepsilon$ for all $n \geq k_0$, where $k_0 = \gamma(\theta, \mu, d(x_0, x_1), \varepsilon)$. On the other hand, by Theorem 24, there exists $k_2 \in \mathbb{N}$ such that $d(x_n, y_n) < \varepsilon$ for all $n \geq k_2$. Therefore we can take $n \geq \max\{k_0, k_1, k_2\}$ to get

$$\begin{aligned} d(y_0, x_{k_0}) &\leq d(y_0, y_n) + d(y_n, x_n) + d(x_n, x_{k_0}) \\ &< k_1\, b + 3\,\varepsilon. \end{aligned}$$

□

Lemma 26. Let (X, d) be a metric space, $T : X \to X$ a map satisfying conditions $(wE_{\alpha,\mu})$ and (1), with μ and θ as in Definition 5 and Definition 3. Let $x, y \in X$ and $b > 0$. If $d(x, Tx) \leq b$ and $d(y, Ty) \leq b$, then

$$d(x, y) \leq 2(b\,\gamma(\theta, \mu, b, \varepsilon) + 3\,\varepsilon)$$

for all $\varepsilon > 0$.

Proof. By Lemma 25. □

Theorem 27. Let (X, d) be a metric space, $x_0 \in X$ and $T : X \to X$ be a map satisfying conditions $(wE_{\alpha,\mu})$ and (1), with μ and θ as in Definition 5 and Definition 3. Define $\psi(\theta, \mu, \cdot\,) : (0, \infty) \to (0, \infty)$ by

$$\psi(\theta, \mu, t) := \min\left\{1, \frac{t\,[1 - \theta(t, 2\,(\gamma(\theta, \mu, 1, 1) + 3))]}{\mu + 1}\right\}.$$

Then for all $\varepsilon \in (0, \infty)$ and for all $x, y \in X$ with $d(x, Tx) \leq \psi(\theta, \mu, \varepsilon)$ and $d(y, Ty) \leq \psi(\theta, \mu, \varepsilon)$, we have that $d(x, y) \leq \varepsilon$.

Proof. Let $\varepsilon > 0$ and $x, y \in X$ with $d(x, Tx) \leq \psi(\theta, \mu, \varepsilon) \leq 1$ and $d(y, Ty) \leq \psi(\theta, \mu, \varepsilon) \leq 1$. By Lemma 26, we have

$$d(x, y) \leq 2\,\big(\gamma(\theta, \mu, 1, 1) + 3\big). \tag{7}$$

Now the result follows by Theorem 21. □

If we exchange being compactly less than 1 for another condition (type Rakotch), we can delete the boundedness hypothesis in Theorem 21 without requiring T to satisfy condition (1).

Theorem 28. Let D a nonempty subset of a metric space (X, d) and $T : D \to X$ a map such that

$$d(x, Ty) \leq \alpha(\varepsilon)\, d(x, y) + \mu\, d(x, Tx) \tag{$wE'_{\alpha,\mu}$}$$

for all $\varepsilon > 0$ and for all $x, y \in D$ with $d(x,y) > \varepsilon$, where $\mu \geq 1$ and $\alpha : (0,\infty) \to [0,\infty)$ satisfies that $0 \leq \alpha(t) < 1$ for all $t > 0$. If we define $\psi : (0,\infty) \to (0,\infty)$ by

$$\psi(t) := \frac{t\,(1-\alpha(t))}{\mu+1},$$

then for all $\varepsilon \in (0,\infty)$ and for all $x, y \in X$, if $d(x,Tx) \leq \psi(\varepsilon)$ and $d(y,Ty) \leq \psi(\varepsilon)$, then $d(x,y) \leq \varepsilon$.

Remark 29. Actually Rakotch [14] considered that α must also be a monotonically decreasing function.

Proof. Assume that $x, y \in X$ such that $d(x,Tx) \leq \psi(\varepsilon)$ and $d(y,Ty) \leq \psi(\varepsilon)$. We will prove that $d(x,y) \leq \varepsilon$. To do this, we will argue by contradiction. Suppose that $d(x,y) > \varepsilon$. We note

$$\begin{aligned} d(x,y) &\leq d(x,Ty) + d(y,Ty) \\ &\leq \alpha(\varepsilon)\, d(x,y) + \mu\, d(x,Tx) + d(y,Ty) \\ &\leq \alpha(\varepsilon)\, d(x,y) + (\mu+1)\psi(\varepsilon). \end{aligned}$$

Since $\alpha(\varepsilon) < 1$, we have that

$$d(x,y) \leq \frac{\mu+1}{1-\alpha(\varepsilon)}\,\psi(\varepsilon) = \varepsilon,$$

which is a contradiction. □

5. A fixed point result

Under the assumption of completeness, we use Theorem 7 and Theorem 24 to obtain the following fixed point theorem.

Theorem 30. Let (X,d) be a nonempty, complete metric space, and let $T : X \to X$ be a map satisfying the conditions $(wE_{\alpha,\mu})$ and (1). Then T has a unique fixed point p in X, and all Picard iteration sequences converge to p, with rates of convergence given by Theorem 7.

Proof. Let $x_0 \in X$. By Theorem 7 we know that $\{T^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and since (X,d) is complete, there exists p in X such that $T^n x_0 \to p$ as $n \to \infty$. Assume that p is not a fixed point of T. Then there exist $\varepsilon > 0$ and $y_0 \in X$ with $Tp = y_0$ and $d(p,y_0) = \varepsilon$. By Theorem 24 we know that

$$\lim_{n\to\infty} y_n = p,$$

where $\{y_n\}_{n\in\mathbb{N}}$ is defined by $y_n = T^n y_0$ for each $n \in \mathbb{N}$. By taking $x = y_n$ and $y = p$ in $(wE_{\alpha,\mu})$ we get that for each $n \in \mathbb{N}$

$$\begin{aligned} d(y_n, Tp) &\le \alpha(y_n, p)\, d(y_n, p) + \mu\, d(y_n, Ty_n) \\ &\le d(y_n, p) + \mu\, d(y_n, y_{n+1}). \end{aligned}$$

Taking limits as $n \to \infty$, we obtain that $\varepsilon \le 0$, which is a contradiction.

Finally, we shall prove the uniqueness of the fixed point p. Suppose that q is another fixed point of T, with $p \neq q$. By Theorem 24 we know that $\lim_{n\to\infty} T^n q = p$, which is a contradiction. □

Remark 31. Note that if we check the above proof carefully, we can show that any map satisfying the conditions $(wE_{\alpha,\mu})$ and (1) is continuous at its unique fixed point.

As a consequence of the previous theorem, we obtain the following local result.

Theorem 32. Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Assume that $T : \overline{B(x_0, r)} \to X$ satisfies the conditions $(wE_{\alpha,\mu})$ and (1). If θ is defined as usual, and

$$d(x_0, Tx_0) \le \frac{r}{\mu} \min\left\{\frac{1}{2}, 1 - \theta\left(\tfrac{r}{2}, r\right)\right\},$$

then T has a unique fixed point in $\overline{B(x_0, r)}$.

Proof. Using the above result, we only have to show that $\overline{B(x_0, r)}$ is invariant under T. In order to do this, let $x \in \overline{B(x_0, r)}$. We shall consider two cases. If $d(x_0, x) \le \frac{r}{2}$, by $(wE_{\alpha,\mu})$, we have

$$\begin{aligned} d(x_0, Tx) &\le \alpha(x_0, x)\, d(x_0, x) + \mu\, d(x_0, Tx_0) \\ &\le \frac{r}{2} + \mu \frac{r}{2\,\mu} = r. \end{aligned}$$

Otherwise, $\frac{r}{2} \le d(x_0, x) \le r$. So, $\alpha(x_0, x) \le \theta(\frac{r}{2}, r) < 1$. Then,

$$\begin{aligned} d(x_0, Tx) &\le \alpha(x_0, x)\, d(x_0, x) + \mu\, d(x_0, Tx_0) \\ &\le \theta(\tfrac{r}{2}, r)\, r + \mu \frac{r}{\mu} \left(1 - \theta(\tfrac{r}{2}, r)\right) = r. \end{aligned}$$

□

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