REMARKS ON MULTIVALUED NONEXPANSIVE MAPPINGS

BY

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Abstract. Convergence of fixed point sets of multivalued nonexpansive mappings is studied under both the Mosco and Hausdorff senses. A characterization for *-nonexpansive multivalued mappings is given. Also a counterexample is constructed to show a negative answer to a question raised by A. Canetti, G. Marino and P. Piriromala.

Let $H$ be a Hilbert space, $C$ a bounded closed convex subset of $H$ and $T: C \to C$ is a (single-valued) nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$). Then for each fixed $x_0 \in C$ and $\lambda \in [0, 1)$, the mapping $T_\lambda : C \to C$ defined by

$$T_\lambda x = (1 - \lambda)x_0 + \lambda Tx, \quad x \in C$$

is a contraction on $C$. Hence, Banach's Contraction Principle yields a unique $x_\lambda \in C$ such that $T_\lambda x_\lambda = x_\lambda$; namely,

$$x_\lambda = (1 - \lambda)x_0 + \lambda Tx_\lambda.$$  

An elegant result in the fixed point theory of (single-valued) nonexpansive mappings is Browder's theorem [1] which states that the approximating curve $x_\lambda$ defined by (2) converges strongly as $\lambda \to 1$ to a fixed point of $T$. This result

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was extended by Reich [11] to a framework of a uniformly smooth Banach space. For recent progress along the line, the reader is referred to [12], [7], [14].

Now we turn to the multivalued case. For a metric space \((X,d)\), we use \(CB(X)\) to denote the family of all nonempty closed bounded subsets of \(X\), \(K(X)\) the family of all nonempty compact subsets of \(X\), and \(H\) the Hausdorff metric on \(CB(X)\) induced by the metric \(d\) of \(X\); that is, for \(A, B \in CB(X)\),

\[
H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},
\]

where \(d(x, E) = \inf \{d(x, y) : y \in E\}\) is the distance from a point \(x \in X\) to a subset \(E \subseteq X\). Now recall that a multivalued mapping \(T : X \to CB(X)\) is said to be nonexpansive if

\[
H(Tx, Ty) \leq d(x, y), \quad x, y \in X.
\]

Recall also that a sequence \(\{A_n\}\) in \(CB(X)\) is said to converge to an element \(A \in CB(X)\) under the Mosco sense if

\[
\limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n = A,
\]

where \(\limsup_{n \to \infty} A_n = \{x \in X : \text{there are subsequences } \{n_k\} \text{ and } \{x_{n_k}\} \text{ with } x_{n_k} \in A_{n_k} \text{ such that } x_{n_k} \to x\}\) and \(\liminf_{n \to \infty} A_n = \{x \in X : \text{there exists } x_n \in A_n \text{ for each } n \text{ such that } x_n \to x\}\). It is not hard to see that if \(H(A_n, A) \to 0 (A_n, A \in CB(X))\), then \(A_n \to A\) under the sense of Mosco.

Assume now \(H\) and \(C\) are as above and \(T : C \to K(C)\) is nonexpansive. For each fixed \(x_0 \in C\) and \(\lambda \in [0, 1)\), we define the mapping \(T_\lambda : C \to K(C)\) by the same formula (1) above. Then \(T_\lambda\) is a multivalued contraction and hence has a (nonunique, in general) fixed point \(x_\lambda \in C\) (see[8]); i.e.,

\[
x_\lambda \in (1 - \lambda)x_0 + \lambda Tx_\lambda.
\]

Let \(y_\lambda \in Tx_\lambda\) be such that

\[
x_\lambda = (1 - \lambda)x_0 + \lambda y_\lambda.
\]
A natural question now gives rise to whether Browder’s theorem can be extended to the multivalued case. The following simple example presents a negative answer.

**Example 1.** [10] Let \( C = [0, 1] \times [0, 1] \) be the square in the real plane and \( T : C \to K(C) \) be defined by

\[
T(a, b) = \text{the triangle with vertices } (0, 0), (a, 0), (0, b), \quad (a, b) \in C.
\]

Then it is easy to see that for any \((a_i, b_i) \in C, i = 1, 2,\)

\[
H(T(a_1, b_1), T(a_2, b_2)) = \max \{|a_1 - a_2|, |b_1 - b_2|\} \leq \| (a_1, b_1) - (a_2, b_2) \|,
\]

showing that \(T\) is nonexpansive. It is also easy to see that the fixed point set of \(T\) is \(F(T) = \{(a, 0) : 0 \leq a \leq 1\} \cup \{(0, b) : 0 \leq b \leq 1\}\). Let \(x_0 = (1, 0)\). Then the map \(T_\lambda\) defined by (1) has the fixed point set

\[
F(T_\lambda) = \{(a, 0) : 1 - \lambda \leq a \leq 1\}.
\]

Let

\[
x_\lambda = \begin{cases} 
\left( \frac{1}{\lambda}, 0 \right), & \text{if } \lambda = 1 - \frac{1}{n}; \\
(1, 0), & \text{otherwise}.
\end{cases}
\]

Then \(\{x_\lambda\}\) satisfies (3) but is not convergent.

The same example also shows that the net \(\{F(T_\lambda)\}\) of fixed point sets of the \(T_\lambda\)’s does not converge as \(\lambda \to 1\) to the fixed point set \(F(T)\) of \(T\) under either the Hausdorff metric or the Mosco sense. However, this will be so if we put some restrictions on the fixed point set \(F(T)\) of \(T\). First recall that a Banach space \(X\) is said to satisfy Opial’s property [9] if for any sequence \(\{x_n\}\) in \(X\), the condition that \(\{x_n\}\) converges weakly to \(x\) implies that

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \quad \forall y \in X, \quad y \neq x.
\]

Spaces satisfying this property include all Hilbert spaces and \(\ell^p\) for \(1 < p < \infty\). Also it is known [3] that any separable Banach space can be equivalently renormed so that it possesses Opial’s property.
Theorem 1. Let $C$ be a nonempty closed bounded convex subset of a Banach space $X$ satisfying Opial’s property and $T : C \to K(C)$ be a nonexpansive mapping such that $F(T) = \{z\}$. Then for any $x_0 \in C$, the net $\{F(T_\lambda)\}$ of fixed point sets of the $T_\lambda$’s weakly converges as $\lambda \to 1$ to the fixed point set $F(T)$ of $T$ under the Mosco sense, i.e.,

$$w - \limsup_{\lambda \to 1} F(T_\lambda) = w - \liminf_{\lambda \to 1} F(T_\lambda) = F(T).$$

Proof. It is sufficient to show that

(i) $F(T) \supseteq w - \limsup_{\lambda \to 1} F(T_\lambda)$ and

(ii) $w - \liminf_{\lambda \to 1} F(T_\lambda) \supseteq F(T)$.

To show (i), we assume that $x \in w - \limsup_{\lambda \to 1} F(T_\lambda)$, which means that there exist a sequence $\lambda_n \in [0, 1)$ converging to 1 and a sequence $\{x_n\}$ such that $x_n \in F(T_{\lambda_n})$ and $x_n \to x$ weakly. Let $y_n \in Tx_n$ be such that $x_n = (1 - \lambda_n)x_0 + \lambda_n y_n$. Choose $z_n \in Tx$ satisfying

$$\|y_n - z_n\| \leq H(Tx_n, Tx) \leq \|x_n - x\|. \quad (5)$$

Since $Tx$ is compact, we may assume that $z_n \to z_\infty \in Tx$ strongly. Noting that $\|x_n - y_n\| \to 0$, we obtain by (5) that

$$\limsup \|x_n - z_\infty\| \leq \limsup \|x_n - x\|. \quad (6)$$

Since $x_n \to x$ weakly, it follows from (6) and Opial’s property that $x = z_\infty$ and $x \in Tx$. This concludes the proof of (i). Next we show (ii). For each $\lambda \in [0, 1)$, choose any $x_\lambda \in F(T_\lambda)$ and $y_\lambda \in Tx_\lambda$ satisfying (4). Then by the same proof as above, we see that every weak cluster point of $\{x_\lambda\}$ is a fixed point of $T$. But, by assumption, $F(T) = \{z\}$. Hence $\{x_\lambda\}$ converges weakly as $\lambda \to 1$ to $z$.

If the unique fixed point $z$ of $T$ is such that $Tz = \{z\}$, then we have the following strong convergence result.

Theorem 2. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $T : C \to K(C)$ be a nonexpansive mapping with a unique fixed point $z$. Suppose in addition that $Tz = \{z\}$. Then $H(F(T_\lambda), F(T)) \to 0$ as $\lambda \to 1$. 

**Proof.** First we observe that \( \{F(T_\lambda)\} \) is uniformly bounded. In fact, given any \( x_\lambda \in F(T_\lambda) \), we have some \( y_\lambda \in Tx_\lambda \) such that \( x_\lambda = (1 - \lambda)x_0 + \lambda y_\lambda \). However,

\[
\|y_\lambda - z\| = d(y_\lambda, Tz) \leq H(Tx_\lambda, Tz) \leq \|x_\lambda - z\|,
\]

Hence

\[
\|x_\lambda - z\| \leq \lambda \|y_\lambda - z\| + (1 - \lambda) \|x_0 - z\| \leq \lambda \|x_\lambda - z\| + (1 - \lambda) \|x_0 - z\|.
\]

This implies that \( \|x_\lambda - z\| \leq \|x_0 - z\| \) and \( \{x_\lambda\} \) is uniformly bounded. Now choose \( x_\lambda \in F(T_\lambda) \) such that

\[
H(F(T_\lambda), F(T)) = \sup_{x \in F(T_\lambda)} \|x - z\| < \|x_\lambda - z\| + 1 - \lambda.
\]

We shall show that \( \|x_\lambda - z\| \to 0 \) as \( \lambda \to 1 \). Indeed, we have \( y_\lambda \in Tx_\lambda \) satisfying (4). Since \( \|y_\lambda - z\| = d(y_\lambda, Tz) \leq H(Tx_\lambda, Tz) \leq \|x_\lambda - z\| \), we obtain

\[
\left\| \frac{x_\lambda - (1 - \lambda)x_0}{\lambda} - z \right\| \leq \|x_\lambda - z\|; \text{ that is},
\]

\[
\left\| \frac{x_\lambda - x_0}{\lambda} + (x_0 - z) \right\|^2 \leq \|(x_\lambda - x_0) + (x_0 - z)\|^2,
\]

which leads to

\[
\|\lambda - x_0\|^2 \leq \frac{2\lambda}{1 + \lambda} \langle x_\lambda - x_0, z - x_0 \rangle \leq \|x_\lambda - x_0\| \|z - x_0\|.
\]

Therefore,

\[
\|x_\lambda - x_0\| \leq \|z - x_0\|. \tag{7}
\]

From the proof of theorem 1, we know that \( x_\lambda \to z \) weakly as \( \lambda \to 1 \). It then easily follows from (7) that \( \limsup_{\lambda \to 1} \|x_\lambda\| \leq \|z\| \). On the other hand, due to the lower weak continuity of the norm of \( H \), we have \( \liminf_{\lambda \to 1} \|x_\lambda\| \geq \|z\| \). Therefore, we have \( \lim_{\lambda \to 1} \|x_\lambda\| = \|z\| \) and

\[
\|x_\lambda - z\|^2 = \|x_\lambda\|^2 - 2\langle x_\lambda, z \rangle + \|z\|^2 \to 0 \text{ as } \lambda \to 1.
\]

This completes the proof of the theorem.
Corollary 1. [10] Let the assumptions of theorem 2 be satisfied. Then

$$w - \limsup_{\lambda \to 1} F(T_\lambda) = \| \cdot \| - \liminf_{\lambda \to 1} F(T_\lambda) = F(T).$$

Remark 1. The example above shows that the conclusions of theorems 1 and 2 are not valid if the fixed point set $F(T)$ of $T$ is not a singleton. However, it is an open question whether the restriction $Tz = \{z\}$ in Theorem 2 can be removed. We also do not know if Theorem 2 is valid outside a Hilbert space.

Next we let $(X,d)$ be a metric space. A multivalued map $f : X \to K(X)$ is said to be *-nonexpansive [4] if for all $x, y \in X$ and $u_x \in f(x)$ with $d(x,u_x) = \inf \{d(x,z) : z \in f(x)\}$, there exists $u_y \in f(y)$ with $d(y,u_y) = \inf \{d(y,w) : w \in f(y)\}$ such that

$$d(u_x,u_y) \leq d(x,y).$$

It is obvious that this notion is identical with the notion of nonexpansiveness for singlevalued mappings. But they are different for multivalued mappings (see [13]). We now give a characterization of multivalued *-nonexpansive mappings. Denote by $P_f$ the map $x \mapsto \{u_x \in f(x) : d(x,u_x) = \inf \{d(u,x) : u \in f(x)\}\}$. Note that $P_f(x)$ is nonempty for $f(x)$ is compact.

Theorem 3. A multivalued map $f : X \to K(X)$ is *-nonexpansive if and only if the associated map $P_f : X \to K(X)$ is nonexpansive.

Proof. First assume that $f$ is *-nonexpansive. Given any $x, y \in X$ and $u_x \in P_f(x)$. By definition, there is $u_x \in f(y)$ such that $d(u_x,u_y) \leq d(x,y)$. It follows that

$$\sup_{u_x \in P_f(x)} d(u_x,P_f(y)) \leq \sup_{u_x \in P_f(x)} d(u_x,u_y) \leq d(x,y).$$

The same argument shows that

$$\sup_{u_y \in P_f(y)} d(u_y,P_f(x)) \leq d(x,y).$$

Hence

$$H(P_f(x),P_f(y)) \leq d(x,y).$$
and $P_f$ is nonexpansive. Conversely, we assume that $P_f$ is nonexpansive. Then given any $x, y \in X$ and $u_x \in f(x)$ with $d(x, u_x) = \inf \{d(x, z) : z \to f(x)\}$ (i.e., $u_x \in P_f(x)$). By compactness, we can choose $u_y \in P_f(y)$ such that $d(u_x, u_y) = d(u_x, P_f(y))$. Hence

$$d(u_x, u_y) \leq H(P_f(x), P_f(y)) \leq d(x, y)$$

and $f$ is $*$-nonexpansive.

**Remark 2.** Theorem 3 indicates that the fixed point theory of multivalued nonexpansive mappings applies to multivalued $*$-nonexpansive mappings; in particular, we have the following results whose nonexpansive counterparts were proved in [5] and [6], respectively.

**Corollary 2** Let $X$ be a Banach space satisfying Opial’s property, $C$ a nonempty weakly compact convex subset of $X$, and $T : C \to K(C)$ a $*$-nonexpansive mapping. Then $T$ has a fixed point.

**Corollary 3.** Let $X$ be a uniformly convex Banach space, $C$ a nonempty closed bounded convex subset of $X$, and $T : C \to K(C)$ a $*$-nonexpansive mapping. Then $T$ has a fixed point.

Corollaries 2 and 3 improve upon the corresponding results of [4] and [13].

We conclude the paper with a counterexample that presents a negative answer to a question raised by A. Canbtti, G. Marino and P. Pibtramala [2].

Suppose that $H$ is a Hilbert space and $K$ is a nonempty closed convex subset of $H$. We denote by $K\mathcal{C}(K)$ the family of all nonempty compact convex subsets of $K$, and by $d(A, B)$ the distance between two subsets $A, B \subset H$, i.e., $d(A, B) = \inf \{\|x - y\| : x \in A, y \in B\}$. With each mapping $T : K \to K\mathcal{C}(H)$ one can associate a multivalued mapping $\hat{T} : K \to K\mathcal{C}(H)$ defined as follows:

$$\hat{T}x := \{y \in Tx : d(y, K) = d(Tx, K)\}.$$

The question raised by A. Canbtti, G. Marino and P. Pibtramala (see [2, Remark 1, p. 207]) is whether the nonexpansiveness of $T$ implies that $\hat{T}$ is nonexpansive. The following example shows that the answer is negative.
Example 2. Let $OABC$ be the unit square $[0,1] \times [0,1]$ in the plane $H$. Let $D$ and $E$ be the midpoints of the segments $AB$ and $OC$, respectively. Let $K$ be the triangle $\Delta ADF$. To each point $P \in K$, let $P'$ be the symmetric point of $P$ with respect to the diagonal segment $AC$. Let $P'_1$ be the projection of $P'$ onto the segment $OA$. Now we define a map $T: K \to K(H)$ by setting (see the figure below)

$$T(P) := \text{The segment } \overline{CP'_1}.$$

It is then easy to see that $T$ is a nonexpansive mapping with the unique fixed point $A$. We also have the following facts:

(i) $T(A) = \overline{AC}$ and hence $d(T(A), K) = 0$;

(ii) $T(A)$ is the segment $\overline{AF}$;

(iii) $T(D) = \overline{CD'_1}$, where $D'_1$ is the midpoint of $\overline{OA}$;

(iv) $T(D) = \{G\}$, where $G$ is the nearest point projection of $F$ onto the segment $\overline{CD'_1}$. Hence

$$H(\hat{T}(A), \hat{T}(D)) = \sup\{d(G, M) : M \in \overline{AF}\}$$

= The length of the segment $\overline{GA}$

> The length of the segment $\overline{AD}$

= $d(A, D)$,

showing that $\hat{T}$ is not nonexpansive.
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