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Cooperative games restricted by fuzzy graphs

Author
Inés Gallego Sánchez

Supervisor
Dr. *Andrés Jiménez Losada*

Co-supervisor
Dr. *Julio Rodrigo Fernández García*

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Preface

The study of the theory of games started in von Neumann and Morgenstern [53] in 1944 with the publication of *Theory of Games and Economic Behavior,* although there is some earlier research in von Neumann [52] in 1928. Game theory is a mathematical discipline which studies situations of competition and cooperation among several agents (players). This is a consistent definition with the large number of applications. These applications come from economy, sociology, engineering, policy, computation, psicology or biology. Game theory is divided into two branches, called the non-cooperative and cooperative branches. They differ in how they formalize interdependence among players. In the non-cooperative theory, a game is a detailed model of all the moves available to the players. By contrast, the cooperative theory abstracts away from this level of detail, and describes only the outcomes that result when players are grouped together (coalitions). This research is focused on cooperative models. In von Neumann and Morgenstern [53] the authors described coalitional games in characteristic funtion form, also known as transferable utility games (games for short). The characteristic function of a game is a real-valued function on the family of coalitions. The real number assigned to each coalition is interpreted as the utility of the cooperation among this group of players. In these cases the worth of a coalition can be allocated among its players in any way. The adjective transferable refers to the assumption that a player can transfer any part of his utility to another player.

Solving a game means determining which coalition or coalitions are formed and obtaining a payoff vector at the end of the game for the players with the corresponding individual payoffs for their cooperation. The classic model of game considers that the grand coalition (the coalition of all the players) will be formed and assumes that there are no restrictions in cooperation, therefore every subset of players can form a different coalition. A value for a game is a function assigning a payoff vector for each game. The most known value was introduced by Shapley [62] in 1953. The Shapley value determines the payoff vector for a game by an explicit formula using the worths of the characteristic function of the game, and it is sustained in a set of reasonable properties called axioms which identify the value uniquely. In the context of political theory another value (index) was introduced by Banzhaf [8] in 1965 and Dubey and Shapley [26] in 1979, with similar properties to the Shapley value. This index was extended for all games by Owen [57] in 1975.

In real life, political, social or economic circumstances may impose certain restraints on coalition formation. This idea has led several authors to develop models of cooperative games with partial cooperation. One of the first approximations to partial cooperation is due to Aumann and Drèze [6] in 1974. A coalition structure is a partition of the set of players such that the cooperation is possible only if the players belong to the same element of the partition. They introduced the concept of value for games with coalition structure. In this case, the final coalitions are the elements of the partition, but inside each of them all coalitions are feasible. In 1977, Myerson [51], in his seminal work *Graphs and Cooperation in Games*, presented a new class of games with partial cooperation structure. A communication structure is a graph on the set of players, where the links represent how the players can define feasible relations in the following sense: a coalition is feasible if an only if the subgraph generated by the vertices in that coalition is connected. This model is also an extension of the model of coalition structures, here the final coalition structure is the set of connected components. The Myerson value (Myerson [51]) determines a payoff vector for each game and each communication structure in the Shapley sense, moreover if the graph is complete this solution coincides with the Shapley value. But other values for games with communication structure were defined from the Shapley value: the position value (Meessen [49] and Borm et al. [14]), the average tree solution (Herings et al. [39],...). Besides the Banzhaf value has been modified to study communication situations. The graph Banzhaf value was defined by Owen [60], and Alonso-Meijide and Fiestras-Janeiro [3] obtained an axiomatization. The Myerson model has been applied in other situations where the feasible coalitions and the final coalition structure are defined using different relations among players, for instance permission structures (Gilles et al. [35]) or partition systems (López [48]). In the Myerson model a new game is defined in order to get together the information from the game and the graph. Faigle and Kern [27] in 1992 proposed to study partial cooperation, but without extending the game. In this way, games on convex geometries were studied (Bilbao [10]), games on closure systems (Jiménez [42]), games on matroids (Jiménez-Losada [43]), interior operator games (Chacón [19]),...

In 2012 Chalkiadakis et al. [20] published a treatise on algorithms and complexity of cooperative games. Fernández et al. [30] and Bilbao et al. [11] [12] analyzed the complexity of algorithms to calculate the Shapley, Myerson and Banzhaf values. Gallego et al. [34] and Fernández et al. [32] extend these results to compute the graph Banzhaf value and the position value.

Owen in his work *Values for games with a priori unions* [58] in 1977 introduced a different model in partial cooperation. In this case the coalition structure is not a final partition. Owen interpreted the coalition structure as a set of a priori unions based on the closeness among the players, but as in the classic model, the final cooperation is the grand coalition. The Owen model defines a payoff vector in two steps, taking a game over the unions and later taking another game inside each union. The Owen value [58] uses the Shapley value in both steps. On the Banzhaf side, Owen [59] in 1982 (first axiomatization in Amer et al. [5]) and Alonso-Meijide and Fiestras-Janeiro [2] consider two different versions for games with a priori unions. The first one considers the Banzhaf value in both steps and the second one uses the Banzhaf value among the unions and the Shapley value inside each union. Following the Myerson model, Casajus [18] raised a graph as a map of the a priori relations among the players in the Owen sense. This model, called cooperation structure, considers that the a priori unions are the connected components of the graph and the subgraph in each component explains the internal bilateral relationships among the players. The Myerson-Owen value (as we will name here) is a two-step value like the Owen value that applies the Shapley value among the unions and the Myerson value inside each union.

Aubin [7] in 1981 considered games with fuzzy coalitions. In a fuzzy coalition the membership of the players is leveled. A critical issue arises when dealing with usual games and fuzzy coalitions: how to assign a worth to a fuzzy coalition from a usual game. In his seminal paper, Aubin proposed an optimal value, also studied by Jiaquan and Qiang [41]. Butnariu [15] assumed that different players should have the same membership grade in order to cooperate and provided a different way to assign a gain to a fuzzy coalition. Tsurumi et al. [69] in 2001, by using the Choquet integral, came up with a reasonable method to extend a game to the set of fuzzy coalitions. Jiménez-Losada et al. [44] in 2010 began to study games with partial cooperation from fuzzy coalition structures. They introduced the concept of fuzzy communication structure in a particular version. In 2015, Gallardo [33] defined authorization structures as an extension of permission structures and analyzed games with fuzzy authorization structures.

The objective of this thesis is to study values for cooperative games restricted by fuzzy graphs.

In Chapter 1 we introduce the basic aspects needed for our study: cooperative games, Shapley and Banzhaf values, graph theory, fuzzy sets and the Choquet integral.

In part I (Chapters 2,3 and 4) we study values for cooperative games with fuzzy communication. In Chapter 2, we present the Myerson model, i.e., a model to construct values for communication situations using classic values and a measure that gives the potential profit for any communication situation given a cooperative game. We also introduce the concept of fuzzy communication structure and define some particular ones that will be useful in subsequent chapters. For our purpose, we needed to define the concept of partition of a fuzzy graph by a sequence of levels and usual graphs. Then, we extend the concept of measure to fuzzy communication structures, which depends on the chosen partition and later introduced a specific partition, the Choquet by graphs partition,based on the Choquet integral. Our later study of values will be centered on fuzzy communication structures with the Choquet by graphs partition.

In Chapters 3 and 4 we recall the definitions of several communication values: Myerson, graph Banzhaf, position and average tree. We show the first axiomatizations that can be found in the literature. Then, we use the Choquet by graphs partition and the fuzzy Myerson model to define fuzzy versions of them. We also give axiomatizations. Besides we study the stability of our cg-values under two aspects: communication and graph stability.

In part II (Chapters 5, 6 and 7), we particularize our study to proximity relations. These fuzzy relations can be useful to express levels of closeness in any aspect.

In Chapter 5, we present the Owen model, that is based on the existence of a set of a priori unions over the set of players, and the model of Casajus extending the previous construction to situations of cooperation. We introduce proximity relations and give the general formula to build proximity values using cooperation values and the Choquet integral. In the last section we present various ways and properties to reduce the image of the proximity relation, which will be useful to define axioms and prove the uniqueness of our proximity values.

In Chapter 6 we recall some known values for games with a priori unions and some of their axiomatizations: Owen, Banzhaf-Owen and symmetric coalitional Banzhaf. Since the only one that has been analyzed for cooperation structures is the Owen value (called Myerson-Owen value in this case), this chapter is dedicated to define and axiomatize the Banzhaf versions for these structures.

Moreover a different axiomatization of the Myerson-Owen value is obtained. Using the Choquet integral and the values of the previous chapter, in Chapter 7 a similar study is done with proximity values.

In the Appendix we use our fuzzy values in the context of politics, specifically in the European Parliament. The Choquet by graphs values will serve to measure the power of the national component, and the power of the ideological aspect will be better measured by the proximity values. We give comparative charts and tables.

Chapter 1

Introduction

As we have said in the preface the aim of this thesis is to study cooperative games with transferable utility with fuzzy relations between the players. In this first chapter we introduce some aspects of cooperative games, graph theory, fuzzy sets and the Choquet integral that will be useful for the study of games with fuzzy graphs.

1.1 Cooperative games

A cooperative game with transferable utility quantifies for a given situation that involves a set of agents the result of their different cooperations. If N is a finite set we represent by 2^N the power set of N. We have followed Driessen [25] to introduce the main concepts about games.

Definition 1.1 *A cooperative game with transferable utility is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is a mapping with $v(\emptyset) = 0$. The elements of $N = \{1, 2, \ldots, n\}$ are called players. The mapping v is named characteristic function of the game. A subset $S \subseteq N$ is named coalition.*

We will use game, instead of cooperative game with transferable utility, hereafter. The quantity $v(S)$ is the worth of coalition S and represents the profit, benefit or cost that players in S can ensure in the game (N, v), independently of how the rest of the players act. The class of all cooperative games will be denoted by $\mathcal{G}$. We will denote by $\mathcal{G}^N \subset \mathcal{G}$ the subfamily of cooperative games with a fixed set of players, N. When we work only over $\mathcal{G}^N$ we can identify the pair (N, v) with the characteristic function v, in those cases we will write $v \in \mathcal{G}^N$ instead of $(N, v) \in \mathcal{G}^N$.

We see now some examples of cooperative games.

Example 1.2 We consider a production economy in which there are several peasants and one or two landowners. This model has been studied in Shapley and Shubik [63] and Chetty et al. [21]. We suppose that both landowners are of the same type. The peasants contribute only with their work and they are also of the same type. The landowners hire the peasants to cultivate their land. If t peasants are hired by a landowner, then the monetary value of the harvest obtained is denoted by $f(t) \in \mathbb{R}$. The mapping $f : \{0, 1, \ldots, m\} \to \mathbb{R}$ is named production function where m is the total number of peasants. In what follows, it is required that f satisfies these two conditions:

1. A landowner by himself does not produce anything, i.e., $f(0) = 0$.
2. Mapping f is nondecreasing, i.e., $f(t+1) \geq f(t) \quad \text{for each } t \in \{0, 1, \ldots, m-1\}$.

Both conditions imply that f is a nonnegative mapping.

When there is only one landowner, we consider him as player 1 and the peasants as players $2, ..., m+1$. Then this situation can be modeled as a cooperative game with $m+1$ players with characteristic function v given by

$$v(S) = \begin{cases} 0, & \text{if } 1 \notin S \\ f(|S|-1), & \text{if } 1 \in S. \end{cases}$$

The value of any coalition that contains only peasants is 0 because they do not have any land. Even more, the worth of each coalition that contains the landowner is equal to the monetary value of the harvest that is obtained by the peasants that are in that coalition. Notice that $v(\{i\}) = 0$ for every $i \in N = \{1, 2, \ldots, m+1\}$.

We propose in the example below a variant of the previous one, in which we have two landowners instead of one.

Example 1.3 Suppose a set of five agents interested in making use of a land. They decide to cooperate getting the maximum feasible profit. Players $2, 3$ are landowners and players $1, 4, 5$ are peasants. The worth of each coalition depends on what landowners it contains. The characteristic function in millions of euros is: $v(S) = 10(|S|-1)$ if $2 \in S$ but $3 \notin S$, $v(S) = 16(|S|-1)$ if $3 \in S$ but $2 \notin S$, $v(S) = 48(|S|-2)$ if $2, 3 \in S$ and $v(S) = 0$, otherwise.

Next example of game was proposed to solve the problem of distributing a quantity among a set of creditors when it is less than or equal to the total demand.

Example 1.4 A bankruptcy problem with a set of creditors N is a pair (E, d), where $E \in \mathbb{R}$ and $d \in \mathbb{R}^N$ are such that, for each $i \in N, d_i \geq 0$ and $0 \leq E \leq \sum_{i=1}^{n} d_i$.
Given the bankruptcy problem (E, d) with a set of players N, O'Neill [56] defined the associated bankruptcy game (N, v) for each $S \subseteq N$ by

$$v(S) = \max \left\{ 0, E - \sum_{i \notin S} d_i \right\}.$$

The set of players N consists of the n creditors and the value of the coalition S is equal to 0 or what remains of E after each creditor of the complementary coalition $N \setminus S$ has been satisfied with its corresponding demand d_i. Thereby, in this game the creditors have a pessimistic point of view.

Next game provides an example in which the worth of a coalition is interpreted as cost.

Example 1.5 Sometimes the expression $v(S)$ does not represent benefit, but the cost that has to be faced by the members of a coalition. Normally in that case it is denoted by $c(S)$ and (N, c) is called cost game. We see a typical example: suppose that planes of several types and sizes must land in an airport. The airport needs to construct landing strips suitable for the size of the planes that will use them. The airlines run with a certain percentage of the costs.
We construct the cost game (N, c). We divide the planes in m types ($m \geq 1$). Let N_j be the set of landings of the planes of types j, $(j = 1, \cdots, m)$ and $N = \cup_{j=1}^{m} N_j$ the set of all landings in the airport. Let C_j be the cost of a suitable landing strip for the planes of type j. Without loss of generality, these types can be ordered like this: $0 = C_0 < C_1 < C_2 < \cdots < C_m$. Let

$S \subseteq N, S \neq \emptyset$. Then the cost $c(S)$ of a suitable landing strip for all landings in S is given by

$$c(S) = \max\{C_j |\, 1 \leq j \leq m,\, S \cap N_j \neq \emptyset\}.$$

Then, the airport game is defined as the pair (N, c). The airport game is associated to the saving game (N, v) with

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S).$$

A first issue to be solved in a cooperative game is which coalitions will be formed in the end. Certain assumptions lead, in the original definition, to consider N as the final coalition. We see now some properties for a game, which are described in terms of profits.

Definition 1.6 *Let $(N, v) \in \mathcal{G}$, it is monotonic if*

$$v(S) \leq v(T)\,, \forall S \subseteq T \subseteq N.$$

That is, the worth of a coalition does not decrease if the coalition becomes greater. These games rewards for cooperation, nevertheless they need not be more beneficial individually.

It is also reasonable that the worth of the union of two disjoint coalitions is greater or equal than the sum of the worth of both coalitions separately, i.e., there is an incentive for cooperation.

Definition 1.7 *Let $(N, v) \in \mathcal{G}$, it is called superadditive if $v(S) + v(T) \leq v(S \cup T)$, for all $S, T \subseteq N$ with $S \cap T = \emptyset$.*

A generalization of the previous concept to pairs of not necessarily disjoint coalitions is convexity.

Definition 1.8 *Let $(N, v) \in \mathcal{G}$, it is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.*

Strictly, cooperative games imply benefits for cooperation, so as in the individual coalitions there is not any cooperation, it can be understood that their worth is zero.

Definition 1.9 *Let $(N, v) \in \mathcal{G}$, it is 0-normalized if $v(\{i\}) = 0$ for all $i \in N$. If (N, v) is not 0-normalized we can obtain its 0-normalization (N, v_0), which is defined by*

$$v_0(S) = v(S) - \sum_{i \in S} v(\{i\}), \text{ for all } S \subseteq N.$$

A special family of games that will play an important role in what follows are the unanimity games and the restricted games.

Definition 1.10 *Let N be a finite set and $T \subseteq N, T \neq \emptyset$. We define the unanimity game (N, u_T) by*

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0, & \text{otherwise.} \end{cases}$$

Then, in the unanimity game u_T, a coalition has worth 1 if it contains all members of T and has worth 0 otherwise.

For a fixed set N of players, $\mathcal{G}^N$ is a vector space with the following operations.

Definition 1.11 *Let $v_1, v_2 \in \mathcal{G}^N$ and $\alpha \in \mathbb{R}$. We define the games $v_1 + v_2, \alpha v_1 \in \mathcal{G}^N$ by*

$$\begin{aligned} (v_1 + v_2)(S) &= v_1(S) + v_2(S), \qquad \forall S \subseteq N \\ (\alpha v_1)(S) &= \alpha v_1(S), \qquad \forall\, S \subseteq N \end{aligned}$$

Shapley [62] proved that given a finite set N, the family of all unanimity games that can be defined, constitutes a basis of the vector space of the games over N.

Proposition 1.12 *The set of unanimity games $\{(N, u_T) : T \subseteq N, T \neq \emptyset\}$ forms a basis of $\mathcal{G}^N$. In particular, the characteristic function of every game $v \in \mathcal{G}^N$ can be expressed as a linear combination* *of unanimity games in N, i.e.,*

$$v = \sum_{\{T \subseteq N : T \neq \emptyset\}} \Delta_T^v u_T \quad \textit{with} \quad \Delta_T^v = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S).$$

The coefficients of the above combination, Δ_T^v for each non-empty coalition $T \subseteq N$, are named *Harsanyi dividends* [38] of the game.

Sometimes it will be necessary to restrict the cooperative game to a particular domain because we may want to leave the rest of the players inactive, by setting them aside or keeping them but subjecting their activity to the chosen coalition. This is the philosophy behind the next two definitions.

Definition 1.13 *If $(N, v) \in \mathcal{G}$ and $S \subseteq N$ we define the restricted game to that coalition S, (S, v) as the restriction of the characteristic function v to 2^S.*

The restricted game can be extended to 2^N in this way.

Definition 1.14 *If $(S, v) \in \mathcal{G}$ and $S \subseteq N$ we define the extended game $(N, v_S) \in \mathcal{G}$ by*

$$v_S(T) = v(T \cap S), \ \forall T \subseteq N.$$

In the classic game theory it is considered that all coalitions make sense and their worths are known. Nonetheless, partial cooperation analyzes situations in which some coalitions are not possible or its worth has to be modified either because of ignorance or additional information.

1.2 The Shapley value and the Banzhaf value

A *payoff vector* for a cooperative game (N, v) is a vector $x \in \mathbb{R}^N$ so that x_i is interpreted as the payment that the player $i \in N$ would receive for its cooperation. To solve a game then corresponds to find a reasonable payoff vector in order to benefit the players according to their possibilities of cooperation.

Definition 1.15 *A value or solution for cooperative games is a mapping f over $\mathcal{G}$ so that it assigns to each game (N, v) a payoff vector $f(N, v) \in \mathbb{R}^N$.*

To elaborate reasonable values, they must be justified by logical conditions. If these conditions lead us to discriminate a value from the rest we say that we have obtained an axiomatization.

Let f be a value for cooperative games. Next we introduce some logical properties that are part of axiomatizations for the values on which we focus this section.

If we suppose that the quantification of the number given by the characteristic function to a coalition is a distributable amount and the result of the game is the cooperation among all players, that is the formation of the *grand coalition*, then it seems logical to assume the next axiom for a value f.

Efficiency. The value f satisfies efficiency if for every game $(N, v) \in \mathcal{G}$ it holds

$$\sum_{i \in N} f_i(N, v) = v(N).$$

The existence of a player whose contribution at every moment is the same implies that this player can be satisfied with this contribution as payment.

Let $(N, v) \in \mathcal{G}$, $S \subseteq N$ a coalition and $i \notin S$ a player. The contribution of i to S is measured by the *marginal contribution* $v(S \cup \{i\}) - v(S)$.

Definition 1.16 *Let $(N, v) \in \mathcal{G}$. We say that $i \in N$ is a dummy player in that game if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for each $S \subseteq N \setminus \{i\}$.*

Dummy player property. A value f satisfies dummy player if for each player i that is dummy in $(N, v) \in \mathcal{G}$ it holds $f_i(N, v) = v(\{i\})$.

A particular case of dummy player is the null player, whose marginal contribution to each coalition is equal to zero.

Definition 1.17 *Let $(N, v) \in \mathcal{G}$. We say that $i \in N$ is a null player in that game if it holds $v(S \cup \{i\}) = v(S)$ for each $S \subseteq N \setminus \{i\}$.*

Null player property. A value f satisfies null player if for each player i that is null in $(N, v) \in \mathcal{G}$ it holds $f_i(N, v) = 0$.

A reasonable property is that players that contribute the same to each coalition receive the same payment.

Equal treatment. A value f satisfies equal treatment if for each $(N, v) \in \mathcal{G}$ and each pair of players $i, j \in N$, such that $v(S \cup \{i\}) = v(S \cup \{j\}),\ \forall S \subseteq N \backslash \{i, j\},\ f_i(N, v) = f_j(N, v)$.

Another desirable condition for a value is that the total payoff for any coalition is greater than or equal to the worth that the coalition receives by means of the characteristic function of the game, that is, no coalition has an incentive to abandon the game and play apart.

Stability. A value f satisfies stability (also known as coalitional rationality) if for each $(N, v) \in \mathcal{G}$, it holds

$$\sum_{i \in S} f_i(N, v) \geq v(S),\ \forall S \subseteq N.$$

The value can behave in a reasonable way if two players are merged into one. It seems logical that the payoffs that players i and j obtained before the merger sum up the same as what obtains the new player resulting from the merger.

We need to introduce the next game that explains the situation after the merger.

Definition 1.18 *The amalgamated game of (N, v) for $i, j \in N$ is another game $\left(N^{ij}, v^{ij}\right)$ where $N^{ij} = N \backslash \{i, j\} \cup \{p\}$, where p is the player resulting from the merger of players i, j and for every $S \subseteq N^{ij}$,*

$$v^{ij}(S) = \begin{cases} v(S \backslash \{p\} \cup \{i, j\}), & \text{if } p \in S \\ v(S), & \text{if } p \notin S. \end{cases}$$

Pairwise merging. The value f satisfies pairwise merging if for each $(N, v) \in \mathcal{G}$ and each pair of players $i, j \in N$

$$f_p\left(N^{ij}, v^{ij}\right) = f_i(N, v) + f_j(N, v).$$

The previous axiom implies in particular, together with the dummy player axiom, efficiency in the family of two-player games, that is why in many occasions it is named 2-efficiency.

The logic of the axiom that we present next is more focused on computational and operational aspects of the value.

Linearity For any pair of games $(N, v_1), (N, v_2) \in \mathcal{G}$ and $a_1, a_2 \in \mathbb{R}$ we have for a value f,

$$f(N, a_1 v_1 + a_2 v_2) = a_1 f(N, v_1) + a_2 f(N, v_2).$$

Shapley [62] defined a value for superadditive cooperative games that has been one of the most used since then.

Definition 1.19 *The Shapley value ϕ is defined for every $(N, v) \in \mathcal{G}$ and every $i \in N$ by*

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \left[v(S \cup \{i\}) - v(S) \right].$$

This value is a weighted average of all marginal contributions, where the different weights depend on the size of each coalition.

Example 1.20 We are going to compute the Shapley value of the game of Example 1.3. First, we are going to express the worth of each coalition

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\emptyset$	0	{1,4}	0	{1,2,3}	48	{2,4,5}	20
{1}	0	{1,5}	0	{1,2,4}	20	{3,4,5}	32
{2}	0	{2,3}	0	{1,2,5}	20	{1,2,3,4}	96
{3}	0	{2,4}	10	{1,3,4}	32	{1,2,3,5}	96
{4}	0	{2,5}	10	{1,3,5}	32	{1,2,4,5}	30
{5}	0	{3,4}	16	{1,4,5}	0	{1,3,4,5}	48
{1,2}	10	{3,5}	16	{2,3,4}	48	{2,3,4,5}	96
{1,3}	16	{4,5}	0	{2,3,5}	48	{1,2,3,4,5}	144

Table 1.1: Game v in Example 1.3

Moreover, we have

$$\text{If } |S| = 0 \rightarrow \frac{|S|!(n-|S|-1)!}{n!} = \frac{0!(5-0-1)!}{5!} = \frac{1}{5}.$$
$$\text{If } |S| = 1 \rightarrow \frac{|S|!(n-|S|-1)!}{n!} = \frac{1!(5-1-1)!}{5!} = \frac{1}{20}.$$
$$\text{If } |S| = 2 \rightarrow \frac{|S|!(n-|S|-1)!}{n!} = \frac{2!(5-2-1)!}{5!} = \frac{1}{30}.$$
$$\text{If } |S| = 3 \rightarrow \frac{|S|!(n-|S|-1)!}{n!} = \frac{3!(5-3-1)!}{5!} = \frac{1}{20}.$$
$$\text{If } |S| = 4 \rightarrow \frac{|S|!(n-|S|-1)!}{n!} = \frac{4!(5-4-1)!}{5!} = \frac{1}{5}.$$

Then $\phi(N, v) = (20.333, 37, 46, 20.333, 20.333)$.

Some of the axioms introduced above permit to axiomatize the Shapley value. A variant of the original axiomatization obtained by Shapley in [62] is

Theorem 1.21 *The Shapley value is the only value that satisfies the axioms of efficiency, null player, equal treatment and linearity.*

In general, the Shapley value does not satisfy stability. It holds that if $(N, v) \in \mathcal{G}$ is convex, then $\phi(N, v)$ is stable.

Banzhaf [8] proposed another value in the context of political theory for games where the quantification of the coalitions has more to do with a discrimination between them. There exist several versions of this value and we will center in the probabilistic version, introduced by Dubey and Shapley [26].

Definition 1.22 *Given a game (N, v), the Banzhaf value β assigns to each player $i \in N$ the real number*

$$\beta_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} \left[v(S \cup \{i\}) - v(S) \right].$$

Example 1.23 We are going to compute now the Banzhaf value of the game of Example 1.3. The worth of each coalition is expressed in Table 1.1. Since $\frac{1}{2^{n-1}} = \frac{1}{16}$, then

$$\beta(N, v) = (18.5, 31.5, 40.5, 17.875, 18.5).$$

The Banzhaf value does not satisfy efficiency except for $|N| = 1$ or $|N| = 2$. We can find different axiomatizations in Lehrer [46], Haller [37] and Nowak [54]. An example of axiomatization is the following theorem.

Theorem 1.24 *The Banzhaf value is the only value that satisfies the axioms of dummy player, equal treatment, pairwise merging and linearity.*

Theorems 1.21 and 1.24 imply that the Shapley value does not satisfy pairwise merging and the Banzhaf value does not satisfy efficiency. The Banzhaf value is not efficient and then can not be interpreted as an allocation of profits. It is rather a quantification of the possible movements of each player that can be used as a proportion index.

The context of the solutions presented above supposes that in a game (N, v), the final coalition is N, however Aumann and Drèze [6] proposed a model where the final result of cooperation in the game is a *structure of coalitions*, i.e., a partition of N. Thus, when setting a payoff vector that is efficient we must take into account that structure.

Coalitional efficiency. Let $(N, v) \in \mathcal{G}$ and $\{S_1, \ldots, S_p\}$ a partition of N. A solution f is coalitionally efficient for that structure if for each $(N, v) \in \mathcal{G}$.

$$\sum_{i \in S_k} f_i(N, v) = v(S_k), \ \forall k = 1, \ldots, p.$$

Evidently, with the trivial partition N, the previous axiom reduces to efficiency.

1.3 Simple games

Up to now, we have interpreted the worth of a coalition as a continuous profit. We are interested now in a different interpretation. In the family of simple games, the worth of a coalition means the ability to win or lose in a decision making.

Definition 1.25 *A game (N, v) is called simple if it is monotonic and*

$$v(S) \in \{0,1\} \text{ for every } S \subseteq N, \quad v(N) = 1.$$

A coalition S is called winning if $v(S) = 1$ and losing if $v(S) = 0$.

A simple game can also be introduced as a pair (N, W) where N is a finite set of agents and $W \subseteq 2^N$ is a family of coalitions that is monotonic, i.e., if $S \in W$ and $T \supseteq S$ then $T \in W$. The coalitions in W are named winning and the rest, losing. The class of all simple games will be denoted by $\mathcal{S}$. We will denote by $\mathcal{S}^N \subset \mathcal{S}$ the subfamily of simple games with a fixed set of players, N.

An example of simple games are the unanimity games of Proposition 1.12.

The condition $v(N) = 1$ guarantees the existence of a winning coalition. Players whose absence leads to losing coalitions are called veto players.

Definition 1.26 *The set of all veto players J^v of a game (N, v) is defined by*

$$J^v = \{i \in N : v(N \setminus \{i\}) = 0\}$$

Generally the unanimity game with respect to a coalition represents the voting system in which the votes of the members of that coalition are essential to approve a proposal. Then, it follows that for the unanimity game u_T, the members of T are precisely the veto players of that game, i.e., $J^{u_T} = T$.

Examples of simple games are the weighted majority games, also known as voting games.

Definition 1.27 *A voting game $v = [q; w_1, \ldots, w_n]$ is a simple game in $\mathcal{S}^N$ given by*

$$v(S) = \begin{cases} 1, & \text{if } \sum_{i \in S} w_i \geq q \\ 0, & \text{if } \sum_{i \in S} w_i < q, \end{cases}$$

where $q \in \mathbb{R}_+$ is called quota and $w_i \in \mathbb{R}_+$ is the weight of each player i with $\sum_{i \in N} w_i \geq q$.

Example 1.28 Let $v = [50; 28, 25, 24, 23]$. In this voting game the first voter is much stronger than the last three since he needs only another one to pass an issue (to form a winning coalition), while the other three must all combine in order to win. The winning coalitions are

$$W = \{12, 13, 14, 123, 124, 134, 234, 1234\}.$$

In this game there are not any veto players, i.e., $J^v = \emptyset$.

One of the contributions of cooperative game theory to political science and decision theory is the quantified determination of the power of the different agents involved. For each simple game $(N, v) \in \mathcal{S}$ a *power vector* is any vector $x \in \mathbb{R}^N$ where x_i represents the power of the player in that game. That is, mathematically, a power vector is a payoff vector with a more particular interpretation. The values applied to simple games are named power indices.

Definition 1.29 *A power index is a mapping f over the class of simple games $\mathcal{S}$ that obtains a power vector $f(N, v) \in \mathbb{R}^N$ for each $(N, v) \in \mathcal{S}$.*

The axioms for values may make sense in the context of power indices except linearity because the operations of sum and product by a real number are not interior in $\mathcal{S}^N$. In this case we introduce the operations maximum and minimum which will be denoted by $\vee$ and $\wedge$ henceforth.

Definition 1.30 *Let $v_1, v_2 \in \mathcal{S}^N$. We define the games $v_1 \vee v_2$, $v_1 \wedge v_2 \in \mathcal{S}^N$ as*

$$(v_1 \vee v_2)(S) = v_1(S) \vee v_2(S),$$

$$(v_1 \wedge v_2)(S) = v_1(S) \wedge v_2(S).$$

An analogous axiom of linearity for simple games is called transference.

Transference. A power index f over $\mathcal{S}$ satisfies transference if $\forall (N, v_1), (N, v_2) \in \mathcal{S}$ it holds

$$f(N, v_1 \vee v_2) + f(N, v_1 \wedge v_2) = f(N, v_1) + f(N, v_2).$$

The Shapley value applied exclusively to simple games is known as the *Shapley-Shubik index*, while the Banzhaf value in that context is called *Banzhaf-Coleman index*. The axiomatizations presented before are valid simply substituting linearity by the transference axiom.

1.4 Graph theory

In this section we are going to see some concepts of graph theory that will be essential to understand the models in next chapters. Let V be a finite set. We denote by $L(V) = \{\{i, j\} : i, j \in V, i \neq j\}$ the set of unordered pairs of different members in V.

Definition 1.31 *An undirected graph $g = (V, L)$ is defined by a finite set V and a subset L of $L(V)$. The elements of V are called vertices and the elements of L are called links or edges.*

Normally a graph is represented by points and lines. The points are the vertices and each link is a line that joins the vertices that form it. When V is fixed g can be identified with L. We will denote link $\{i, j\} \in L$ by ij, in order to distinguish it from coalition $\{i, j\}$.

Definition 1.32 *If $i \in V$ then the degree of i is the number of edges in which i takes part, i.e.,*

$$d_i(g) = |\{ij \in L\}|.$$

Definition 1.33 *A vertex i is isolated in the graph $g = (V, L)$ if and only if $d_i(g) = 0$.*

In the graph of Figure 1.1 vertex 6 is isolated, and $d_1(g) = 2$.

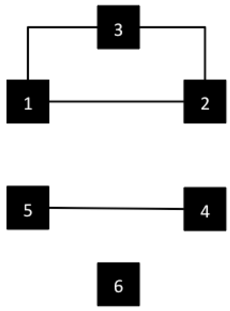


Figure 1.1: Graph

If we fix a set of vertices V, the two extreme cases of graphs are the complete graph and the empty graph.

Definition 1.34 *A graph is named complete if $L = L(V)$, and it is denoted by LV, that is, $LV = (V, L(V))$. The graph in which all its vertices are isolated is called the empty graph and it is denoted by $(V, \emptyset)$ and the graph without vertices and links is denoted by $g = \emptyset$.*

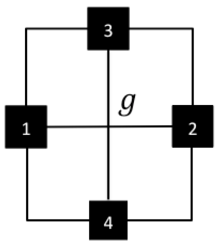


Figure 1.2: Complete graph

The graph of Figure 1.2 is a complete graph.

If $g = (V, L)$ is complete then $d_i(g) = |V| - 1, \ \forall i \in V$. If $g = (V, \emptyset)$ then $d_i(g) = 0, \ \forall i \in V$.

We introduce now the concept of subgraph that permits the restriction of the original graph.

Definition 1.35 *Let $g = (V, L)$ be a graph. A subgraph $g' = (V', L')$ of a graph g is another graph satisfying that $V' \subseteq V$ and $L' \subseteq L$.*

The concept of subgraph establishes a relation of partial order among graphs. We see now three particular types of subgraphs that are interesting to us: the graph generated by a set of vertices, the graph generated by a set of links and the graph resulting of eliminating a link.

Definition 1.36 *Let $g = (V, L)$ be a graph. If $S \subseteq V$ we denote by g_S the subgraph of g that only contains the vertices of S and the links among them. If $B \subseteq L$ we denote by g_B the subgraph of g that only contains the links in B and the vertices joined by them.*

Definition 1.37 *Let $g = (V, L)$ be a graph. If $ij \in L$ we denote by g_{-ij} the subgraph of g that consists of eliminating the link ij in g.*

We can see examples of a graph generated by a set of vertices or restricted graph in Figure 1.3, a graph generated by a set of links in Figure 1.4 and a graph resulting from eliminating an edge in Figure 1.5.

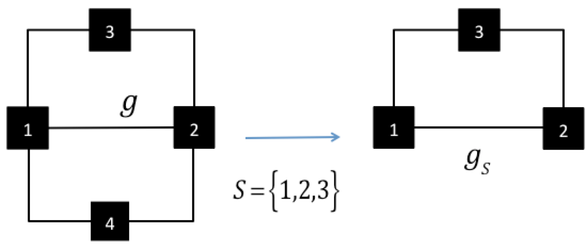


Figure 1.3: Graph generated by a set of vertices

Sometimes the reduction of a graph does not use vertices or links from the original sets. We introduce the operation of merging of vertices in graphs.

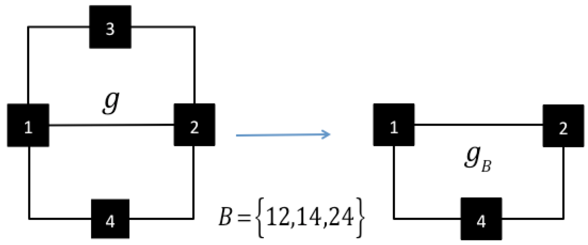


Figure 1.4: Graph generated by a set of links

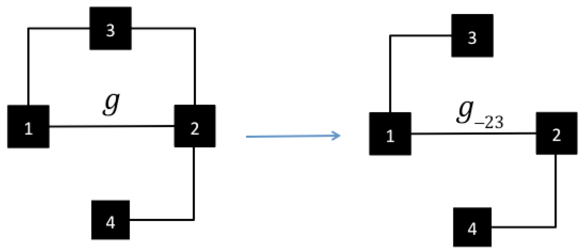


Figure 1.5: Graph resulting from eliminating an edge

Definition 1.38 *For each link $ij \in L$ in $g = (V, L)$ we define the next graph*

$$g^{ij} = (V^{ij}, L^{ij})$$

donde $V^{ij} = V \backslash \{i, j\} \cup \{p\}$ and

$$L^{ij} = \{i'j' \in L : i', j' \neq i, j\} \cup \{i'p : i' \neq i, j \text{ and } [i'i \in L \text{ or } i'j \in L]\}$$

In Figure 1.6 we can see an amalgamated graph.

One of the principal aspects of graph theory is connection.

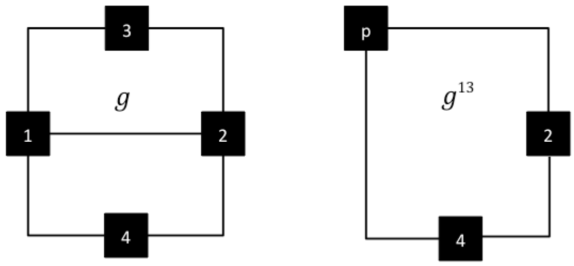


Figure 1.6: Amalgamated graph

Definition 1.39 *A path in a graph $g = (V, L)$ is defined by a sequence of vertices $(i_k)_{k=1}^{k=m}$ satisfying that $i_k i_{k+1} \in L$ is a link in g for each $k = 1, \ldots, m-1$. A path is a cycle if $i_1 = i_m$. Two vertices $i, j \in V$ are connected in g if there exists a path that contains both of them.*

In the graph of Figure 1.1, (1,2,3,1) is a cycle.

Definition 1.40 *Given a graph $g = (V, L)$ when any pair of vertices in g is connected the graph g is called connected. The connected components of g are the maximal connected subgraphs of g. The family of subsets*

$$V/g = \{T \subseteq V : g_T \text{ is a connected component of } g\}$$

is a partition of V.

By abuse of language, we will use for the subsets in V/g, the words connected components of g. In terms of the above definition we can say that i is isolated if and only if $\{i\} \in V/g$.

In the graph of Figure 1.1 the set of connected components is $V/g = \{\{1,2,3\},\{4,5\},\{6\}\}$.

Definition 1.41 *A graph is called tree if it is connected and does not have any cycles. We will use the notation $t = (V, L)$. A graph is called forest if it is a disjoint union of trees.*

In a tree $t = (V, L)$ it holds $|L| = |V| - 1$.

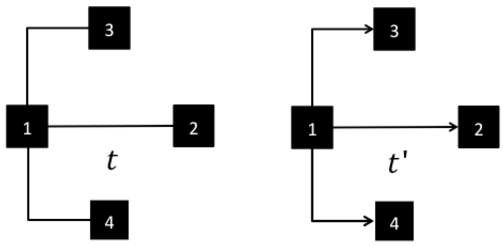


Figure 1.7: Tree and rooted tree

Definition 1.42 *A tree $t = (V, L)$ is rooted if we fix a vertex $r \in V$ and consider that the links have a direction from the vertex nearest to r to the furthest. Vertex r is named the root and for the vertices in each link $ij \in t$, the one closest to the root is called father and the other is named son. If a vertex is not the father of any other then it is a leaf.*

In figure 1.7 we can see a tree t and a rooted tree t' that represents t rooted at 1.

Graph theory, as we will comment in next chapters, has permitted to introduce additional information over the set of players and their relationships. In this way, important progress has been made in the field of partial cooperation.

1.5 Fuzzy sets and the Choquet integral

In the classic set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition, an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set. In this section we are going to introduce some concepts related to fuzzy sets and the Choquet integral that will be useful in subsequent chapters.

Definition 1.43 *A fuzzy set of a finite set K is a mapping $\tau : K \to [0,1]$. The support of τ is the set $supp(\tau) = \{i \in K : \tau(i) \neq 0\}$. The image of τ is the set of the non-null images of the function, $im(\tau) = \{\lambda \in (0,1] : \exists i \in K, \tau(i) = \lambda\}$. The family of fuzzy sets over a finite set K will be denoted by $[0,1]^K$.*

Sometimes, for convenience, the image of a fuzzy set is expressed as an ordered set, i.e.,

$$im(\tau) = \{\lambda_1 < \cdots < \lambda_p\}.$$

Definition 1.44 *Two fuzzy sets τ, τ' are comonotone if for all $i, j \in K$ it holds*

$$(\tau(i) - \tau(j))(\tau'(i) - \tau'(j)) \geq 0.$$

Proposition 1.45 *Comonotony is an equivalence relation in $[0,1]^K$.*

Definition 1.46 *Each subset $Q \subseteq K$ is associated to the fuzzy set $e^Q \in [0,1]^K$ with $e^Q(i) = 1$ if $i \in Q$ and $e^Q(i) = 0$ otherwise. We denote $e^\emptyset = 0$.*

Usually, each $Q \subseteq K$ is named crisp subset, in contrast with the other elements in $[0,1]^K$.

A fundamental tool for the analysis of fuzzy sets are the so-called cuts, that relate each fuzzy set with a collection of crisp sets.

Definition 1.47 *For each $t \in (0,1]$ the t-cut of the fuzzy set τ is*

$$[\tau]_t = \{i \in K : \tau(i) \geq t\}.$$

Example 1.48 Let $K = \{1,2,3,4\}$ and τ be the fuzzy set given by $\tau(1) = 0.5$, $\tau(2) = 0.7$, $\tau(3) = 0$ and $\tau(4) = 0.3$. The support of τ is $supp(\tau) = \{1,2,4\}$ and the image of τ is $im(\tau) = \{0.3 < 0.5 < 0.7\}$.

Let τ' be the fuzzy set given by $\tau'(1) = 0.4$, $\tau'(2) = 0.6$, $\tau'(3) = 0$ and $\tau'(4) = 0.2$. It is easy to see that τ and τ' are comonotone, since both fuzzy sets order the elements of K in the same way according to the images.

If we try to compute the *t-cuts*, $\forall t \in (0,1]$, we realise that there is a finite number of them for every $\tau \in [0,1]^K$. If we take τ, the cuts are the following: $[\tau]_t = \{1,2,4\}$, $t \in (0,0.3]$, $[\tau]_t = \{1,2\}$, $t \in (0.3,0.5]$, $[\tau]_t = \{2\}$, $t \in (0.5,0.7]$ and $[\tau]_t = \{\emptyset\}$, $t \in (0.7,1]$.

The Choquet integral is an aggregation operator defined by Choquet in [22]. It was initially used in other fields, but found its way into decision theory in the 1980s, where it is used as a way of measuring the expected utility of an uncertain event.

Definition 1.49 *Given $f : 2^K \to \mathbb{R}$ and τ a fuzzy set over K, the (signed) Choquet integral of τ with respect to f is defined as*

$$\int \tau\, df = \sum_{k=1}^{p} (\lambda_k - \lambda_{k-1})\, f\left([\tau]_{\lambda_k}\right),$$

where $im(\tau) = \{\lambda_1 < \cdots < \lambda_p\}$ and $\lambda_0 = 0$.

Although the Choquet integral was defined at the beginning only for monotonic set functions (capacities), later Schmeidler [66] extended the concept and next properties.

Proposition 1.50 *The following properties of the Choquet integral are known*

(C1) $\int e^S\, df = f(S)$, *for all* $S \subseteq K$.

(C2) $\int t\tau\, df = t \int \tau\, df$, *for all* $t \in [0,1]$.

(C3) $\int \tau\, d(a_1 f_1 + a_2 f_2) = a_1 \int \tau\, df_1 + a_2 \int \tau\, df_2$, *when* $a_1, a_2 \in \mathbb{R}$.

(C4) $\int (\tau + \tau')\, df = \int \tau\, df + \int \tau'\, df$, *when* $\tau + \tau' \leq e^K$ *and* τ, τ' *are comonotone.*

(C5) $\int \tau\, df = A \bigvee_{i \in N} \tau(i)$, *if* $f([\tau]_t) = A$, *for all* $t \in im(\tau)$.

(C6) The Choquet integral is a continuous operator.

Example 1.51 *We take the fuzzy set τ from Example 1.48 and the mapping $f(S) = |S|,\ \forall S \in 2^K$. Then*

$$\begin{aligned}\int \tau\, df &= \sum_{k=1}^{3} (\lambda_k - \lambda_{k-1})\, f\left([\tau]_{\lambda_k}\right) = (0.7-0.5)f(\{2\}) + (0.5-0.3)f(\{1,2\}) \\ &+ (0.3-0)f(\{1,2,4\}) = 0.2\cdot 1 + 0.2\cdot 2 + 0.3\cdot 3 = 1.5.\end{aligned}$$

The Choquet integral is an essential tool for our later study of games with fuzzy communication among the players.

Aubin [7] and Butnariu [15] used fuzzy sets to introduce fuzzy coalitions. Thus, a *game with fuzzy coalitions* is a pair $([0,1]^N, v)$ where $v(e^{\emptyset}) = 0$. Butnariu [15] and Tsurumi et al. [69] showed the interest in analyzing processes of fuzziness for usual games. That is, if (N, v) is a game, how to establish the worth of $\tau \in [0,1]^N$ from N. In this thesis we consider fuzziness in classic games, which comes from the existence of additional fuzzy information over the agents and their situation.

Tsurumi et al. [69] introduced an extension of a game by means of a partition by levels that uses the Choquet integral.

Definition 1.52 *The* Choquet extension *of a game (N, v), is denoted as $([0,1]^N, v_{ch})$ and defined by*

$$v_{ch}(\tau) = \int \tau dv,$$

for each $\tau \in [0,1]^N$, i.e., the Choquet integral of τ with respect to the characteristic function v.

This Choquet behaviour of the players means that they can allocate their capacities and they try to achieve the biggest coalition.

They also introduced an extension of the Shapley value for this family of games. This value determines a payoff vector for the Choquet extensions given a fuzzy set of players.

Definition 1.53 *The Choquet Shapley value for the Choquet extension of game v fixed $\tau \in [0,1]^N$*

is

$$\phi^{ch}(\tau, v) = \sum_{k=1}^{m} [\lambda_k - \lambda_{k-1}] \, \phi\left(N, v_{S_k}\right),$$

where $im(\tau) = \{\lambda_1 < \cdots < \lambda_m\}, \lambda_0 = 0$ *and* $S_k = [\tau]_{\lambda_k}$.

Part I

Fuzzy Communication

Chapter 2

Cooperative Games with Fuzzy Communication Structure

This chapter will be devoted to developing the theory of games with fuzzy communication structures. The treatment of fuzzy communication is similar to that developed by Aubin [7] for fuzzy coalitions, although we focus on a particular case. In games with fuzzy coalitions, membership of players can be leveled. In games with fuzzy communication structures, the relationships among the players can also be leveled. First we present the model of Myerson as a way of obtaining values for games with communication structure. We then introduce the concept of fuzzy graph and some related graphs. Next we introduce some operations and properties with examples and finally we extend that model to games with fuzzy communication structures.

2.1 The Myerson model

Myerson [51] thought that, although when we establish the characteristic function of a game we consider all the relations among the players, this situation can be altered if not all the communications are possible. To determine the relationships among the players he thought of a graph.

Definition 2.1 *Let (N, v) be a game. A communication structure for this game is a graph $g = (N, L)$ where the links of L represent the feasible communications among the players. The set of all communication structures for that game is denoted by CS^N.*

Although Myerson [51] introduced for (N, v) communication structures only for graphs with all the players, we will also use graphs in any subset of players. We denote as CS_0^N the set of communication structures over any coalition of N, i.e.,

$$CS_0^N = \{g \in CS^T : T \subseteq N\}. \tag{2.1}$$

Definition 2.2 *A game with communication structure is a triple (N, v, g) where (N, v) is a game and $g \in CS^N$ is a communication structure for their players. The family of games with communication structure is $\mathcal{G}_{com}$.*

Then it is necessary to find a solution depending on which communications were indeed feasible.

Definition 2.3 *A communication value is a mapping f over the games with communication structure that assigns to each $(N, v, g) \in \mathcal{G}_{com}$ a payoff vector $f(N, v, g) \in \mathbb{R}^N$.*

The Myerson model permitted the elaboration of communication values by applying traditional values for cooperative games over the so-called vertex game. That vertex game incorporates the information of the communication structure in the characteristic function. That incorporation was made possible because he was able to evaluate the communication structure through the game. This model considers that players, following the classic idea, try to cooperate at the maximum extent of their possibilities, which are restricted by the communication structure.

For that purpose, given a game $v \in \mathcal{G}^N$, Myerson [51] defined a "measure" of the potential profit by v obtained in any communication substructure $g = (T, L) \in CS_0^N$ by

$$r^{(N,v)}(g) = \sum_{H \in T/g} v(H), \quad r^{(N,v)}(\emptyset) = 0. \tag{2.2}$$

We will use $r^{(N,v)} = r$ from now on. This measure allows to compare different communication structures and satisfies for each $g = (T, L) \in CS_0^N$ the following logical properties:

1. If g is connected then $r(g) = v(T)$. Particularly, if $g \in CS^N$ and g is connected then $r(g) = v(N)$.
2. If (N, v) is superadditive, it is monotone by links, that is $r(g) \geq r(g_{-ij})$ for each $ij \in L$.
3. It is component additive, that is $r(g) = \sum_{H \in T/g} r(g_H)$, for all $g \neq \emptyset$.

Moreover, this is the only mapping satisfying these conditions.

In the model described by Myerson players lack of incentives to form disconnected graphs, what leads to think that the final coalition structure in $(N, v, g) \in \mathcal{G}_{com}$ would be the partition formed by the connected components N/g (see Definition 1.40). This model describes then a situation similar to that in Aumann and Drèze [6], where the structure of each component in the graph also intervenes.

In order to get communication values from this measure, Myerson [51] elaborated a new cooperative game that included the information of the communication structure in the characteristic function.

Definition 2.4 *Given a game with communication structure (N, v, g) the vertex game is (N, v^g) where*

$$v^g(S) = r(g_S),\ \forall S \subseteq N.$$

Therefore players are supposed to form the biggest feasible coalitions in the subgraph generated by each coalition.

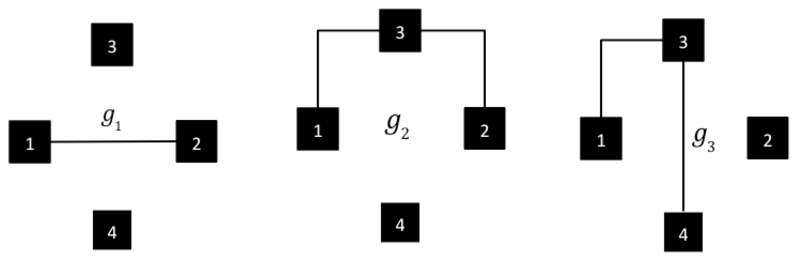


Figure 2.1: Graphs g_1, g_2, g_3

Example 2.5 Let $(N, v) \in \mathcal{G}$ with $N = \{1, 2, 3, 4\}$ and v given by

$$v(T) = 2,\ \text{if } \{1, 2\} \subseteq T \text{ and } |T| \leq 3,\ v(N) = 4,\ v(T) = 0 \text{ otherwise.}$$

Consider first the communication structure g_1 of Figure 2.1. The vertex game (N, v^{g_1}) of

(N, v, g_1) is

$$v^{g_1}(T) = 2 \text{ if } \{1,2\} \subseteq T,\ v^{g_1}(T) = 0 \text{ otherwise.}$$

Then the vertex game assigns to the grand coalition

$$v^{g_1}(N) = \sum_{R \in N/g} v(R) = v(\{1,2\}) + v(\{3\}) + v(\{4\}) = 2.$$

If we consider the communication structure g_2 of Figure 2.1 the vertex game (N, v^{g_2}) would be

$$v^{g_2}(T) = 2 \text{ if } \{1,2,3\} \subseteq T,\ v^{g_2}(T) = 0 \text{ otherwise.}$$

In this case players 1 and 2 need player 3 to communicate.

Finally consider the communication structure g_3 of Figure 2.1. The vertex game (N, v^{g_3}) is the null game $v^{g_3}(T) = 0$ for each $T \subseteq N$ because now players 1 and 2 are not able to communicate.

Myerson [51] proposed to define communication values by applying any value f to each vertex game. Namely, if f is a value for cooperative games, a communication value can be defined by $f(N, v^g)$ for each (N, v, g).

2.2 Fuzzy communication structures

Fuzzy cooperation allows players to participate even partially in several coalitions at a time, depending on the selected partition. In Jiménez-Losada et al. [44] they present situations of cooperation in a game that come from fuzzy communications among the players. Thereby, following Myerson [51] a fuzzy communication structure will be a fuzzy graph in the set of players. We will use then the fuzzy graph theory [50].

Definition 2.6 *A fuzzy graph for a finite set N is a pair $\gamma = (\tau, \rho)$ where $\tau \in [0,1]^N$ is the fuzzy set of vertices and $\rho \in [0,1]^{L(N)}$ is the fuzzy set of links, satisfying the condition $\rho(ij) \leq \tau(i) \wedge \tau(j)$, for all $ij \in L(N)$.*

The *null graph* will be denoted as $\gamma = 0$ where $\tau = 0$ and $\rho = 0$. It is equivalent to the graph $\emptyset$. Any graph, $g = (S, L)$ with $S \subseteq N$, that will be named *crisp graph* from now on, will be identified with the fuzzy graph $g = (\tau, \rho)$, where $\tau = e^S$ and $\rho = e^L$ (see Definition 1.46).

It is also possible to associate a crisp graph to a given fuzzy graph representing the set of active elements. The set of vertices of a fuzzy graph $\gamma = (\tau, \rho)$ is $vert(\gamma) = supp(\tau)$ and the set of links is $link(\gamma) = supp(\rho)$.

Definition 2.7 *The crisp version of a fuzzy graph $\gamma = (\tau, \rho)$ is the graph $g^\gamma = (vert(\gamma), link(\gamma))$.*

Using the crisp version several concepts of graph theory are extended to fuzzy graphs. Therefore, we say that the fuzzy graph γ is *connected* if and only if its crisp version g^γ is connected. The *connected components* of γ are the connected components of g^γ. We introduce the notation

$$N/\gamma = vert(\gamma)/g^\gamma. \tag{2.3}$$

Definition 2.8 *The fuzzy degree of a vertex $i \in N$ in a fuzzy graph $\gamma = (\tau, \rho)$ is*

$$\delta_i(\gamma) = \sum_{j \in N \setminus \{i\}} \rho(ij).$$

In particular $i \in N$ is isolated in γ if $\delta_i(\gamma) = 0$.

Observe that in a fuzzy graph γ, if $i \notin vert(\gamma)$ then $\delta_i(\gamma) = 0$ and i is an isolated vertex.

Definition 2.9 *The minimal level in a fuzzy graph $\gamma = (\tau, \rho)$ is*

$$\wedge\gamma = \bigwedge_{i \in vert(\gamma)} \tau(i) \wedge \bigwedge_{ij \in link(\gamma)} \rho(ij)$$

and the maximal level of γ is defined as

$$\vee\gamma = \bigvee_{i \in N} \tau(i).$$

Observe that, despite the inequality included in Definition 2.6, the minimal level must take into account the levels of the isolated vertices, but the maximal level in the structure is defined only from the vertices.

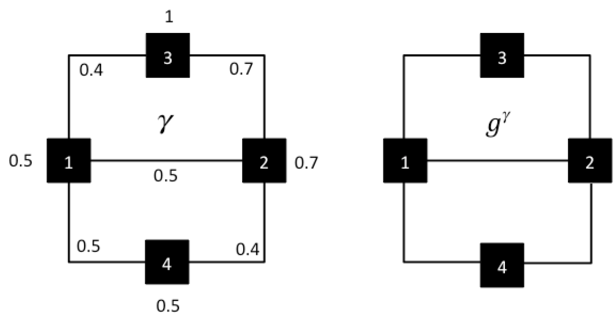


Figure 2.2: Fuzzy graph and crisp version

Example 2.10 Let $\gamma = (\tau, \rho)$ with $N = \{1, 2, 3, 4\}$ where $\rho(13) = \rho(24) = 0.4,\ \rho(12) = \rho(14) = 0.5,\ \rho(23) = 0.7,\ \rho(34) = 0,\ \tau(1) = \tau(4) = 0.5,\ \tau(2) = 0.7,\ \tau(3) = 1$. Figure 2.2 represents this fuzzy graph and its crisp version. It is a connected fuzzy graph whose minimal level is $\wedge\gamma = 0.4$ and maximal level $\vee\gamma = 0.7$ and, for instance, $\delta_2(\gamma) = 1.6$.

The concept of subgraph is highly important to study the communication among the players. So we consider this definition for fuzzy graphs.

Definition 2.11 *A fuzzy graph $\gamma' = (\tau', \rho')$ over N is a subgraph of the fuzzy graph $\gamma = (\tau, \rho)$ if and only if $\tau' \leq \tau$ and $\rho' \leq \rho$. We will denote in that case $\gamma' \leq \gamma$.*

Particularly, it is possible to define in the fuzzy context the concepts of restricted subgraph by vertices or links.

Definition 2.12 *If $S \subseteq N$ and $\gamma = (\tau, \rho)$ is a fuzzy graph, then $\gamma_S = (\tau_S, \rho_S)$ is the subgraph of γ defined as*

$$\tau_S(i) = \begin{cases} \tau(i), & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \rho_S(ij) = \begin{cases} \rho_S(ij), & \text{if } i, j \in S \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.13 *Let $\gamma = (\tau, \rho)$ be a fuzzy graph. If $A \subseteq L(N)$ then $\gamma_A = (\tau_A, \rho_A)$ is a new fuzzy graph given by*

$$\tau_A\left(i\right) = \begin{cases} \tau\left(i\right), & \text{if } \exists ij \in A \cap link(\gamma) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \rho_A\left(ij\right) = \begin{cases} \rho\left(ij\right), & \text{if } ij \in A \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.14 If $S = \{1, 2, 3\}$ we obtain the subgraph γ_S in Figure 2.3 for the fuzzy graph γ in Figure 2.2.

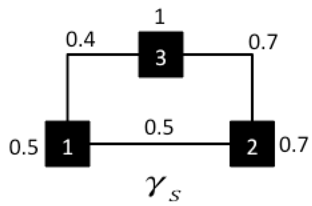


Figure 2.3: Subgraph generated by a set of vertices

Jiménez-Losada et al. introduced three operations for fuzzy graphs in [44].

Definition 2.15 *Let $\gamma = (\tau, \rho)$ and $\gamma' = (\tau', \rho')$ be two fuzzy graphs over N,*

1. $\gamma + \gamma' = (\tau + \tau', \rho + \rho')$, *if $\tau(i) + \tau'(i) \leq 1$ for all $i \in N$.*
2. $\gamma - \gamma' = (\tau - \tau', \rho -^* \rho')$, *if $\gamma' \leq \gamma$ where for all $i, j \in N$,*

$$(\rho -^* \rho')(ij) = [\rho(ij) - \rho'(ij)] \wedge [\tau(i) - \tau'(i)] \wedge [\tau(j) - \tau'(j)].$$

3. $t\gamma = (t\tau, t\rho)$ *if* $t \in [0, 1]$, *where* $(t\tau)(i) = t\tau(i)$ *and* $(t\rho)(ij) = t\rho(ij)$, *for all* $i, j \in N$.

We can see that the sum and the substraction of fuzzy graphs are not opposite operations because of the special definition of the substraction.

Example 2.16 In Figure 2.4 we compute the difference γ'' of a fuzzy graph γ with a fuzzy subgraph γ'. Notice that $\gamma'' + \gamma' \neq \gamma$.

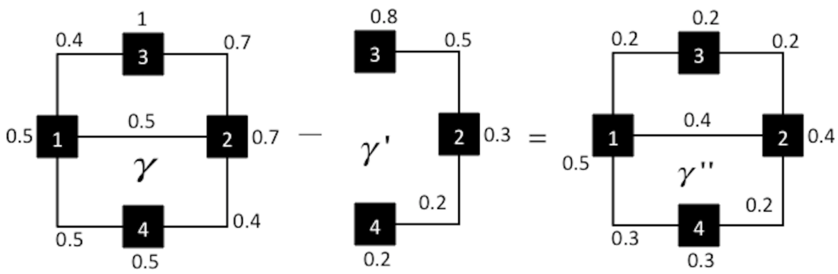


Figure 2.4: Substraction of fuzzy graphs

However, the following equality holds [44].

Proposition 2.17 *If* $\gamma, \gamma', \gamma''$ *are fuzzy graphs over* N *such that* $\gamma'' \leq \gamma - \gamma'$ *and* $\gamma' \leq \gamma$ *then*

$$(\gamma - \gamma') - \gamma'' = \gamma - (\gamma' + \gamma'').$$

Fuzzy graphs that permit the maximum possible relation among their vertices are called complete by links.

Definition 2.18 *A fuzzy graph* $\gamma = (\tau, \rho)$ *is called complete by links if and only if* $\rho(ij) = \tau(i) \wedge \tau(j)$ *for all* $i, j \in N$.

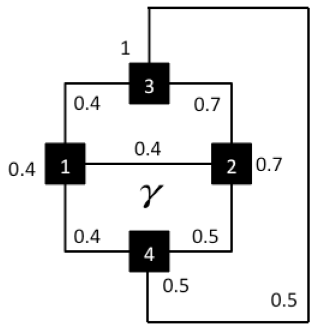


Figure 2.5: Fuzzy graph complete by links

This kind of graph permits to identify a fuzzy set with a fuzzy graph by vertices. So, if $\gamma = (\tau, \rho)$ is complete by links then ρ is defined by τ and we can associate γ with τ. Observe that if γ is complete by links then g^{γ} is a complete graph.

Example 2.19 Figure 2.5 shows a complete by links graph γ that can be identified with the fuzzy set $\tau = (0.4, 0.7, 1, 0.5)$.

As we said at the beginning of the section we mean a fuzzy communication structure for a cooperative game $(N, v) \in \mathcal{G}$ as a fuzzy graph over N.

Definition 2.20 *A fuzzy communication structure for a set of players N is a fuzzy graph $\gamma = (\tau, \rho)$ over N. The set of all fuzzy communication structures over N will be denoted as FCS^N.*

The number $\tau(i)$ can be interpreted as the level of involvement of player $i \in N$ in the game (N, v), while $\rho(ij)$ represents the maximum level at which the edge ij can be used.

Every communication structure $g \in CS_0^N$ is also a fuzzy communication structure over N and therefore we will also use $g \in FCS^N$. We call them crisp communication structures.

We extend now the concept of game with communication structure and communication value for fuzzy graphs.

Definition 2.21 *A game with fuzzy communication structure is a triple* (N, v, γ) *where* (N, v) *is a cooperative game and* γ *is a fuzzy communication structure over* N*. The class of games with fuzzy communication structure will be denoted by* $\mathcal{G}_{fcom}$.

Definition 2.22 *A fuzzy communication value is a mapping* F *that for each game with fuzzy communication structure* $(N, v, \gamma) \in \mathcal{G}_{fcom}$ *gives a payoff vector* $F(N, v, \gamma) \in \mathbb{R}^N$.

2.3 The fuzzy Myerson model

In Jiménez-Losada et al. [45], they extend the Myerson model to games with fuzzy communication structure. The measure r for a game evaluated the graphs with logical properties: connection, link monotonicity and component additivity. We try to extend this idea of measure in this section. The property of link monotonicity indicates that the more communication we have, the more profit we obtain. For a fuzzy extension of this property we introduce the next subgraph.

Definition 2.23 *Let* $\gamma = (\tau, \rho) \in FCS^N$*. If* $ij \in link(\gamma)$ *and* $t \in [0, \rho(ij)]$ *then* γ^t_{-ij} *represents the same fuzzy graph but reducing by* t *the capacity of the link* ij.

Example 2.24 Figure 2.6 presents the graph γ reducing the link 13 by 0.4.

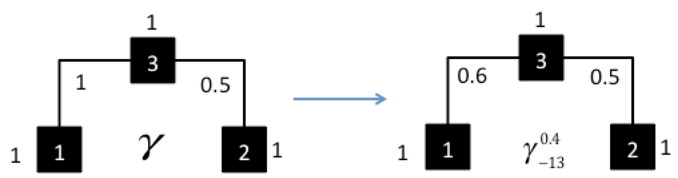


Figure 2.6: Fuzzy graph with reduced link

So, following Myerson, we define the next concept.

Definition 2.25 *Let (N, v) be a game. A mapping $\epsilon : FCS^N \to \mathbb{R}$ is called a profit measure for fuzzy graphs by v if it satisfies:*

1. $\epsilon(g) = r(g)$ *for all* $g \in CS_0^N$.
2. *If* (N, v) *is superadditive,* $\epsilon(\gamma) \geq \epsilon\left(\gamma_{-ij}^t\right)$ *for all* $\gamma = (\tau, \rho) \in FCS^N$ *and* $t \in [0, \rho(ij)]$.
3. $\epsilon(\gamma) = \sum_{H \in N/\gamma} \epsilon(\gamma_H)$ *for all* $\gamma \in FCS^N$, $\gamma \neq 0$.

Nevertheless, as it will become clear at the end of the section, there is not a unique profit measure for fuzzy graphs. Therefore we obtain different forms of incorporating the information of the fuzzy communication structure in the game.

Now, following Aubin [7], we establish a particular method for obtaining profit measures for fuzzy graphs. Let us first introduce the concept of partition by levels of a fuzzy graph.

Definition 2.26 *A partition by levels for a fuzzy communication structure $\gamma \in FCS^N$ is a finite sequence $(g_k, s_k)_{k=1}^m$ with $s_k \in (0, 1]$ and $g_k \in CS_0^N$ such that*

1. $s_k g_k \leq \gamma - \sum_{l=1}^{k-1} s_l g_l, \ \forall k \in \{1, \ldots, m\}$.
2. $\gamma - \sum_{k=1}^m s_k g_k = 0$.

A fuzzy partition election over N is a mapping pe that for each fuzzy communication structure γ obtains a partition by levels $pe(\gamma)$.

Particularly, any partition by levels $(g_k, s_k)_{k=1}^m$ of $\gamma \in FCS^N$ satisfies that for each $i \in N$ and $\gamma = (\tau, \rho)$,

$$\sum_{\{k : i \in vert(g_k)\}} s_k = \tau(i), \tag{2.4}$$

but it does not hold in general that $\sum_{\{k : ij \in link(g_k)\}} s_k = \rho(ij)$.

A way of obtaining profit measures for fuzzy graphs is selecting a partition for each one. Let pe be a fuzzy partition election for N. We use the following "measure" defined by the game (N, v) for all $\gamma \in FCS^N$

$$\epsilon^{pe}(\gamma) = \sum_{k=1}^{m} s_k r(g_k), \tag{2.5}$$

where $pe(\gamma) = (g_k, s_k)_{k=1}^m$. However not all of these measures are admissible because not all of them satisfy the conditions of Definition 2.25, as we will see in the forthcoming example.

Example 2.27 Let $\gamma = (\tau, \rho) \in FCS^N$ a fuzzy communication structure. We take for each $t \in (0, 1]$ the set of links $A(t) = \{ij \in L(N) : \rho(ij) = t\}$. Consider this algorithm

***pl*-Algorithm**

Take $k = 0, pl = \emptyset$ and $\gamma = \gamma$
While $\gamma \neq 0$ do
$k = k + 1$
$t = \vee\{\rho(ij) : i, j \in N\}$
If $t = 0$, then
$s_k = \wedge\{\tau(i) : i \in vert(\gamma)\}$
$g_k = g^\gamma$
else $s_k = t, g_k = g^\gamma_{A(t)}$
$pl = pl \cup \{(g_k, s_k)\}$
$\gamma = \gamma - s_k g_k$

The partition by levels is pl.

This algorithm has only one possible result. The proportional by links election obtains for each fuzzy graph γ the partition by levels $pl(\gamma)$ given by the pl-algorithm, $pl(0) = (\emptyset, 0)$.
We will see that this election does not always generate a profit measure. Consider the fuzzy communication structure γ of Figure 2.7 with $N = \{1, 2, 3\}$. Take the superadditive game v such that $v(N) = 5$, $v(\{2, 3\}) = 3$ and $v(S) = 0$ otherwise. In the same figure we can see how

the pl-algorithm works over γ. The measure ϵ^{pl} is not link monotonic, since

$$\epsilon^{pl}(\gamma) = 0 < 1.5 = \epsilon^{pl}\left(\gamma^{1}_{-13}\right).$$

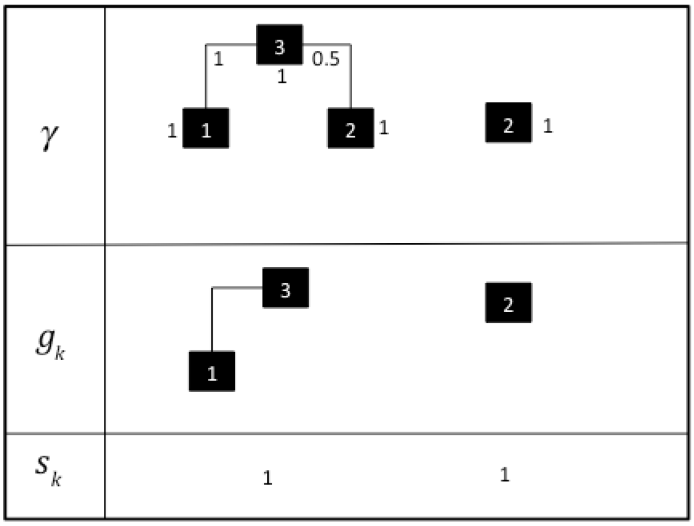


Figure 2.7: pl-partition

In Figure 2.8 the reader can see γ^{1}_{-13}.

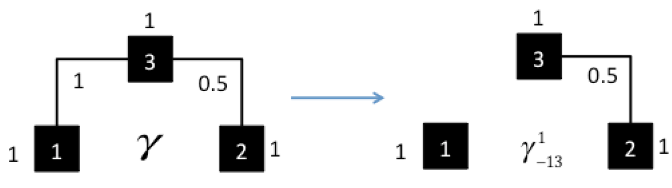


Figure 2.8: Elimination of link 13

Definition 2.28 *Let (N, v) be a game. A fuzzy partition election pe for N is v-admissible if ϵ^{pe} is a profit measure for fuzzy graphs by v.*

In Jiménez-Losada et al. [45] they present several partitions. In this thesis we have analyzed fuzzy communication values using a particular fuzzy partition election. The Choquet by graphs behavior (whose philosophy is based on the Choquet integral [22]) says that players and links can allocate their capacities and they try to get the biggest crisp graph. This behavior will be the base in the construction of our values for games with fuzzy communication structures. We apply the following algorithm.

cg-Algorithm

Take $k = 0, cg = \emptyset$ and $\gamma = \gamma$
While $\gamma \neq 0$ do
$k = k + 1$
$s_k = \wedge\gamma$
$g_k = g^{\gamma}$
$cg = cg \cup \{(g_k, s_k)\}$
$\gamma = \gamma - s_k g_k$

The partition by levels is cg.

Definition 2.29 *The Choquet by graphs election obtains for each fuzzy graph γ the unique partition by levels $cg(\gamma)$ according to the cg-algorithm, named the cg-partition of γ. Particularly $cg(0) = (\emptyset, 0)$.*

Example 2.30 In Figure 2.9, you can see the cg-partition of a fuzzy graph γ.

This election is admissible as next proposition says.

Proposition 2.31 *The Choquet by graphs election is v-admissible for every game (N, v).*

Proof. Let $g \in CS_0^N$ be a non-null graph. In that case $s_1 = 1$ and $g_1 = g$. There are not more steps

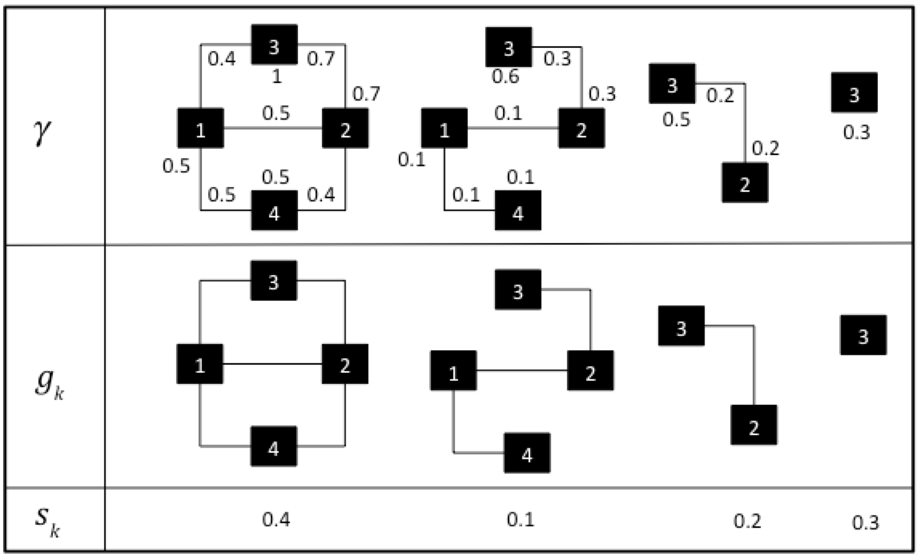


Figure 2.9: cg-partition

in the previous algorithm, then we obtain by (2.5) $\epsilon^{cg}(g) = r(g)$.

Suppose (N, v) superadditive and consider $\gamma = (\tau, \rho) \in FCS^N$ and $ij \in link(\gamma)$. We apply the previous algorithm with its notation to compute cg-partitions. Let $t \in (0, \rho(ij)]$. The cg-partitions $(g_p, s_p)_p, (g'_q, s'_q)_q$ obtained by the algorithm for the fuzzy graphs γ, γ^t_{-ij} respectively are compared in two different situations:

1. First we suppose $t = \rho(ij)$. Notice that we update the notation in each step. While $\wedge\gamma < \rho(ij)$, we obtain the same number of steps in both algorithms and $(g'_p, s'_p) = ((g_p)_{-ij}, s_p)$. Then $\sum_{p<k} s_p r(g'_p) \leq \sum_{p<k} s_p r(g_p)$. Let k be the step in which $\wedge\gamma = \rho(ij)$. In that moment there are two possibilities: either $\wedge\gamma^t_{-ij} = \rho(ij)$ or $\wedge\gamma^t_{-ij} > \rho(ij)$. If $\wedge\gamma^t_{-ij} = \rho(ij)$ then $s_k = s'_k$ and $g'_k = (g_k)_{-ij}$. This fact implies that $\gamma = \gamma^t_{-ij}$ in the step $k+1$ of the algorithm. Since $r(g'_k) \leq r(g_k)$ we have by (2.5),

$$\epsilon^{cg}(\gamma^t_{-ij}) = \sum_{p \leq k} s_p r(g'_p) + \sum_{p>k} s_p r(g_p) \leq \epsilon^{cg}(\gamma).$$

If $\wedge\gamma^t_{-ij} > \rho(ij)$ then $s_k < s'_k$ and g_k is g'_k adding link ij again. In addition, we can observe that s'_k is just the following minimal level to s_k in γ. Unfortunately the equality $\gamma = \gamma^t_{-ij}$ is not true in the step $k+1$ of the algorithm. But in that step we get $s_{k+1} = s'_k - s_k$ and $g_{k+1} = g'_k$ because we discarded the link ij in the last step. Thus, we need to compare two steps of the algorithm for γ with only one of the algorithm for γ^t_{-ij}. Beginning in step k for both fuzzy graphs, we have $\gamma = \gamma - (s_k g_k + s_{k+1} g_{k+1}) = \gamma^t_{-ij} - s'_k g'_k$, i.e., the fuzzy graphs are the same in step $k+2$ for γ and in step $k+1$ for γ^t_{-ij}. Since $s'_k r(g'_k) \leq s_k r(g_k) + s_{k+1} r(g_{k+1})$ we get by (2.5) $\epsilon^{cg}(\gamma^t_{-ij}) \leq \epsilon^{cg}(\gamma)$.

2. On the other hand we consider in the beginning $t < \rho(ij)$. In this other case the first different step k in the algorithm for γ, γ^t_{-ij} occurs when $\wedge\gamma^t_{-ij} = \alpha - t$, where $\alpha = \rho(ij)$ in that step. We obtain (g_k, s_k) in step k following the algorithm for γ with $\alpha - t \leq s_k \leq \alpha$. In the same step k for the other fuzzy graph we obtain $s'_k = \alpha - t$ and $g'_k = g_k$ because the link ij is eliminated. We start next step with $\gamma^t_{-ij} = \gamma^t_{-ij} - (\alpha - t) g_k$. The algorithm chooses $s'_{k+1} = s_k - \alpha + t$ and g'_{k+1} as g_k deleting maybe (if $s_k < \alpha$) the link ij. Since $g'_{k+1} \leq g_k$ using the same vertices we have $r(g'_{k+1}) \leq r(g_k)$ and then we obtain $s'_k r(g'_k) + s'_{k+1} r(g'_{k+1}) \leq s_k r(g_k)$. If γ is the fuzzy graph in step $k+1$ and γ^t_{-ij} is the other fuzzy graph in step $k+2$, then $t = \rho(ij)$. Now the algorithm can continue as in the first case obtaining $\epsilon^{cg}(\gamma^t_{-ij}) \leq \epsilon^{cg}(\gamma)$.

Finally we can see that the election is component additive. If $g \in CS^N_0$ then $r(g) = \sum_{H \in vert(g)/g} r(g_H)$. Let $\gamma = (\tau, \rho) \in FCS^N$, $\gamma \neq 0$, and $cg(\gamma) = (g_k, s_k)_{k=1}^{k=m}$. For each $H \in N/\gamma$ we consider the indices $(k_p)_{p=1}^{p=q}$ such that $\left(g_{k_p+1}\right)_H \neq \left(g_{k_p}\right)_H$ (we take $k_q = m$ if $(g_m)_H \neq \emptyset$). It is easy to check that $cg(\gamma_H) = (g'_p, s'_p)_{p=1}^{p=q}$ with

$$g'_p = (g_{k_p})_H \text{ and } s'_p = \sum_{k=k_{p-1}+1}^{k_p} s_k.$$

Moreover, the measure of the connected component is

$$\epsilon^{cg}(\gamma_H) = \sum_{p=1}^{q} \left(\sum_{k=k_{p-1}+1}^{k_p} s_k \right) r\left(\left(g_{k_p}\right)_H \right) = \sum_{k=1}^{m} s_k r\left((g_k)_H \right),$$

where $(g_k)_H = 0$ for each $k > k_q$. Thus, by (2.2)

$$\begin{aligned}\sum_{H\in N/\gamma} \epsilon^{cg}(\gamma_H) &= \sum_{H\in N/\gamma}\sum_{k=1}^{m} s_k r\left((g_k)_H\right) = \sum_{H\in N/\gamma}\sum_{k=1}^{m} s_k \sum_{H'\in N/(g_k)_H} v(H') \\ &= \sum_{k=1}^{m} s_k \sum_{H\in N/\gamma}\sum_{H'\in vert((g_k)_H)/(g_k)_H} v(H') = \sum_{k=1}^{m} s_k r(g_k) = \epsilon^{cg}(\gamma),\end{aligned}$$

since $vert(g_k)/g_k = \bigcup_{H\in N/\gamma} vert((g_k)_H)/(g_k)_H$ for each k. □

Moreover, the cg-algorithm uses all edges adjacent to a vertex while using this vertex, then you obtain that if $cg(\gamma) = (s_k, g_k)_{k=1}^{m}$,

$$\sum_{\{k: ij\in link(g_k)\}} s_k = \rho(ij). \tag{2.6}$$

Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$ and ϵ a v-admissible profit measure for fuzzy graphs given by v. We define a fuzzy version of the vertex game $(N, v^{\gamma}_{\epsilon})$ as

$$v^{\gamma}_{\epsilon}(S) = \epsilon(\gamma_S), \quad \text{for all } S \subseteq N.$$

Therefore, each v-admissible election defines a fuzzy vertex game. We particularize in the case of the cg-partition by means of the next definition.

Definition 2.32 *Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$. The cg-vertex game (N, v^{γ}) is defined by:*

$$v^{\gamma}(S) = \epsilon^{cg}(\gamma_S).$$

Example 2.33 We compute $v^{\gamma}(N)$, for the graph γ of Figure 2.9 with $v(S) = |S|$, for all $S \subseteq N$. Then

$$v^{\gamma}(N) = 0.4v(N) + 0.1v(N) + 0.2v(23) + 0.3v(3) = 2.7.$$

We study now the transmission of properties from the original game to the cg-vertex game.

Proposition 2.34 *Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$. It holds*

1. *If* (N, v) *monotonic then* (N, v^γ) *monotonic.*

2. *If* (N, v) *superadditive then* (N, v^γ) *superadditive.*

3. *If* (N, v) *0-normalized then* (N, v^γ) *0-normalized.*

Proof. 1) Let (N, v) be a monotonic game. Let $T \subseteq S \subseteq N$. Then $v^\gamma(S) = \epsilon(\gamma_S)$. By property 2) of Definition 2.25, $\epsilon(\gamma_S) \geq \epsilon(\gamma_T)$. And $\epsilon(\gamma_T) = v^\gamma(T)$, by definition of fuzzy vertex game.

2) We mimic the proof of Proposition 1 in Jiménez-Losada et al. [45]. Let $S, T \subseteq N$ be two disjoint coalitions. If $H \in N/(\gamma_S + \gamma_T)$ then either $H \in N/\gamma_S$ or $H \in N/\gamma_T$. Using the third condition in Definition 2.25 we obtain

$$v^\gamma(S) + v^\gamma(T) = \epsilon^{cg}(\gamma_S) + \epsilon^{cg}(\gamma_T) = \epsilon^{cg}(\gamma_S + \gamma_T),$$

because $S \cap T = \emptyset$. Finally, the fuzzy graph $\gamma_S + \gamma_T$ is a subgraph of $\gamma_{S\cup T}$ which can be obtained by deleting links. Applying successively the second condition in Definition 2.25 we get $\epsilon^{cg}(\gamma_S + \gamma_T) \leq \epsilon^{cg}(\gamma_{S\cup T})$ and so $v^\gamma(S) + v^\gamma(T) \leq v^\gamma(S \cup T)$.

3) We know that $v(\{i\}) = 0$, for all $i \in N$. Then $v^\gamma(\{i\}) = s_i v(\{i\}) = 0$, where $s_i = \tau(\{i\})$. □

It is not true that in general if (N, v) convex, (N, v^γ) convex. Notice that if we have a crisp graph $g = (N, L)$, then its cg-partition is $cg(g) = (1, g)$ and $v^\gamma = v^g$. We can find a counterexample in this way in van den Nouweland et al. [55].

The fuzzy Myerson model supposes the application of a classic value for crisp games f to the cg-vertex game to define a fuzzy communication value as

$$F(N, v, \gamma) = f(N, v^\gamma), \ \forall (N, v, \gamma) \in \mathcal{G}_{fcom}.$$

Chapter 3

Choquet by graphs values I

In this chapter we propose fuzzy communication values that follow the fuzzy Myerson model: the cg-Myerson and cg-Banzhaf values. In Section 3.1 we introduce their crisp versions and present their axiomatizations. In Sections 3.2 and 3.3 we present their fuzzy versions following the Choquet by graphs model.

3.1 Myerson and graph Banzhaf values

Myerson [51] defined a communication value extending the Shapley value using the previous model, that is, the Myerson value of a game with communication structure is the Shapley value of its vertex game.

Definition 3.1 *The Myerson value μ is a communication value defined for a game with communication structure (N, v, g) as*

$$\mu(N, v, g) = \phi(N, v^g).$$

Communication values are axiomatized by means of properties concerning the game and the communication structure.

If we suppose that players will form the biggest feasible coalitions then they will search for, taking into account the communication structure, the connected components. Myerson proposed this type of efficiency.

Component efficiency. A communication value f satisfies component efficiency if for all $(N, v, g) \in \mathcal{G}_{com}$

$$\sum_{i \in S} f_i(N, v, g) = v(S), \ \forall S \in N/g.$$

It seems logical that two players that reach a bilateral agreement should benefit equally for it.

Fairness. A communication value f satisfies fairness if for all $(N, v, g) \in \mathcal{G}_{com}$ and for any $ij \in g$

$$f_i(N, v, g) - f_i(N, v, g_{-ij}) = f_j(N, v, g) - f_j(N, v, g_{-ij}).$$

Myerson [51] proved the next theorem.

Theorem 3.2 (Myerson [51]) *The Myerson value is the only communication value that satisfies component efficiency and fairness.*

Following Myerson's model again, Owen [60] proposed an extension of the Banzhaf value.

Definition 3.3 *The graph Banzhaf value assigns to each game with communication structure* (N, v, g) *the vector*

$$\eta(N, v, g) = \beta(N, v^g).$$

Alonso-Meijide and Fiestras-Janeiro [3] characterized the value given by Owen by introducing the following axioms.

A structural extension of the null player axiom (Section 1.2) is the case of an isolated player. If he cannot communicate, his payoff would be the one obtained by his individual coalition.

Isolation. A communication value f satisfies isolation if for each $(N, v, g) \in \mathcal{G}_{fcom}$ and $i \in N$ isolated player in g

$$f_i(N, v, g) = v(\{i\}).$$

Next axiom supposes a structural extension of the pairwise merging axiom (Section 1.2). Suppose that two players can merge if their bilateral communication is feasible, and in that case the result is the amalgamated graph of Definition 1.38.

Graph pairwise merging. For each $(N, v, g) \in \mathcal{G}_{com}$ and $i, j \in N$ with ij a link in g, it holds for a fuzzy communication value f,

$$f_i(N, v, g) + f_j(N, v, g) = f_p\left(N^{ij}, v^{ij}, g^{ij}\right).$$

Theorem 3.4 (Alonso-Meijide and Fiestras-Janeiro [3]) *The graph Banzhaf value is the only communication value that satisfies isolation, fairness and graph pairwise merging.*

Notice that the Myerson value does not satisfy graph pairwise merging but it satisfies isolation. The graph Banzhaf value does not satisfy component efficiency.

Remark 3.5 *The Myerson and the graph Banzhaf values are component decomposable, i.e., the solution can be computed by restricting to each component. For $f = \mu$ and $f = \eta$, it holds*

$$f_i(N, v, g) = f_i\left(S, v, g_S\right),$$

where $S \in N/g$ is such that $i \in S$.

3.2 The cg-Myerson value

This section will be dedicated in particular to study one value that applies the model proposed in the previous chapter: the Choquet by graphs model. We obtain an axiomatization of the cg-Myerson value.

If (N, v, γ) is a game with fuzzy communication structure then by (2.2) and Definition 2.32 we have

$$v^{\gamma}(S) = \epsilon^{cg}\left(\gamma_S\right) = \sum_{k=1}^{m} s_k \sum_{H \in vert(g_k)/g_k} v(H), \tag{3.1}$$

where $cg(\gamma_S) = (g_k, s_k)_{k=1}^{m}$.

Definition 3.6 *The cg-Myerson value is the fuzzy communication value M defined for each $(N, v, \gamma) \in \mathcal{G}_{fcom}$ as*

$$M(N, v, \gamma) = \phi(N, v^{\gamma}).$$

An axiomatization of the Myerson value for all admissible elections was given in Jiménez-Losada et al. [45]. We present here another proof of this axiomatization which is special for the cg-model. We will see that this value can be written using a Choquet-type formula depending on the Myerson values of the graphs in the cg-partition. Previously, using extended games (Definition 1.14), we relate the cg-vertex game to the vertex games of the graphs of the cg-partition.

Lemma 3.7 *Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$ and $cg(\gamma) = (g_k, s_k)_{k=1}^m$. It holds*

$$v^{\gamma} = \sum_{k=1}^{m} s_k (v^{g_k})_{vert(g_k)}.$$

Proof. Let $S \subseteq N$ be a coalition. Observe that each graph that appears in the application of the cg-algorithm to γ_S is the subgraph restricted to S of one of the graphs that are in the cg-partition of γ. Nevertheless not all the steps for γ affect to γ_S since there are vertices and links that are not there. Therefore if $cg(\gamma_S) = (g'_p, s'_p)_{p=1}^q$, then there are indices $(k_p)_{p=1}^q$ with $\left(g_{k_p+1}\right)_S \neq \left(g_{k_p}\right)_S$ ($k_q = m$ if $(g_m)_S \neq 0$) such that

$$g'_p = (g_k)_S, \quad \forall k_p \leq k < k_{p+1} \text{ and } s'_p = \sum_{k=k_p}^{k_{p+1}-1} s_k.$$

So, we obtain using the games $(vert(g_k), v^{g_k})$, for all $k \in \{1, \ldots, m\}$,

$$\begin{aligned} v^{\gamma}(S) &= \sum_{p=1}^{q} s'_p r(g'_p) = \sum_{p=1}^{q} \sum_{k=k_p}^{k_{p+1}-1} s_k r\left((g_k)_S\right) \\ &= \sum_{k=1}^{k_q} s_k r\left((g_k)_S\right) = \sum_{k=1}^{k_q} s_k v^{g_k}(S \cap vert(g_k)). \end{aligned}$$

Notice that $S \cap vert\left((g_k)_S\right) = \emptyset$ for every $k > k_q$. Consequently, from Definition 1.14

$$v^{\gamma} = \sum_{k=1}^{m} s_k (v^{g_k})_{vert(g_k)}.$$

□

Remark 3.8 *In the proof of the previous theorem we have reasoned that if* $cg(\gamma) = (g_k, s_k)_{k=1}^{m}$ *then*

$$v^{\gamma}(S) = \sum_{k=1}^{m} s_k r\left((g_k)_S\right), \forall S \subseteq N.$$

If $T \subseteq N$ and $x \in \mathbb{R}^T$ we denote as $x^0 \in \mathbb{R}^N$ the vector which satisfies $x_i^0 = x_i,$ if $i \in T$ and $x_i^0 = 0,$ if $i \notin T.$

Lemma 3.9 *Let* $(T, v) \in \mathcal{G}$ *and* $T \subseteq N$. *The Shapley value of the extended game satisfies*

$$\phi(N, v_T) = \phi^0(T, v).$$

Proof. If $i \in N \setminus T$ then i is a null player for game (N, v_T), since for all $S \subseteq N \setminus \{i\}$ it holds

$$v_T(S \cup \{i\}) = v(S \cap T) = v_T(S).$$

Since the Shapley value satisfies the null player axiom (Theorem 1.21) then $\phi_i(N, v_T) = 0.$
We suppose then that $i \in T$. By Definition 1.19 we have

$$\phi_i(N, v_T) = \sum_{H \subseteq N \setminus \{i\}} c_h^n [v_T(H \cup \{i\}) - v_T(H)] = \sum_{H \subseteq N \setminus \{i\}} c_h^n [v(T \cap H \cup \{i\}) - v(T \cap H)],$$

where $c_h^n = \dfrac{h!(n-h-1)!}{n!}, h = |H|$. If we take $R = H \cap T$ and $S = H \setminus T$ we obtain

$$\phi_i(N, v_T) = \sum_{R \subseteq T \setminus \{i\}} \left(\sum_{S \subseteq N \setminus T} c_{s+r}^n \right) [v(R \cup \{i\}) - v(R)],$$

with $r = |R|, t = |T|, s = |S|$. We see what happens with the coefficients of the expression:

$$\sum_{S \subseteq N \setminus T} c^n_{s+r} = \sum_{s=0}^{n-t} c^n_{s+r} \binom{n-t}{s} = \sum_{s=0}^{n-t} \frac{(s+r)!(n-s-r-1)!}{n!} \frac{(n-t)!s!}{(n-s-t)!}.$$

Multiplying and dividing by $(t-r-1)!r!$ we have

$$\begin{aligned} \sum_{S \subseteq N \setminus T} c^n_{s+r} &= \frac{(n-t)!r!(t-r-1)!}{n!} \sum_{s=0}^{n-t} \binom{s+r}{s} \binom{n-s-r-1}{n-t-s} \\ &= \frac{(n-t)!r!(t-r-1)!}{n!} \binom{n}{n-t} = c^t_r, \end{aligned}$$

where we have used a variant of the identity of Vandermonde (see [36]). We conclude that

$$\phi_i(N, v_T) = \sum_{R \subseteq T \setminus \{i\}} c^t_r \left[v(R \cup \{i\}) - v(R)\right] = \phi_i(T, v). \square$$

Theorem 3.10 *The cg-Myerson value satisfies, for each* $(N, v, \gamma) \in \mathcal{G}_{fcom}$

$$M(N, v, \gamma) = \sum_{k=1}^{m} s_k \mu^0(vert(g_k), v, g_k)$$

where $cg(\gamma) = (g_k, s_k)_{k=1}^m$.

Proof. If for each $\gamma \in FCS^N$ we have that $cg(\gamma) = (g_k, s_k)_{k=1}^m$ then by Lemma 3.7 it holds

$$v^\gamma = \sum_{k=1}^{m} s_k (v^{g_k})_{vert(g_k)}.$$

By the linearity of the Shapley value (Theorem 1.21) and Lemma 3.9 we obtain

$$\begin{aligned} M(N, v, \gamma) = \phi\left(N, v^\gamma\right) &= \sum_{k=1}^{m} s_k \phi\left(N, (v^{g_k})_{vert(g_k)}\right) = \sum_{k=1}^{m} s_k \phi^0\left(vert(g_k), v^{g_k}\right) \\ &= \sum_{k=1}^{m} s_k \mu^0\left(vert(g_k), v, g_k\right). \end{aligned}$$

□

We search for an axiomatization of the cg-Myerson value. The component efficiency axiom can be applied to fuzzy situations taking into account the chosen profit measure.

***cg*-Component efficiency.** The fuzzy communication value F satisfies cg-component efficiency if $\forall(N,v,\gamma)$ and $H \in N/\gamma \cup \{\{i\} : i \notin vert(\gamma)\}$ it holds

$$\sum_{i\in H} F_i(N,v,\gamma) = \epsilon^{cg}(\gamma_H),$$

The fairness axiom can be applied by levels. So, if the level of a communication is reduced then both players in the link have the same loss in their payments. We will use the fuzzy graph of Definition 2.23.

Fuzzy fairness. The fuzzy communication value F satisfies fuzzy fairness if for each (N,v,γ) with $\gamma = (\tau,\rho)$ it holds

$$F_i(N,v,\gamma) - F_i\left(N,v,\gamma^t_{-ij}\right) = F_j(N,v,\gamma) - F_j\left(N,v,\gamma^t_{-ij}\right),$$

$\forall i,j \in N$ and $t \in [0,\rho(ij)]$.

Theorem 3.11 *The cg-Myerson value satisfies cg-component efficiency and fuzzy fairness.*

Proof. We will prove that M satisfies the axioms.

cg-Component efficiency. If $i \notin vert(\gamma)$ it is straightforward. Suppose now$vert(\gamma) = N$. We will prove the cg-component efficiency by using Theorem 3.10 by which, for $(N,v,\gamma) \in \mathcal{G}_{fcom}$ it holds

$$M(N,v,\gamma) = \sum_{k=1}^{m} s_k\mu^0\left(vert(g_k),v,g_k\right),$$

where $cg(\gamma) = (g_k,s_k)_{k=1}^m$. Since the Myerson value satisfies component efficiency (Theorem 3.2) we obtain for each $H \in N/\gamma$,

$$\begin{aligned}\sum_{i\in H} M_i(N,v,\gamma) &= \sum_{k=1}^{m} s_k \sum_{i\in H\cap vert(g_k)} \mu_i\left(vert(g_k),v,g_k\right)\\ &= \sum_{k=1}^{m} s_k \sum_{T\in[H\cap vert(g_k)]/(g_k)_H} \sum_{i\in T}\mu_i\left(vert(g_k),v,g_k\right)\\ &= \sum_{k=1}^{m} s_k \sum_{T\in[H\cap vert(g_k)]/(g_k)_H} v(T) = \sum_{k=1}^{m} s_k r\left((g_k)_H\right) = \epsilon^{cg}(\gamma_H),\end{aligned}$$

using (2.3) and Remark 3.8.

Fuzzy fairness. In order to prove the fuzzy fairness axiom, we can follow the proof of Theorem 3.2, building up the game $\left(N, v^{\gamma} - v^{\gamma^t_{-ij}}\right)$, where i,j are symmetric so using that the Shapley value satisfies equal treatment (Section 1.2),

$$\phi_i\left(N, v^{\gamma} - v^{\gamma^t_{-ij}}\right) = \phi_j\left(N, v^{\gamma} - v^{\gamma^t_{-ij}}\right).$$

The linearity axiom of ϕ and Definition 3.6 conclude the proof.

$$M_i(N,v,\gamma) - M_i\left(N,v,\gamma^t_{-ij}\right) = M_j(N,v,\gamma) - M_j\left(N,v,\gamma^t_{-ij}\right).$$

□

Theorem 3.12 *The cg-Myerson value is the only fuzzy communication value that satisfies cg-component efficiency and fuzzy fairness.*

Proof. The existence was proven in the above theorem. Let F^1 and F^2 be two fuzzy communication values that satisfy cg-component efficiency and fuzzy fairness. Let $(N,v,\gamma)\in\mathcal{G}_{fcom}$. We will prove that over this game both values coincide by induction in $|link(\gamma)|$.

If $link(\gamma)=\emptyset$, then each player $i\in N$ satisfies that i is isolated in γ and by the cg-component efficiency axiom,

$$F^1_i(N,v,\gamma) = F^2_i(N,v,\gamma) = \tau(\{i\})v(\{i\}).$$

We suppose that if $|link(\gamma)| < p$ then $F^1(N,v,\gamma) = F^2(N,v,\gamma)$. Now we consider γ with

$|link(\gamma)| = p$.
If we take $t = \rho(ij) > 0$, by definition of γ, we have

$$F_i^1\left(N, v, \gamma_{-ij}^t\right) = F_i^2\left(N, v, \gamma_{-ij}^t\right), F_j^1\left(N, v, \gamma_{-ij}^t\right) = F_j^2\left(N, v, \gamma_{-ij}^t\right).$$

Using the fuzzy fairness axiom,

$$\begin{aligned} F_i^1(N,v,\gamma) - F_j^1(N,v,\gamma) &= F_i^1\left(N, v, \gamma_{-ij}^t\right) - F_j^1\left(N, v, \gamma_{-ij}^t\right) \\ &= F_i^2\left(N, v, \gamma_{-ij}^t\right) - F_j^2\left(N, v, \gamma_{-ij}^t\right) \\ &= F_i^2(N,v,\gamma) - F_j^2(N,v,\gamma), \end{aligned}$$

and we obtain $F_i^1(N,v,\gamma) - F_i^2(N,v,\gamma) = F_j^1(N,v,\gamma) - F_j^2(N,v,\gamma)$. Now let $H \in N/\gamma$. If $H = \{i\}$ then $F_i^1(N,v,\gamma) = F_i^2(N,v,\gamma)$ because both values are cg-component efficient. If $|H| > 1$ then there exists a constant K such that $F_i^1(N,v,\gamma) - F_i^2(N,v,\gamma) = K$ for every $i \in H$. Since the values are cg-component efficient and fuzzy fair, these sums are equal, i.e., $\sum_{i\in H} F_i^1(N,v,\gamma) = \sum_{i\in H} F_i^2(N,v,\gamma)$. Consequently,

$$\sum_{i\in H} F_i^1(N,v,\gamma) - F_i^2(N,v,\gamma) = |H|K = 0,$$

and this implies $K = 0$. We obtain $F_i^1(N,v,\gamma) = F_i^2(N,v,\gamma)$. So the uniqueness is proven. □

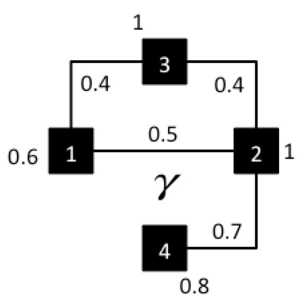


Figure 3.1: Graph γ

Although in the Appendix we analyze the calculation of the cg-Myerson value, next axiom shows how the levels in the fuzzy model modify the value.

Example 3.13 Let us see an example of the computation of the cg-Myerson value. Let (N, v) with $N = \{1, 2, 3, 4\}$ and $v(S) = 2|S| - 1$, $S \neq \emptyset$. The Shapley value of this game is

$$\phi(N, v) = (1.75, 1.75, 1.75, 1.75).$$

We take γ the fuzzy graph of Figure 3.1. If the communication structure is considered without levels then we would obtain the usual Myerson value

$$\mu\left(N, v, g^{\gamma}\right) = (1.67, 1.67, 2.17, 1.5).$$

If we take into account the levels of the fuzzy graph

$$M(N, v, \gamma) = (1.27, 0.92, 1.67, 1.15).$$

Next proposition shows the different extensions of the value defined in this section.

Proposition 3.14 *Let* $(N, v) \in \mathcal{G}$.

1. *If* $g = (S, L) \in CS_0^N$ *then* $M(N, v, g) = \mu^0(S, v, g)$.
2. *If* $g = (N, L(N))$ *then* $M(N, v, g) = \phi(N, v)$.
3. *If* $\gamma = (\tau, \rho)$ *is complete by links then* $M(N, v, \gamma) = \phi^{ch}(\tau, v)$.

Proof.

1) We have that $cg(g) = (g, 1)$. By Theorem 3.10,

$$M(N, v, g) = \sum_{k=1}^{m} s_k \mu^0(vert(g_k), v, g_k) = \mu^0(S, v, g).$$

2) Analogously to 1) $M(N, v, g) = \mu(N, v, g) = \phi(N, v^g) = \phi(N, v)$, since $v^g = v$ because g is a

complete graph.
3) Let $\gamma = (\tau, \rho) \in FCS^N$ be complete by links (Definition 2.18) and let the non-null different values in τ be $h_1 < \cdots < h_m$. We denote by

$$S_k = \{i \in N : \tau(i) \geq h_k\}, \ \forall k \in \{1, \ldots, m\}.$$

We observe that for every fuzzy graph γ complete by links, it holds that $\gamma - tg^\gamma$ is also complete by links if $tg^\gamma \leq \gamma$. In the first step of the cg-algorithm we obtain $s_1 = h_1 = h_1 - 0 = \bigwedge_{i \in N} \tau(i)$ and $vert(g_1) = N = S_1$. By the previous observation $\gamma - s_1 g_1$ is also complete by links. Suppose that for $k' < k$ we have $s_{k'} = h_{k'} - h_{k'-1}$ and $vert(g_{k'}) = S_{k'}$. Once again, $\gamma' = \gamma - \sum_{p=1}^{k-1} s_p g_p$ is also complete by links, then all g_k are complete. Now,

$$s_k = \bigwedge_{i \in vert(\gamma')} \tau'(i) = h_k - \sum_{p=1}^{k-1} s_p = h_k - \sum_{p=1}^{k-1} [h_p - h_{p-1}] = h_k - h_{k-1}$$

and $i \in vert(g_k)$ if and only if $\tau(i) \geq h_k$, then $vert(g_k) = S_k$. Therefore, $\forall k$ it holds

$$v^{g_k}(S) = v\,(vert(g_k) \cap S) = v_{S_k}(S), \ \forall S \subseteq N.$$

We have that

$$\begin{aligned} M(N, v, \gamma) &= \sum_{k=1}^{m} s_k \mu^0\,(vert(g_k), v, g_k) &= \sum_{k=1}^{m} s_k \phi\,(N, v^{g_k}) \\ &= \sum_{k=1}^{m} [h_k - h_{k-1}] \phi\,(N, v_{S_k}) &= \phi^{ch}(\tau, v). \end{aligned}$$

□

The value that we have introduced has good properties with respect to the information about the communication.

The cg-Myerson value can be described as a Choquet integral and then we can take advantage of its good mathematical properties. Each fuzzy graph $\gamma = (\tau, \rho) \in FCS^N$ can be identified with a fuzzy set over $\overline{LN} = L(N) \cup \{ii : i \in N\}$ by

$$\gamma\,(ij) = \begin{cases} \rho(ij), & \text{if } ij \in L(N) \\ \tau(i), & \text{if } i = j. \end{cases} \tag{3.2}$$

On the other hand consider the fuzzy Myerson set function for each game $(N, v) \in \mathcal{G}$ and $i \in N$ defined by $\mu_i(N, v) : CS_0^N \to \mathbb{R}$ with

$$\mu_i(N, v)(g) = \mu_i^0(vert(g), v, g).$$

Since every $g \in CS_0^N$ is also a subset of $\overline{LN}$ with the previous identification we can enunciate the following results.

Theorem 3.15 *The cg-Myerson value satisfies for every $(N, v, \gamma) \in \mathcal{G}_{fcom}$ and each $i \in N$*

$$M_i(N, v, \gamma) = \int \gamma \; d\mu_i(N, v).$$

Proof. Let $im(\gamma) = \{\lambda_1 < \ldots \lambda_m\}$ and $\lambda_0 = 0$. If we apply the cg-algorithm to the fuzzy graph γ we see that $cg(\gamma) = (g_k, s_k)_{k=1}^m$ with $g_k = [\gamma]_{\lambda_k}, s_k = \lambda_k - \lambda_{k-1}$, by recurrence in k. If we consider $k = 1$ then $s_1 = \wedge\gamma = \lambda_1$ and $g_1 = g^\gamma = [\gamma]_{\lambda_1}$. Moreover we observe that $\gamma - s_1 g_1$ substracts a fixed amount to each element of the graph, then $s_2 = \wedge(\gamma - s_1 g_1) = \lambda_2 - s_1 = \lambda_2 - \lambda_1$ and $g_2 = [\gamma]_{\lambda_2}$. If we suppose that $s_{k-1} = \lambda_{k-1} - \lambda_{k-2}$ for $k-1$ and $g_{k-1} = [\gamma]_{\lambda_{k-1}}$. It suffices to call now

$$\gamma = \gamma - \sum_{p=1}^{k-2} s_p g_p,$$

and repeat the previous reasoning, since step k would be the first of the new γ. Therefore, using Theorem 3.10

$$\begin{aligned} \int \gamma \, d\mu_i(N, v) &= \sum_{k=1}^m (\lambda_k - \lambda_{k-1})\, \mu_i(N, v)\, ([\gamma]_{\lambda_k}) = \\ &= \sum_{k=1}^m s_k\, \mu_i^0(N, v, g_k) = M_i(N, v, \gamma). \end{aligned}$$

□

We say that two fuzzy graphs $\gamma, \gamma' \in FCS^N$ are comonotone if they are as fuzzy sets of $\overline{LN}$ (Definition 1.43).

Proposition 3.16 *The cg-Myerson value M satisfies the following properties*

1. *M is a linear function with respect to v.*
2. *M is continuous with respect to γ.*
3. *M is comonotonous with respect to γ, i.e., if γ and γ' are comonotonous fuzzy communication structures and $\alpha \in [0,1]$ then*

$$M(N, v, \alpha\gamma + (1-\alpha)\gamma') = \alpha M(N, v, \gamma) + (1-\alpha)M(N, v, \gamma')$$

Proof. 1) Let $(N, v), (N, w) \in \mathcal{G}, \alpha, \beta \in \mathbb{R}$. Then by definition of M,

$$M(N, \alpha v + \beta w, \gamma) = \int \gamma \, d\mu(N, \alpha v + \beta w).$$

The Myerson value is linear because $(\alpha v + \beta w)^g = \alpha v^g + \beta w^g$ and ϕ is linear (Therorem 1.21). Now by property (C3) of Proposition 1.50,

$$M(N, \alpha v + \beta w, \gamma) = \alpha \int \gamma \, d\mu(N, v) + \beta \int \gamma \, d\mu(N, w) = \alpha M(N, v, \gamma) + \beta M(N, w, \gamma).$$

2) It is straightforward using the previous theorem and (C6) in Proposition 1.50.
3) Since γ and γ' are comonotonous and $\alpha \in [0,1]$ then we have $\alpha\gamma + (1-\alpha)\gamma', \alpha\gamma, (1-\alpha)\gamma' \in FCS^N$, and moreover $\alpha\gamma$ and $(1-\alpha)\gamma'$ are comonotonous. Using properties (C2) and (C4) of Proposition 1.50 we have that $\forall i \in N$,

$$\begin{aligned} M_i(N, v, \alpha\gamma + (1-\alpha)\gamma') &= \int (\alpha\gamma + (1-\alpha)\gamma') \, d\mu_i(N, v) \\ &= \int \alpha\gamma \, d\mu_i(N, v) + \int (1-\alpha)\gamma' d\mu_i(N, v) \\ &= \alpha \int \gamma \, d\mu_i(N, v) + (1-\alpha) \int \gamma' d\mu_i(N, v). \end{aligned}$$

□

3.3 The cg-Banzhaf value

Following once again the cg-model we define a Banzhaf value for games with fuzzy communication structure. For each $\gamma \in FCS^N$ we will also use the fuzzy vertex game (Definition 2.32).

Definition 3.17 *The cg-Banzhaf value is the fuzzy communication value defined as*

$$B(N, v, \gamma) = \beta(N, v^\gamma),$$

for every game with fuzzy communication structure (N, v, γ).

The cg-Banzhaf value can be expressed in terms of the graph Banzhaf values of the graphs in the cg-partition, like the cg-Myerson value. We see first that the Banzhaf value satisfies with respect to the extended game the same relation that the Shapley value.

Lemma 3.18 *Let $(T, v) \in \mathcal{G}$ and $T \subseteq N$. The Banzhaf value of the extended game is*

$$\beta(N, v_T) = \beta^0(T, v).$$

Proof. If $i \notin T$ then i is a null player and the null player property of the Banzhaf value (Theorem 1.24) implies $\beta_i(N, v_T) = 0$. If $i \in T$ and $t = |T|$ then

$$\begin{aligned} \beta_i(N, v_T) &= \frac{1}{2^{n-1}} \sum_{\{S \subseteq N : i \in S\}} [v_T(S \cup \{i\}) - v_T(S)] \\ &= \frac{1}{2^{n-1}} \sum_{\{S \subseteq N : i \in S\}} [v(S \cup \{i\} \cap T) - v(S \cap T)] \\ &= \frac{1}{2^{n-1}} \sum_{\{R \subseteq T : i \in R\}} 2^{n-t}[v(R \cup \{i\}) - v(R)] = \beta_i(T, v). \end{aligned}$$

□

Theorem 3.19 *Let (N, v, γ) be a game with fuzzy communication structure and cg-partition*

$cg(\gamma) = (g_k, s_k)_{k=1}^m$. *It holds*

$$B(N, v, \gamma) = \sum_{k=1}^{m} s_k \eta^0 \left(vert(g_k), v, g_k \right).$$

Proof.

If for each $\gamma \in FCS^N$ we have $cg(\gamma) = (g_k, s_k)_{k=1}^m$ then Lemma 3.7 implies

$$v^\gamma = \sum_{k=1}^{m} s_k (v^{g_k})_{vert(g_k)}.$$

By linearity of the Banzhaf value (Theorem 1.24) and Lemma 3.18 we obtain

$$\begin{aligned} B(N, v, \gamma) = \beta\left(N, v^\gamma\right) &= \sum_{k=1}^{m} s_k \beta\left(N, (v^{g_k})_{vert(g_k)}\right) = \sum_{k=1}^{m} s_k \beta^0\left(vert(g_k), v^{g_k}\right) \\ &= \sum_{k=1}^{m} s_k \eta^0\left(vert(g_k), v, g_k\right). \end{aligned}$$

□

We search for an axiomatization of the cg-Banzhaf value. We are going to extend the properties of the graph Banzhaf value (Section 3.1) for a fuzzy communication value F.

Similar to the isolation axiom (Section 3.1) in this case the level of the vertex marks the maximum participation of the corresponding player.

Fuzzy isolation. A fuzzy communication value F satisfies the fuzzy isolation axiom if for every $(N, v, \gamma) \in \mathcal{G}_{fcom}$ with $\gamma = (\tau, \rho)$ and $i \in N$ isolated in γ it holds

$$F_i(N, v, \gamma) = \tau(i) v(\{i\}).$$

For the next axiom we define the following fuzzy communication structure, which represents the amalgamation of players i, j in the structure up to level t, extending the analogous crisp concept given in Definition 1.6.

Definition 3.20 *Let* $\gamma \in FCS^N, i, j \in N$ *and* $t \in [0, \rho(ij)]$. *We define the fuzzy graph*

$\gamma_t^{ij} = \left(\tau_t^{ij}, \rho_t^{ij}\right) \in FCS^{N^{ij}}$ *where for all* $i', j' \in N^{ij} = N \backslash \{i, j\} \cup \{p\}$,

$$\tau_t^{ij}(i') = \begin{cases} t, & \text{if } i' = p \\ \tau(k) \wedge t, & \text{if } i' \neq p \end{cases} \text{ and } \rho_t^{ij}(i'j') = \begin{cases} \rho(i'j') \wedge t, & \text{if } i', j' \neq p \\ (\rho(i'i) \vee \rho(i'j)) \wedge t, & \text{if } j' = p. \end{cases}$$

If we suppose the amalgamation of players i, j until level t, next fuzzy subgraph shows the situation after overcoming level t.

Definition 3.21 *Let* $\gamma \in FCS^N$. *If* $t \in [0, 1]$, *then* $\gamma^t = \left(\tau^t, \rho^t\right) \in FCS^N$ *is the subgraph*

$$\tau^t(i) = (\tau(i) - t) \vee 0, \qquad \rho^t(ij) = (\rho(ij) - t) \vee 0.$$

Example 3.22 Consider the fuzzy graph γ in Figure 3.2 with $N = \{1, 2, 3, 4\}$. In this Figure we show the amalgamation of players $2, 3$ until level 0.5, $\gamma_{0.5}^{23}$ and the subgraph $\gamma^{0.5}$.

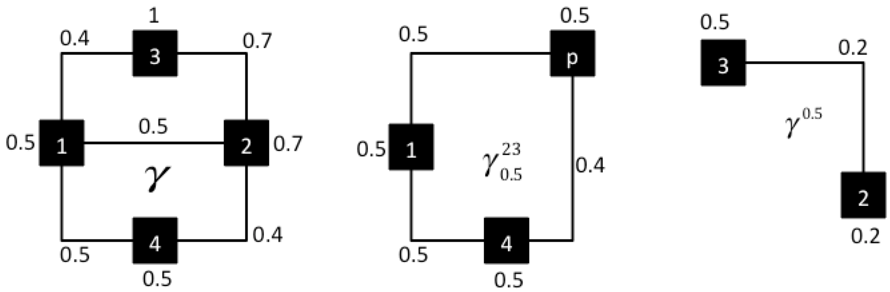


Figure 3.2: Amalgamation up to level 0.5 in γ

We can now propose a merger or amalgamation axiom similar to the axiom of graph pairwise merging.

Pairwise fuzzy merging. The fuzzy communication value F satisfies pairwise fuzzy merging if $\forall (N, v, \gamma) \in \mathcal{G}_{fcom}$, $i, j \in N$ and $t \in [0, \rho(ij)]$ it holds

$$F_i(N, v, \gamma) + F_j(N, v, \gamma) = F_p\left(N^{ij}, v^{ij}, \gamma_t^{ij}\right) + F_i\left(N, v, \gamma^t\right) + F_j\left(N, v, \gamma^t\right).$$

If we take a usual communication structure, $\gamma = g \in CS^N$ and $t = 1$ as the merging level of the link ij then $\gamma_t^{ij} = g^{ij}$ and $\gamma^t = 0$. Therefore, in this situation the pairwise fuzzy merging axiom coincides with the graph pairwise merging axiom (Section 3.1).

Theorem 3.23 *The cg-Banzhaf value satisfies fuzzy isolation, fuzzy fairness and pairwise fuzzy merging.*

Proof. We prove that our value satisfies the axioms.
Fuzzy isolation. Let (N, v, γ) with $\gamma = (\tau, \rho)$ and $cg(\gamma) = (s_k, g_k)_{k=1}^m$. Let $i \in N$ be an isolated player in γ. Since $g_k \leq g^\gamma$ for every k then i is an isolated player in g_k when $i \in vert(g_k)$. Consequently, in this case by the isolation property, we obtain $\eta_i\,(vert(g_k), v, g_k) = v(\{i\})$. Theorem 3.19 and (2.4) imply

$$\begin{aligned} B_i(N, v, \gamma) &= \sum_{k=1}^{m} s_k \eta_i^0\,(vert(g_k), v, g_k) \\ &= v(\{i\}) \sum_{\{k: i \in vert(g_k)\}} s_k = \tau(i) v(\{i\}), \end{aligned}$$

so B satisfies fuzzy isolation.

Fuzzy fairness. Now take $i, j \in N$ with $\rho(ij) > 0$. Let $t \in (0, \rho(ij)]$. We define the game (N, w) as $w = v^\gamma - v^{\gamma_{-ij}^t}$. If i or j are not in S then $w(S) = 0$, using $\gamma_S = \left(\gamma_{-ij}^t\right)_S$. So,

$$w(S) - w(S\backslash\{i\}) = \left\{ \begin{array}{ll} w(S), & \text{if } i, j \in S \\ 0, & \text{otherwise.} \end{array} \right\} = w(S) - w(S\backslash\{j\}).$$

So, the equal treatment axiom of the Banzhaf value implies $\beta_i(N, w) = \beta_j(N, w)$. By linearity we have

$$\begin{aligned} B_i(N, v, \gamma) - B_i\left(N, v, \gamma_{-ij}^t\right) &= \beta_i\,(N, v^\gamma) - \beta_i\left(N, v^{\gamma_{-ij}^t}\right) \\ &= \beta_i(N, w) = \beta_j(N, w) \\ &= B_j(N, v, \gamma) - B_j\left(N, v, \gamma_{-ij}^t\right). \end{aligned}$$

Pairwise fuzzy merging. Finally let us see that pairwise fuzzy merging is satisfied. Consider $i, j \in N$ and $t \in [0, \rho(ij)]$. There exists $q \in \{1, \ldots, m\}$ such that $s_q \leq t < s_{q+1}$. The Choquet by graphs partition of γ_t^{ij} and γ^t are respectively

$$cg\left(\gamma_t^{ij}\right) = \left(s_k, (g_k)^{ij}\right)_{k=1}^{q} \cup \left(t - s_q, (g_{q+1})^{ij}\right)$$

$$cg(\gamma^t) = \left(s_k, (g_k)^{ij}\right)_{k=q+2}^{m} \cup \left(s_{q+1} - t + s_q, g_{q+1}\right).$$

Using Theorem 3.19 and the pairwise fuzzy merging axiom, the sum $B_i(N, v, \gamma) + B_j(N, v, \gamma)$ is

$$\begin{aligned}
\sum_{k=1}^{m} s_k \left[\eta_i^0\left(vert(g_k), v, g_k\right) + \eta_j^0\left(vert(g_k), v, g_k\right)\right] &= \\
\sum_{k=1}^{q} s_k \left[\eta_i^0\left(vert(g_k), v, g_k\right) + \eta_j^0\left(vert(g_k), v, g_k\right)\right] &+ \\
(t - s_q)\left[\eta_i^0\left(vert(g_{q+1}), v, g_{q+1}\right) + \eta_j^0\left(vert(g_{q+1}), v, g_{q+1}\right)\right] &+ \\
(s_{q+1} - t + s_q)\left[\eta_i^0\left(vert(g_{q+1}), v, g_{q+1}\right) + \eta_j^0\left(vert(g_{q+1}), v, g_{q+1}\right)\right] &+ \\
\sum_{k=q+2}^{m} s_k \left[\eta_i^0\left(vert(g_k), v, g_k\right) + \eta_j^0\left(vert(g_k), v, g_k\right)\right] &= \\
\sum_{k=1}^{q} s_k \eta_p^0\left((vert(g_k))^{ij}, v^{ij}, (g_k)^{ij}\right) &+ \\
(t - s_q)\eta_p^0\left((vert(g_{q+1}))^{ij}, v^{ij}, (g_{q+1})^{ij}\right) &+ \\
B_i\left(N, v, \gamma^t\right) + B_j\left(N, v, \gamma^t\right) &= \\
B_p\left(N^{ij}, v^{ij}, \gamma_t^{ij}\right) + B_i\left(N, v, \gamma^t\right) + B_j\left(N, v, \gamma^t\right). &\qquad \square
\end{aligned}$$

Theorem 3.24 *The cg-Banzhaf value is the only fuzzy communication value satisfying fuzzy isolation, fuzzy fairness and pairwise fuzzy merging.*

Proof. The above theorem showed the existence. Let us prove the uniqueness by induction on the cardinality of $link(\gamma)$. Suppose F^1, F^2 two fuzzy communication values satisfying the three axioms and (N, v, γ) a game with fuzzy communication structure, where $\gamma = (\tau, \rho)$.
If $|link(\gamma)| = 0$ then $\rho = 0$ and all vertices are isolated. Therefore the fuzzy isolation property implies

$$F_i^1(N, v, \gamma) = F_i^2(N, v, \gamma) = \tau(i)v(\{i\}),$$

for every $i \in N$.
Suppose that $F^1 = F^2$ if $|link(\gamma)| < k$.
Let $|link(\gamma)| = k$ and $i \in N$. If i is isolated then the fuzzy isolation property implies again that both values are the same. So, choose $ij \in link(\gamma)$. Applying the fuzzy fairness axiom to this link and $t = \rho(ij)$, like in Theorem 3.12, we have

$$F_i^1(N, v, \gamma) - F_j^1(N, v, \gamma) = F_i^1\left(N, v, \gamma_{-ij}^t\right) - F_j^1\left(N, v, \gamma_{-ij}^t\right) \tag{3.3}$$

$$= F_i^2\left(N, v, \gamma_{-ij}^t\right) - F_j^2\left(N, v, \gamma_{-ij}^t\right) = F_i^2(N, v, \gamma) - F_j^2(N, v, \gamma),$$

since $\left|link\left(\gamma_{-ij}^t\right)\right| < k$. Finally, we use the pairwise fuzzy merging condition with players i, j and $t = \rho(ij)$. Then,

$$F_i^1(N, v, \gamma) + F_j^1(N, v, \gamma) = F_p^1\left(N^{ij}, v^{ij}, \gamma_t^{ij}\right) + F_i^1\left(N, v, \gamma^t\right) + F_j^1\left(N, v, \gamma^t\right) \tag{3.4}$$

$$= F_p^2\left(N^{ij}, v^{ij}, \gamma_t^{ij}\right) + F_i^2\left(N, v, \gamma^t\right) + F_j^2\left(N, v, \gamma^t\right) = F_i^2(N, v, \gamma) + F_j^2(N, v, \gamma),$$

because $\left|link\left(\gamma_t^{ij}\right)\right|, \left|link\left(\gamma^t\right)\right| < k$. Adding (3.3) and (3.4) we have

$$F_i^1(N, v, \gamma) = F_i^2(N, v, \gamma).$$

□

Remark 3.25 *We can particularize to γ complete by links obtaining a fuzzy Banzhaf mapping similar to the Shapley one given in Definition 1.53, that we denote by β^{ch}. We have a property in this case in the following proposition.*

Proposition 3.26 *Let $(N, v) \in \mathcal{G}$.*

1. *If $g = (S, L) \in CS_0^N$ then $B(N, v, g) = \eta^0(S, v, g)$.*
2. *If $g = (N, L(N))$ then $B(N, v, g) = \beta(N, v)$.*
3. *If $\gamma = (\tau, \rho)$ is complete by links then $B(N, v, \gamma) = \beta^{ch}(\tau, v)$.*

Proof. It is analogous to that of Proposition 3.14 changing the Shapley value for the Banzhaf value and the Myerson value for the graph Banzhaf value. □

Proposition 3.27 *The cg-Banzhaf value B satisfies the following properties*

1. *B is linear with respect to v.*
2. *B is continuous with respect to γ.*
3. *B is comonotonous with respect to γ, i.e., if γ and γ' are comonotonous fuzzy communication structures and $\alpha \in [0, 1]$ then*

$$B(N, v, \alpha\gamma + (1 - \alpha)\gamma') = \alpha B(N, v, \gamma) + (1 - \alpha)B(N, v, \gamma')$$

Proof. The proof is analogous to that of Theorem 3.16. □

Example 3.28 Consider the fuzzy communication structure γ of Figure 2.2 among four agents in the simple game (N, v) with $v(S) = 1$ if $|S| \geq 3$ and $v(S) = 0$ otherwise.

We use the Banzhaf value to describe the power of each agent. In this game the important coalitions are those with cardinality 3 because they are winning but if a player leaves they are losing. If we forget the structure then the usual Banzhaf value is

$$\beta(N, v) = (0.375, 0.375, 0.375, 0.375).$$

Moreover, if we suppose the structure without levels, g^γ, then the graph Banzhaf value coincides with the previous result,

$$\eta\,(N, v, g^\gamma) = \beta(N, v),$$

because all the important coalitions are still winning. Now we compute the cg-Banzhaf value. In Figure 2.9 we can see the cg-partition $cg(\gamma)$ of our fuzzy graph. The power indices are

$$B(N, v, \gamma) = (0.2625, 0.2625, 0.1875, 0.1875).$$

We can see that the leveled relations among the players imply asymmetry between players $\{1, 2\}$ and players $\{3, 4\}$. We also observe that actually in the leveled situation is more difficult for those players to form winning coalitions. This is reflected in our value if we compare the indices with the other versions of Banzhaf.

Chapter 4

Choquet by graph values II

In this chapter we analyze two examples of communication values that do not follow exactly the Myerson model: the position value and the average tree value. We present them and then we study their fuzzy versions.

4.1 The position value

The Myerson value for $g \in CS^N$ is defined using the vertex game (Definition 2.4). In other way, the link game is defined by Borm et al. [14] using the graph restricted by links (Definition 1.36) and the measure of Myerson (2.2).

Definition 4.1 *Let $(N, v, g) \in \mathcal{G}_{com}$ with $g = (N, L)$. The link game is a new game $\left(L, v^{Lg}\right)$ defined by*

$$v^{Lg}(B) = r(g_B), \quad \forall B \subseteq L.$$

Observe that the link game does not use the isolated vertices and therefore it loses information from the active isolated players. That is why Borm et al. [14] proposed the link game only for 0-normalized games. From now on, we write v^{Lg} instead of $(v_0)^{Lg}$ to simplify the notation.

The position value is a communication value defined as follows.

Definition 4.2 *For any game (N, v) and graph $g = (N, L) \in CS^N$ the position value is defined by*

$$\pi_i(N, v, g) = v(\{i\}) + \frac{1}{2} \sum_{\{j \in N \setminus \{i\} : ij \in L\}} \phi_{ij}\left(L, v^{Lg}\right), \ \forall i \in N.$$

We present now an axiom that serves to characterize this value together with the component efficiency axiom (3.1).

Balanced total threats. For all $(N, v, g) \in \mathcal{G}_{com}$ and $i, j \in N$ it holds

$$\sum_{ih \in link(g)} [f_j(N, v, g) - f_j(N, v, g_{-ih})] = \sum_{jh \in link(g)} [f_i(N, v, g) - f_i(N, v, g_{-jh})].$$

This property means that if we take two players, the total loss in payment for one of them if we break all the links in which the other takes part is the same as the total loss for the other player if we break all the links in which the first is involved.

Next characterization appears in [68].

Theorem 4.3 (Slikker [68]) *The position value is the only communication value that satisfies component efficiency and balanced total threats.*

4.2 The cg-position value

Following again the Choquet by graphs model we define a fuzzy position value. We define first a fuzzy version of the link game in order to introduce the cg-position value, where we use Definition 2.32 and Definition 2.13.

Definition 4.4 *Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$. The cg-link game $\left(L(N), v^{L\gamma}\right)$ is defined by taking the links as players, that is, for any $A \subseteq L(N)$*

$$v^{L\gamma}(A) = \epsilon^{cg}(\gamma_A).$$

For the same reason that the crisp version, the cg-link game loses information from the worths of the individual coalitions and therefore it is only useful for 0-normalized games. By analogy with the crisp version, we will use $v^{L\gamma} = (v_0)^{L\gamma}$. We introduce the fuzzy version of the position value.

Definition 4.5 *The cg-position value is the fuzzy communication value defined for each game $(N, v, \gamma) \in \mathcal{G}_{fcom}$ with $\gamma = (\tau, \rho)$ and every player $i \in N$ by*

$$P_i(N, v, \gamma) = \tau(i)v(\{i\}) + \frac{1}{2} \sum_{j \in N \setminus \{i\}} \phi_{ij} \left(L(N), v^{L\gamma} \right).$$

Now an axiomatization for the cg-position value is presented. First, we are going to relate the cg-position value to the crisp position value π. The following lemma provides a formula in Choquet form in order to compute the cg-position value by means of a linear combination of crisp position values.

Lemma 4.6 *Let $(N, v) \in \mathcal{G}$ and $\gamma \in FCS^N$. If the cg-partition of γ is $cg(\gamma) = (g_k, s_k)_{k=1}^m$ then*

1. $v^{L\gamma} = \sum_{k=1}^{m} s_k \left(v^{Lg_k} \right)_{link(g_k)}.$

2. $P(N, v, \gamma) = \sum_{k=1}^{m} s_k \pi^0(vert(g_k), v, g_k).$

Proof. 1) It is analogous to Lemma 3.7, substituting S for A and v^{g_k} for v^{Lg_k}
2) By the linearity of the Shapley value and 1) we get

$$\phi \left(N, v^{L\gamma} \right) = \sum_{k=1}^{m} s_k \phi \left(link(g_k), v^{Lg_k} \right).$$

We calculate the cg-position value for any player $i \in N$. We denote by k_i the last level in the cg-algorithm such that $i \in vert\left(g_{k_i}\right)$. If $k > k_i$ then for each $j \in N \setminus \{i\}$ the link ij is a null player for the link game v^{Lg_k} and then using the null player property of ϕ,

$$\begin{aligned}
P_i(N,v,\gamma) &= \tau(i)v(\{i\}) + \frac{1}{2}\sum_{j\in N\setminus\{i\}} \phi_{ij}\left(L(N), v^{L\gamma}\right) \\
&= \sum_{k=1}^{k_i} s_k v(\{i\}) + \frac{1}{2}\sum_{j\in N\setminus\{i\}} \sum_{k=1}^{k_i} s_k \phi_{ij}\left(link(g_k), v^{Lg_k}\right) \\
&= \sum_{k=1}^{k_i} s_k \left[v(\{i\}) + \frac{1}{2}\sum_{j\in N\setminus\{i\}} \phi_{ij}\left(link(g_k), v^{Lg_k}\right)\right].
\end{aligned}$$

We obtain from Definition 4.5, $P_i(N,v,\gamma) = \sum_{k=1}^{m} s_k \pi_i^0 \left(vert(g_k), v, g_k\right)$. □

We look for an axiomatization of the cg-position value following Slikker [68]. Let (N,v) be a cooperative game and $\gamma \in FCS^N$. Consider this axiom: the payoffs obtained for the players must be efficient for each connected component with respect to the measure of the fuzzy graph with this model. We presented this axiom in Section 3.2 when we axiomatized the cg-Myerson value. We will denote as $\wedge_i\gamma = \bigwedge_{ik\in link(\gamma)} \rho(ik)$ the lowest level of communication for a non-isolated player $i \in N$, that is, the minimal degree of $i \in N$. For two non-isolated players in γ we consider the notation $\wedge_{ij}\gamma = (\wedge_i\gamma) \wedge (\wedge_j\gamma) > 0$, that is, the *minimal common degree* of the players.

Balanced total fuzzy threats. Let $i, j \in N$ be two different non-isolated players and $t \in [0, \wedge_{ij}\gamma]$. Then

$$\sum_{ih\in link(\gamma)} F_j\left(N,v,\gamma\right) - F_j\left(N,v,\gamma_{-ih}^{\rho(ih)-t}\right) = \sum_{jh\in link(\gamma)} F_i\left(N,v,\gamma\right) - F_i\left(N,v,\gamma_{-jh}^{\rho(jh)-t}\right).$$

This axiom means that the total loss for a player if we reduce to t all the communications of another player is the same as in the reciprocal case. As a consequence anyone can threaten someone in these terms.

Now we are going to prove that the cg-position value satisfies the axioms of cg-component efficiency and balanced total fuzzy threats.

Theorem 4.7 *The cg-position value satisfies cg-component efficiency and balanced total fuzzy threats.*

Proof. Let (N,v) be a game. We take $\gamma \in FCS^N$ with cg-partition $cg(\gamma)=(g_k,s_k)_{k=1}^m$.

cg-Component efficiency. By Theorem 4.3 we know that π is efficient by components, i.e., if $g \in CS^N$ then for all $S \in N/g$ we get

$$\sum_{i\in S}\pi_i(N,v,g)=v(S).$$

Let $S \in N/\gamma$. We have by Lemma 4.6, (2.3) and Remark 3.8,

$$\begin{aligned}\sum_{i\in S}P_i(N,v,\gamma) &= \sum_{k=1}^m\sum_{i\in S}s_k\pi_i^0(vert(g_k),v,g_k)=\sum_{k=1}^m s_k\sum_{T\in(S\cap vert(g_k))/(g_k)_S}v(T)\\ &= \sum_{k=1}^m s_k r((g_k)_S)=\epsilon^{cg}(\gamma_S).\end{aligned}$$

Balanced total fuzzy threats. Slikker [68] proved that π satisfies the balanced total threats axiom, that is for all pair of players $i,j\in N$ and $g\in CS^N$ it holds

$$\sum_{ih\in link(g)}\left[\pi_j(N,v,g)-\pi_j(N,v,g_{-ih})\right]=\sum_{jh\in link(g)}\left[\pi_i(N,v,g)-\pi_i(N,v,g_{-jh})\right].$$

Let $i,j\in N$ and $t\in[0,\wedge_{ij}\gamma]$. There exists $k_t\in\{1,...,m\}$ such that $0\le t-\sum_{k=1}^{k_t-1}s_k<s_{k_t}$ supposing $s_0=0$. We consider the following partition by levels for γ equivalent to $cg(\gamma)$,

$$\left\{(g_k,s_k)_{k=1}^{k_t-1},\left(g_{k_t},t-\sum_{k=1}^{k_t-1}s_k\right),(g_{k_t},s_{k_t}-t),(g_k,s_k)_{k=k_t+1}^{k_{ih}},(g_k,s_k)_{k=k_{ih}+1}^m\right\}.$$

For each $ih\in link(\gamma)$ (or jh) there is $k_{ih}\in\{k_t,...,m\}$ with

$$\sum_{k=1}^{k_{ih}}s_k=\rho(ih),$$

by (2.6). We take for $\gamma_{-ih}^{\rho(ih)-t}$ the partition

$$\left\{(g_k,s_k)_{k=1}^{k_t-1},\left(g_{k_t},t-\sum_{k=1}^{k_t-1}s_k\right),((g_{k_t})_{-ih},s_{k_t}-t),((g_k)_{-ih},s_k)_{k=k_t+1}^{k_{ih}},(g_k,s_k)_{k=k_{ih}+1}^m\right\} \quad (4.1)$$

Observe that this partition is equivalent to the cg-partition of the corresponding fuzzy graph. Hence using Lemma 4.6,

$$\sum_{ih\in link(\gamma)} P_j(N,v,\gamma) - P_j\left(N,v,\gamma_{-ih}^{\rho(ih)-t}\right) =$$

$$(s_{k_t}-t)\sum_{ih\in link(\gamma)}\left[\pi_j^0\left(vert(g_k),v,g_{k_t}\right)-\pi_j^0\left(vert(g_k),v,(g_{k_t})_{-ih}\right)\right] +$$

$$\sum_{ih\in link(\gamma)}\sum_{k=k_t+1}^{k_{ih}} s_k\left[\pi_j^0\left(vert(g_k),v,g_k\right)-\pi_j^0\left(vert(g_k),v,((g_k)_{-ih}\right)\right] =$$

$$(s_{k_t}-t)\sum_{ih\in link(\gamma)}\left[\pi_j^0\left(vert(g_k),v,g_{k_t}\right)-\pi_j^0\left(vert(g_k),v,(g_{k_t})_{-ih}\right)\right] +$$

$$\sum_{k=k_t+1}^{m} s_k\sum_{ih\in link(g_k)}\left[\pi_j^0\left(vert(g_k),v,g_k\right)-\pi_j^0\left(vert(g_k),v,(g_k)_{-ih}\right)\right] =$$

$$(s_{k_t}-t)\sum_{jh\in link(\gamma)}\left[\pi_i^0\left(vert(g_k),v,g_{k_t}\right)-\pi_i^0\left(vert(g_k),v,(g_{k_t})_{-jh}\right)\right] +$$

$$\sum_{k=k_t+1}^{m} s_k\sum_{jh\in link(g_k)}\left[\pi_i^0\left(vert(g_k),v,g_k\right)-\pi_i^0\left(vert(g_k),v,(g_k)_{-jh}\right)\right] =$$

$$\sum_{jh\in link(\gamma)} P_i(N,v,\gamma) - P_i\left(N,v,\gamma_{-jh}^{\rho(jh)-t}\right).$$

□

Next theorem says that our fuzzy communication value is the only one satisfying these two axioms.

Theorem 4.8 *There is at most one fuzzy communication value satisfying cg-component efficiency and balanced total fuzzy threats.*

Proof. As our fuzzy communication value P satisfies both axioms (see Theorem 4.7) then it is only necessary to prove the uniqueness. Consider F another fuzzy communication value satisfying these axioms. The proof of the uniqueness is by recurrence in $K_\gamma = |link(\gamma)|$. If $K_\gamma = 0$ then all the players are isolated in γ and the cg-component efficiency implies $F_i(N,v,\gamma) = \tau(i)v(\{i\})$, for all $i \in N$. We suppose that $F = P$, for all $\gamma \in FCS^N$ with $K_\gamma < p$. Now let $\gamma \in FCS^N$ with $K_\gamma = p$. We will find a unique feasible payoff for the players in each connected component. If $S \in N/\gamma$ with $S = \{i\}$ then by cg-component efficiency we get $P_i(N, v, \gamma) = \tau(i)v(\{i\})$. Suppose $|S| > 1$. We take $i \in S$ and the other players in the component $S \backslash \{i\} = \{j_1, \ldots, j_q\}$. We look for $F_i(N, v, \gamma), F_{j_1}(N, v, \gamma), \ldots, F_{j_q}(N, v, \gamma)$. Applying the balanced total fuzzy threats axiom with $t = 0$ to every pair of players i, j_k with $k = 1, \ldots, q$ we obtain

$$
\begin{aligned}
\sum_{ih \in link(\gamma)} F_{j_1}(N, v, \gamma) - F_{j_1}\left(N, v, \gamma_{-ih}^{\rho(ih)}\right) &= \sum_{j_1 h \in link(\gamma)} F_i(N, v, \gamma) - F_i\left(N, v, \gamma_{-j_1 h}^{\rho(j_1 h)}\right) \\
&\vdots \\
&\vdots \\
\sum_{ih \in link(\gamma)} F_{j_q}(N, v, \gamma) - F_{j_q}\left(N, v, \gamma_{-ih}^{\rho(ih)}\right) &= \sum_{j_q h \in link(\gamma)} F_i(N, v, \gamma) - F_i\left(N, v, \gamma_{-j_q h}^{\rho(j_q h)}\right).
\end{aligned}
$$

We denote $Q_i = |\{ih \in link(\gamma)\}|$ and Q_{j_k} in the same way. These numbers $Q_i, Q_{j_k} \neq 0$ because these vertices are in the same connected component. Each link $j_k h$ (or ih) satisfies that $K_{\gamma_{-j_k h}^{\rho(j_k h)}} < p$, thus $F = P$ over them. Adding the equations for S by the cg-component efficiency we get the following linear system,

$$
\begin{aligned}
Q_i F_{j_1}(N, v, \gamma) - Q_{j_1} F_i(N, v, \gamma) &= \sum_{ih \in link(\gamma)} P_{j_1}\left(N, v, \gamma_{-ih}^{\rho(ih)}\right) - \sum_{j_1 h \in link(\gamma)} P_i\left(N, v, \gamma_{-j_1 h}^{\rho(j_1 h)}\right) \\
&\vdots \\
&\vdots \\
Q_i F_{j_q}(N, v, \gamma) - Q_{j_q} F_i(N, v, \gamma) &= \sum_{ih \in link(\gamma)} P_{j_q}\left(N, v, \gamma_{-ih}^{\rho(ih)}\right) - \sum_{j_q h \in link(\gamma)} P_i\left(N, v, \gamma_{-j_q h}^{\rho(j_q h)}\right) \\
F_{j_1}(N, v, \gamma) + \cdots + F_{j_q}(N, v, \gamma) &+ F_i(N, v, \gamma) = \epsilon^{cg}(\gamma).
\end{aligned}
$$

This is a set of $|S|$ linear equations with $|S|$ unknowns: $F_{j_1}(N, v, \gamma), \ldots, F_{j_q}(N, v, \gamma), F_i(N, v, \gamma)$, whose coefficient matrix is

$$\begin{bmatrix} Q_i & 0 & 0 & \cdots & 0 & -Q_{j_1} \\ 0 & Q_i & 0 & \cdots & 0 & -Q_{j_2} \\ 0 & 0 & Q_i & \cdots & 0 & -Q_{j_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

Looking at its first row it is easy to check that it is a nonsingular matrix and then $F = P$ is unique. □

Borm et al. [14] showed that the position value is proportional to the centrality measure of the players, the degree of the vertex in this case, if the link game only depends on the number of links. The cg-position value is too.

Theorem 4.9 *Let (N, v) be a game such that $v^{Lg}(A) = |link(g_A)|$ for all $g \in CS^N$. It holds that*

$$P_i(N, v, \gamma) = \delta_i(\gamma) + \tau(\{i\})v(\{i\}).$$

Proof. Suppose that (N, v) is a game satisfying the condition of the theorem. For each $g \in CS_0^N$ and (N, v), Borm et al. [14] proved that $\pi_i(vert(g), v, g) = d_i(g)$, for all $i \in vert(g)$. Thus for our game

$$\pi_i(vert(g), v, g) = v(\{i\}) + d_i(g), \forall i \in vert(g).$$

Given $\gamma \in FCS^N$, by Lemma 4.6 and (2.6)

$$\begin{aligned} P_i(N, v, \gamma) &= \sum_{k=1}^{m} s_k \pi_i^0(vert(g_k), v, g_k) = \sum_{k=1}^{m} s_k d_i(g_k) + \sum_{k=1}^{m} s_k v(\{i\}) \\ &= \sum_{ij \in link(\gamma)} \sum_{\{k \in \{1,\dots,m\} : ij \in link(g_k)\}} s_k + \sum_{k=1}^{m} s_k v(\{i\}) = \delta_i(\gamma) + \tau(\{i\})v(\{i\}). \end{aligned}$$

□

The cg-position value can be written in Choquet integral form like the cg-Myerson and cg-Banzhaf values

$$P_i(N, v, \gamma) = \int \gamma \, d\pi_i(N, v), \ \forall i \in N,$$

where $\pi_i(N, v)(g) = \pi_i^0(vert(g), v, g)$.

This value has similar properties to the cg-Myerson value.

Proposition 4.10 *The cg-position value P satisfies the following properties*

1. *P is a linear function with respect to v.*

2. *P is continuous with respect to γ.*

3. *P is comonotonous with respect to γ, i.e., if γ and γ' are comonotonous fuzzy communication structures and $\alpha \in [0, 1]$ then*

$$P(N, v, \alpha\gamma + (1 - \alpha)\gamma') = \alpha P(N, v, \gamma) + (1 - \alpha)P(N, v, \gamma')$$

Proof. Analogous to that of Proposition 3.16. □

Proposition 4.11 *Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$ and $i \in N$. Then if $\gamma = (\tau, \rho)$ is complete by links then*

$$P_i(N, v, \gamma) = \vee\gamma \, v(\{i\}) + \frac{1}{2} \sum_{ij \in link(g_k)} \phi_{ij}^{ch}\left(\rho, v^{Lg^\gamma}\right).$$

Proof. Let $\gamma = (\tau, \rho) \in FCS^N$ be complete by links (Definition 2.18). Since it is complete by links then the images of ρ are the same than the ones of τ, perhaps except for the last one, that besides must correspond only to a vertex. If $cg(\gamma) = (s_k, g_k)_{k=1}^m$ then we know that all g_k are complete (see the proof of Proposition 3.14), so only the last graph might not have any links (it would be formed by only one vertex). In the same proof we saw that the s_k coincide with the differences of levels in τ. For these reasons we conclude that $im(\rho) = \{h_1 < \cdots < h_{m-1}\}$ (even it could reach to h_m) with $s_k = h_k = h_{k-1}, \ \forall k = 1, \ldots, m - 1$. If we denote

$$A_k = \{ij \in L(N) : \rho(ij) \geq h_k\}, \ \forall k \in \{1, \ldots, m - 1\},$$

it holds following Lemma 4.6

$$P_i(N,v,\gamma) = \sum_{k=1}^{m} s_k\, v(\{i\}) + \sum_{k=1}^{m} s_k\, \frac{1}{2} \sum_{ij\in link(g_k)} \phi_{ij}\left(link(g_k), v^{Lg_k}\right)$$

$$= \vee\gamma\, v(\{i\}) + \sum_{k=1}^{m-1} (h_k - h_{k-1})\, \frac{1}{2} \sum_{ij\in link(g_k)} \phi_{ij}\left(A_k, v_{A_k}^{Lg^\gamma}\right)$$

$$= \vee\gamma\, v(\{i\}) + \frac{1}{2} \sum_{ij\in link(g_k)} \phi_{ij}^{ch}(\rho, v^{Lg^\gamma}).$$

We used that $\vee\gamma = \sum_{k=1}^{m} s_k$, g_m does not exist or does not have any link and $\forall A \subseteq L(N)$,

$$v^{Lg_k}(A) = v^{Lg^\gamma}\left(link(g_k)\cap A\right) = v^{Lg^\gamma}\left(A_k \cap A\right) = v_{A_k}^{Lg^\gamma}(A).$$

□

4.3 Stability and the average tree value

In a game with communication structure the players determine their payoffs depending on the chosen communication graph. They often try to connect using the minimum number of links, i.e, using a tree. Hence it is interesting to study this particular subfamily of games with communication structure. A game with communication structure (N,v,g) is named *game with forest communication structure* if g is a forest. The family of games with forest communication structure will be denoted by $\mathcal{G}_{comf}$. In this family the convexity property of the original game is transmitted to the vertex and link games, as it is proven in van den Nouweland and Borm [55].

Proposition 4.12 *If $(N,v,g) \in \mathcal{G}_{comf}$ with $g = (N,L)$ and (N,v) convex, then (N,v^g) and $\left(L, v^{Lg}\right)$ are convex games.*

One of the main properties that is transmitted due to the above proposition is stability. As we said in Chapter 1, the Shapley value is stable when the game is convex. The definition of stability for games with communication structure (written in terms of the measure of Myerson) is the following.

Communication stability. A communication value f satisfies communication stability if for every $(N, v, g) \in \mathcal{G}_{com}$ and for all $S \subseteq N$ it holds

$$\sum_{i \in S} f_i(N, v, g) \geq r\left(g_S\right).$$

Theorem 4.13 (Van den Nouweland and Borm [55]) *The Myerson value and the position value satisfy communication stability in the family of convex games with forest communication structure.*

That is to say, when the communication structure is a forest, the same condition of stability of the Shapley value holds.

Herings et al. [40] introduced the average tree value as a communication value only on forest communication structures over N. Suppose first that $(N, v, g) \in \mathcal{G}_{com}$ with $g = (N, L)$ a tree. For each player $i \in N$, we focus on the directed tree rooted at $i \in N$. We denote as $C_i^g(j)$ the set of *successors* of j in the directed tree rooted at i, namely those players $h \in N$ such that the only directed path in g from h to i contains j and the set $N^g(j) = \{i \in N : ij \in L\}$ is the family of *neighbors* of j. Notice that $C_i^g(i) = N$. So we consider the following payoff vector associated to player i,

$$t_j^i(N, v, g) = v\left(C_i^g(j)\right) - \sum_{h \in C_i^g(j) \cap N^g(j)} v\left(C_i^g(h)\right), \quad \forall j \in N.$$

Definition 4.14 *The average tree value is defined for each $(N, v, g) \in \mathcal{G}_{com}$ with g a tree by*

$$\alpha(N, v, g) = \frac{1}{|N|} \sum_{i \in N} t^i(N, v, g).$$

If $(N, v, g) \in \mathcal{G}_{comf}$ then we repeat the process inside each connected component.

There exist several axiomatizations of the average tree value for forests. Herings et al. [40] introduced the following axiom. If g is a tree and $ij \in link(g)$, then N_{ij}^i, N_{ij}^j denote the connected components that contain i, j respectively, after eliminating the link ij.

Component fairness. Let $(N,v,g) \in \mathcal{G}_{comf}$ and $ij \in link(g)$. Then

$$\frac{1}{|N_{ij}^i|}\sum_{h\in N_{ij}^i} f_h(N,v,g) - f_h\left(N,v,g_{-ij}\right) = \frac{1}{|N_{ij}^j|}\sum_{h\in N_{ij}^j} f_h(N,v,g) - f_h\left(N,v,g_{-ij}\right).$$

Theorem 4.15 (Herings et al. [40]) *The average tree value is the only communication value over $\mathcal{G}_{comf}$ that satisfies component efficiency and component fairness.*

Herings et al. [39] also proved the following result.

Proposition 4.16 *If (N,v) is superadditive then the average tree value over tree communication structures satisfies communication stability.*

Another concept of stability was given by Myerson in [51]. This stability means that the communication between two players is beneficial.

Graph stability. A value for games with communication structure f satisfies graph stability if for every communication structure g and $ij \in link(g)$

$$f_i(N,v,g) \geq f_i\left(N,v,g_{-ij}\right) \text{ and } f_j(N,v,g) \geq f_j\left(N,v,g_{-ij}\right).$$

If the game is superadditive, the Myerson value satisfies graph stability even when the graph is not a tree, as we can see in the only theorem in Myerson [51]. The same proof is valid for the graph Banzhaf value, substituting the Shapley value for the Banzhaf value. It is easy to see that v superadditive implies v^{Lg} superadditive, then again by definition of the position value and the same proof, π also satisfies graph stability. We prove now that the average tree value is graph stable, but in this case the proof is different from Myerson's.

Proposition 4.17 *Let $(N,v,g) \in \mathcal{G}_{comf}$ with (N,v) a superadditive game. Then the average tree value is graph stable.*

Proof. Suppose without loss of generality a tree rooted at i and $pq \in link(g)$ where p is the father of q. Take vertex p. It holds $\{h \in C_i^g(p) \cap N^g(p)\} = \left\{h \in C_i^{g-pq}(p) \cap N^g(p)\right\} \cup \{q\}$ and $C_i^{g-pq}(p) = C_i^g(p) \setminus C_i^g(q)$. Then

$$t_p^i(N, v, g) = v\left(C_i^g(p)\right) - \sum_{h \in C_i^{g-pq}(p) \cap N^g(p) \cup \{q\}} v\left(C_i^g(h)\right)$$

and

$$t_p^i\left(N, v, g_{-pq}\right) = v\left(C_i^g(p) \setminus C_i^g(q)\right) - \sum_{h \in C_i^{g-pq}(p) \cap N^g(p)} v\left(C_i^g(h)\right).$$

Then if we impose $t_p^i(N, v, g) \geq t_p^i\left(N, v, g_{-pq}\right)$ the inequality that results is

$$v\left(C_i^g(p)\right) \geq v\left(C_i^g(p) \setminus C_i^g(q)\right) + v\left(C_i^g(q)\right)$$

and it is true if v is superadditive.
Now we are going to take vertex q. It holds $\{h \in C_i^g(q) \cap N^g(q)\} = \left\{h \in C_i^{g-pq}(q) \cap N^g(q)\right\}$ and $C_i^{g-pq}(q) = C_i^g(q)$. Then

$$t_q^i(N, v, g) = v\left(C_i^g(q)\right) - \sum_{h \in C_i^g(q) \cap N^g(q)} v\left(C_i^g(h)\right)$$

and

$$t_q^i\left(N, v, g_{-pq}\right) = v\left(C_i^g(q)\right) - \sum_{h \in C_i^g(q) \cap N^g(q)} v\left(C_i^g(h)\right).$$

Therefore in this case $t_q^i(N, v, g) = t_q^i\left(N, v, g_{-pq}\right)$. □

4.4 Fuzzy stability and the cg-average tree value

Several concepts of tree in fuzzy graphs have been defined, for example in Rosenfeld [61] and in Delgado et al. [23]. In this section we will focus on one of them introduced by Rosenfeld.

Definition 4.18 *A fuzzy graph γ is called a full tree if its crisp version g^γ is a tree. It is named full forest if g^γ is a forest. The family of games with full forest communication structure will be denoted* *by $\mathcal{G}_{fcomf}$.*

We present now the concepts of fuzzy stability that we are going to study.

Fuzzy communication stability. A value F over games with fuzzy communication structure is fuzzy communication stable if $\forall S \subseteq N$,

$$\sum_{i\in S} F_i(N, v, \gamma) \geq \epsilon^{cg}(\gamma_S).$$

Fuzzy graph stability. A value F over games with fuzzy communication structure is fuzzy graph stable if for every $ij \in link(\gamma)$ and $t \in [0, \rho(ij)]$,

$$F_i(N, v, \gamma) \geq F_i\left(N, v, \gamma^t_{-ij}\right) \text{ and } F_j(N, v, \gamma) \geq F_j\left(N, v, \gamma^t_{-ij}\right).$$

Theorem 4.19 *Let $(N, v, \gamma) \in \mathcal{G}_{fcomf}$.*

1. *If (N, v) is convex then the cg-Myerson and cg-position values are fuzzy communication stable.*

2. *If (N, v) is superadditive then the cg-Myerson and cg-position values are fuzzy graph stable.*

Proof. Since γ is a full forest all the g_k in its cg-partition are forests.
1) Let $S \subseteq N$, then

$$\begin{aligned}\sum_{i\in S} M_i(N, v, \gamma) &= \sum_{i\in S}\sum_{k=1}^{m} s_k \mu_i^0(vert(g_k), v, g_k) \\ &= \sum_{k=1}^{m} s_k \sum_{i\in vert(g_k)\cap S} \mu_i(vert(g_k), v, g_k) \\ &\geq \sum_{k=1}^{m} s_k v^{g_k}(S \cap vert(g_k)) = v^{\gamma}(S),\end{aligned}$$

where the first equality comes from Theorem 3.10, the inequality is due to Theorem 4.13 and the last equality is obtained by Lemma 3.7. The proof of the fuzzy communication stability for the cg-position value is analogous.
2) Consider $t' \in [0, \rho(ij)]$. For the cg-Myerson value, the result follows using again Theorem 3.10,

the partition (4.1) with $t = \rho(ij) - t'$ and the fact that the Myerson value is graph stable. Like in the proof for the crisp graph, this is valid for any γ. The proof of the fuzzy graph stability for the cg-position value is analogous. □

We define now a fuzzy version of the average tree value using the Choquet by graphs model and following the process of Herings et al. [40]. A hierarchical fuzzy outcome associated to each player is defined on the class of fuzzy communication games such that the fuzzy communication structure is a full tree. If $\gamma = (\tau, \rho)$ is a full tree then for each player $i \in N$ we use $C_i^{\gamma}(j) = C_i^{g^{\gamma}}(j)$ for all $j \in N$ and $N^{\gamma}(i) = N^{g^{\gamma}}(i)$. Notice that $C_i^{\gamma}(i)= supp(\tau)$.

Definition 4.20 *Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$ with γ a full tree. For each player $i \in supp(\tau)$ the i-hierarchical outcome is*

$$t_j^i(N, v, \gamma) = v^{\gamma}\left(C_i^{\gamma}(j)\right) - \sum_{h \in C_i^{\gamma}(j) \cap N^{\gamma}(j)} v^{\gamma}\left(C_i^{\gamma}(h)\right),$$

for all $j \in supp(\tau)$ and $t_j^i(N, v, \gamma) = 0$ otherwise.

Definition 4.21 *The cg-average tree value is the fuzzy communication value defined for each $(N, v, \gamma) \in \mathcal{G}_{fcom}$ with $\gamma = (\tau, \rho)$ a full tree by*

$$A(N, v, \gamma) = \frac{1}{|supp(\tau)|} \sum_{i \in supp(\tau)} t^i(N, v, \gamma).$$

If $(N, v, \gamma) \in \mathcal{G}_{fcomf}$ then we repeat the process for each component $K \in N/\gamma$.

Example 4.22 We calculate the cg-average tree value for the full tree of Figure 4.1 and the game $v(S) = |S|^2$. First we construct the cg-game. Figure 4.2 shows the cg-partition of γ.

Next table determines the restricted game v^{γ} for our game v.

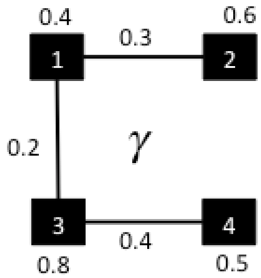


Figure 4.1: Full tree

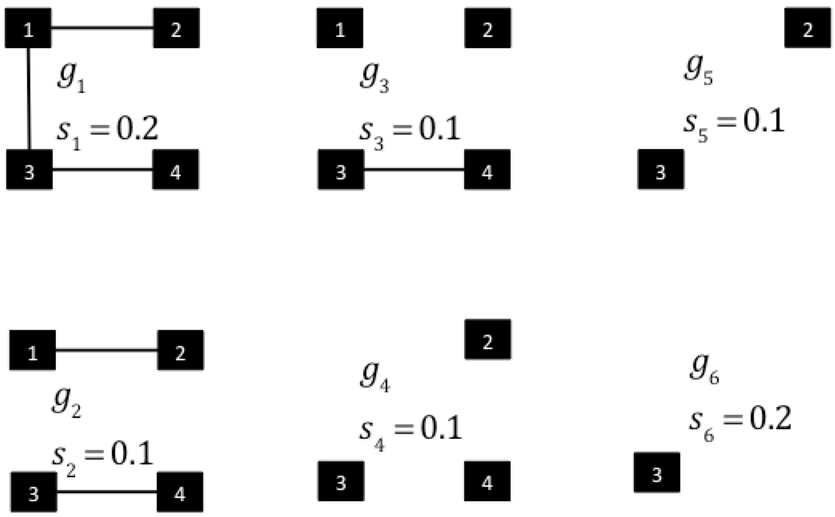


Figure 4.2: cg-partitition of γ

S	$v^\gamma(S)$	S	$v^\gamma(S)$	S	$v^\gamma(S)$	S	$v^\gamma(S)$
$\emptyset$	0	$\{1\}$	0.4	$\{2\}$	0.6	$\{3\}$	0.8
$\{4\}$	0.5	$\{1,2\}$	1.6	$\{1,3\}$	1.6	$\{1,4\}$	0.9
$\{2,3\}$	1.4	$\{2,4\}$	1.1	$\{3,4\}$	2.1	$\{1,2,3\}$	3.2
$\{1,2,4\}$	2.1	$\{1,3,4\}$	3.3	$\{2,3,4\}$	2.7	N	5.3

Now we describe the construction of $t^1(N, v, \gamma)$. The tree rooted at 1 and the sets of successors $C_1^\gamma(j)$ with $j \in supp(\gamma)$ are showed in Figure 4.3,

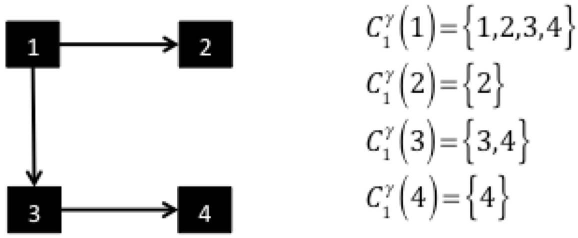


Figure 4.3: Sets of successors in the 1-rooted tree of g^γ

The 1-hierarchical payoff for each player is calculated as

$$\begin{aligned}
t_1^1(N, v, \gamma) &= v^\gamma(N) - v^\gamma(\{2\}) - v^\gamma(\{3, 4\}) = 2.6, \\
t_2^1(N, v, \gamma) &= v^\gamma(\{2\}) = 0.6, \\
t_3^1(N, v, \gamma) &= v^\gamma(\{3, 4\}) - v^\gamma(\{4\}) = 1.6, \\
t_4^1(N, v, \gamma) &= v^\gamma(\{4\}) = 0.5.
\end{aligned}$$

We get the 1-hierarchical outcome, $t^1(N, v, \gamma) = (2.6, 0.6, 1.6, 0.5)$. Notice that this vector is efficient because the sum of its coordinates is $v^\gamma(N)$ (in this graph we have only one component). The other vectors can be computed in the same way changing the hierarchy.

We compute now these quantities

$$\begin{aligned} t_1^1(N,v,g_1) &= v(N) - v(\{2\}) - v(\{3,4\}) = 11, \\ t_1^1(N,v,g_2) &= v(\{1,2\}) - v(\{1\}) = 3, \\ t_1^1(N,v,g_3) &= v(\{1\}) = 1, \\ t_1^1(N,v,g_k) &= 0, \ \forall k \geq 4. \end{aligned}$$

We observe that, in this case

$$2.6 = t_1^1(N,v,\gamma) = \sum_{k=1}^{6} s_k \, t_1^1(N,v,g_k) = 0.2 \cdot 11 + 0.1 \cdot 3 + 0.1 \cdot 1$$

Now we look for an axiomatization of the cg-average tree value following Herings et al. [40]. Consider the following axioms for a given fuzzy communication value F.

For next axiom we use the modified fuzzy graph of Definition 2.23. We denote by K_i^{ij} the component of $\gamma_{-ij}^{\rho(ij)-t}$ that contains player i (analogously K_j^{ij}).

Fuzzy components fairness. A fuzzy communication value F satisfies fuzzy components fairness if for any link $ij \in \gamma$ with $\rho(ij) > 0$ and $t \in [0, \rho(ij)]$ it holds that

$$\frac{1}{|K_i^{ij}|} \sum_{h \in K_i^{ij}} F_h(N,v,\gamma) - F_h\left(N,v,\gamma_{-ij}^{\rho(ij)-t}\right) = \frac{1}{|K_j^{ij}|} \sum_{r \in K_j^{ij}} F_r(N,v,\gamma) - F_r\left(N,v,\gamma_{-ij}^{\rho(ij)-t}\right).$$

This axiom means that when we reduce to t the level of a link $ij \in \gamma$, the resulting average change in payoff of the players in K_i^{ij} is equal to the average change in payoff of the players in K_j^{ij}. Notice that when we reduce to $0 < t < \rho(ij)$ the axiom is satisfied trivially because the fuzzy communication structure is not disconnected, and then $|K_i^{ij}| = |K_j^{ij}| = N$.

We are going to prove that the cg-average tree value satisfies these axioms but previously we need this result.

Lemma 4.23 *Let (N,v) be a game. We take γ a full tree with cg-partition $cg(\gamma) = (g_k, s_k)_{k=1}^m$,* *then*

$$\sum_{h \in C_i^\gamma(j)} t_h^i(N, v, \gamma) = v^\gamma\left(C_i^\gamma(j)\right).$$

Proof. The payoff of player j in the full tree rooted at i is equal to the contribution of player j when he joins his subordinates in the hierarchy. Clearly, the set $C_i^\gamma(j)$ itself is connected, so when γ joins his subordinates, player j connects all the subsets of subordinates of his successors into one connected set and receives his marginal contribution to them. Observe that a player $j \in N$ receives his own worth $v(j)$ when he has no subordinates in γ. More generally, the total payoff of a player j and all his subordinates in γ is equal to the worth of the coalition $C_i^\gamma(j)$, i.e.

$$\sum_{h \in C_i(j)} t_h^i\left(N, v, \gamma\right) = v^\gamma\left(C_i^\gamma(j)\right).$$

□

Theorem 4.24 *The cg-average tree value satisfies cg-component efficiency and fuzzy components fairness on $\mathcal{G}_{fcomf}$.*

Proof. Let $(N, v, \gamma) \in \mathcal{G}_{fcom}$. We take γ a full tree with cg-partition $cg(\gamma) = (g_k, s_k)_{k=1}^m$. If γ is a full forest the proof is repeated in each component.
Then

$$\sum_{h \in N} A_h(N, v, \gamma) = \frac{1}{|N|}\sum_{h \in N}\sum_{i \in N} t_h^i(N, v, \gamma) = \frac{1}{|N|}\sum_{i \in N}\sum_{h \in N} t_h^i(N, v, \gamma).$$

cg-Component efficiency. Since $C_i^\gamma(i) = N$ and by the above Lemma, it holds

$$\frac{1}{|N|}\sum_{i \in N}\sum_{h \in N} t_h^i(N, v, \gamma) = \frac{1}{|N|}\sum_{i \in N} v^\gamma\left(C_i^\gamma(i)\right) = v^\gamma(N) = \epsilon^{cg}\left(\gamma_N\right).$$

Fuzzy components fairness. Now we check fuzzy components fairness for $t = 0$. Let $h \in K_i^{ij}$. By definition of $t^h\left(N, v, \gamma\right)$ it holds

$$\sum_{h' \in K_j^{ij}} t_{h'}^h\left(N, v, \gamma\right) = v^\gamma\left(K_j^{ij}\right).$$

Notice that we have $\left|K_i^{ij}\right|$ equations of this type. On the other hand, applying cg-component efficiency and the above equality, for $h \in K_j^{ij}$

$$\sum_{h' \in K_j^{ij}} t_{h'}^{h}(N, v, \gamma) = v^{\gamma}(N) - v^{\gamma}\left(K_i^{ij}\right).$$

Notice also that we have $\left|K_j^{ij}\right|$ equations of this type. Therefore, we have

$$\sum_{h \in K_j^{ij}} A_h(N, v, \gamma) = \frac{\left|K_i^{ij}\right| v^{\gamma}\left(K_j^{ij}\right) + \left|K_j^{ij}\right| \left[v^{\gamma}(N) - v^{\gamma}\left(K_i^{ij}\right)\right]}{|N|}.$$

Then, as $\left|K_i^{ij}\right| + \left|K_j^{ij}\right| = |N|$ and $A_h\left(N, v, \gamma_{-ij}^{\rho(ij)}\right) = v^{\gamma}\left(K_j^{ij}\right)$ we obtain

$$\sum_{h \in K_i^{ij}} A_h(N, v, \gamma) - A_h\left(N, v, \gamma_{-ij}^{\rho(ij)}\right) = \frac{\left|K_i^{ij}\right| \left[v^{\gamma}(N) - v^{\gamma}\left(K_i^{ij}\right) - v^{\gamma}\left(K_j^{ij}\right)\right]}{|N|}$$

Analogously, for $h \in K_j^{ij}$, the axiom is satisfied.□

Next theorem says that our fuzzy communication value is the only one satisfying these two axioms.

Theorem 4.25 *There is only one value over $\mathcal{G}_{fcomf}$ satisfying cg-component efficiency and fuzzy components fairness*

Proof. As our fuzzy communication value A satisfies both axioms then it is only necessary to prove the uniqueness. Consider F another fuzzy communication value satisfying these axioms. If we prove uniqueness for full trees then we obtain it also for full forests, then suppose γ a full tree. Since γ is connected, by the fuzzy components fairness axiom with $t = 0$, we have $|N| - 1$ equations as follows

$$\frac{1}{\left|K_i^{ij}\right|}\left(\sum_{h \in K_i^{ij}} F_h(N, v, \gamma) - v^{\gamma}\left(K_i^{ij}\right)\right) = \frac{1}{\left|K_j^{ij}\right|}\left(\sum_{h \in K_j^{ij}} F_h(N, v, \gamma) - v^{\gamma}\left(K_j^{ij}\right)\right)$$

what implies

$$\frac{1}{\left|K_i^{ij}\right|}\sum_{h\in K_i^{ij}} F_h\left(N,v,\gamma\right)-\frac{1}{\left|K_j^{ij}\right|}\sum_{h\in K_j^{ij}} F_h\left(N,v,\gamma\right)=\frac{v^{\gamma}\left(K_i^{ij}\right)}{\left|K_i^{ij}\right|}-\frac{v^{\gamma}\left(K_j^{ij}\right)}{\left|K_j^{ij}\right|}.$$

Together with the cg-component efficiency axiom we have $|N|$ linearly independent equations (because the coefficient matrix is the same as the one that appears in Herings et al. [39] for the crisp case), yielding a unique solution. □

We see now some more properties of the cg-average tree value.

Theorem 4.26 *If (N,v) is superadditive then the cg-average tree value is fuzzy communication stable.*

Proof. We take $p\in N$ and an order π on N compatible with g^{γ} rooted at p. Take a coalition $S\subseteq N$ connected with respect to π, i.e., for each $i,j\in N$ with $i<j$, and $\pi(i),\pi(j)\in S$, it holds that $\pi(k)\in S$ for all $k\in\{i,\ldots,j\}$, where the notation $i<j$ means that there exists a path in the directed graph g^{γ} from j to i. We also have that the game v^{γ} is component additive by condition 3) in Definition 2.25. Then it is easy to see that $t^p(N,v,\gamma)=m^{\pi}(N,v^{\gamma})$, where $m^{\pi}_{\pi(i)}(N,v^{\gamma})=v^{\gamma}(\{\pi(1),\ldots,\pi(i)\})-v^{\gamma}(\{\pi(1),\ldots,\pi(i-1)\})$. Now applying Theorem 3.1 in van Velzen et al. [70], we conclude that $t^p(N,v,\gamma)$ is fuzzy communication stable. □

Theorem 4.27 *If (N,v) is superadditive then the cg-average tree value is fuzzy graph stable.*

Proof. Let $t\in[0,\rho(ij)]$ and $\gamma'=\gamma^t_{-ij}$. If $t=0$ it is straightforward. If $0<t<\rho(ij)$, suppose without loss of generality that both γ,γ' are rooted at p and that i is the father of j. It holds $\{h\in C^{\gamma}_p(i)\cap N^{\gamma}(i)\}=\left\{h\in C^{\gamma'}_p(i)\cap N^{\gamma'}(i)\right\}$ and $C^{\gamma'}_p(i)=C^{\gamma}_p(i)$. Then

$$t^p_i(N,v,\gamma)=v^{\gamma}\left(C^{\gamma}_p(i)\right)-\sum_{h\in C^{\gamma}_p(i)\cap N^{\gamma}(i)} v^{\gamma}\left(C^{\gamma}_p(h)\right)$$

and

$$t^p_i\left(N,v,\gamma'\right)=v^{\gamma'}\left(C^{\gamma}_p(i)\right)-\sum_{h\in C^{\gamma}_p(i)\cap N^{\gamma}(i)} v^{\gamma'}\left(C^{\gamma}_p(h)\right).$$

Besides, if $i, j \notin S, v^\gamma(S) = v^{\gamma'}(S)$. Thus if we impose $t_i^p(N, v, \gamma) \geq t_i^p(N, v, \gamma')$ the inequality that results is

$$v^\gamma \left(C_p^\gamma(i)\right) - v^\gamma \left(C_p^\gamma(j)\right) \geq v^{\gamma'} \left(C_p^\gamma(i)\right) - v^{\gamma'} \left(C_p^\gamma(j)\right).$$

Since $C_p^\gamma(j) \subseteq C_p^\gamma(i)$, if the game $w = v^\gamma - v^{\gamma'}$ is monotonous, we have the desired inequality. But by condition 2) in Definition 2.25 and using that v is superadditive, it follows that w is monotonous. Now we take vertex j. It also holds $C_p^{\gamma'}(j) = C_p^\gamma(j)$ and

$$\left\{h \in C_p^\gamma(j) \cap N^\gamma(j)\right\} = \left\{h \in C_p^{\gamma'}(j) \cap N^{\gamma'}(j)\right\}.$$

Then again

$$t_j^p(N, v, \gamma) = v^\gamma \left(C_p^\gamma(j)\right) - \sum_{h \in C_p^\gamma(j) \cap N^\gamma(j)} v^\gamma \left(C_p^\gamma(h)\right)$$

and

$$t_j^p\left(N, v, \gamma'\right) = v^{\gamma'} \left(C_p^\gamma(j)\right) - \sum_{h \in C_p^\gamma(j) \cap N^\gamma(j)} v^{\gamma'} \left(C_p^\gamma(h)\right).$$

Thus if we impose $t_j^p(N, v, \gamma) \geq t_j^p(N, v, \gamma')$ the inequality that results is

$$v^\gamma \left(C_p^\gamma(j)\right) \geq v^{\gamma'} \left(C_p^\gamma(j)\right),$$

taking into account that $i, j \notin \left(C_p^\gamma(h)\right)$, for any $h \in C_p^\gamma(j) \cap N^\gamma(j)$. But this is true by condition 2) in Definition 2.25.

If $t = \rho(ij)$ the proof is analogous to the one of Proposition 4.17 using the cg-games and condition 2) of Proposition 2.34. □

If we look at the previous values, an analogous formula for the cg-average tree value would be a Choquet expression, i.e.,

$$F(N, v, \gamma) = \sum_{k=1}^{m} s_k \alpha^0 \left(vert(g_k), v, g_k\right),$$

for all $(N, v, \gamma) \in \mathcal{G}_{fcomf}$ and $cg(\gamma) = (s_k, g_k)_{k=1}^m$. The average tree value α is evaluated in the game restricted to the vertices of each g_k and then it is extended by zeros. But next example shows that this option is not consistent with Definition 4.21.

Example 4.28 The cg-partition of the previous example is in Figure 4.2. We calculate the average tree value for each communication game of the partition.

g_k	$\alpha(N,v,g_k)$	g_k	$\alpha(N,v,g_k)$	g_k	$\alpha(N,v,g_k)$
g_1	$(5.5,2.5,5.5,2.5)$	g_2	$(2,2,2,2)$	g_3	$(1,1,2,2)$
g_4	$(0,1,1,1)$	g_5	$(0,1,1,0)$	g_6	$(0,0,1,0)$

Thus,

$$F(N,v,\gamma)=(1.4,1,1.9,1)\neq(1.45,0.95,2,0.9)=A(N,v,\gamma).$$

Remark 4.29 *The value $A'(N,v,\gamma) = \sum_{k=1}^{m} s_k\alpha^0\left(vert(g_k),v,g_k\right)$ satisfies cg-component efficiency, fuzzy communication stability and fuzzy graph stability (if v is superadditive). The cg-component efficiency follows from the component efficiency of α. Fuzzy communication stability follows from Proposition 4.16 and fuzzy graph stability in the same way as in Theorem 4.19 using that the average tree value is graph stable (see Proposition 4.17).*
The problem is that it does not satisfy fuzzy components fairness, since the number of elements in the components changes in each level. We cannot give an analogous axiom that fits all levels at once.

Remark 4.30 *Delgado et al. [23] introduced another definition of fuzzy tree that is called fuzzy tree by levels. They say that γ is a* fuzzy tree by levels *if $\exists t\in(0,1]$ such that $g_t=(supp(\tau),[\rho]_t)$ is a tree. We denote $t_0=\wedge\{t : g_t$ is a tree $\}$, take the fuzzy graph γ^{t_0} from Definition 3.21, and construct the value $G(N,v,\gamma)=A\left(N,v,\gamma^{t_0}\right)$. This value satisfies the previous axioms for A but it does not satisfy fuzzy communication stability. Instead of that, it satisfies this condition that we call fuzzy stability*

$$\exists t\in[0,1] \text{ s.t. } \forall S\subseteq N,\ \sum_{i\in S}F_i\left(N,v,\gamma^t\right)\geq\epsilon^{cg}\left(\gamma_S^t\right)$$

with $t=t_0$. We can give a definition of fuzzy forest by levels by changing the word tree for forest.

Part II

Proximity Relations

Chapter 5

Games with a proximity relation among the players

Aumann and Drèze [6] introduced coalition structures. A coalition structure is a partition of the set of players representing the different coalitions obtained at the end of game. Hence there should be non-side payments between these coalitions. This was improved by Myerson [51] considering communication structures. In this case the final coalition structure is the set of connected components in the graph but we can also use the information given by the graph about the formation of these coalitions. Owen [58] proposed a different model from that of Aumann and Drèze based on another interpretation of the coalition structures.

5.1 The Owen model

The Owen's approach supposes that the players are organized in a priori unions that have common interests in the game. But these unions are not considered as a final structure but as a starting point for further negotiations. So each union negotiates as a whole with the other unions to achieve a fair payoff. Nevertheless, as in the original model of cooperative games, the grand coalition is the final structure.
We focus now on the Owen variation. We next introduce some definitions to explain the model, following Owen [58].

Definition 5.1 *A game with a priori unions is a triple* $(N, v, \mathcal{P})$ *where* (N, v) *is a game and* $\mathcal{P} = \{N_1, \ldots, N_m\}$ *is a partition of* N*. We will denote the set of games with a priori unions by* $\mathcal{G}_{un}$*.*

Players in N_k for each k have similar interests in the game and they bargain as a whole in order to get an acceptable payoff.

Definition 5.2 *A value for games with a priori unions is a mapping f that assigns a payoff vector $f(N,v,\mathcal{P}) \in \mathbb{R}^N$ to each $(N,v,\mathcal{P}) \in \mathcal{G}_{un}$.*

Owen [58] proposed a method to obtain values for games with a priori unions, which is defined in two steps. It is supposed that players are interested in the grand coalition N but considering the a priori unions as bargaining elements.

Definition 5.3 *Let $(N,v,\mathcal{P}) \in \mathcal{G}_{un}$ with $\mathcal{P} = \{N_1,\ldots,N_m\}$. The quotient game is a game $\left(M, v^{\mathcal{P}}\right)$ with set of players $M = \{1,\ldots,m\}$ defined by*

$$v^{\mathcal{P}}(Q) = v\left(\bigcup_{q\in Q} N_q\right), \forall Q \subseteq M.$$

Let $(N,v,\mathcal{P}) \in \mathcal{G}_{un,}$ $\mathcal{P} = \{N_1,\ldots,N_m\}$ and $k \in M$. For each $S \subset N_k$ the partition $\mathcal{P}_S$ of $N \setminus (N_k \setminus S)$ consists of replacing N_k with S, i.e.,

$$\mathcal{P}_S = \left\{N_1,\ldots,\overset{k)}{S},\ldots,N_m\right\}.$$

Let f^1 be a classic value for cooperative games. The first step consists of a negotiation among unions that is focused on S. The result of the quotient game generates a new game in N_k. We define the game (N_k, v_k) as

$$v_k(S) = f_k^1\left(M, v^{\mathcal{P}_S}\right), \forall S \subseteq N_k. \tag{5.1}$$

In the second step we solve the game in every group using another classic value f^2. So, for each player $i \in N$, if $k(i)$ is such that $i \in N_{k(i)}$ then the new value f is defined by

$$f_i\left(N,v,\mathcal{P}\right) = f_i^2\left(N_{k(i)}, v_{k(i)}\right). \tag{5.2}$$

There are other values in addition to the Owen value based on the Owen model, like the Banzhaf-Owen value defined by Owen in [59] (that applies the Banzhaf value in both steps) and the symmetric coalitional Banzhaf value (that applies the Shapley value among the unions and the Banzhaf value inside each union). The first axiomatic characterization for the Banzhaf-Owen value was given by Albizuri [1], but only on the class of simple games. Amer et al. [5] were the first that provided a characterization on the class of all cooperative games. The symmetric coalitional Banzhaf value was introduced by Alonso-Meijide and Fiestras-Janeiro [2]. In this article several characterizations were provided and two political examples illustrating the differences with respect to the Owen value and the Banzhaf-Owen value were given.

In the Owen model players are organized in a priori unions but there is no information about the internal structure of these unions. Later Casajus [17] proposed a modification of the Owen model in the Myerson sense. We call this model games with cooperation structure. A cooperation structure is a graph where the connected components represent the a priori unions, but the edges give us additional information about how they are formed.

Definition 5.4 *A game with cooperation structure is a triple* (N, v, L) *with* $(N, v) \in \mathcal{G}$ *and* $L \subseteq L(N)$. *The family of games with cooperation structure is denoted by* $\mathcal{G}_{coop}$.

By definition $\mathcal{G}_{com} = \mathcal{G}_{coop}$; nevertheless the interpretation is completely different. Moreover we have $\mathcal{G}_{un} \subsetneq \mathcal{G}_{coop}$, because an a priori union structure can be identified with a cooperation structure with complete components.

Definition 5.5 *A value for games with cooperation structure is a mapping* f *that assigns a payoff vector* $f(N, v, L) \in \mathbb{R}^N$ *to each* $(N, v, L) \in \mathcal{G}_{coop}$.

Casajus [17] proposed to follow the model of Owen to get an allocation rule for games with cooperation structure. Given $(N, v, L) \in \mathcal{G}_{coop}$, we consider the partition of N by its connected components N/L ($N/L = N/g$ with $g = (N, L)$ by abuse of notation). Therefore N/L is a set of a priori unions for the players in N but the links in L tell us how these unions are formed. We use again the quotient game (Definition 5.3) with the partition $N/L = \{N_1, \ldots, N_m\}$. Given a value

f^1, now for all $k \in M$ with $M = \{1, \ldots, m\}$,

$$v_k(S) = f_k^1\left(M, v^{(N/L)_S}\right), \forall S \subseteq N_k. \tag{5.3}$$

In the second step we consider a communication value (Definition 2.3) f^2 to allocate the profit inside each component.
For each $i \in N$ the new value f is defined by

$$f_i^2\left(N_{k(i)}, v_{k(i)}, L_{N_{k(i)}}\right),$$

where $k(i)$ is such that $i \in N_{k(i)}$.

5.2 Proximity relations

Owen [58] considered that the players in a game are organized in a priori unions depending on their common interests. Now we suppose that it is possible to measure the closeness of the ideas of the players. In this order we are going to think of a proximity function describing the closeness among them. Mordeson and Nair [50] were the first to introduce proximity relations.

Definition 5.6 *A bilateral fuzzy relation (Mordeson and Nair [50]) over N is a mapping $\rho : N \times N \to [0,1]$ satisfying the condition $\rho(i,j) \leq \rho(i,i) \wedge \rho(j,j)$. A proximity relation over N, is a fuzzy relation ρ satisfying: (Reflexivity) $\rho(i,i) = 1$ for all $i \in N$, and (Symmetry) $\rho(i,j) = \rho(j,i)$ for all $i, j \in N$. Similarity relations are particular fuzzy versions of equivalence relations. A similarity relation over N is a proximity relation ρ satisfying besides: (Transitivity) $\rho(i,j) \geq \rho(i,k) \wedge \rho(k,j)$ for all $i, j, k \in N$.*

Let (N, v) be a game. If ρ is a proximity relation over the set of players N, $\rho(i,j)$ represents the closeness level between players $i, j \in N$. Then a proximity relation is identified with a fuzzy graph where the vertices always have level 1. We will use the notation $\rho(ij)$ instead of $\rho(i,j)$.

Definition 5.7 *A game with a proximity relation among the players is a triple (N, v, ρ) such that (N, v) is a game and ρ is a proximity relation over N. The set of games with a proximity relation among the players is denoted as $\mathcal{G}_{prox}$.*

Next we see an example of game with a proximity relation

Example 5.8 We take the same game (N, v) of Example 1.3 but in this case the relations are: players 1,2 are relatives, players 2,3 are owners, players 1,4,5 are workers, 1,2,5 have been working together for a long time and 1,5 are beer friends. We can define the relationships among the players as the following proximity relation that considers all the relations with same importance: $\rho(ii) = 1$ for all i, $\rho(15) = 0.6$, $\rho(12) = 0.4$, $\rho(14) = \rho(23) = \rho(25) = \rho(45) = 0.2$ and $\rho(ij) = 0$ otherwise. We represent the situation by a fuzzy graph, a graph with weighted edges (Figure 5.1).

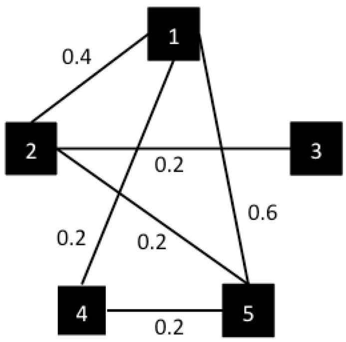


Figure 5.1: Proximity relation

Now we extend the Owen model in a fuzzy way. We can see a proximity relation as a cooperation structure by levels of the players. Let $(N, v, \rho) \in \mathcal{G}_{prox}$. For each $t \in (0, 1]$ we suppose that a set of players forms a cooperation structure if they are connected at least at level t and this set is maximal. It is like doing an analysis of the grouping of the players depending on a fixed closeness level.

Example 5.9 In Figure 5.2 we can see the different groups formed at each level $t \in (0, 1]$ in the above example. Every group has a specific cooperation structure which determines how the union is formed. The reader can see for instance that if our demand to form a group is to connect them with level at least $t = 0.3$ then $\{1, 2, 5\}$ is a union. But in this group the position of player 1 is not the same as in the others.

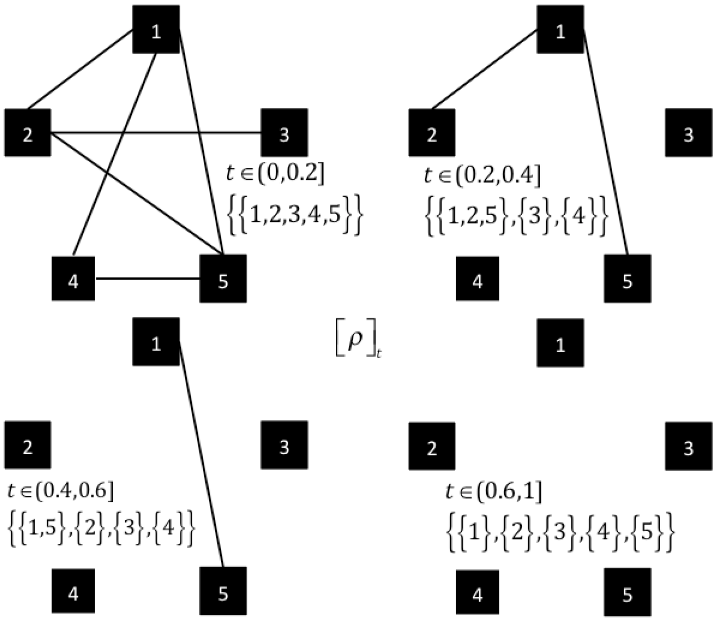


Figure 5.2: Cooperation structure partition.

A proximity relation ρ over N can be seen as a fuzzy set over $\overline{LN} = L(N) \cup \{ii : i \in N\}$ where $\rho(ij) = \rho(i, j)$, taking into account symmetry. Therefore we can calculate t-cuts and Choquet integrals of proximity relations. But not all the fuzzy sets ρ over $\overline{LN}$ are proximity relations because we need $\rho(ii) = 1$ for each $i \in N$. Proximity relations form the family of the fuzzy sets over $\overline{LN}$ which t-cuts contain $\{ii : i \in N\}$ for all $t \in (0, 1]$.

We say that a proximity relation ρ is *crisp* if $im(\rho) = \{1\}$. Cooperation structures are identified with the family of crisp proximity relations. Each cooperation structure $L \subseteq L(N)$ is identified with the crisp proximity relation ρ^L such that $\rho^L(ij) = 1$, if $i = j$ or $ij \in L$, and $\rho^L(ij) = 0$, otherwise. On the other hand, if ρ is a crisp proximity relation then we take the communication structure $L^\rho = \{ij \in L(N) : \rho(ij) = 1\}$. Particularly the t-cuts of a proximity relation are cooperation structures. If we consider a similarity relation, transitivity means here that if the level of closeness between players i, k is $\rho(ik)$ and the one between players k, j is $\rho(kj)$, then i, j can assume at least (in the worst case) $\rho(ik) \wedge \rho(kj)$ level of closeness. Games with crisp similarity relations are games with a priori unions.

We can only consider set functions over $L(N)$ for Choquet integrals of proximity relations. Each $f : 2^{L(N)} \to \mathbb{R}$ is identified with another set function over $\overline{LN}$, denoted with the same letter f, given by $f(A) = f(A \cap L(N))$ for all $A \subseteq \overline{LN}$, and then we use the Choquet integral of a proximity relation with respect to the first f as the one with respect to the second f.

Definition 5.10 *A proximity value is a mapping F that assigns a payoff vector $F(N, v, \rho) \in \mathbb{R}^N$ to each $(N, v, \rho) \in \mathcal{G}_{prox}$.*

Our model to get proximity values consists of taking a cooperation value f and a set function from it defined by

$$f_i(N, v)(L) = f_i(N, v, L), \ \forall L \subseteq L(N).$$

Then the proximity value is

$$F_i(N, v, \rho) = \int \rho \, df_i(N, v).$$

In the definition above we include a two-step construction in a similar way to the Owen model. Then, if we consider a value for cooperative games f^1 and a communication value f^2, we define the functional over $\overline{LN}$

$$f_i^2(v)(L) = f_i^2 \left(N_{k(i)}, v_{k(i)}, L_{N_{k(i)}} \right), \tag{5.4}$$

where $v_{k(i)}$ is constructed from f^1 as in (5.3). In that case

$$F_i(N, v, \rho) = \int \rho \, df_i^2(v). \tag{5.5}$$

In a cooperation structure $L \subseteq L(N)$ the coalitions which determine the a priori unions among the players are the connected components, the family N/L. In a proximity relation this role is played by the groups as we define now.

Definition 5.11 *Let ρ be a proximity relation over N. A coalition $S \subseteq N$ is a t-group for ρ with $t \in (0, 1]$ if $S \in N/[\rho]_t$. The family of groups of ρ is the set $N/\rho = \bigcup_{t \in (0,1]} N/[\rho]_t$.*

A group in a proximity relation is a coalition which can be considered as a cooperation structure when we establish a minimum relation level. If ρ is a crisp proximity relation (a cooperation structure) then $S \in N/\rho$ if and only if S is a connected component in the graph.

Definition 5.12 *Let ρ be a proximity relation over N. Coalitions $S_1, \ldots, S_r \subseteq N$ are leveled groups if there is a number $t \in (0,1]$ such that $S_1, \ldots, S_r$ are t-groups.*

For each set of leveled groups $S_1, \ldots, S_r,\ (r \geq 1)$ we denote

$$t_{S_1 \ldots S_r} = \bigwedge \{t \in (0,1], S_1, \ldots, S_r \in N/[\rho]_t\} \tag{5.6}$$

$$t^{S_1 \ldots S_r} = \bigvee \{t \in (0,1], S_1, \ldots, S_r \in N/[\rho]_t\} \tag{5.7}$$

Observe that number $t^{S_1 \ldots S_r}$ is a maximum but number $t_{S_1 \ldots S_r}$ is an infimum. Moreover $0 \leq t_{S_1 \ldots S_r} < t^{S_1 \ldots S_r} \leq 1$. Obviously, we can say then that groups $S_1, \ldots, S_r \in N/[\rho]_t$ for all $t \in \left(t_{S_1 \ldots S_r}, t^{S_1 \ldots S_r}\right]$. If ρ is a crisp proximity relation then $t_{S_1 \ldots S_r} = 0$ and $t^{S_1 \ldots S_r} = 1$ for all sets of components.

Proposition 5.13 *Let ρ be a proximity relation over N. If $S, T \in N/\rho$ are groups with $S \cap T \neq \emptyset$ then $S \subseteq T$ or $T \subseteq S$. Particularly, if S, T are leveled groups then $S \cap T = \emptyset$.*

Proof. Suppose $S, T \in N/\rho$ with ρ a proximity relation. If they are leveled then there exists $t \in (0,1]$ with $S, T \in N/[\rho]_t$, thus $S \cap T = \emptyset$. If $t_S = t_T$ then they are leveled groups. Hence we consider $t_S > t_T$. There is a number $t > t_S$ such that $S \in N/[\rho]_t$ and T is union of components in $N/[\rho]_t$, therefore either $S \cap T = \emptyset$ or S is one of these components.

5.3 Reducing a proximity relation.

In this section we introduce several ways of reducing a proximity relation, the set of elements affected or the set of levels of the image. We also show several properties of the proximity relations related to the Choquet integral. These properties will serve us to present the axioms of the following chapter.

Definition 5.14 *Let ρ be a proximity relation over N. If $S \subseteq N$ then the proximity relation restricted to S is ρ_S, a new proximity relation over S with $\rho_S(ij) = \rho(ij)$ for all $i, j \in S$.*

Obviously, for each $S \subseteq N$ we have $|im(\rho_S)| \leq |im(\rho)|$. Now we see a relation with the Choquet integral of the restriction.

Proposition 5.15 *Let ρ be a proximity relation over N. If $f : 2^{L(N)} \to \mathbb{R}$ is such that there is $S \subseteq N$ with $f(L) = f(L_S)$ for all $L \subseteq L(N)$ then*

$$\int \rho \, df = \int \rho_S \, df|_{L_S}.$$

Proof. Consider $S \subseteq N$ and ρ a proximity relation. For all $t \in (0,1]$ we have the equality $([\rho]_t)_S = [\rho_S]_t$. Let $f : 2^{L(N)} \to \mathbb{R}$ be a set function with $f(L) = f(L_S)$ for all $L \subseteq L(N)$. If $im(\rho) = \{\lambda_1, \ldots, \lambda_p\}$ then $im(\rho_S) = \{\lambda'_1, \ldots, \lambda'_{p'}\} \subseteq im(\rho)$. For each $q' \in \{1, \ldots, p'\}$ and $q \in \{1, \ldots, p\}$ with $\lambda'_{q'} \leq \lambda_q < \lambda'_{q'+1}$ we obtain $[\rho_S]_{\lambda_q} = [\rho_S]_{\lambda'_{q'}}$. So,

$$\begin{aligned} \int \rho \, df &= \sum_{q=1}^{p} (\lambda_q - \lambda_{q-1}) f\left([\rho]_{\lambda_q}\right) = \sum_{q=1}^{p} (\lambda_q - \lambda_{q-1}) f\left(\left([\rho]_{\lambda_q}\right)_S\right) \\ &= \sum_{q=1}^{p} (\lambda_q - \lambda_{q-1}) f\left([\rho_S]_{\lambda_q}\right) = \sum_{q'=1}^{p'} \left(\lambda'_{q'} - \lambda'_{q'-1}\right) f\left([\rho_S]_{\lambda'_{q'}}\right) = \int \rho_S \, df|_{L_S}. \end{aligned}$$

□

Now we define a scaling of a proximity relation which considers insignificant the levels out of an interval.

Definition 5.16 *Let ρ be a proximity relation over N. If $a, b \in [0,1]$ with $a < b$ then ρ_a^b is the interval scaling of ρ, a new proximity relation over N defined as*

$$\rho_a^b(ij) = \begin{cases} 1, & \text{if } \rho(ij) \geq b \\ \dfrac{\rho(ij) - a}{b - a}, & \text{if } \rho(ij) \in (a, b) \\ 0, & \text{if } \rho(ij) \leq a. \end{cases}$$

Observe that it holds $\left|im\left(\rho_a^b\right)\right| \le |im(\rho)|$ and particularly $\rho_0^1 = \rho$. The interval scaling of a proximity relation and the original proximity relation are comonotone as fuzzy sets.

Proposition 5.17 *Let ρ be a proximity relation over N and $a, b \in [0,1]$ with $a < b$. The interval scaling ρ_a^b and ρ are comonotone.*

Proof. We prove that ρ_a^b, ρ are comonotone as fuzzy sets over $\overline{LN}$. Let $ij, kl \in \overline{LN}$. We suppose $\rho(ij) \ge \rho(kl)$ without loss of generality. If $\rho(ij) \ge b$ then $\rho_a^b(ij) = 1 \ge \rho_a^b(kl)$. If $\rho(kl) \le a$ then $\rho_a^b(kl) = 0 \le \rho_a^b(ij)$. Otherwise, $a < \rho(kl) \le \rho(ij) \le b$, we get

$$\frac{\rho(ij) - a}{b - a} \ge \frac{\rho(kl) - a}{b - a}.$$

□

The above proposition implies the next result for the Choquet integral.

Proposition 5.18 *Let ρ be a proximity relation over N and $a_1, \ldots, a_r \in [0,1]$ with $a_1 < \cdots < a_r$. It holds for all $f : 2^{L(N)} \to \mathbb{R}$ that*

$$\int \rho \, df = \sum_{p=1}^{r+1} (a_p - a_{p-1}) \int \rho_{a_{p-1}}^{a_p} \, df,$$

with $a_0 = 0$ and $a_{r+1} = 1$.

Proof. Suppose ρ a proximity relation and consider numbers $a_1 < \cdots < a_r$ in $[0,1]$, $a_0 = 0$ and $a_{r+1} = 1$. Remember that comonotony is a transitive property. Hence as $(a_p - a_{p-1}) \ge 0$ for every $p \in \{1, \ldots, r+1\}$, Proposition 5.17 implies that $(a_p - a_{p-1}) \, \rho_{a_{p-1}}^{a_p}$ and $(a_q - a_{q-1}) \, \rho_{a_{q-1}}^{a_q}$ are comonotone for all $p, q \in \{1, \ldots, r+1\}$.

We also prove that

$$\rho = \sum_{p=1}^{r+1} (a_p - a_{p-1}) \, \rho_{a_{p-1}}^{a_p}.$$

Let $ij \in \overline{LN}$. We suppose $\rho(ij) \ne 0$ because otherwise $\rho_{a_{p-1}}^{a_p}(ij) = 0$ for all p . In that case there exists $q \in \{1, \ldots, r+1\}$ with $\rho(ij) \in (a_{q-1}, a_q]$. For each $p < q$ we have $\rho^{a_p}_{a_{p-1}}(ij) = 1$ and for each $p > q$ we get $\rho^{a_p}_{a_{p-1}}(ij) = 0$. If $p = q$,

$$\rho^{a_q}_{a_{q-1}}(ij) = \frac{\rho(ij) - a_{q-1}}{a_q - a_{q-1}}.$$

So, we obtain

$$\sum_{p=1}^{r+1} (a_p - a_{p-1})\, \rho^{a_p}_{a_{p-1}}(ij) = \sum_{p=1}^{q-1} (a_p - a_{p-1}) + (\rho(ij) - a_{q-1}) = \rho(ij).$$

Now we use properties (C4) and (C2) of Proposition 1.50 to get for a set function f

$$\begin{aligned} \int \rho\, df &= \int \sum_{p=1}^{r+1} (a_p - a_{p-1})\, \rho^{a_p}_{a_{p-1}}\, df \\ &= \sum_{p=1}^{r+1} \int (a_p - a_{p-1})\, \rho^{a_p}_{a_{p-1}}\, df = \sum_{p=1}^{r+1} (a_p - a_{p-1}) \int \rho^{a_p}_{a_{p-1}}\, df. \end{aligned}$$

□

Now we define a scaling of a proximity relation where the insignificant levels are those within the interval.

Definition 5.19 *Let ρ be a proximity relation over N. Let $a, b \in [0,1]$ be numbers with $a < b$ and $a \neq 0$ or $b \neq 1$. The dual interval scaling of ρ is a new proximity relation over N given by*

$$\overline{\rho}^b_a(ij) = \begin{cases} \dfrac{\rho(ij) + a - b}{1 + a - b}, & \text{if } \rho(ij) \geq b \\ \dfrac{a}{1 + a - b}, & \text{if } \rho(ij) \in (a, b) \\ \dfrac{\rho(ij)}{1 + a - b}, & \text{if } \rho(ij) \leq a. \end{cases}$$

Remark 5.20 *If $a = 0$ and $b = 1$ then the dual interval scaling is not well-defined. Suppose $a = 0$*

and $b \in (0,1]$, *then* $\overline{\rho}_a^1(ij) = 0$, *if* $\rho(ij) < b$. *Then we can define* $\overline{\rho}_0^1$ *in the same way,*

$$\overline{\rho}_0^1(ij) = \begin{cases} 1, & \textit{if } \rho(ij) = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

Observe that it also holds $\left|im\left(\overline{\rho}_a^b\right)\right| \leq |im(\rho)|$. Next result about the Choquet integral is obtained from Proposition 5.18.

Proposition 5.21 *Let* ρ *be a proximity relation over* N. *For every pair of numbers* $a, b \in [0,1]$ *with* $a < b$ *and for every set function* $f : 2^{L(N)} \to \mathbb{R}$ *it holds*

$$\int \rho \, df = (b-a) \int \rho_a^b \, df + (1+a-b) \int \overline{\rho}_a^b \, df.$$

Proof. Consider ρ a proximity relation and numbers $a, b \in [0,1]$ with $a < b$. If $a = 0$ and $b = 1$ we have a trivial equality. Otherwise, Proposition 5.18 says

$$\int \rho \, df = a \int \rho_0^a \, df + (b-a) \int \rho_a^b \, df + (1-b) \int \rho_b^1 \, df.$$

Since comonotony is a transitive property we get that $a\rho_0^a$ and $(1-b)\rho_b^1$ are comonotone using Proposition 5.17. Therefore (C2) and (C4) of Proposition 1.50 imply

$$\int \rho \, df = (b-a) \int \rho_a^b \, df + \int \left[a\rho_0^a + (1-b)\rho_b^1\right] \, df.$$

Now we prove the next equality of fuzzy sets $(1+a-b)\overline{\rho}_a^b = a\rho_0^a + (1-b)\rho_b^1$. Suppose $i, j \in N$. If $\rho(ij) \leq a$ then

$$a\rho_0^a(ij) + (1-b)\rho_b^1(ij) = a\frac{\rho(ij)}{a} = \rho(ij).$$

If $\rho(ij) \geq b$ then

$$a\rho_0^a(ij) + (1-b)\rho_b^1(ij) = a + (1-b)\frac{\rho(ij) - b}{1-b} = \rho(ij) + a - b.$$

Finally, if $\rho(ij) \in (a,b)$ then $a\rho_0^a(ij) + (1-b)\rho_b^1(ij) = a$. We finish the proof using (C2) again. □

Proposition 5.18 allows to write the values in two steps following (5.5) in terms of the t-groups. If

f^2 is the communication value of the second step, $N_r = R \in N/\rho$ and L is any graph such that $R \in N/L$ we define the functional

$$f_i^2(R,v)(L) = f_i^2\left(R, v_r, L_R\right),$$

where v_r is like (5.3) with a classic value f^1. In that case, for each $i \in N$

$$F_i(N,v,\rho) = \sum_{R \in N/\rho, i \in R} \left(t^R - t_R\right) \int \rho_{t_R}^{t^R} df_i^2(R,v).$$

Chapter 6

Values for games with cooperation structure

Although our aim is the analysis of values for games with proximity relations, we previously need, as we saw in the preceding chapter, to introduce some values for games with cooperation structure. We follow the logical sequence of the construction. Then in next section we present the main values for games with a priori unions that follow the Owen model. We present also some axiomatizations found in the literature. In the following sections we present our values for games with cooperation structures together with their axiomatizations.

6.1 Values for games with a priori unions

In this section we are going to recall some known values for games with a priori unions that follow the Owen model. We are also going to present one of their existent axiomatizations for each one. The first one is the Owen value.

Definition 6.1 *The Owen value ω is defined, for each $(N, v, \mathcal{P})$ with $\mathcal{P}$ a set of a priori unions and $i \in N$, by*

$$\omega_i\left(N, v, \mathcal{P}\right) = \phi_i\left(N_{k(i)}, v_{k(i)}\right),$$

where $k(i)$ is such that $i \in N_{k(i)}$ and $v_{k(i)}(S) = \phi_{k(i)}\left(M, v^{\mathcal{P}_S}\right)$, $\forall S \subseteq N_k$.

If we look at the Owen model, in this case, $f^1 = \phi$ in (5.3) and $f^2 = \phi$ in (5.2).

We show the axiomatization of Owen [58], but first we introduce some axioms for games with a priori union structure.

Equal treatment within the unions. If $i, j \in N_k, k \in M$ are substitutable in (N, v), i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$, $\forall S \subseteq N \setminus \{i, j\}$ then $f_i(N, v, \mathcal{P}) = f_j(N, v, \mathcal{P})$.

A similar condition among the unions is expressed in the following axiom.

Coalitional symmetry. If $k_1, k_2 \in M$ satisfy that $v\left(N_{k_1} \cup \bigcup_{q \in Q} N_q\right) = v\left(N_{k_2} \cup \bigcup_{q \in Q} N_q\right)$ for every $Q \subseteq M \setminus \{k_1, k_2\}$ then

$$\sum_{i \in N_{k_1}} f_i(N, v, \mathcal{P}) = \sum_{j \in N_{k_2}} f_j(N, v, \mathcal{P}).$$

Theorem 6.2 (Owen [58]) *The Owen value is the only value over $\mathcal{G}_{un}$ satisfying efficiency, linearity, null player, equal treatment within the unions and coalitional symmetry.*

The axioms of efficiency, linearity, null player and dummy player are defined for games with a priori unions in the same way as for usual games (see Section 1.2).

Remark 6.3 *Owen [58] used another axiom called symmetry in each union instead of equal treatment within the unions, but both axioms are equivalent in a context with efficiency, linearity and null player.*

Next value based on the Owen model is the Banzhaf-Owen value defined by Owen in [59].

Definition 6.4 *The Banzhaf-Owen value ψ is defined, for each $(N, v, \mathcal{P})$ with $\mathcal{P}$ a set of a priori unions and $i \in N$, by*

$$\psi_i(N, v, \mathcal{P}) = \beta_i\left(N_{k(i)}, v_{k(i)}\right),$$

where $k(i)$ is such that $i \in N_{k(i)}$ and $v_{k(i)}(S) = \beta_{k(i)}\left(M, v^{\mathcal{P}_S}\right)$, $\forall S \subseteq N_k$.

If we look at the Owen model, in this case, $f^1 = \beta$ in (5.3) and $f^2 = \beta$ in (5.2).

Amer et al. [5] were the first that provided a characterization on the class of all cooperative games. They defined a new game in order to express all the axioms using the same player set, N.

Definition 6.5 *Let $(N,v) \in \mathcal{G}$ and $i,j \in N$. The delegation game $(N, v_{i \triangleright j}) \in \mathcal{G}$ is defined by*

$$v_{i \triangleright j}(S) = \begin{cases} v(S \cup \{j\}), & \text{if } i \in S \\ v(S \setminus \{j\}), & \text{if } i \notin S. \end{cases}$$

Delegation neutrality. If $i,j \in N_k$, $l \notin N_k$ then

$$f_l\left(N, v_{i \triangleright j}, \mathcal{P}\right) = f_l\left(N, v, \mathcal{P}\right).$$

Delegation transfer. If $i,j \in N_k$, then

$$f_i\left(N, v_{i \triangleright j}, \mathcal{P}\right) = f_i\left(N, v, \mathcal{P}\right) + f_j\left(N, v, \mathcal{P}\right).$$

Many null players. If $\mathcal{P}$ is an a priori union structure having, at most, one non-null player for game (N,v) in each union, then

$$f\left(N, v, \mathcal{P}\right) = f\left(N, v, \mathcal{P}'\right),$$

where $\mathcal{P}' = \{N\}$.

Theorem 6.6 (Amer et al. [5]) *The Banzhaf-Owen value is the only value over $\mathcal{G}_{un}$ satisfying linearity, dummy player, equal treatment within the unions, delegation neutrality, delegation transfer and many null players.*

The symmetric coalitional Banzhaf value was introduced by Alonso-Meijide and Fiestras-Janeiro [2]. In this article several characterizations were provided and two political examples illustrating the differences with respect to the Owen value and the Banzhaf-Owen value were given.

Definition 6.7 *The symmetric coalitional Banzhaf value φ is defined, for each $(N,v,\mathcal{P}) \in \mathcal{G}_{un}$ and $i \in N$, by*

$$\varphi_i\left(N, v, \mathcal{P}\right) = \phi_i\left(N_{k(i)}, v_{k(i)}\right),$$

where $k(i)$ is such that $i \in N_{k(i)}$ and $v_{k(i)}(S) = \beta_{k(i)}\left(M, v^{\mathcal{P}_S}\right)$, $\forall S \subseteq N_k$.

If we look at the Owen model, in this case, $f^1 = \beta$ in (5.3) and $f^2 = \phi$ in (5.2).

The first axiomatization of this value was given by Alonso-Meijide and Fiestras-Janeiro in [2]. The difference between this first axiomatization and that of Theorem 6.2 is analogous to that between the axiomatizations of the Shapley value and the Banzhaf value that appear in Feltkamp [28]. But they have another characterization that also appears in Alonso-Meijide and Fiestras-Janeiro [2]. We need some more definitions and axioms to introduce it.

Definition 6.8 *Let N be a set of players. The trivial coalition structure is $\mathcal{P}^n = \{\{1\},\{2\},\ldots,\{n\}\}$, where each union is a singleton.*

Coalitional Banzhaf value. A value f is a coalitional Banzhaf value if

$$f\left(N,v,\mathcal{P}^n\right) = \beta(N,v), \text{ for all } (N,v) \in \mathcal{G}.$$

Definition 6.9 *Let $\mathcal{P} = \{P_1,\ldots,P_m\}$ be a coalition structure. If $i \in P_k$, $\mathcal{P}_{-i}$ denotes the partition*

$$\mathcal{P}_{-i} = \{P_h \in P : h \neq k\} \cup \{P_k \setminus \{i\},\{i\}\}.$$

Balanced contributions within unions. For all $(N,v,\mathcal{P}) \in \mathcal{G}_{un}$ and $i,j \in P_k$

$$f_i(N,v,\mathcal{P}) - f_i\left(N,v,\mathcal{P}_{-j}\right) = f_j(N,v,\mathcal{P}) - f_j\left(N,v,\mathcal{P}_{-i}\right).$$

Quotient game property. For all $(N,v,\mathcal{P}) \in \mathcal{G}_{un}$ and $P_k \in \mathcal{P}$

$$\sum_{i \in P_k} f_i(N,v,\mathcal{P}) = f_k\left(M,v^{\mathcal{P}},\mathcal{P}^m\right).$$

Theorem 6.10 (Alonso-Meijide and Fiestras-Janeiro [2]) *The symmetric coalitional Banzhaf value is the only coalitional Banzhaf value over $\mathcal{G}_{un}$ satisfying balanced contributions within unions and the quotient game property.*

In Alonso-Meijide et al. [4] there is a comparative between the properties of the previous coalitional values.

6.2 The Myerson-Owen value

Casajus [17] proposed an allocation rule for games with cooperation structure following the sense of the Owen value that we name the Myerson-Owen value.

Definition 6.11 *The Myerson-Owen value is an allocation rule over $\mathcal{G}_{coop}$ defined for each (N, v, L) with $N/L = \{N_1, \ldots, N_m\}$ and $i \in N$ as*

$$\xi_i(N, v, L) = \mu_i \left(N_{k(i)}, v_{k(i)}, L_{N_{k(i)}} \right),$$

where $k(i)$ is such that $i \in N_{k(i)}$ and $v_{k(i)}(S) = \phi_{k(i)} \left(M, v^{(N/L)_S} \right)$, $\forall S \subseteq N_k$.

If we look at the Casajus model, in this case, $f^1 = \phi$ and $f^2 = \mu$.

Notice that the unions have a structure, so instead of the Shapley value inside the unions, Casajus applied the Myerson value in order not to lose this additional information.

Observe that the Myerson-Owen solution is a generalization of other values presented so far.

(a) If $(N, v, L) \in \mathcal{G}_{coop}$ satisfies that L is connected then $\xi(N, v, L) = \mu(N, v, L)$.

(b) If $(N, v, L) \in \mathcal{G}_{coop}$ satisfies that $L_S = L(S)$ for all $S \in N/L$ then we identify (N, v, L) with $(N, v, N/L) \in \mathcal{G}_{un}$ and $\xi(N, v, L) = \omega(N, v, N/L)$.

(c) If $(N, v, L) \in \mathcal{G}_{coop}$ with $L = L(N)$ then $\xi(N, v, L) = \phi(N, v)$.

In Casajus [17] there is an axiomatization of the Myerson-Owen value, but we provided the value with another one in Fernández et al. [31] with the purpose of defining all the axioms from the data (the game and the graph) and obtaining a better analogy with the first axiomatization of the Owen value given by Owen [58]. We present now some axioms that will be useful in our new axiomatization.

A null player can obtain profit due to his position in the graph if the players are asymmetric in the structure of the component. But if all the players in the component are null then it is impossible to get profits despite the strategic position of each player. We say that a coalition $S \subseteq N$ is a *null* *coalition* in a game $(N, v) \in \mathcal{G}$ if each player $i \in S$ is a null player for the game.

Null component. Let $(N, v, L) \in \mathcal{G}_{coop}$ and $S \in N/L$ a null coalition, then $f_i(N, v, L) = 0$ for all $i \in S$.

Two coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$ are *substitutable* in a game (N, v) if $v(R \cup S) = v(R \cup T)$ for all $R \subseteq N \setminus (S \cup T)$. We can suppose that two substitutable components obtain the same total payoff.

Substitutable components. Let $(N, v, L) \in \mathcal{G}_{coop}$. If $S, T \in N/L$ are substitutable components in (N, v) then

$$\sum_{i \in S} f_i(N, v, L) = \sum_{j \in T} f_j(N, v, L).$$

Now (following the axiomatization of the Owen value) we see that here the equal treatment property within the unions axioms depends on the structure in each component because they are asymmetric. The Myerson fairness cannot be used to explain this asymmetry because the deletion of a link can cause a change in the number of components (unions). So, we use the modified fairness proposed by Casajus [17]. This axiom says that the difference of payoffs when we break a link, placing the players disconnected by this fact out of the game, is the same for both of the players in the link. Let $(N, v, L) \in \mathcal{G}_{coop}$ and $ij \in L$. If $S \in N/L$ with $i, j \in S$ and $S_i \in N/(L \setminus \{ij\})$ with $i \in S_i$ (in the same way S_j) then $N^i_{ij} = (N \setminus S) \cup S_i$ (in the same way N^j_{ij}).

Modified fairness. Let $(N, v, L) \in \mathcal{G}_{coop}$ and $ij \in L$, it holds

$$f_i(N, v, L) - f_i\left(N^i_{ij}, v, L_{N^i_{ij}} \setminus \{ij\}\right) \quad = \quad f_j(N, v, L) - f_j\left(N^j_{ij}, v, L_{N^j_{ij}} \setminus \{ij\}\right).$$

We prove in the next theorem that the Myerson-Owen value is the only one satisfying all these axioms.

Theorem 6.12 *The Myerson-Owen value satisfies the following axioms: efficiency, linearity, null component, substitutable components and modified fairness.*

Proof. We will test that each one of the axioms is satisfied by the Myerson-Owen value. Let $(N, v, L) \in \mathcal{G}_{coop}$, $N/L = \{N_1, ..., N_m\}$ and $M = \{1, \dots, m\}$.

Efficiency. The quotient game satisfies $v^{(N/L)_{N_k}} = v^{N/L}$, for every $k \in M$. Using that the Myerson value is efficient by components and the Shapley value is efficient we get

$$\begin{aligned}\sum_{i\in N}\xi_i(N,v,L) &= \sum_{i\in N}\mu_i\left(N_{k(i)},v_{k(i)},L_{N_{k(i)}}\right)=\sum_{k=1}^{m}\sum_{i\in N_k}\mu_i\left(N_k,v_k,L_{N_k}\right)=\sum_{k=1}^{m}v_k\left(N_k\right)\\ &= \sum_{k=1}^{m}\phi_k\left(M,v^{(N/L)_{N_k}}\right)=\sum_{k=1}^{m}\phi_k\left(M,v^{N/L}\right)=v^{N/L}(M)=v(N).\end{aligned}$$

Linearity. Suppose now another game with the same cooperation structure, (N,w,L), and two numbers $\alpha,\beta\in\mathbb{R}$. As the Shapley value is a linear function (Section 1.2), for each $k\in M$ we have for all $S\subseteq N_k$,

$$(\alpha v+\beta w)_k(S)=\phi_k\left(M,(\alpha v+\beta w)^{(N/L)_S}\right)=\alpha v_k(S)+\beta w_k(S)$$

because by definition of the quotient game $(\alpha v+\beta w)^{(N/L)_S}=\alpha v^{(N/L)_S}+\beta w^{(N/L)_S}$. Since the graph L_{N_k} is the same for both games then by definition of the vertex game $(\alpha v+\beta w)_k^{L_{N_k}} = \alpha v_k^{L_{N_k}}+\beta w_k^{L_{N_k}}$. Using the linearity of the Shapley value again

$$\begin{aligned}\xi_i(N,\alpha v+\beta w,L) &= \mu_i\left(N_{k(i)},(\alpha v+\beta w)_{k(i)},L_{N_{k(i)}}\right)=\phi_i\left(N_{k(i)},(\alpha v+\beta w)_{k(i)}^{L_{N_{k(i)}}}\right)\\ &= \alpha\,\xi_i(N,v,L)+\beta\,\xi_i(N,w,L).\end{aligned}$$

Null component. Suppose $N_1\in N/L$ a null coalition for the game (N,v) and $N_1=S$. If $Q\subseteq M$ with $1\notin Q$ then we use $N_Q=\bigcup_{q\in Q}N_q$. For each $T=\{i_1,\ldots,i_p\}\subseteq S$ we have that $i_1,\ldots,i_p$ are null players for the game and by Definition 5.3

$$\begin{aligned}v^{(N/L)_T}(Q\cup\{1\})-v^{(N/L)_T}(Q) &= v\left(N_Q\cup T\right)-v\left(N_Q\right)\\ &= \sum_{l=2}^{p}\left[v\left(N_Q\cup\{i_1,...,i_l\}\right)-v\left(N_Q\cup\{i_1,...,i_{l-1}\}\right)\right]\\ &\quad+\left[v\left(N_Q\cup\{i_1\}\right)-v\left(N_Q\right)\right]=0.\end{aligned}$$

Hence 1 is a null player in $\left(M,v^{(N/L)_T}\right)$. As the Shapley value satisfies the null player axiom (see Section 1.2) we get $\phi_1\left(M,v^{(N/L)_T}\right)=0$. So using (5.3), $v_1(T)=0$ for all $T\subseteq N_1$. But if $v_1=0$

then $v_1/L_{N_1} = 0$ in N_1. For all $i \in N_1$ we have

$$\xi_i(N,v,L) = \mu_i\left(N_1, 0, L_{N_1}\right) = \phi_i\left(N_1, 0\right) = 0.$$

Substitutable components. Let $S,T \subseteq N$ be two substitutable coalitions in the game (N,v) such that $S,T \in N/L$. Consider $N_1 = S, N_2 = T$. For each $Q \subseteq M$ we denote $N_Q = \bigcup_{q\in Q} N_q$ again. We test that $1,2$ are substitutable players for the quotient game $\left(M, v^{N/L}\right)$. Let $Q \subseteq M \setminus \{1,2\}$,

$$v^{N/L}(Q \cup \{1\}) = v\left(N_Q \cup S\right) = v\left(N_Q \cup T\right) = v^{N/L}(Q \cup \{2\}),$$

because S,T are substitutable in (N,v). It is known that the Shapley value satisfies the equal treatment axiom (see Theorem 1.21), thus

$$v_1(S) = \phi_1\left(M, v^{N/L}\right) = \phi_2\left(M, v^{N/L}\right) = v_2(T).$$

The Myerson value is efficient by components (Theorem 3.2) so

$$\begin{aligned}\sum_{i\in S} \xi_i(N,v,L) &= \sum_{i\in S} \mu_i\left(S, v_1, L_S\right) = v_1(S) \\ &= v_2(T) = \sum_{j\in T} \mu_j\left(T, v_2, L_T\right) = \sum_{j\in T} \xi_j(N,v,L).\end{aligned}$$

Modified fairness. Let $ij \in L$ and suppose $i,j \in N_1$. We have

$$N^i_{ij}/(L_{N^i_{ij}} \setminus \{ij\}) = \left\{(N_1)_i, N_2, ..., N_m\right\}.$$

Although the quotient game depends on the graph we get $v^{\left(N^i_{ij}/L_{N^i_{ij}\setminus\{ij\}}\right)_S} = v^{(N/L)_S}$ for each $S \subseteq (N_1)_i$. Now we use two properties of the Myerson value: decomposability (Remark 3.5) and fairness (Theorem 3.2),

$$\begin{aligned}\xi_i(N,v,L)-\xi_i\left(N_{ij}^i,v,L_{N_{ij}^i}\setminus\{ij\}\right) &= \mu_i\left(N_1,v_1,L_{N_1}\right)-\mu_i\left((N_1)_i,v_1,L_{(N_1)_i}\right)\\ &= \mu_i\left(N_1,v_1,L_{N_1}\right)-\mu_i\left(N_1,v_1,L_{N_1}\setminus\{ij\}\right)\\ &= \mu_j\left(N_1,v_1,L_{N_1}\right)-\mu_j\left(N_1,v_1,L_{N_1}\setminus\{ij\}\right)\\ &= \xi_j(N,v,L)-\xi_j\left(N_{ij}^j,v,L_{N_{ij}^j}\setminus\{ij\}\right).\square\end{aligned}$$

Theorem 6.13 *The Myerson-Owen value is the only allocation rule for games with cooperation structure satisfying the following axioms: efficiency, linearity, null component, substitutable components and modified fairness.*

Proof. The existence was proved in the previous theorem. Suppose f^1, f^2 different values over $\mathcal{G}_{coop}$ satisfying the five axioms. We take the smallest N and L such that $f^1 \neq f^2$. Hence there is a characteristic function v with $f^1(N,v,L) \neq f^2(N,v,L)$. Linearity and Proposition 1.12 imply that there exists a unanimity game u_T with $T \subseteq N$ such that

$$f^1\left(N,u_T,L\right) \neq f^2\left(N,u_T,L\right).$$

The family N/L is a partition of N. We set $M_T = \{S \in N/L : S \cap T \neq \emptyset\}$. If $S \notin M_T$ then all the players in S are null players for the unanimity game (N,u_T). The null group property says that for all $i \in S$

$$f_i^1\left(N,u_T,L\right) = f_i^2\left(N,u_T,L\right) = 0.$$

If $S \in M_T$ with $|S| > 1$ then for each $i \in S$ there is $j \in S \setminus \{i\}$ with $ij \in L$. Taking into account the minimal election of N and L and the modified fairness

$$\begin{aligned}f_i^1\left(N,u_T,L\right)-f_j^1\left(N,u_T,L\right) &= f_i^1\left(N_{ij}^i,u_T,L_{N_{ij}^i}\setminus\{ij\}\right)-f_j^1\left(N_{ij}^j,u_T,L_{N_{ij}^j}\setminus\{ij\}\right)\\ &= f_i^2\left(N_{ij}^i,u_T,L_{N_{ij}^i}\setminus\{ij\}\right)-f_j^2\left(N_{ij}^j,u_T,L_{N_{ij}^j}\setminus\{ij\}\right)\\ &= f_i^2\left(N,u_T,L\right)-f_j^2\left(N,u_T,L\right).\end{aligned}$$

Therefore $f_i^1\left(N,u_T,L\right)-f_i^2\left(N,u_T,L\right) = f_j^1\left(N,u_T,L\right)-f_j^2\left(N,u_T,L\right)$. Since L_S is connected there exists $B \in \mathbb{R}$ with $f_i^1\left(N,u_T,L\right)-f_i^2\left(N,u_T,L\right) = B$ for all $i \in S$. If $S, S' \in M_T$ then $S \cap S' = \emptyset$

and

$$u_T(S \cup R) = 0 = u_T\left(S' \cup R\right)$$

for all $R \subseteq N \setminus (S \cup S')$. Hence S and S' are substitutable for (N, u_T). The substitutable components axiom implies that there exist two numbers $A, A' \in \mathbb{R}$ such that for all $S \in M_T$

$$\sum_{i \in S} f_i^1(N, u_T, L) = A \text{ and } \sum_{i \in S} f_i^2(N, u_T, L) = A'.$$

Now we apply efficiency using that $u_T(N) = 1$,

$$\sum_{i \in N} f_i^1(N, u_T, L) = |M_T|A = 1 = |M_T|A' = \sum_{i \in N} f_i^2(N, u_T, L).$$

Thus $A = A'$ and

$$\sum_{i \in S} f_i^1(N, u_T, L) = \sum_{i \in S} f_i^2(N, u_T, L), \quad \forall S \in M_T.$$

For each $S \in M_T$ we use the above equality. If $S = \{i\}$ (a component with an isolated player in L) then

$$f_i^1(N, u_T, L) = f_i^2(N, u_T, L).$$

Otherwise we obtain

$$0 = \sum_{i \in S} f_i^1(N, u_T, L) - f_i^2(N, u_T, L) = |S|B,$$

thus $B = 0$ and

$$f_i^1(N, u_T, L) = f_i^2(N, u_T, L), \quad \forall i \in N_k.$$

Hence we get the contradiction $f_i^1(N, u_T, L) = f_i^2(N, u_T, L)$ for all $i \in N$. □

6.3 The coalitional graph Banzhaf value

In this section we are going to define and axiomatize a value following the same scheme of the Myerson-Owen value. The differences are that in the negotiation among unions, the Banzhaf value is applied, and inside the unions we allocate the profit using the graph Banzhaf value.

There is a concept analogous to that for games with a priori unions when the unions are singletons.

Now the situation is that the cooperation structure is the empty graph.

Definition 6.14 *A coalitional value of Banzhaf is an allocation rule f over $\mathcal{G}_{coop}$ that satisfies*

$$f(N, v, \emptyset) = \beta(N, v),$$

where $\emptyset$ denotes the empty graph, i.e., the graph without links.

Definition 6.15 *The coalitional graph Banzhaf value θ is defined by*

$$\theta_i(N, v, L) = \eta_i\left(N_{k(i)}, v_{k(i)}, L_{N_{k(i)}}\right),$$

where $k(i)$ is such that $i \in N_{k(i)},\ v_{k(i)}(S) = \beta_{k(i)}\left(M, v^{(N/L)_S}\right)$ for each $S \subseteq N_{k(i)}$ and η denotes the graph Banzhaf value of Definition 3.3.

If we look at the Casajus model, in this case, $f^1 = \beta$ and $f^2 = \eta$.

The coalitional graph Banzhaf value is a generalization of the Banzhaf-Owen value defined in Owen [59] but taking into account the inner structure of the components of the a priori unions, in this case N/L. It satisfies the following coincidences.

(a) If $(N, v, L) \in \mathcal{G}_{coop}$ satisfies that L is connected then $\theta(N, v, L) = \eta(N, v, L)$.

(b) If $(N, v, L) \in \mathcal{G}_{coop}$ satisfies that $L_S = L(S)$ for all $S \in N/L$ then we identify (N, v, L) to $(N, v, N/L) \in \mathcal{G}_{un}$ and $\theta(N, v, L) = \psi(N, v, N/L)$.

(c) If $(N, v, L) \in \mathcal{G}_{coop}$ with $L = L(N)$ then $\theta(N, v, L) = \beta(N, v)$.

We are going to give an axiomatization for θ but first we need to present an axiom which says that if we merge two players in a component, this merging does not affect the payoffs of the players outside that component.

Amalgamation neutrality. Given $ij \in L$ with $i, j \in N_k$ and $l \notin N_k$,

$$f_l(N, v, L) = f_l\left(N^{ij}, v^{ij}, L^{ij}\right).$$

Theorem 6.16 *The coalitional graph Banzhaf value is a coalitional value of Banzhaf that satisfies graph pairwise merging (Section 3.1), modified fairness (Section 6.2) and amalgamation neutrality.*

Proof. We will test each one of the axioms, but let us see first that θ is a coalitional value of Banzhaf.

If $L = \emptyset$ all players are isolated, then $N/L = N$ and $M = N$, consequently

$$\theta_i(N, v, \emptyset) = \eta_i\left(i, v_i, L_i\right) = v_i(i) = \beta_i\left(N, v^{(N/L)_i}\right) = \beta_i(N, v)$$

because $(N/L)_i = N$, so $v^{(N/L)_i} = v$ and η satisfies the isolation property by Theorem 3.4.

Graph pairwise merging. Let $ij \in L$ with $i, j \in N_k$. Then

$$\theta_i(N, v, L) + \theta_j(N, v, L) = \eta_i\left(N_k, v_k, L_{N_k}\right) + \eta_j\left(N_k, v_k, L_{N_k}\right) = \eta_p\left(\left(N_k\right)^{ij}, \left(v_k\right)^{ij}, \left(L_{N_k}\right)^{ij}\right),$$

because η satisfies graph pairwise merging by Theorem 3.4. On the other hand

$$\theta_p\left(N^{ij}, v^{ij}, L^{ij}\right) = \eta_p\left(\left(N^{ij}\right)_k, \left(v^{ij}\right)_k, \left(L^{ij}\right)_{N_k}\right).$$

Therefore if we check that $\left(N^{ij}\right)_k = \left(N_k\right)^{ij}, \left(v_k\right)^{ij} = \left(v^{ij}\right)_k$ and $\left(L_{N_k}\right)^{ij} = \left(L^{ij}\right)_{N_k}$ we have the desired equality. Since $i, j \in N_k$, the first and last equalities are straightforward. Observe that the merger of two players i, j that are connected by an edge does not change the number of components of the graph. If we merge two players in a component $i, j \in N_k$, we have $N^{ij}/L^{ij} = \{N_1, N_2, \ldots, \overset{k)}{(N_k)^{ij}}, \ldots, N_m\}$, and $M = \{1, \ldots, m\} = M^{ij}$ because we only focus on the number of components.

It only remains to see $\left(v_k\right)^{ij} = \left(v^{ij}\right)_k$.

We denote by p or ij indistinctly the player resulting from the merger of players i, j.

$\forall T \subseteq \left(N_k\right)^{ij}$ we have that $v_k\left(T \backslash \{p\} \cup \{i, j\}\right) = \beta_k\left(M, v^{(N/L)_{T \backslash \{p\} \cup \{i,j\}}}\right)$, if $p \in T$ and

$v_k(T) = \beta_k\left(M, v^{(N/L)_T}\right)$, if $p \notin T$. Then

$$(v_k)^{ij}(T) = \begin{cases} \beta_k\left(M, v^{(N/L)_{T\setminus\{p\}\cup\{i,j\}}}\right), & \text{if } p \in T \\ \\ \beta_k\left(M, v^{(N/L)_T}\right), & \text{if } p \notin T. \end{cases}$$

Moreover, $\forall T \subseteq (N_k)^{ij}$, $\left(v^{ij}\right)_k(T) = \beta_k\left(M, \left(v^{ij}\right)^{(N/L)_T}\right)$.

We distinguish two cases

1) If $p \in T$, we prove the claim

$$\left(v^{ij}\right)^{(N/L)_T} = v^{(N/L)_{T\setminus\{p\}\cup\{i,j\}}}.$$

Let $Q \subseteq M$, then $v^{ij}\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q} N_q\right)$, if $k \notin Q$ and

$v^{ij}\left(T \cup \left(\bigcup_{q\in Q\setminus\{k\}} N_q\right)\right) = v\left((T\setminus\{p\} \cup \{i,j\}) \cup \bigcup_{q\in Q\setminus\{k\}} N_q\right)$, if $k \in Q$.

Thus,

$$\left(v^{ij}\right)^{(N/L)_T}(Q) = \left\{ \begin{array}{ll} v\left(\bigcup_{q\in Q} N_q\right), & \text{if } k \notin Q \\ \\ v\left((T\setminus\{p\} \cup \{i,j\}) \cup \bigcup_{q\in Q\setminus\{k\}} N_q\right), & \text{if } k \in Q \end{array} \right\} = v^{(N/L)_{T\setminus\{p\}\cup\{i,j\}}}(Q).$$

2) If $p \notin T$, we prove the claim

$$\left(v^{ij}\right)^{(N/L)_T} = v^{(N/L)_T}.$$

Let $Q \subseteq M$, then $v^{ij}\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q} N_q\right)$, if $k \notin Q$ and

$v^{ij}\left(T \cup \left(\bigcup_{q\in Q\setminus\{k\}} N_q\right)\right) = v\left(T \cup \left(\bigcup_{q\in Q\setminus\{k\}} N_q\right)\right)$, if $k \in Q$.

$$\left(v^{ij}\right)^{(N/L)_T}(Q) = \left\{ \begin{array}{ll} v\left(\bigcup_{q\in Q} N_q\right), & \text{if } k \notin Q, \\ \\ v\left(T \cup \left(\bigcup_{q\in Q\setminus\{k\}} N_q\right)\right), & \text{if } k \in Q \end{array} \right\} = v^{(N/L)_T}(Q).$$

Therefore in all cases the games coincide, then $(v_k)^{ij} = \left(v^{ij}\right)_k$ and θ satisfies graph pairwise merging.

Modified fairness. This proof is analogous to Theorem 6.12 using two properties of the graph Banzhaf value η: decomposability and fairness (see Alonso-Meijide and Fiestras-Janeiro [3]).

Amalgamation neutrality. Let $ij \in N_k, l \notin N_k, l \in N_s$. We have, by definition of θ,

$$\theta_l(N, v, L) = \eta_l\left(N_s, v_s, L_{N_s}\right)$$

and

$$\theta_l\left(N^{ij}, v^{ij}, L^{ij}\right) = \eta_l\left(\left(N^{ij}\right)_s, \left(v^{ij}\right)_s, \left(L^{ij}\right)_{N_s}\right) = \eta_l\left(N_s, \left(v^{ij}\right)_s, L_{N_s}\right),$$

where the last equality comes from the fact that $ij \notin N_s$ and then the merging takes place out of the component N_s. So if we see that $\left(v^{ij}\right)_s = v_s$ we have the desired equality. Let $R \subseteq N_s$. By definition, $\left(v^{ij}\right)_s(R) = \beta_s\left(M, \left(v^{ij}\right)^{\left(N^{ij}/L^{ij}\right)_R}\right)$ and $v_s(R) = \beta_s\left(M, v^{(N/L)_R}\right)$. Let $Q \subseteq M$.

1) If $s \notin Q$, then

$v^{ij}\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q\setminus\{k\}} N_q \cup N_k\right)$, if $k \in Q$, and

$v^{ij}\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q} N_q\right)$, if $k \notin Q$.

2) If $s \in Q$, then

$v^{ij}\left(\bigcup_{q\in Q\setminus\{s\}} N_q \cup R\right) = v\left(\bigcup_{q\in Q\setminus\{k,s\}} N_q \cup N_k \cup R\right)$, if $k \in Q$ and

$v^{ij}\left(\bigcup_{q\in Q\setminus\{s\}} N_q \cup R\right) = v\left(\bigcup_{q\in N\setminus\{s\}} N_q \cup R\right)$, if $k \notin Q$.

Therefore

$$\left(v^{ij}\right)^{\left(N^{ij}/L^{ij}\right)_R}(Q) = \begin{cases} v\left(\bigcup_{q\in Q\setminus\{k\}} N_q \cup N_k\right), & \text{if } s \notin Q, k \in Q \\ v\left(\bigcup_{q\in Q} N_q\right), & \text{if } s, k \notin Q \\ v\left(\bigcup_{q\in Q\setminus\{k,s\}} N_q \cup N_k \cup R\right), & \text{if } s, k \in Q \\ v\left(\bigcup_{q\in N\setminus\{s\}} N_q \cup R\right), & \text{if } s \in Q, k \notin Q. \end{cases}$$

On the other hand,

1) If $s \notin Q$, then

$v\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q\setminus\{k\}} N_q \cup N_k\right)$, if $s \notin Q, k \in Q$, and

$v\left(\bigcup_{q\in Q} N_q\right) = v\left(\bigcup_{q\in Q} N_q\right)$, if $s, k \notin Q$.

2) If $s \in Q$, then

$v\left(\bigcup_{q\in Q\setminus\{s\}} N_q \cup R\right) = v\left(\bigcup_{q\in Q\setminus\{k,s\}} N_q \cup R \cup N_k\right)$, if $s, k \in Q$ and

$v\left(\bigcup_{q\in Q\setminus\{s\}} N_q \cup R\right) = v\left(\bigcup_{q\in Q\setminus\{s\}} N_q \cup R\right)$, if $s \in Q, k \notin Q$.

$$v^{(N/L)_R}(Q) = \begin{cases} v\left(\bigcup_{q\in Q\setminus\{k\}} N_q \cup N_k\right), & \text{if } s \notin Q, k \in Q \\ v\left(\bigcup_{q\in Q} N_q\right), & \text{if } s, k \notin Q \\ v\left(\bigcup_{q\in Q\setminus\{k,s\}} N_q \cup R \cup N_k\right), & \text{if } s, k \in Q \\ v\left(\bigcup_{q\in Q\setminus\{s\}} N_q \cup R\right), & \text{if } s \in Q, k \notin Q. \end{cases}$$

In all cases the games coincide, therefore θ satisfies amalgamation neutrality. □

Theorem 6.17 *The coalitional graph Banzhaf value θ is the only coalitional value of Banzhaf that satisfies the previous axioms.*

Proof. It remains to prove the uniqueness. Let $f^1 \neq f^2$ be two cooperation values that satisfy the axioms. We will use an induction in the number of links and players. If there are not any edges in L then, since θ is a coalitional value of Banzhaf, we have the uniqueness. If there is at least one component with one or more edges, we can apply amalgamation neutrality for the players outside that component and inside the component we apply modified fairness and graph pairwise merging in this way:

If $ij \in N_k$, then $\forall l \notin N_k$,

$$f_l^1(N,v,L) = f_l^1\left(N^{ij},v^{ij},L^{ij}\right) = f_l^2\left(N^{ij},v^{ij},L^{ij}\right) = f_l^2(N,v,L),$$

where the second equality comes from the induction hypothesis because $\left|N^{ij}\right| < |N|$ and $\left|link\left(L^{ij}\right)\right| < |link(L)|$.

Moreover, for every link $ij \in L$ with $i,j \in N_k$, by the modified fairness axiom

$$\begin{aligned} f_i^1(N,v,L) - f_j^1(N,v,L) &= f_i^1\left(N_{ij}^i, v, L_{N_{ij}^i} \setminus \{ij\}\right) - f_j^1\left(N_{ij}^j, v, L_{N_{ij}^j} \setminus \{ij\}\right) = \\ &= f_i^2\left(N_{ij}^i, v, L_{N_{ij}^i} \setminus \{ij\}\right) - f_j^2\left(N_{ij}^j, v, L_{N_{ij}^j} \setminus \{ij\}\right) = \\ &= f_i^2(N,v,L) - f_j^2(N,v,L), \end{aligned}$$

where the first equality comes from applying modified fairness, the second one is because of the induction hypothesis and the last is again by modified fairness. On the other hand, applying graph pairwise merging, we have

$$\begin{aligned} f_i^1(N,v,L) + f_j^1(N,v,L) &= f_p^1\left(N^{ij},v^{ij},L^{ij}\right) = f_p^2\left(N^{ij},v^{ij},L^{ij}\right) \\ &= f_i^2(N,v,L) + f_j^2(N,v,L). \end{aligned}$$

Adding together the previous equalities we obtain $f_i^1(N,v,L) = f_i^2(N,v,L)$. $\square$

6.4 The Banzhaf-Myerson value

The last cooperation value that we present is a mix of the previous values that applies the Banzhaf value among the unions and the Myerson value within the unions.

Definition 6.18 *The Banzhaf-Myerson value δ is an allocation rule defined over the class of games with cooperation structure by*

$$\delta_i(N,v,L) = \mu_i\left(N_{k(i)}, v_{k(i)}, L_{N_{k(i)}}\right),$$

where $k(i)$ is such that $i \in N_{k(i)}$ and $v_{k(i)}(S) = \beta_{k(i)}\left(M, v^{(N/L)_S}\right)$ for each $S \subseteq N_{k(i)}$.

If we look at the Casajus model, in this case, $f^1 = \beta$ and $f^2 = \mu$.

The Banzhaf-Myerson value is a generalization of the symmetric coalitional Banzhaf value defined in Alonso-Meijide and Fiestras-Janeiro [2], but taking into account the inner structure of the a priori unions, in this case N/L.

The Banzhaf-Myerson solution satisfies the following coincidences.

(a) If $(N, v, L) \in \mathcal{G}_{coop}$ satisfies that L is connected then $\delta(N, v, L) = \mu(N, v, L)$.

(b) If $(N, v, L) \in \mathcal{G}_{coop}$ satisfies that $L_S = L(S)$ for all $S \in N/L$ then we identify (N, v, L) with $(N, v, N/L) \in \mathcal{G}_{un}$ and $\delta(N, v, L) = \varphi(N, v, N/L)$.

(c) If $(N, v, L) \in \mathcal{G}_{coop}$ with $L = L(N)$ then $\delta(N, v, L) = \phi(N, v)$.

With the purpose of obtaining an axiomatization we introduce some more axioms.

Connected efficiency. A cooperation value f satisfies connected efficiency if

$$\sum_{i \in N} f_i(N, v, L) = v(N),$$

for every L that is connected.

Definition 6.19 *Let $(N, v, L) \in \mathcal{G}_{coop}$, $N/L = \{N_1, \ldots, N_m\}$ and $i_k \in N_k$, $k = r, s \in \{1, \ldots, m\}$. If we add the edge $\{i_r i_s\}$ we define the graph $L_{N_r N_s} = L \cup \{i_r i_s\}$.*

Component merging. A cooperation value f satisfies component merging if for every $r, s \in M$,

$$\sum_{k \in N_r \cup N_s} f_k(N, v, L) = \sum_{k \in N_r \cup N_s} f_k\left(N, v, L_{N_r N_s}\right).$$

Theorem 6.20 *The Banzhaf-Myerson value δ is a coalitional value of Banzhaf that satisfies connected efficiency, component merging, null component, substitutable components, modified* *fairness and linearity.*

Proof. We will test that each one of the axioms is satisfied by the Banzhaf-Myerson value but we prove first that it is a coalitional value of Banzhaf.

Let $(N, v, L) \in \mathcal{G}_{coop}$, $N/L = \{N_1, ..., N_m\}$ and $M = \{1, ..., m\}$.
If $L = \emptyset$ all players are isolated, then $N/L = N$ and $M = N$, consequently

$$\delta_i(N, v, \emptyset) = \mu_i\left(i, v_i, L_i\right) = v_i(i) = \beta_i\left(N, v^{(N/L)_i}\right) = \beta_i(N, v),$$

because $(N/L)_i = N$, so $v^{(N/L)_i} = v$ and μ satisfies component efficiency by Theorem 3.2.

Connected efficiency. Using that L is connected and that the Myerson value is efficient by components we get

$$\sum_{i\in N} \delta_i(N, v, L) = \sum_{i\in N} \mu_i(N, v, L) = v(N).$$

Component merging. Let $(N, v, L) \in \mathcal{G}_{coop}$, $N/L = \{N_1, \ldots, N_m\}$ and $i_k \in N_k, k = r, s \in M$. It holds

$$\begin{aligned}\sum_{k\in N_r} \delta_k(N, v, L) + \sum_{k\in N_s} \delta_k(N, v, L) &= \sum_{k\in N_r} \mu_k\left(N_r, v_r, L_{N_r}\right) + \sum_{k\in N_s} \mu_k\left(N_s, v_s, L_{N_s}\right) \\ &= v_r\left(N_r\right) + v_s\left(N_s\right),\end{aligned}$$

applying the component efficiency property of the Myerson value. Now we have

$$v_r\left(N_r\right) + v_s\left(N_s\right) = \beta_r\left(M, v^{N/L}\right) + \beta_s\left(M, v^{N/L}\right) = \beta_{rs}\left(M^{rs}, \left(v^{N/L}\right)^{rs}\right),$$

where the last equality comes from the pairwise merging axiom of the Banzhaf value. Now by component efficiency and definition of δ we have

$$\beta_{rs}\left(M^{rs}, \left(v^{N/L}\right)^{rs}\right) = \sum_{k\in N^{rs}} \mu_k\left(N^{rs}, v_{rs}, L_{N_r N_s}\right) = \sum_{k\in N_r\cup N_s} \delta_k\left(N, v, L_{N_r N_s}\right),$$

where $N^{rs} = N_r \cup N_s$ and $\left(v^{N/L}\right)^{rs} = v^{N/L_{N_r N_s}}$, because if $rs \in Q$ and $Q \subseteq M^{rs}$ with

$M^{rs} = \{N_1, \ldots, N^{rs}, \ldots, N_m\}$, then

$$\left(v^{N/L}\right)^{rs}(Q) = v^{N/L}(Q \cup \{rs\}) = v\left(\bigcup_{q \in Q \setminus \{rs\}} N_q \cup N_r \cup N_s\right) = v^{N/L_{N_r N_s}}(Q).$$

Linearity. It is analogous to Theorem 6.12, taking into account the linearity of the Shapley and Banzhaf values.

Null component. It is analogous to Theorem 6.12, taking into account that the Banzhaf value also satisfies the null player axiom.

Substitutable components. It is analogous to Theorem 6.12, taking into account that the Banzhaf value also satisfies the equal treatment axiom.

Modified fairness. It is analogous to Theorem 6.12, taking into account two properties of the Myerson value: decomposability (Remark 3.5) and fairness (Theorem 3.2).

□

Theorem 6.21 *The Banzhaf-Myerson value δ is the only cooperation value that satisfies connected efficiency, component merging, null component, substitutable components, modified fairness and linearity.*

Proof. It remains to prove the uniqueness. We prove it by induction in $|N/L| = m, |N|$ and $|L|$. If $m = 1$ it means that L is connected. Suppose f^1, f^2 different values over $\mathcal{G}_{coop}$ satisfying connected efficiency and modified fairness (we only need these two axioms in this case). Let L be the graph with the minimum number of edges such that $f^1(N, v, L) \neq f^2(N, v, L)$. Notice that L must have at least one link, otherwise, as L is connected, it would be a singleton and by connected efficiency, we have uniqueness. Taking into account the minimality of L, if ij is a link in L, then $f^1(N, v, L \setminus \{ij\}) = f^2(N, v, L \setminus \{ij\})$. Then, by modified fairness

$$f_i^1(N, v, L) - f_j^1(N, v, L) = f_i^2(N, v, L) - f_j^2(N, v, L),$$

so $f_i^1(N,v,L) - f_i^2(N,v,L) = B$ for every $i \in N$. Then

$$B|N| = \sum_{i\in N} f_i^1(N,v,L) - f_i^2(N,v,L) = v(N) - v(N) = 0,$$

therefore $B = 0$ and $f_i^1(N,v,L) = f_i^2(N,v,L)$, for every $i \in N$.

We suppose that $f^1 = f^2$ with $|N/L| = p-1$.

Now suppose that $|N/L| = p > 1$. We take the smallest N and L such that $f^1 \neq f^2$. Hence there is a characteristic function v with $f^1(N,v,L) \neq f^2(N,v,L)$. Linearity implies that there exists a unanimity game u_T with $T \subseteq N$ such that

$$f^1(N,u_T,L) \neq f^2(N,u_T,L).$$

The family N/L is a partition of N. We set $M_T = \{S \in N/L : S \cap T \neq \emptyset\}$. If $S \notin M_T$ then all the players in S are null players for the unanimity game (N,u_T). The null component property says that for all $i \in S$

$$f_i^1(N,u_T,L) = f_i^2(N,u_T,L) = 0.$$

If $S \in M_T$ with $|S| > 1$ then for each $i \in S$ there is $j \in S\backslash\{i\}$ with $ij \in L$. Taking into account the minimal election of N and L and the modified fairness

$$\begin{aligned} f_i^1(N,u_T,L) - f_j^1(N,u_T,L) &= f_i^1\left(N_{ij}^i, u_T, L_{N_{ij}^i} \setminus \{ij\}\right) - f_j^1\left(N_{ij}^j, u_T, L_{N_{ij}^j} \setminus \{ij\}\right) \\ &= f_i^2\left(N_{ij}^i, u_T, L_{N_{ij}^i} \setminus \{ij\}\right) - f_j^2\left(N_{ij}^j, u_T, L_{N_{ij}^j} \setminus \{ij\}\right) \\ &= f_i^2(N,u_T,L) - f_j^2(N,u_T,L). \end{aligned}$$

Therefore $f_i^1(N,u_T,L) - f_i^2(N,u_T,L) = f_j^1(N,u_T,L) - f_j^2(N,u_T,L)$. Since L_S is connected there exists $B_S \in \mathbb{R}$ with $f_i^1(N,u_T,L) - f_i^2(N,u_T,L) = B_S$ for all $i \in S$. If $S, S' \in M_T$ then $S \cap S' = \emptyset$ and

$$u_T(S \cup R) = 0 = u_T(S' \cup R)$$

for all $R \subseteq N\backslash(S \cup S')$. Hence S and S' are substitutable for (N,u_T). The substitutable components axiom implies that there exist two numbers $A, A' \in \mathbb{R}$ s.t. for all $S \in M_T$

$$\sum_{i\in S} f_i^1(N, u_T, L) = A \text{ and } \sum_{i\in S} f_i^2(N, u_T, L) = A'.$$

If $M_T = \emptyset$ we have finished. If $S \in M_T$ and $S' \notin M_T$ with $S' \in N/L$ then by connected efficiency

$$\begin{aligned} A = \sum_{i\in S} f_i^1(N, u_T, L) &= \sum_{i\in S\cup S'} f_i^1(N, u_T, L_{SS'}) \\ = \sum_{i\in S\cup S'} f_i^2(N, u_T, L_{SS'}) &= \sum_{i\in S} f_i^2(N, u_T, L) = A', \end{aligned}$$

where the third equality comes from the induction hypothesis. If $M_T = N/L$ then again by connected efficiency with $S, S' \in M_T$,

$$\begin{aligned} 2A = \sum_{i\in S\cup S'} f_i^1(N, u_T, L) &= \sum_{i\in S\cup S'} f_i^1(N, u_T, L_{SS'}) \\ = \sum_{i\in S\cup S'} f_i^2(N, u_T, L_{SS'}) &= \sum_{i\in S\cup S'} f_i^2(N, u_T, L) = 2A', \end{aligned}$$

where the third equality comes from the induction hypothesis. This implies $A = A'$.

Thus $A = A'$ and then $\forall S \in M_T$,

$$\sum_{i\in S} f_i^1(N, u_T, L) - \sum_{i\in S} f_i^2(N, u_T, L) = |S| B_S = A - A' = 0,$$

for all $S \in M_T$. Then $B_S = 0$ and $f_i^1(N, u_T, L) = f_i^2(N, u_T, L)$ for all $i \in S$. Hence we get the contradiction $f_i^1(N, u_T, L) = f_i^2(N, u_T, L)$ for all $i \in N$. □

Remark 6.22 *In fact, when L is connected, the Banzhaf-Myerson value δ coincides with the Myerson value μ. Moreover, connected efficiency coincides with component efficiency and modified fairness with fairness. This fact explains why we only need these two axioms to prove the uniqueness in this case.*

If we compare the axiomatizations of the Myerson-Owen value in Fernández et al. [31] and the Banzhaf-Myerson value, the latter differs from the first in the fact that connected efficiency and component merging replace efficiency. This is a logical consequence of the axiomatizations of the Shapley value and the Banzhaf value in Feltkamp [28]. They have in common linearity, symmetry and null player. Nevertheless, the Shapley value is efficient, whereas the Banzhaf value satisfies pairwise merging.

Chapter 7

Values for games with proximity relations

The relationships of closeness among the players could modify the bargaining among them and consequently their payoffs. Often this closeness has been studied using a priori unions or undirected graphs. Now we propose to use proximity relations to represent leveled closeness among the players. Our values for games with proximity relations are computed by means of Choquet integrals of the proximity relations with respect to values for cooperation structures

Let $(N, v, \rho) \in \mathcal{G}_{prox}$. Following Section 5.2, for each $t \in (0, 1]$ we suppose that a set of players forms an a priori union with communication structure (which we have called cooperation structure) if they are connected at least at level t and this set is maximal. Hence we have to use a cooperation value in each level.

In Examples 5.8 and 5.9 we can see an example of game with proximity relation and its partition in cooperation structures.

7.1 The prox-Owen value

Let ρ be a proximity relation over N (Definition 5.6). We define the set function over $L(N)$ for each player $i \in N$ given by $\xi_i(N, v)(L) = \xi_i(N, v, L),\ \forall L \subseteq L(N)$, where ξ is the Myerson-Owen value. Now we introduce the solution proposed for games with a proximity relation among the players.

Definition 7.1 *The prox-Owen value is the allocation rule defined for all* $(N, v, \rho) \in \mathcal{G}_{prox}$ *and* $i \in N$ *as*

$$W_i(N, v, \rho) = \int \rho \, d\xi_i(N, v).$$

Example 7.2 Suppose the game of example 5.8. Depending on the assumed information we obtain the following solutions. If we omit the relationships among the players the Shapley value is

$$\phi(N,v) = (20.333, 37, 46, 20.333, 20.333).$$

If we consider the graph without the numbers on the links we apply the Myerson-Owen value of the game (which coincides with the Myerson value because the graph is connected),

$$\xi(N,v,L) = (20.4, 50.9, 36.733, 15.566, 20.4).$$

Finally we calculate the prox-Owen value. We have to consider the different cooperation structures in Figure 5.2 to determine the Choquet integral. Then

$$\begin{aligned} W(N,v,\rho) &= (0.2-0)\,\xi(N,v)([\rho]_{0.2}) + (0.4-0.2)\,\xi(N,v)([\rho]_{0.4}) \\ &+ (0.6-0.4)\,\xi(N,v)([\rho]_{0.6}) + (1-0.6)\,\xi(N,v)([\rho]_{1}) \\ &= (21.38, 38.346, 45.613, 19.38, 19.28). \end{aligned}$$

We propose an axiomatization for the prox-Owen value inspired by the axioms of the Owen value and the Myerson-Owen value given in Section 6.2. Let F be a proximity value. Consider the following axioms.

Efficiency. For all $(N,v,\rho) \in \mathcal{G}_{prox}$ it holds

$$\sum_{i\in N} F_i(N,v,\rho) = v(N).$$

Remember that a coalition $S \subseteq N$ is a null coalition in a game $(N,v) \in \mathcal{G}$ if each player $i \in S$ is a null player in (N,v). Players in a null coalition do not obtain profit when they are considered as a union or a partition of unions, therefore we can take as insignificant these levels and rescale.

Null group. Let $(N,v,\rho) \in \mathcal{G}_{prox}$ and $S \in N/\rho$ a group which is null for the game (N,v) then

$$F_i(N,v,\rho) = t_S F_i\left(N,v,\rho_0^{t_S}\right),\ \forall i \in S.$$

Particularly if we consider a crisp proximity relation ρ (a cooperation structure) the axiom says: if S is a component for ρ which is a null coalition for the game (N, v) then $F_i(N, v, \rho) = 0$ for all $i \in S$, i.e., it coincides with the null component axiom in Section 6.2.

Two coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$ are substitutable in a game (N, v) if $v(R \cup S) = v(R \cup T)$ for all $R \subseteq N \backslash (S \cup T)$. We can suppose that while both coalitions are groups the total payoff for each group is the same, that is

$$\sum_{i \in S} F_i \left(N, v, \rho_{t_{ST}}^{t^{ST}}\right) = \sum_{j \in T} F_j \left(N, v, \rho_{t_{ST}}^{t^{ST}}\right). \quad (7.1)$$

But we can get a similar condition using the next axiom, the part of the payoffs for each group which is not obtained in the common interval must be the same.

Substitutable leveled groups Let $(N, v, \rho) \in \mathcal{G}_{prox}$. If $S, T \in N/\rho$ are leveled groups and they are substitutable in (N, v) then

$$\sum_{i \in S} F_i(N,v,\rho) - \left(1 + t_{ST} - t^{ST}\right) F_i \left(N, v, \overline{\rho}_{t_{ST}}^{t^{ST}}\right) = \sum_{j \in T} F_j(N,v,\rho) - \left(1 + t_{ST} - t^{ST}\right) F_j \left(N, v, \overline{\rho}_{t_{ST}}^{t^{ST}}\right).$$

When we take a crisp proximity relation ρ the axiom says: if S, T are substitutable components of ρ for a game (N, v) then $\sum_{i \in S} F_i(N, v, \rho) = \sum_{j \in T} F_j(N, v, \rho)$, i.e., it coincides with the substitutable components axiom of Section 6.2. Observe that, by Proposition 5.21, our prox-Owen value satisfies the substitutable leveled groups axiom if and only if (7.1) holds.

We extend the modified fairness axiom from Casajus [17] to a fuzzy situation. In this case, we take into account the mere reduction of the relation between two players. So we have to consider that this reduction of level only concerns to the interval between the reduced level and the original one. Let ρ be a proximity relation over a set of players N with $im(\rho) = \{\lambda_1 < \cdots < \lambda_m\}$ and $\lambda_0 = 0$. Consider $i, j \in N$ two different players with $\rho(ij) = \lambda_k > 0$. The number $\rho^*(ij) = \lambda_{k-1}$ satisfies that for all $t \in (\rho^*(ij), \rho(ij)]$ the set N_{ij}^i (or N_{ij}^j) in the cooperation structure $[\rho]_t$ is the same. We denote also as N_{ij}^i (or N_{ij}^j) this common set for ρ. Now the modified fuzzy fairness says that the modified fairness is true if we reduce by t the level of a link ij for the payoffs in $(\rho(ij) - t, \rho(ij)]$, adding those payoffs obtained out of this interval.

Modified fuzzy fairness Let $(N, v, \rho) \in \mathcal{G}_{prox}$ and $i, j \in N$ with $\rho(ij) > 0$. For each $t \in$

$(0, \rho(ij) - \rho^*(ij)]$ it holds

$$\begin{aligned} F_i(N,v,\rho) \quad - \quad & F_j(N,v,\rho) = (1-t)\left[F_i\left(N,v,\overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right) - F_j\left(N,v,\overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right)\right] \\ + \quad & t\left[F_i\left(N^i_{ij}, v, \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^i_{ij}}\right) - F_j\left(N^j_{ij}, v, \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^j_{ij}}\right)\right]. \end{aligned}$$

If we consider a crisp proximity relation and we take $t = 1$ then the last axiom coincides with the modified fairness for games with cooperation structure (Section 6.2). Finally, we suppose a common axiom of the Shapley-type values: linearity.

Linearity For all games $(N, v), (N, w) \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$ and ρ proximity relation over N,

$$F(N, \alpha v + \beta w, \rho) = \alpha F(N, v, \rho) + \beta F(N, w, \rho).$$

Next theorem proves that the prox-Owen value satisfies all these axioms.

Theorem 7.3 *The prox-Owen value W satisfies the following axioms: efficiency, null group, substitutable leveled groups, modified fuzzy fairness and linearity.*

Proof. We will test each one of the axioms.

Efficiency. Theorem 6.12 showed that the Myerson-Owen value ξ satisfies efficiency. Hence,

$$\sum_{i \in N} \xi_i(N, v)(L) = \sum_{i \in N} \xi_i(N, v, L) = v(N),$$

for all $(N, v, L) \in \mathcal{G}_{coop}$. Thus the set function $\sum_{i \in N} \xi_i(N, v)$ is constant over $2^{L(N)}$. Now, applying the properties (C3) and (C5) of the Choquet integral and $\bigvee_{ij \in \overline{LN}} \rho(ij) = 1$

$$\begin{aligned} \sum_{i \in N} W_i(N, v, \rho) &= \sum_{i \in N} \int \rho \, d\xi_i(N, v) = \int \rho \, d \sum_{i \in N} \xi_i(N, v) \\ &= \sum_{k=1}^{r} (\lambda_k - \lambda_{k-1}) \sum_{i \in N} \xi_i(N, v)\left([\rho]_{\lambda_k}\right) = v(N), \end{aligned}$$

if $im(\rho) = \{\lambda_1 < \cdots < \lambda_r\}$.

Null group. Let S be a null coalition for a game (N, v). We consider ρ a proximity relation over N with $S \in N/\rho$ and $i \in S$. We have for the number t_S (5.6) that for all $r > t_S$ there exists a partition $\{S_1, \ldots, S_m\}$ of S such that $S_1, \ldots, S_m \in N/[\rho]_r$. Obviously, these coalitions are also null coalitions and then $\xi_i(N, v, [\rho]_r) = 0, \forall i \in S$, since the Myerson-Owen value satisfies null component (Theorem 6.12). If $t_S = 0$ then $W_i(N, v, \rho) = 0$. Otherwise, by Proposition 5.21 we get

$$\begin{aligned} W_i(N, v, \rho) &= \int \rho \, d\xi_i(N, v) = (t_S - 0) \int \rho_0^{t_S} \, d\xi_i(N, v) + (1 - t_S) \int \rho_{t_S}^1 \, d\xi_i(N, v) \\ &= t_S \, W_i\left(N, v, \rho_0^{t_S}\right) + (1 - t_S) \int \rho_{t_S}^1 \, d\xi_i(N, v). \end{aligned}$$

If $t \in im\left(\rho_{t_S}^1\right)$ then there is $r > t_S$ with

$$t = \frac{r - t_S}{1 - t_S}.$$

We can see from Definition 5.16 that $\rho(ij) \geq r$ if and only if $\rho_{t_S}^1(ij) \geq t$. Hence, $\left[\rho_{t_S}^1\right]_t = [\rho]_r$ and $\xi_i(N, v)\left(\left[\rho_{t_S}^1\right]_t\right) = 0$ for all t. By (C5) of Proposition 1.50 we have

$$\int \rho_{t_S}^1 \, d\xi_i(N, v) = 0.$$

Substitutable leveled groups. Let $S, T \subseteq N$ be two substitutable coalitions in a game (N, v). Consider now ρ a proximity relation over N with $S, T \in N/\rho$ leveled groups. We take numbers t_{ST} (5.6) and t^{ST} (5.7). Applying Proposition 5.21 for any player $i \in N$,

$$\begin{aligned} W_i(N, v, \rho) &= \int \rho \, d\xi_i(N, v) \\ &= \left(1 + t_{ST} - t^{ST}\right) \int \overline{\rho}_{t_{ST}}^{t^{ST}} \, d\xi_i(N, v) + \left(t^{ST} - t_{ST}\right) \int \rho_{t_{ST}}^{t^{ST}} \, d\xi_i(N, v) \\ &= \left(1 + t_{ST} - t^{ST}\right) W_i\left(N, v, \overline{\rho}_{t_{ST}}^{t^{ST}}\right) + \left(t^{ST} - t_{ST}\right) \int \rho_{t_{ST}}^{t^{ST}} \, d\xi_i(N, v). \end{aligned}$$

So, for groups S and T we have by (C3) of Proposition 1.50.

$$\begin{aligned}\sum_{i\in S} W_i(N,v,\rho) - \left(1+t_{ST}-t^{ST}\right) W_i\left(N,v,\overline{\rho}_{t_{ST}}^{t^{ST}}\right) &= \left(t^{ST}-t_{ST}\right)\int \rho_{t_{ST}}^{t^{ST}}\, d\sum_{i\in S}\xi_i(N,v).\\ \sum_{j\in T} W_j(N,v,\rho) - \left(1+t_{ST}-t^{ST}\right) W_j\left(N,v,\overline{\rho}_{t_{ST}}^{t^{ST}}\right) &= \left(t^{ST}-t_{ST}\right)\int \rho_{t_{ST}}^{t^{ST}}\, d\sum_{j\in T}\xi_j(N,v).\end{aligned}$$

If $t \in im\left(\rho_{t_{ST}}^{t^{ST}}\right)$ then there exists a number r with $t_{ST} < r \leq t^{ST}$ and

$$t = \frac{r-t_{ST}}{t^{ST}-t_{ST}}.$$

We can check using Definition 5.19 that $\left[\rho_{t_{ST}}^{t^{ST}}\right]_t = [\rho]_r$. So, as $S,T \in N/[\rho]_r$ for all $r \in (t_{ST}, t^{ST}]$ then we obtain from the substitutable components axiom of the Myerson-Owen value (Theorem 6.12),

$$\left[\sum_{i\in S}\xi_i(N,v)\right]\left(\left[\rho_{t_{ST}}^{t^{ST}}\right]_t\right) = \left[\sum_{j\in T}\xi_j(N,v)\right]\left(\left[\rho_{t_{ST}}^{t^{ST}}\right]_t\right).$$

Hence,

$$\left(t^{ST}-t_{ST}\right)\int \rho_{t_{ST}}^{t^{ST}}\, d\sum_{i\in S}\xi_i(N,v) = \left(t^{ST}-t_{ST}\right)\int \rho_{t_{ST}}^{t^{ST}}\, d\sum_{j\in T}\xi_j(N,v).$$

Modified fuzzy fairness. Let $i,j \in N$. Theorem 6.12 showed that the Myerson-Owen value satisfies modified fairness. Hence if $L \subseteq L(N)$ is such that $ij \in L$ then

$$\xi_i(N,v)(L) - \xi_j(N,v)(L) = \xi_i\left(N_{ij}^i, v\right)\left(L_{N_{ij}^i}\setminus\{ij\}\right) - \xi_j\left(N_{ij}^j, v\right)\left(L_{N_{ij}^j}\setminus\{ij\}\right).$$

We consider ρ a proximity relation with $\rho(ij) > 0$ and $t \in (0, \rho(ij)-\rho^*(ij)]$. Using Proposition 5.21 for numbers $\rho(ij)-t, \rho(ij)$ and (C3) in Proposition 1.50,

$$\begin{aligned} W_i(N,v,\rho) \quad - \quad & W_j(N,v,\rho) = \int \rho \, d[\xi_i(N,v) - \xi_j(N,v)] \\ = \quad & (1-t)\int \overline{\rho}^{\rho(ij)}_{\rho(ij)-t} \, d[\xi_i(N,v) - \xi_j(N,v)] + t\int \rho^{\rho(ij)}_{\rho(ij)-t} \, d[\xi_i(N,v) - \xi_j(N,v)] \\ = \quad & (1-t)\left[W_i\left(N,v,\overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right) - W_j\left(N,v,\overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right)\right] \\ + \quad & t\int \rho^{\rho(ij)}_{\rho(ij)-t} \, d[\xi_i(N,v) - \xi_j(N,v)]. \end{aligned}$$

For each $x \in im\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)$ there exists $r \in (\rho(ij)-t, \rho(ij)]$ with

$$x = \frac{r - \rho(ij) + t}{t}.$$

Moreover using Definition 5.16, $\left[\rho^{\rho(ij)}_{\rho(ij)-t}\right]_x = [\rho]_r$. Since $r \le \rho(ij)$ then $ij \in [\rho]_r$, thus the modified fairness of the Myerson-Owen value (Theorem 6.12) implies

$$\xi_i(N,v)\left(\left[\rho^{\rho(ij)}_{\rho(ij)-t}\right]_x\right) - \xi_j(N,v)\left(\left[\rho^{\rho(ij)}_{\rho(ij)-t}\right]_x\right) =$$

$$\xi_i\left(N^i_{ij},v\right)\left(\left(\left[\rho^{\rho(ij)}_{\rho(ij)-t}\right]_x\right)_{N^i_{ij}} \setminus \{ij\}\right) - \xi_j\left(N^j_{ij},v\right)\left(\left(\left[\rho^{\rho(ij)}_{\rho(ij)-t}\right]_x\right)_{N^j_{ij}} \setminus \{ij\}\right).$$

Hence, we obtain by (C3) and Proposition 5.15,

$$\int \rho^{\rho(ij)}_{\rho(ij)-t}\, d[\xi_i(N,v) - \xi_j(N,v)] =$$

$$= \int \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^i_{ij}} d\xi_i\left(N^i_{ij}, v\right)|_{N^i_{ij}} - \int \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^j_{ij}} d\xi_j\left(N^j_{ij}, v\right)|_{N^j_{ij}} =$$

$$W_i\left(N^i_{ij}, v, \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^i_{ij}}\right) - W_j\left(N^j_{ij}, v, \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^j_{ij}}\right).$$

Linearity. Suppose now another game with the same proximity relation (N, w, ρ), and two numbers $a, b \in \mathbb{R}$. As the Shapley value is a linear function (Theorem 1.21), (C3) implies

$$\begin{aligned} W_i(N, av + bw, \rho) &= \int \rho\, d\xi_i(N, av + bw) \\ &= a \int \rho\, d\xi_i(N, v) + b \int \rho\, d\xi_i(N, w) \\ &= a\, W_i(N, v, \rho) + b\, W_i(N, w, \rho). \end{aligned}$$

□

Theorem 7.4 *There is at most one value over $\mathcal{G}_{prox}$ satisfying the previous axioms: efficiency, null group, substitutable leveled groups, modified fuzzy fairness and linearity.*

Proof. Suppose F^1, F^2 different values over $\mathcal{G}_{prox}$ satisfying the five axioms. We prove the result by induction on the cardinality of the image of the proximity relation ρ.
Let $|im(\rho)| = 1$. Of course $im(\rho) = \{1\}$ and ρ is a crisp proximity relation. Hence in this case we obtain the uniqueness for the family of cooperation structures (Theorem 6.12). We suppose that there is only one value for all the games with a proximity relation ρ with $|im(\rho)| < d, d > 1$. Consider now a proximity relation ρ over a set of players N with $|im(\rho)| = d$. If $F^1 \neq F^2$ linearity implies that there exists a unanimity game u_T satisfying

$$F^1(N, u_T, \rho) \neq F^2(N, u_T, \rho).$$

The family $N/[\rho]_1$ is a partition of N. We set $M_T = \{S \in N/[\rho]_1 : S \cap T \neq \emptyset\}$. If $S \notin M_T$ then S is a null group for (N, u_T). We apply the null group property, if $t_S = 0$ then $F_i^1(N, u_T, \rho) = 0 = F_i^2(N, u_T, \rho)$. Otherwise, as $0 < t_S < 1$ then $t_S \in im(\rho)\backslash\{1\}$ but for all $i, j \in N$ with $\rho(ij) = t_S$ it holds $\rho_0^{t_S}(ij) = 1$. Hence $\left|im\left(\rho_0^{t_S}\right)\right| \leq |im(\rho)| - 1 < d$. The null group property implies now that for all $i \in S$,

$$F_i^1(N, u_T, \rho) = t_S F_i^1\left(N, u_T, \rho_0^{t_S}\right) = t_S F_i^2\left(N, u_T, \rho_0^{t_S}\right) = F_i^2(N, u_T, \rho).$$

Let $S, S' \in M_T$. We have several cases depending on the numbers $t_{SS'}, t^{SS'}$. If $t_{SS'} = 0$ and $t^{SS'} = 1$ then $\left|im\left(\overline{\rho}_0^1\right)\right| = 1 < d$. If $t_{SS'} > 0$ and $t^{SS'} = 1$ then $t_{SS'} \in im(\rho)\backslash\{1\}$ but for all $i, j \in N$ with $\rho(ij) = t_{SS'}$ it holds $\overline{\rho}_{t_{SS'}}^1(ij) = 1$, therefore $\left|im\left(\overline{\rho}_{t_{SS'}}^1\right)\right| \leq |im(\rho)| - 1 < d$. Otherwise $0 < t_{SS'} < t^{SS'} < 1$, then $t_{SS'}, t^{SS'} \in im(\rho)$ but for all i, j with $\rho(ij) = t_{SS'}$ and for all i', j' with $\rho(i'j') = t^{SS'}$ it holds $\overline{\rho}_{t_{SS'}}^{t^{SS'}}(ij) = \overline{\rho}_{t_{SS'}}^{t^{SS'}}(i'j')$, therefore $\left|im\left(\overline{\rho}_{t_{SS'}}^{t^{SS'}}\right)\right| \leq |im(\rho)| - 1 < d$. So, applying the substitutable leveled groups axiom

$$\begin{aligned}
\sum_{i \in S} F_i^1(N, u_T, \rho) &- \sum_{j \in S'} F_j^1(N, u_T, \rho) \\
&= \left(1 + t_{SS'} - t^{SS'}\right)\left[\sum_{i \in S} F_i^1\left(N, u_T, \overline{\rho}_{t_{SS'}}^{t^{SS'}}\right) - \sum_{j \in S'} F_j^1\left(N, u_T, \overline{\rho}_{t_{SS'}}^{t^{SS'}}\right)\right] \\
&= \left(1 + t_{SS'} - t^{SS'}\right)\left[\sum_{i \in S} F_i^2\left(N, u_T, \overline{\rho}_{t_{SS'}}^{t^{SS'}}\right) - \sum_{j \in S'} F_j^2\left(N, u_T, \overline{\rho}_{t_{SS'}}^{t^{SS'}}\right)\right] \\
&= \sum_{i \in S} F_i^2(N, u_T, \rho) - \sum_{j \in S'} F_j^2(N, u_T, \rho).
\end{aligned}$$

Hence,

$$\sum_{i \in S} F_i^1(N, u_T, \rho) - F_i^2(N, u_T, \rho) = \sum_{j \in S'} F_j^1(N, u_T, \rho) - F_j^2(N, u_T, \rho) = H.$$

Now, using efficiency

$$\begin{aligned}\sum_{i\in N} F_i^1(N,u_T,\rho) - F_i^2(N,u_T,\rho) &= \sum_{S\in M_T}\sum_{i\in S} F_i^1(N,u_T,\rho) - F_i^2(N,u_T,\rho)\\ &= |M_T|H = 0,\end{aligned}$$

thus $H = 0$. If $S = \{i\}$ with $i \in T$ then $F_i^1(N,u_T,\rho) = F_i^2(N,u_T,\rho)$. Suppose then $S \in M_T$ with $i,j \in S$ two different players with $\rho(ij) = 1$. We apply modified fuzzy fairness to this link reducing by $1-\rho^*(ij)$,

$$F_i^1(N,u_T,\rho) - F_j^1(N,u_T,\rho) = \rho^*(ij)\left[F_i^1\left(N,u_T,\overline{\rho}^1_{\rho^*(ij)}\right) - F_j^1\left(N,u_T,\overline{\rho}^1_{\rho^*(ij)}\right)\right]$$

$$+(1-\rho^*(ij))\left[F_i^1\left(N^i_{ij},u_T,\left(\left(\rho^1_{\rho^*(ij)}\right)^1_{-ij}\right)_{N^i_{ij}}\right) - F_j^1\left(N^j_{ij},u_T,\left(\left(\rho^1_{\rho^*(ij)}\right)^1_{-ij}\right)_{N^j_{ij}}\right)\right]$$

$$= F_i^2(N,u_T,\rho) - F_j^2(N,u_T,\rho).$$

because similar to a previous reasoning $\rho^*(ij) \in im(\rho)\backslash\{1\}$ and then

$$\left|im\left(\overline{\rho}^1_{\rho^*(ij)}\right)\right|, \left|im\left(\left(\left(\rho^1_{\rho^*(ij)}\right)^1_{-ij}\right)_{N^i_{ij}}\right)\right| \leq |im(\rho)| - 1 < d.$$

Coalition S is connected in $[\rho]_1$, this fact implies that we can connect two players in S by $\{i = i_0, i_1, \ldots, i_p = j\} \subseteq S$ with $\rho(i_q i_{q-1}) = 1$ for all $q = 1,\ldots,p$. Thus $F_i^1(N,u_T,\rho) - F_i^2(N,u_T,\rho) = K$ for all $i \in S$ and

$$0 = \sum_{i\in S} F_i^1(N,u_T,\rho) - F_i^2(N,u_T,\rho) = |S|K.$$

We get $K = 0$ and $F_i^1(N,u_T,\rho) = F_i^2(N,u_T,\rho)$ for all $i \in S$. □

The prox-Owen value can be seen as a fuzzy version of the Myerson-Owen value for games with cooperation structure. Similarity relations is the subfamily of proximity relations associated to the a priori unions structures of Owen, because the bilateral relations among the players are transitive. Moreover if ρ is a similarity relation then $[\rho]_t$ is a structure of a priori unions for each $t \in (0,1]$. We can obtain an axiomatization for the prox-Owen value over this subfamily. Obviously the prox-Owen value satisfies efficiency and linearity within this subfamily. As the restriction, the interval scaling and the dual interval scaling of a similarity relation are similarity relations then null group and substitutable leveled groups are also feasible axioms for similarity relations. Observe that the modified fuzzy fairness is not feasible because if we reduce the level of a pair of players we can break up the transitivity. In exchange, we introduce this other axiom used for the Owen value. For a similarity relation ρ and for two different players $i, j \in N$ such that there is a group $S \in N/\rho$ with $i, j \in S$ we denote

$$t^{ij} = \bigvee \left\{ t^S : S \in N/\rho, i, j \in S \right\}.$$

Substitutable players in a group Let ρ be a similarity relation over N. If i, j are substitutable for the game (N, v) (as individual coalitions) and there exists a group $S \in N/\rho$ with $i, j \in S$ then

$$F_i(N, v, \rho) - F_j(N, v, \rho) = \left(1 - t^{ij}\right) \left[F_i\left(N, v, \rho^1_{t^{ij}}\right) - F_j\left(N, v, \rho^1_{t^{ij}}\right)\right].$$

Theorem 7.5 *The prox-Owen value is the only value over $\mathcal{G}_{sim}$ (the set of games with a similarity relation among the players) which satisfies efficiency, null group, substitutable leveled groups, substitutable players in a group and linearity.*

Proof. The uniqueness part is similar to Theorem 7.4 using substitutable players in a group instead of modified fuzzy fairness.

Hence we only have to check that the prox-Owen value satisfies substitutable players in a group over similarity relations. Let $i, j \in N$ be two substitutable players in a game (N, v). As we said in Section 5.1 an a priori union structure is actually a communication structure L where every component is a complete graph, and $\xi = \omega$. Suppose L so. If i, j are in the same component in L the equal treatment for players within the unions axiom (see Section 6.1) of the Owen value implies

$$\xi_i(N, v)(L) = \xi_j(N, v)(L).$$

Let ρ be a similarity relation with a group containing players i, j. Using Proposition 5.21 with number t^{ij} we have

$$W_i(N,v,\rho) - \left(1 - t^{ij}\right) W_i\left(N,v,\rho^1_{t^{ij}}\right) = t^{ij} \int \rho_0^{t^{ij}}\, d\xi_i(N,v),$$

$$W_j(N,v,\rho) - \left(1 - t^{ij}\right) W_j\left(N,v,\rho^1_{t^{ij}}\right) = t^{ij} \int \rho_0^{t^{ij}}\, d\xi_j(N,v).$$

For each $t \in \left(\rho_0^{t^{ij}}\right)$ there exists $r \in \left(0, t^{ij}\right]$ with $t = \frac{r}{t^{ij}}$. Moreover, $\left[\rho_0^{t^{ij}}\right]_t = [\rho]_r$. As $r \in \left(0, t^{ij}\right]$ then i, j are contained in the same connected component of $[\rho]_r$. Therefore

$$\int \rho_0^{t^{ij}}\, d\xi_i(N,v) = \int \rho_0^{t^{ij}}\, d\xi_j(N,v).\square$$

7.2 The prox-Banzhaf value

This new fuzzy value allocates the benefits in a game with a proximity relation using the coalitional graph Banzhaf value.

Definition 7.6 *Let ρ be a proximity relation over N, and θ the coalitional graph Banzhaf value. We define the prox-Banzhaf value for each $i \in N$ by*

$$D_i(N,v,\rho) = \int \rho\, d\theta_i(N,v).$$

We are going to see some axioms for D that are a fuzzy extension of the axioms already presented for θ.

In the next axiom we use the fuzzy merging graph given in Definition 3.20, considering ρ as the fuzzy graph $\left(e^N, \rho\right)$.

Example 7.7 We see first an example of this proximity relation in Figure 7.1 based on the proximity relation of Figure 5.1.

Fuzzy merging. A proximity value F satisfies fuzzy merging (without restrictions) if given a link ij with $\rho(ij) = t$ then

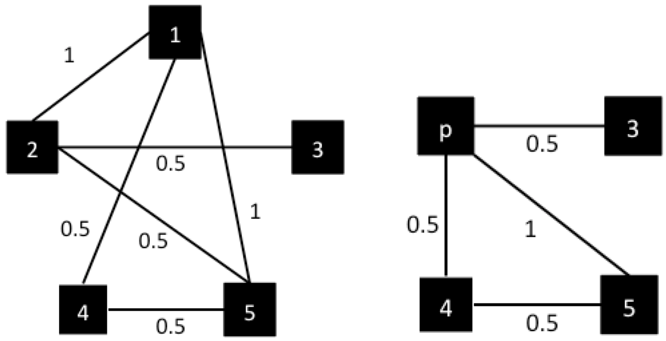


Figure 7.1: Proximity relations $\rho_0^{0.4}$ (left) and $(\rho_0^{0.4})_1^{12}$ (right)

$$F_i(N,v,\rho)+F_j(N,v,\rho)=tF_p\left(N^{ij},v^{ij},\left(\rho_0^t\right)_1^{ij}\right)+(1-t)\left[F_i\left(N,v,\overline{\rho}_0^t\right)+F_j\left(N,v,\overline{\rho}_0^t\right)\right],$$

where p is the player resulting from the merger at level t.

We say fuzzy merging without restrictions because the players are merged at the maximum level of the edge, t. The axiom is different although of a similar nature to pairwise fuzzy merging. In fact, $\left(\rho_0^t\right)_1^{ij}$ is related to the fuzzy graph ρ_t^{ij} of Definition 3.20 and $\overline{\rho}_0^t$ to the fuzzy graph ρ^t of Definition 3.21.

If $t=1$ and $\rho=L$ is a cooperation structure the axiom can be reduced to the graph pairwise merging axiom because $\rho_0^1=\rho$

We present now a fuzzy version of amalgamation neutrality which divides the payoffs between the levels in which amalgamation neutrality is satisfied and the levels in which is not.

Fuzzy amalgamation neutrality. Let $i,j,l\in N$, $\rho(ij)>0$, $l\in N\backslash\{i,j\}$ and

$$r_{\{i,j,l\}}=\left(\bigvee_{\{\exists T\in N/[\rho]_t;\,i,j,l\in T\}}t\right),$$

with $\rho(ij)>r_{\{i,j,l\}}$. We say that a fuzzy communication value over a proximity relation ρ satisfies fuzzy amalgamation neutrality if

$$F_l(N,v,\rho) = \left(\rho(ij) - r_{\{i,j,l\}}\right) \, F_l\left(N^{ij}, v^{ij}, \left(\rho^{\rho(ij)}_{r_{\{i,j,l\}}}\right)^{ij}_1\right) + \left(1 + r_{\{i,j,l\}} - \rho(ij)\right) \, F_l\left(N, v, \overline{\rho}^{\rho(ij)}_{r_{\{i,j,l\}}}\right).$$

If we have $\rho = L$ a cooperation structure and $ij \in L$ then fuzzy amalgamation neutrality and amalgamation neutrality (Section 6.3) coincide.

Theorem 7.8 *The prox-Banzhaf value D is a coalitional value of Banzhaf that satisfies fuzzy merging, modified fuzzy fairness and fuzzy amalgamation neutrality.*

Proof. We will test each one of the axioms. But we first prove that it is a coalitional value of Banzhaf.

Let $\rho = \emptyset$. As we have said before $\emptyset$ is in particular a cooperation structure. Let $i \in N$,

$$\begin{aligned} D_i(N,v,\emptyset) &= \int \rho \, d\theta_i(N,v) = (1-0)\,\theta_i(N,v,[\rho]_1) = (1-0)\,\theta_i(N,v,\emptyset) \\ &= (1-0)\beta_i(N,v) = \beta_i(N,v). \end{aligned}$$

Fuzzy merging. Let $\rho(ij) = t$. We have by Definition 7.6 and Proposition 5.21

$$\begin{aligned} D_i(N,v,\rho) + D_j(N,v,\rho) &= \int \rho \, d(\theta_i(N,v) + \theta_j(N,v)) \\ &= (t-0) \int \rho^t_0 \, d(\theta_i(N,v) + \theta_j(N,v)) \\ &+ (1-t) \int \overline{\rho}^t_0 \, d(\theta_i(N,v) + \theta_j(N,v)). \end{aligned}$$

We have $\int \rho^t_0 \, d(\theta_i(N,v) + \theta_j(N,v)) = D_i\left(N,v,\rho^t_0\right) + D_j\left(N,v,\rho^t_0\right)$, by definition. It remains to prove that $D_i\left(N,v,\rho^t_0\right) + D_j\left(N,v,\rho^t_0\right) = D_p\left(N^{ij}, v^{ij}, \left(\rho^t_0\right)^{ij}_1\right)$.

Let $im\left(\rho^t_0\right) = \{\lambda_1 < \cdots < \lambda_s\}$, then taking into account Definition 1.38,

$$
\begin{aligned}
D_i\left(N,v,\rho_0^t\right)+D_j\left(N,v,\rho_0^t\right) &= \sum_{k=1}^{s}\left(\lambda_k-\lambda_{k-1}\right)\left[\theta_i\left(N,v,\left[\rho_0^t\right]_{\lambda_k}\right)+\theta_j\left(N,v,\left[\rho_0^t\right]_{\lambda_k}\right)\right] \\
&= \sum_{k=1}^{s}\left(\lambda_k-\lambda_{k-1}\right)\theta_p\left(N^{ij},v^{ij},\left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}\right),
\end{aligned}
$$

where the second equality comes from the graph pairwise merging axiom. Besides,

$$
im\left(\rho_0^t\right)_1^{ij}\subseteq im\left(\rho_0^t\right).
$$

If we have $im\left(\rho_0^t\right)_1^{ij}=im\left(\rho_0^t\right)$ and we prove $\left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}=\left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_k}$, it is done. Suppose without loss of generality $\left|im\left(\rho_0^t\right)_1^{ij}\right|=\left|im\left(\rho_0^t\right)\right|-1$, i.e., $\exists\lambda_m\notin im\left(\rho_0^t\right)_1^{ij}$ but $\lambda_m\in im\left(\rho_0^t\right)$ and the rest of elements are the same:

$$
im\left(\rho_0^t\right)=\{\lambda_1<\lambda_2<\cdots<\lambda_m<\cdots<\lambda_s\} \text{ and } im\left(\rho_0^t\right)_1^{ij}=\{\lambda_1<\lambda_2\overset{m}{<\curlyvee}<\cdots<\lambda_s\}.
$$

That occurs due to the definition of $\left(\rho_0^t\right)_1^{ij}$. Then

$$
\begin{aligned}
\sum_{k=1}^{s}\left(\lambda_k-\lambda_{k-1}\right)\theta_p\left(N^{ij},v^{ij},\left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}\right) &= \sum_{k=1}^{m-1}\left(\lambda_k-\lambda_{k-1}\right)\theta_p\left(N^{ij},v^{ij},\left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}\right) \\
+\left(\lambda_m-\lambda_{m-1}\right)\theta_p\left(N^{ij},v^{ij},\left(\left[\rho_0^t\right]_{\lambda_m}\right)^{ij}\right) &+ \left(\lambda_{m+1}-\lambda_m\right)\theta_p\left(N^{ij},v^{ij},\left(\left[\rho_0^t\right]_{\lambda_{m+1}}\right)^{ij}\right) \\
+\sum_{k=m+2}^{s}\left(\lambda_k-\lambda_{k-1}\right)\theta_p\left(N^{ij},v^{ij},\left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}\right).
\end{aligned}
$$

On the other hand,

$$\begin{aligned} D_p\left(N^{ij}, v^{ij}, \left(\rho_0^t\right)_1^{ij}\right) &= \sum_{k=1}^{m-1}(\lambda_k - \lambda_{k-1})\theta_p\left(N^{ij}, v^{ij}, \left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_k}\right) \\ &+ \sum_{k=m+2}^{s}(\lambda_k - \lambda_{k-1})\theta_p\left(N^{ij}, v^{ij}, \left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_k}\right) \\ &+ (\lambda_{m+1} - \lambda_{m-1})\theta_p\left(N^{ij}, v^{ij}, \left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_{m+1}}\right). \end{aligned}$$

Let us see now $\left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_k} = \left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}, \ \forall k \neq m.$

We have $vz \in \left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_k}$ if and only if $\left(\rho_0^t\right)_1^{ij}(vz) \geq \lambda_k$, but following Definition 3.20,

$$\left(\rho_0^t\right)_1^{ij}(vz) = \begin{cases} \rho_0^t(vz), & \text{if } v, z \in N\backslash\{i,j\} \\ \rho_0^t(iv) \vee \rho_0^t(jv), & \text{if } z = p \\ \rho_0^t(iz) \vee \rho_0^t(jz), & \text{if } v = p. \end{cases}$$

If we are in the first case $\left(\rho_0^t\right)_1^{ij}(vz) = \rho_0^t(vz) \geq \lambda_k$ if and only if $vz \in \left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}$. If we are in the second case $z = p$, $\left(\rho_0^t\right)_1^{ij}(vp) = \rho_0^t(iv) \vee \rho_0^t(jv) \geq \lambda_k$, since $vp \in \left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}$ if and only if $vi \in \left[\rho_0^t\right]_{\lambda_k}$ or $vj \in \left[\rho_0^t\right]_{\lambda_k}$. The third case is similar.

Then

$$\sum_{k=1}^{m-1}(\lambda_k - \lambda_{k-1})\,\theta_p\left(N^{ij}, v^{ij}, \left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}\right) = \sum_{k=1}^{m-1}(\lambda_k - \lambda_{k-1})\theta_p\left(N^{ij}, v^{ij}, \left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_k}\right)$$

and

$$\sum_{k=m+2}^{s}(\lambda_k - \lambda_{k-1})\,\theta_p\left(N^{ij}, v^{ij}, \left(\left[\rho_0^t\right]_{\lambda_k}\right)^{ij}\right) = \sum_{k=m+2}^{s}(\lambda_k - \lambda_{k-1})\theta_p\left(N^{ij}, v^{ij}, \left[\left(\rho_0^t\right)_1^{ij}\right]_{\lambda_k}\right).$$

It remains to prove that

$$(\lambda_m - \lambda_{m-1})\,\theta_p\left(N^{ij}, v^{ij}, \left([\rho_0^t]_{\lambda_m}\right)^{ij}\right) + (\lambda_{m+1} - \lambda_m)\theta_p\left(N^{ij}, v^{ij}, \left([\rho_0^t]_{\lambda_{m+1}}\right)^{ij}\right)$$

$$= (\lambda_{m+1} - \lambda_{m-1})\theta_p\left(N^{ij}, v^{ij}, \left[(\rho_0^t)_1^{ij}\right]_{\lambda_{m+1}}\right).$$

But $\left([\rho_0^t]_{\lambda_m}\right)^{ij} = \left([\rho_0^t]_{\lambda_{m+1}}\right)^{ij}$, because λ_m is a level in ρ_0^t that is not present in $(\rho_0^t)_1^{ij}$. It has been lost because it only appeared once in ρ_0^t and it is equal to $\rho_0^t(ih)$ or $\rho_0^t(jh)$ with $h \neq i, j$; let us suppose that the minimum is $\rho_0^t(ih)$, then $\rho_0^t(jh) \geq \lambda_{m+1}$ and the edge ph remains the same in both cases, and so does the rest. Then, we have

$$(\lambda_m - \lambda_{m-1})\,\theta_p\left(N^{ij}, v^{ij}, \left([\rho_0^t]_{\lambda_m}\right)^{ij}\right) + (\lambda_{m+1} - \lambda_m)\theta_p\left(N^{ij}, v^{ij}, \left([\rho_0^t]_{\lambda_{m+1}}\right)^{ij}\right)$$

$$= (\lambda_m - \lambda_{m-1} + \lambda_{m+1} - \lambda_m)\theta_p\left(N^{ij}, v^{ij}, \left([\rho_0^t]_{\lambda_{m+1}}\right)^{ij}\right)$$

$$= (\lambda_{m+1} - \lambda_{m-1})\theta_p\left(N^{ij}, v^{ij}, \left([\rho_0^t]_{\lambda_{m+1}}\right)^{ij}\right),$$

and the axiom is satisfied because we have proved before $\left([\rho_0^t]_{\lambda_{m+1}}\right)^{ij} = \left[(\rho_0^t)_1^{ij}\right]_{\lambda_{m+1}}$.

Modified fuzzy fairness. The proof is the same as in the prox-Owen value because θ also satisfies modified fairness.

Fuzzy amalgamation neutrality. Let $i, j, l \in N$, $\rho(ij) > 0$ and $l \in N\backslash\{i, j\}$, $r_{\{i,j,l\}} < \rho(ij)$.
We have

$$\begin{aligned} D_l(N, v, \rho) &= \left(\rho(ij) - r_{\{i,j,l\}}\right) \int \rho^{\rho(ij)}_{r_{\{i,j,l\}}}\, d\theta_l(N, v) + \left(1 + r_{\{i,j,l\}} - \rho(ij)\right) \int \overline{\rho}^{\rho(ij)}_{r_{\{i,j,l\}}}\, d\theta_l(N, v) \\ &= \left(\rho(ij) - r_{\{i,j,l\}}\right) D_l\left(N, v, \rho^{\rho(ij)}_{r_{\{i,j,l\}}}\right) + \left(1 + r_{\{i,j,l\}} - \rho(ij)\right) \int \overline{\rho}^{\rho(ij)}_{r_{\{i,j,l\}}}\, d\theta_l(N, v). \end{aligned}$$

We have to prove that $D_l\left(N, v, \rho^{\rho(ij)}_{r_{\{i,j,l\}}}\right) = D_l\left(N^{ij}, v^{ij}, \left(\rho^{\rho(ij)}_{r_{\{i,j,l\}}}\right)_1^{ij}\right)$.

As we saw before the images of $\rho_{r_{\{i,j,l\}}}^{\rho(ij)}$ and $\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)_1^{ij}$ are the same except for the worst case in which the image of $\rho_{r_{\{i,j,l\}}}^{\rho(ij)}$ has one more element, λ_m.

$$D_l\left(N,v,\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)=\sum_{k=1}^{s}(\lambda_k-\lambda_{k-1})\,\theta_l\left(N,v,\left[\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right]_{\lambda_k}\right)$$

$$=\sum_{k=1}^{m-1}(\lambda_k-\lambda_{k-1})\,\theta_l\left(N,v,\left[\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right]_{\lambda_k}\right)+(\lambda_m-\lambda_{m-1})\theta_l\left(N,v,\left[\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right]_{\lambda_m}\right)$$

$$+(\lambda_{m+1}-\lambda_m)\theta_l\left(N,v,\left[\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right]_{\lambda_{m+1}}\right)$$

$$+\sum_{k=m+2}^{s}(\lambda_k-\lambda_{k-1})\theta_l\left(N,v,\left[\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right]_{\lambda_k}\right).$$

On the other hand

$$D_l\left(N^{ij},v^{ij},\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)_1^{ij}\right)=\sum_{k=1}^{m-1}(\lambda_k-\lambda_{k-1})\theta_l\left(N^{ij},v^{ij},\left[\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)_1^{ij}\right]_{\lambda_k}\right)$$

$$+(\lambda_{m+1}-\lambda_{m-1})\theta_l\left(N^{ij},v^{ij},\left[\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)_1^{ij}\right]_{\lambda_{m+1}}\right)$$

$$+\sum_{k=m+2}^{s}(\lambda_k-\lambda_{k-1})\theta_l\left(N^{ij},v^{ij},\left[\left(\rho_{r_{\{i,j,l\}}}^{\rho(ij)}\right)_1^{ij}\right]_{\lambda_k}\right).$$

Now we have to do the same as we did to prove fuzzy merging. Notice that in this axiom there are two summands, one part where amalgamation neutrality is satisfied (from $r_{\{i,j,l\}}$ to $\rho(ij)$) and another one where it is not. Number $r_{\{i,j,l\}}$ is the minimum between the level of the link ij and the maximum level t where i,j,l are connected in $[\rho]_t$. So for $t\in\left(r_{\{i,j,l\}},\rho(ij)\right)$, ij is a link so we can merge both players, but l is disconnected from its component. □

Theorem 7.9 *The prox-Banzhaf value D is the only coalitional value of Banzhaf for games with* *a proximity relation that satisfies fuzzy merging, modified fuzzy fairness and fuzzy amalgamation neutrality.*

Proof. Let F^1, F^2 be two different values for games with a proximity relation over N that satisfy all the axioms. We will use induction on $|im(\rho)|$ and on the number of players. Let (N, v, ρ). If $|im(\rho)| = 1$ then $im(\rho) = \{1\}$ and ρ is a communication structure. In this case we have the uniqueness by Theorem 6.17 because the fuzzy axioms are a generalization of the crisp ones.
Let (N, v, ρ) and $|im(\rho)| = k + 1$. We know, applying the induction hypothesis that for every (N', v, ρ') with $|im(\rho')| < k + 1$ or $|N'| < n = |N|$, $F^1 = F^2$.

Let $\rho(ij) > 0$,

$$\begin{aligned} F_i^1(N, v, \rho) + F_j^1(N, v, \rho) &= F_p^1\left(N^{ij}, v^{ij}, \rho^{ij}\right) = F_p^2\left(N^{ij}, v^{ij}, \rho^{ij}\right) = \\ &= F_i^2(N, v, \rho) + F_j^2(N, v, \rho), \end{aligned}$$

where the first equality comes from the fuzzy merging axiom and the second one is due to the induction hypothesis.

Besides, using modified fuzzy fairness

$$\begin{aligned} F_i^1(N, v, \rho) \quad - \quad & F_j^1(N, v, \rho) = (1-t)\left[F_i^1\left(N, v, \overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right) - F_j^1\left(N, v, \overline{\rho}^{\rho(ij)}_{\rho(ij)-t}\right)\right] \\ + \quad & t\left[F_i^1\left(N^i_{ij}, v, \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^i_{ij}}\right) - F_j^1\left(N^j_{ij}, v, \left(\left(\rho^{\rho(ij)}_{\rho(ij)-t}\right)^1_{-ij}\right)_{N^j_{ij}}\right)\right] \\ = \quad & F_i^2(N, v, \rho) - F_j^2(N, v, \rho), \end{aligned}$$

where the second equality comes from the induction hypothesis in the same way as in the prox-Owen value.

Adding up the two equalities we have that:

$$F_i^1(N,v,\rho) = F_i^2(N,v,\rho)$$

$\forall i \in N$ that belongs to an edge.

It remains to prove the case of the components that have only one element, but not when $\rho = \emptyset$ or $\rho = L$ (in this case it would be $r_{\{i,j,l\}} = 0$ and we take $ij \;/\; \rho(ij) > 0$).Then

$$F_l^1(N,v,\rho) = (\rho(ij) - 0)\, F_l^1\left(N^{ij}, v^{ij}, \left(\rho_0^{\rho(ij)}\right)_1^{ij}\right)$$

$$+(1+0-\rho(ij))\, F_l^1\left(N, v, \overline{\rho}_0^{\rho(ij)}\right) = \rho(ij)\, F_l^2\left(N^{ij}, v^{ij}, \left(\rho_0^{\rho(ij)}\right)_1^{ij}\right)$$

$$+(1-\rho(ij))\, F_l^2\left(N, v, \overline{\rho}_0^{\rho(ij)}\right) = F_l^2(N,v,\rho).$$

□

Since we know that the interval scaling and the dual interval scaling of similarity relations are similarity relations, it suffices to prove next proposition to ensure that we have the same previous theorems of existence and uniqueness of the prox-Banzhaf value over the family of games with similarity relations.

Proposition 7.10 *Let ρ be a similarity relation over N. Then the proximity relation that represents the merging at level 1 of any $ij \in link(\rho)$ with $\rho(ij) = 1$ is also a similarity relation.*

Proof. We use Definition 2.23. It is known that in $[0,1]$ it holds

$$(a \vee b) \wedge (c \vee d) = (a \wedge c) \vee (b \wedge d), \tag{7.2}$$

because $[0,1]$ is a distributive lattice. Reflexivity and symmetry are satisfied trivially. We see now that this proximity relation preserves transitivity.

Take $k, l, q \in N^{ij}$.

- If $k, l, q \neq p$ then it is straightforward.

• If $k = p$ (or $l = p$) then
$\rho_1^{ij}(pl) = \rho(il) \vee \rho(jl) \geq [\rho(iq) \wedge \rho(ql)] \vee [\rho(jq) \wedge \rho(ql)] = [\rho(iq) \vee \rho(jq)] \wedge \rho(ql) = \rho_1^{ij}(pq) \wedge \rho_1^{ij}(ql)$, where the equality is by (7.2).

• If $q = p$ then
$\rho_1^{ij}(kl) = \rho(kl) \geq \rho(ki) \wedge \rho(il)$ and also $\rho_1^{ij}(kl) = \rho(kl) \geq \rho(kj) \wedge \rho(jl)$, by definition.
Now $\rho_1^{ij}(kl) \geq [\rho(ki) \wedge \rho(il)] \vee [\rho(kj) \wedge \rho(jl)] = [\rho(ki) \vee \rho(kj)] \wedge [\rho(il) \vee \rho(jl)] = \rho_1^{ij}(kp) \wedge \rho_1^{ij}(pl)$, where the first equality is again by (7.2). □

As a consequence, it holds

Theorem 7.11 *The prox-Banzhaf value D is the only coalitional value of Banzhaf for games with a similarity relation among the players that satisfies fuzzy merging, substitutable players in a group and fuzzy amalgamation neutrality.*

7.3 The prox-Banzhaf-Myerson value

This new fuzzy value allocates the benefits in a game with a proximity relation using the Banzhaf-Myerson value.

Definition 7.12 *Let ρ be a proximity relation over N, and δ the Banzhaf-Myerson value. We define the prox-Banzhaf-Myerson value for each $i \in N$ by*

$$Z_i(N, v, \rho) = \int \rho \, d\delta_i(N, v).$$

Remember that each proximity relation can be identified with a fuzzy graph of the form $\left(e^N, \rho\right)$ that we also denote by ρ. We say that ρ is connected if $\exists t \in (0, 1]$ such that $[\rho]_t$ is connected. In that case

$$t^\rho = \bigvee \{t \in [0, 1] : [\rho]_t \text{ connected}\} \tag{7.3}$$

is called connection level of ρ.

We are going to see some axioms for Z that are a fuzzy extension of the axioms already presented for δ.

Fuzzy connected efficiency. A proximity value F satisfies fuzzy connected efficiency if $\forall (N, v, \rho) \in \mathcal{G}_{prox}$ with ρ connected it holds

$$\sum_{i \in N} F_i(N, v, \rho) - (1 - t^{\rho}) F_i\left(N, v, \rho^1_{t^{\rho}}\right) = t^{\rho} v(N).$$

If $|im(\rho)| = 1$ and ρ is connected then $t^{\rho} = 1$ and the axiom reduces to connected efficiency.

Let $(N, v, \rho) \in \mathcal{G}_{prox}$. If $t \in [0, 1 - \rho(ij)]$ with $i, j \in N$ then in a similar way to Definition 2.23 we can introduce the proximity relation ρ^t_{+ij}, where

$$\rho^t_{+ij}(kl) = \begin{cases} \rho(kl), & \text{if } kl \neq ij \\ \rho(ij) + t, & \text{if } kl = ij. \end{cases}$$

Then the fuzzy extension of component merging is constructed using this proximity relation.

Group merging. A proximity value F satisfies group merging if for every pair of leveled groups S, T and each pair $i \in S, j \in T$ it holds

$$\begin{aligned} \sum_{k \in S \cup T} F_k(N, v, \rho) \quad &- \quad \left(1 + t_{ST} - t^{ST}\right) F_k\left(N, v, \overline{\rho}^{t^{ST}}_{t_{ST}}\right) = \\ &= \sum_{k \in S \cup T} F_k\left(N, v, \rho^{t^{ST} - \rho(ij)}_{+ij}\right) - \left(1 + t_{ST} - t^{ST}\right) F_k\left(N, v, \left(\overline{\rho^{t^{ST} - \rho(ij)}_{+ij}}\right)^{t^{ST}}_{t_{ST}}\right). \end{aligned}$$

Notice that $\rho(ij) \leq t_{ST}$ by (5.6).

If $|im(\rho)| = 1$ group merging reduces to component merging.

Theorem 7.13 *The prox-Banzhaf-Myerson value Z satisfies null group, substitutable leveled groups, modified fuzzy fairness, linearity, fuzzy connected efficiency and group merging.*

Proof. The proof of Z satisfying the first four axioms is the same as in Theorem 7.3, because the Banzhaf-Myerson value satisfies these axioms in their crisp versions. It remains to prove that Z satisfies fuzzy connected efficiency and group merging.

Fuzzy connected efficiency. Let ρ be a connected proximity relation. It holds that $[\rho]_{t^{\rho}}$ is a connected graph by definition of t^ρ. Moreover $ij \in [\rho]_{t^\rho}$ if and only if $\rho(ij) \geq t^\rho$ if and only if $\rho_0^{t^\rho}(ij) \geq 1$ if and only if $ij \in \left[\rho_0^{t^\rho}\right]_1$, by (7.3). This fact means that $[\rho]_{t^\rho} = \left[\rho_0^{t^\rho}\right]_1$ as crisp graphs and therefore $\left[\rho_0^{t^\rho}\right]_1$ is connected. Then $\left[\rho_0^{t^\rho}\right]_t$ is also connected $\forall t \in (0,1]$ and using (C3) and (C4) of Proposition 1.50,

$$\sum_{i\in N} Z_i\left(N, v, \rho_0^{t^\rho}\right) = \int \rho_0^{t^\rho}\, d \sum_{i\in N} \delta_i(N,v) = v(N).$$

In the last equality we have used Theorem 6.20 to deduce $\sum_{i\in N} \delta_i(N,v)\left(\left[\rho_0^{t^\rho}\right]_t\right) = v(N)$ for each t. Then by Proposition 5.21,

$$\sum_{i\in N}\left[Z_i(N,v,\rho) - (1-t^\rho)\, Z_i\left(N,v,\rho^1_{t^\rho}\right)\right] = t^\rho \sum_{i\in N} Z_i\left(N,v,\rho_0^{t^\rho}\right) = t^\rho v(N).$$

Group merging. Let S,T be leveled groups in ρ. Observe that by Proposition 5.18 it holds

$$\sum_{k\in S\cup T}\left[Z_k(N,v,\rho) - \left(1 + t_{ST} - t^{ST}\right) Z_k\left(N,v,\overline{\rho}^{t^{ST}}_{t_{ST}}\right)\right] = \sum_{k\in S\cup T}\left(t^{ST} - t_{ST}\right) Z_k\left(N,v,\rho^{t^{ST}}_{t_{ST}}\right).$$

Again, (C3) implies that the previous expression is equivalent to

$$\left(t^{ST} - t_{ST}\right)\int \rho^{t^{ST}}_{t_{ST}}\, d \sum_{k\in S\cup T} \delta_k(N,v).$$

But we have

$$\left(\rho_{+ij}^{t^{ST}-\rho(ij)}\right)^{t^{ST}}_{t_{ST}} = \left(\rho^{t^{ST}}_{t_{ST}}\right)^1_{+ij}.$$

- If $kl \neq ij$ then it is straightforward because $\rho_{+ij}^{t^{ST}-\rho(ij)}(kl) = \rho(kl)$ and $\left(\rho^{t^{ST}}_{t_{ST}}\right)^1_{+ij}(kl) = \rho^{t^{ST}}_{t_{ST}}(kl)$.
- If $kl = ij$ then $\rho_{+ij}^{t^{ST}-\rho(ij)}(ij) = t^{ST}$ and then $\left(\rho_{+ij}^{t^{ST}-\rho(ij)}\right)^{t^{ST}}_{t_{ST}}(ij) = 1$

On the other hand, $\rho^{t^{ST}}_{t_{ST}}(ij) = 0$ because $\rho(ij) \leq t_{ST}$, so $\left(\rho^{t^{ST}}_{t_{ST}}\right)^1_{+ij}(ij) = 1$. This means that for each $t \in (0,1]$ we have

$$\left[\left(\rho_{+ij}^{t^{ST}-\rho(ij)}\right)^{t^{ST}}_{t_{ST}}\right]_t = \left[\rho^{t^{ST}}_{t_{ST}}\right]_t \cup \{ij\}.$$

Therefore we have that for each t, using that δ satisfies component merging by Theorem 6.20,

$$\sum_{k\in S\cup T}\delta_k(N,v)\left(\left[\rho_{t_{ST}}^{t^{ST}}\right]_t\right)=\sum_{k\in S\cup T}\delta_k(N,v)\left(\left[\rho_{t_{ST}}^{t^{ST}}\right]_t\cup\{ij\}\right).$$

As

$$im\left(\rho_{t_{ST}}^{t^{ST}}\right)=im\left(\left(\rho_{t_{ST}}^{t^{ST}}\right)^1_{+ij}\right)=im\left(\left(\rho_{+ij}^{t^{ST}-\rho(ij)}\right)^{t^{ST}}_{t_{ST}}\right),$$

since the only relation that differs changes the level from 0 to 1, we have that both integrals are equal

$$\int\rho_{t_{ST}}^{t^{ST}}\,d\sum_{k\in S\cup T}\delta_k(N,v)=\int\left(\rho_{+ij}^{t^{ST}-\rho(ij)}\right)^{t^{ST}}_{t_{ST}}\,d\sum_{k\in S\cup T}\delta_k(N,v).$$

□

Theorem 7.14 *There is only one proximity value that satisfies null group, substitutable leveled groups, modified fuzzy fairness, linearity, fuzzy connected efficiency and group merging.*

Proof. The existence was proven in the previous theorem. It remains to prove the uniqueness. Suppose F^1 and F^2 two proximity values satisfying the axioms of the statement. We will prove that they are equal by induction on $|im(\rho)|$. If $|im(\rho)|=1$ then ρ is a cooperation structure and since the axioms coincide with their crisp versions we have $F^1(N,v,\rho)=F^2(N,v,\rho)$. Suppose that $F^1=F^2$ if $|im(\rho)|<d$.
Let ρ be a proximity relation over N with $|im(\rho)|=d$. It is possible to repeat the reasoning of Theorem 7.4 using linearity, null group, modified fuzzy fairness and substitutable leveled groups. Consequently it suffices to prove the uniqueness for a unanimity game $u_T, T\neq\emptyset$. If we define

$$M_T=\{S\in N/[\rho]_1 : S\cap T\neq\emptyset\},$$

it holds that for every $i\in S\in N/[\rho]_1$ with $S\notin M_T$ both values are equal, i.e., $F_i^1(N,u_T,\rho)=F_i^2(N,u_T,\rho),\ \forall i\in S$. Moreover, there exists $H\in\mathbb{R}$ with

$$\sum_{i\in S}F_i^1(N,u_T,\rho)-F_i^2(N,u_T,\rho)=H,\ \forall S\in M_T.$$

Suppose that ρ is connected; $N/[\rho]_1$ is a partition of N. We have by fuzzy connected efficiency

$$\begin{aligned}\sum_{i\in N} F_i^1(N,u_T,\rho) &- F_i^2(N,u_T,\rho)\\ =|M_T|H &= t^\rho v(N)+(1-t^\rho)\sum_{i\in N}F_i^1\left(N,u_T,\rho^1_{t^\rho}\right)\\ -t^\rho v(N) &- (1-t^\rho)\sum_{i\in N}F_i^2\left(N,u_T,\rho^1_{t^\rho}\right)=0,\end{aligned}$$

because $\left|im\left(\rho^1_{t^\rho}\right)\right|<d.$

If ρ is not connected then $\exists S,S'\in N/[\rho]_1$ with $S\neq S'$. Suppose $S\in M_T$. If $S'\notin M_T$ then $t^{SS'}=1$ and we apply group merging with $i\in S, j\in S'$,

$$\begin{aligned}H &= \sum_{i\in S}F_i^1(N,u_T,\rho)-F_i^2(N,u_T,\rho)=\sum_{i\in S\cup S'}F_i^1\left(N,u_T,\rho_{+ij}^{1-\rho(ij)}\right)-F_i^2\left(N,u_T,\rho_{+ij}^{1-\rho(ij)}\right)\\ &+ t_{SS'}\left[\sum_{i\in S\cup S'}F_i^1\left(N,u_T,\overline{\rho}^1_{t_{SS'}}\right)-F_i^2\left(N,u_T,\overline{\rho}^1_{t_{SS'}}\right)\right]\\ &- t_{SS'}\left[\sum_{i\in S\cup S'}F_i^1\left(N,u_T,\left(\rho_{+ij}^{1-\rho(ij)}\right)^1_{t_{SS'}}\right)-F_i^2\left(N,u_T,\left(\rho_{+ij}^{1-\rho(ij)}\right)^1_{t_{SS'}}\right)\right]=0,\end{aligned}$$

since $t_{SS'}<1$ and $\rho(ij)<1$, all the proximity relations above different from ρ have a smaller image.

If $S'\in M_T$ then $t^{SS'}=1$ but now

$$\begin{aligned}2H &= \sum_{i\in S\cup S'}F_i^1(N,u_T,\rho)-F_i^2(N,u_T,\rho)=\sum_{i\in S\cup S'}F_i^1\left(N,u_T,\rho_{+ij}^{1-\rho(ij)}\right)-F_i^2\left(N,u_T,\rho_{+ij}^{1-\rho(ij)}\right)\\ &+ t_{SS'}\left[\sum_{i\in S\cup S'}F_i^1\left(N,u_T,\overline{\rho}^1_{t_{SS'}}\right)-F_i^2\left(N,u_T,\overline{\rho}^1_{t_{SS'}}\right)\right]\\ &- t_{SS'}\left[\sum_{i\in S\cup S'}F_i^1\left(N,u_T,\left(\rho_{+ij}^{1-\rho(ij)}\right)^1_{t_{SS'}}\right)-F_i^2\left(N,u_T,\left(\rho_{+ij}^{1-\rho(ij)}\right)^1_{t_{SS'}}\right)\right]=0.\square\end{aligned}$$

Application: The power of the political groups in the European Parliament

In this appendix we illustrate the calculation of the values studied in this work. We have used Wolfram Language Mathematica to implement the algorithms, in particular the packages Cooperat (see Carter [16]) and Combinatorica (see Skiena [67]).

The EP game

The Treaties of Maastricht (1992) and Lisbon (2009) regulate the functions of the European Parliament in a context of the co-decision procedure with the Council of the European Union. The European Parliament pretends to be the ideologic representation of the european citizens, but currently the channel of voting is the set of national political parties in each member state. Hence, the relations among these groups are partial because of the national interests. The European Parliament is organized in political groups depending on the ideologic feeling. The different political parties of the member countries present a list of candidates in their own countries and later they assume the membership to a specific group in the chamber. Therefore, the behavior of a group is not homogeneous because it is made conditional on the countries relationships. A group needs to verify two conditions: it must contain at least twenty five seats and it must represent at least one-quarter of the member countries. Those members of the chamber who do not belong to any political group are known as non-attached members.

In the seventh legislature there were seven political groups in the European Parliament plus the non-attached seats. So, we consider in our example the following groups corresponding to 2012:

1. European People's Party (Christian Democrats), 265 members.

2. Progressive Alliance of Socialists and Democrats, 183 members.

3. Alliance of Liberals and Democrats for Europe, 84 members.

4. European Conservatives and Reformists, 55 members.

5. Greens/European Free Alliance, 55 members.

6. European United Left - Nordic Green Left, 35 members.

7. Europe of Freedom and Democracy, 29 members.

8. Non-attached Members, 29 members.

We consider the game of the political representation of the groups in the European Parliament in 2012 with 735 seats and a quota of 368. The corresponding weighted voting game over $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, called the *EP-game*, is represented by $v(S) = 1$ if the sum of the number of seats of the groups in S is greater or equal to 368, and $v(S) = 0$ otherwise.

After the Lisbon Treaty the European Parliament has more power and responsibility. Usually, in the national Parliaments the power depends on the "ideological" relationships among the groups. However, in the European Parliament this power is difussed by the structure of the groups, consisting of heterogeneous groups of deputies of various nationalities. The "national" factor favors a representation as a fuzzy communication structure while the "ideological" factor favors a representation as a proximity relation. We can summarize the bilateral relations among the different groups and the degree of cohesion of every group using two fuzzy graphs. We take the fuzzy graphs "ad hoc" to describe the calculation procedure, of course these pictures depend on each moment.

Calculating cg-values

In order to determine the "national" component of the power we use the fuzzy graph $\gamma = (\tau, \rho)$ over N in Figure 1. In this case $\tau(i)$ is interpreted as the membership capacity of the groups the voting day. Number $\rho(ij)$ means the maximal level of agreement that groups i and j can reach in this voting. So, $\rho(ij) = 1$ if all members of both groups are willing to cooperate in a coalition.

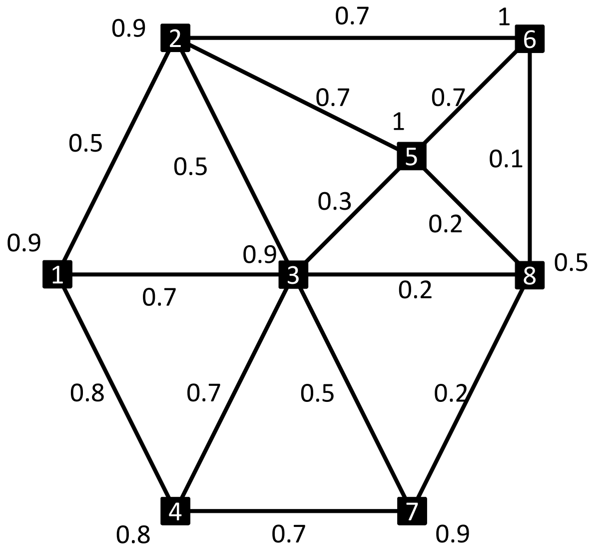


Figure 1. EP fuzzy communication structure

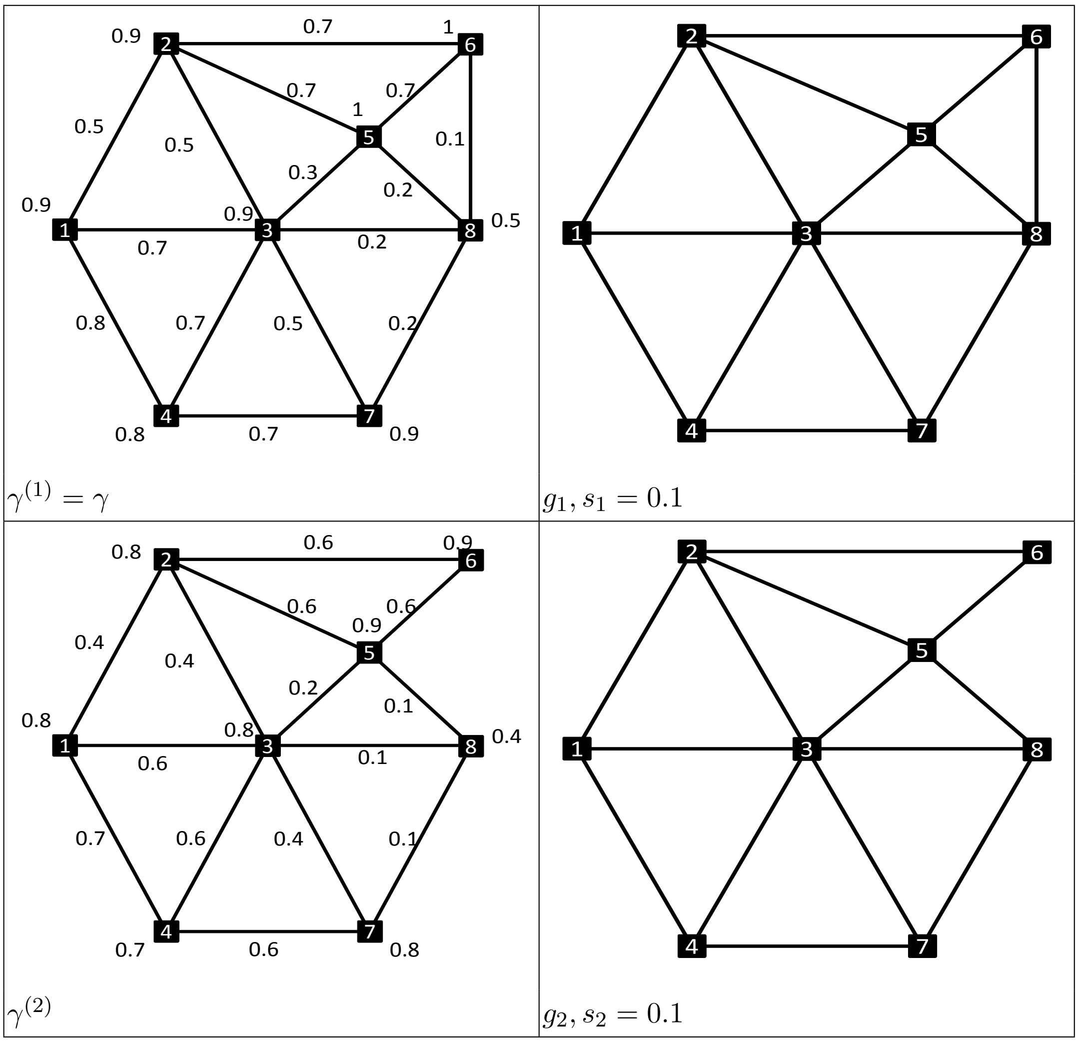
If $\gamma = (\tau, \rho) \in FCS^N$ is a fuzzy communication structure, we can store the fuzzy set of vertices and the fuzzy set of edges in an upper triangular matrix $\gamma = [\gamma(i,j)]_{N \times N}$ where $\gamma(i,i) = \tau(i)$ for every $i \in N$ and $\gamma(i,j) = \rho(ij)$ when $i < j$. We can also represent the crisp graph g^γ corresponding to the graph γ by a matrix g^γ such that $g^\gamma(i,j) = \lceil \gamma(i,j) \rceil$ for all $i, j \in N$. In our case, the matrix representation of the EP fuzzy communication structure γ is:

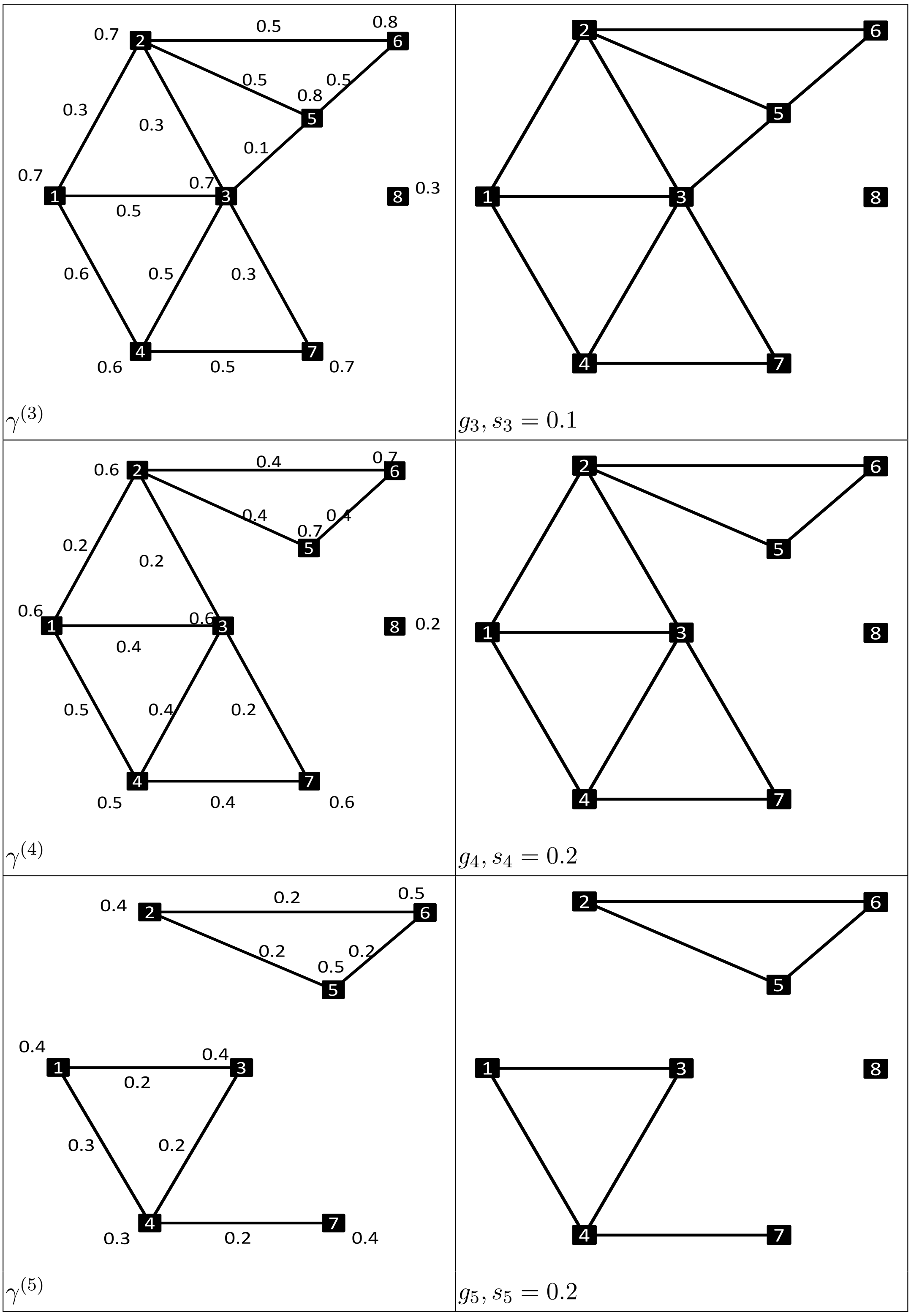
$$\gamma = \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0.5 & 0 & 0.7 & 0.7 & 0 & 0 \\ 0 & 0 & 0.9 & 0.7 & 0.3 & 0 & 0.5 & 0.2 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.7 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

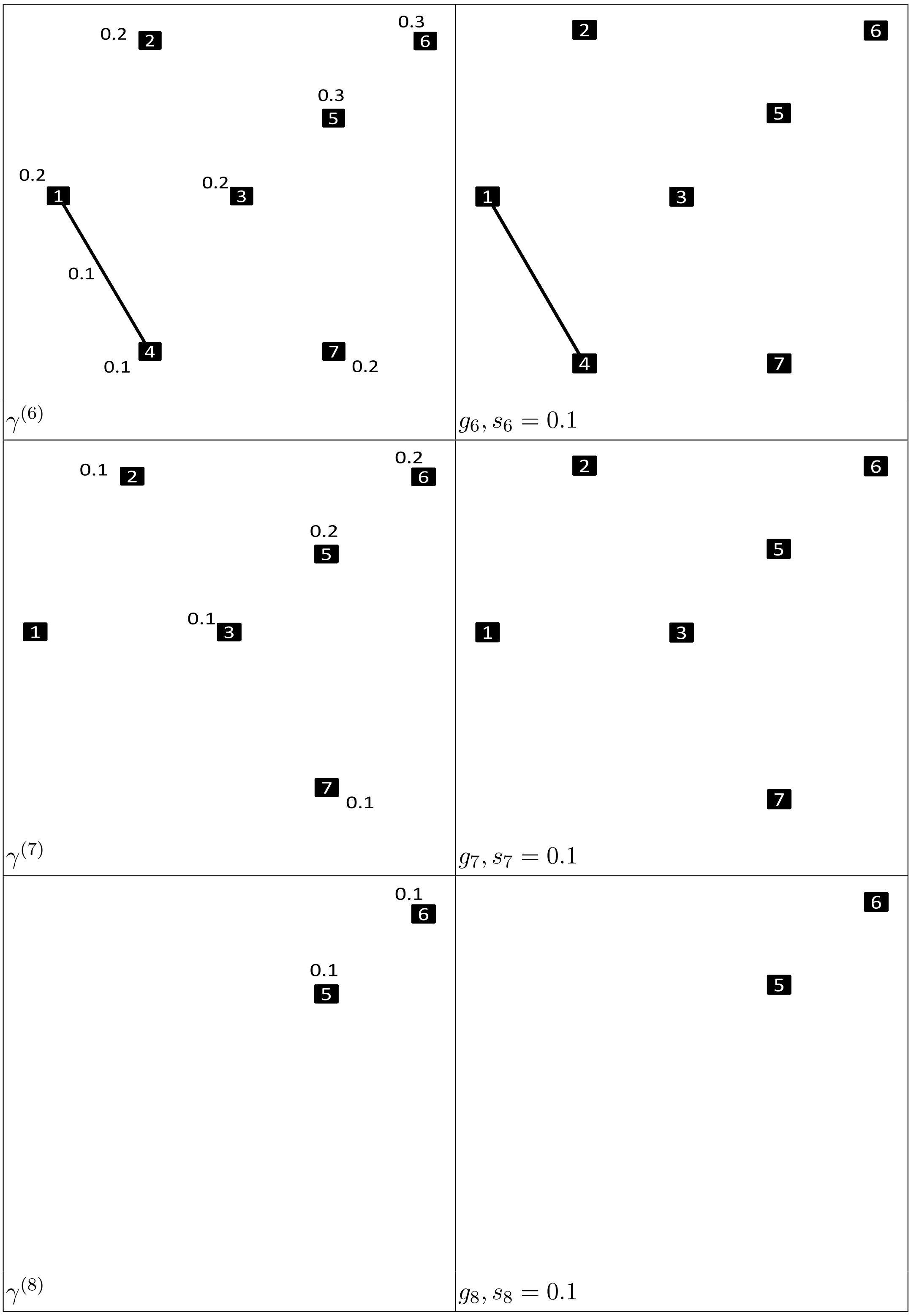
As our fuzzy graph is not a tree we do not calculate the cg-average tree value. For all the other values it is possible to use a Choquet formula: Theorem 3.10 for the cg-Myerson value, Theorem 3.19 for the cg-Banzhaf value and Lemma 4.6 for the cg-position value. The common procedure for all of them is the following.

1. We get the cg-partition of the fuzzy graph.
2. We get the crisp value for the corresponding graph in each level.
3. We calculate the cg-value using the Choquet formula.

The cg-partition of a fuzzy graph is obtained by the cg-algorithm of Section 2.3. The complexity of this algorithm depends in polinomial time on the size of the images of the vertex set and the link set (see Theorem 6 in Gallego et al. [34]). Next figure shows the cg-algorithm applied to the EP fuzzy communication structure.




$\gamma^{(3)}$
$g_3, s_3 = 0.1$
$\gamma^{(4)}$
$g_4, s_4 = 0.2$
$\gamma^{(5)}$
$g_5, s_5 = 0.2$



0.2 2
0.3 6
0.3 5
0.2 1
0.2 3
0.1
4 0.1
7 0.2
γ(6)
2
6
5
1
3
4
7
g6, s6 = 0.1
0.1 2
0.2 6
0.2 5
1
0.1 3
7 0.1
γ(7)
2
6
5
1
3
7
g7, s7 = 0.1
0.1 6
0.1 5
γ(8)
6
5
g8, s8 = 0.1

Now we have to obtain for each graph in the partition the chosen crisp value. In order to calculate directly the crisp values in a situation of communication $(N, v, g) \in \mathcal{G}_{com}$, first it is necessary to determine the characteristic function of the corresponding induced game v^g in the Myerson or graph Banzhaf values, or the game v^{Lg} in the position value. In both cases, its computational cost is very high, since to compute v^g or v^{Lg} we previously need to obtain the set of all the connected components in the induced subgraph.

Let (N, v, g) be a communication situation. A coalition $S \subseteq N$ is feasible in $g \in CS^N$ if g_S is connected. We denote as $\mathcal{F}^g$ the set of feasible coalitions in g. In [60], Owen proved that the set of unanimity games $\{u_T : T \in \mathcal{F}^g,\ T \neq \emptyset\}$ forms a basis of the family $\mathcal{G}_{com}$. As a consequence, the restricted game can be written as a linear combination of the unanimity games corresponding to the feasible coalitions in the graph g,

$$v^g = \sum_{\{T \in \mathcal{F}^g:\ T \neq \emptyset\}} \Delta_T^{v^g} u_T, \quad \text{with} \quad \Delta_T^{v^g}(\emptyset) = 0.$$

For every feasible coalition S it holds

$$v(S) = v^g(S) = \sum_{\{T \in \mathcal{F}^g : T \subseteq S\}} \Delta_T^{v^g}.$$

From this expression we can compute, for every connected coalition S, the dividends of feasible coalitions in the restricted game with the recurrent formula (Fernández [29], Gallego et al. [34]),

$$\Delta_S^{v^g} = v(S) - \sum_{\{T \in \mathcal{F}^g : T \subset S\}} \Delta_T^{v^g}.$$

Notice that it is only necessary to compute the dividends for the feasible coalitions and this provides an alternative to the proposal made by Owen [60] to compute the dividends of the restricted game (N, v^g) from the dividends of game (N, v).

Let $d = \bigvee_{T \in N/g} |T|$. We denote as t_p the number of feasible coalitions in g with cardinal p for all $p = 1, ..., d$ and then the set of these coalitions is $\mathcal{F}_p^g = \left\{S_1^p, ..., S_{t_p}^p\right\}$. Next algorithm describes the process to get the dividends of the vertex game.

Algorithm dividends $(v, g, \mathcal{F}^g)$

$\Delta_{\emptyset}^{v^g} \leftarrow 0$

for p **from** 1 **to** d

for q **from** 1 **to** t_p

$\Delta_{S_q^p}^{v^g} \leftarrow v\left(S_q^p\right) - \sum_{\left\{T \in \mathcal{F}^g : T \subset S_q^p\right\}} \Delta_T^{v^g}$

end

end

Hence we can calculate the dividends of a vertex game directly by using only the feasible coalitions. The Myerson and the graph Banzhaf value for each player $i \in N$ are:

$$\mu_i(N, v, g) = \phi_i(N, v^g) = \sum_{\{S \in \mathcal{F}^g : S \neq \emptyset, i \in S\}} \frac{\Delta_S^{v^g}}{|S|},$$

$$\eta_i(N, v, g) = \beta_i(N, v^g) = \sum_{\{S \in \mathcal{F}^g : S \neq \emptyset, i \in S\}} \frac{\Delta_S^{v^g}}{2^{|S|-1}}.$$

In the next tables we can see the Myerson values and the graph Banzhaf values (extending by zeros) of the graphs in the cg-partition of the EP fuzzy communication structure.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	0.370238	0.370238	0.385714	0.369048	0.333333	0.	0.	0.
2	0.232143	0.232143	0.252381	0.269048	0.	0.	0.	0.
3	0.175	0.175	0.202381	0.185714	0.333333	0.	0.	0.
4	0.0630952	0.0630952	0.052381	0.0690476	0.333333	0.	0.	0.
5	0.0464286	0.0464286	0.052381	0.0357143	0.	0.	0.	0.
6	0.0202381	0.0202381	0.0190476	0.0190476	0.	0.	0.	0.
7	0.0464286	0.0464286	0.0357143	0.052381	0.	0.	0.	0.
8	0.0464286	0.0464286	0.	0.	0.	0.	0.	0.

Table 1. Myerson values in the cg-partition of the EP fuzzy communication structure

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	0.632813	0.632813	0.62500	0.59375	0.25	0.	0.	0.
2	0.367188	0.367188	0.37500	0.40625	0.	0.	0.	0.
3	0.320313	0.320313	0.31250	0.28125	0.25	0.	0.	0.
4	0.117188	0.117188	0.09375	0.12500	0.25	0.	0.	0.
5	0.0859375	0.0859375	0.09375	0.06250	0.	0.	0.	0.
6	0.0390625	0.0390625	0.03125	0.03125	0.	0.	0.	0.
7	0.0859375	0.0859375	0.06250	0.09375	0.	0.	0.	0.
8	0.0859375	0.0859375	0.	0.	0.	0.	0.	0.

Table 2. Graph Banzhaf values in the cg-partition of the EP fuzzy communication structure

In the case of the position value we need the dividends of the link game to get the Shapley value of this game and then use the definition of the position value (Definition 4.2). The dual graph of a graph g is another graph Lg such that links in g are vertices in Lg and there is a link in Lg between each two adjacent links of g. We show the concept using the following example.

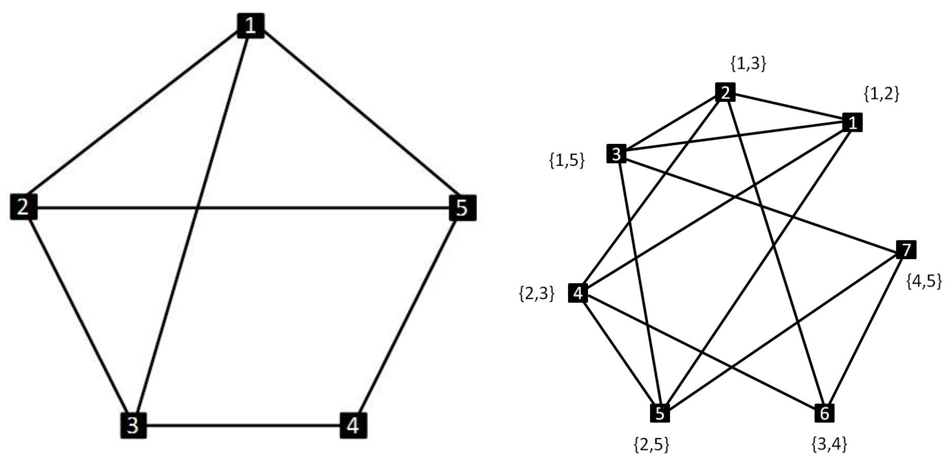


Figure 1: Graph g (left) and dual graph Lg (right)

Suppose known the family of feasible sets of links $\mathcal{F}^{Lg}$, we can obtain the dividends of the link game by a recurrence formula. The Harsanyi dividend of the link game $\left(L, v^{Lg}\right)$ for each $E \in \mathcal{F}^{Lg}$ is

$$\Delta_E^{v^{Lg}} = v\left(\bigcup_{ij \in E} \{i,j\}\right) - \sum_{\{B \in \mathcal{F}^{Lg} : B \subset E\}} \Delta_B^{v^{Lg}}, \tag{1}$$

and $\Delta_E^{v^{Lg}} = 0$ otherwise. We observe that the link game for the graph g coincides with the vertex game for the dual graph Lg. If $E \in \mathcal{F}^{Lg}$ then

$$v^{Lg}(E) = v\left(\bigcup_{ij \in E} \{i,j\}\right).$$

We apply next algorithm to determine the dividends of the link game (L, v^{Lg}). Let $g = (N, L)$, $l = |A|$ and $\{E_k^h : h = 1, ..., E(k)\}$ the set of elements in $\mathcal{F}^{Lg}$ of cardinality k with $k = 1, ..., l$.

Algorithm dividends-link $\left(L, v, \mathcal{F}^{Lg}\right)$

$\Delta_\emptyset^{v^{Lg}} \leftarrow 0$

for k **from** 1 **to** l

  for h **from** 1 **to** $E\left(k\right)$

  $\Delta_{E_k^h}^{v^{Lg}} \leftarrow v\left(\bigcup_{ij \in E_k^h} \{i,j\}\right) - \sum_{\left\{B \in \mathcal{F}^{Lg} : B \subset E_k^h\right\}} \Delta_B^{v^{Lg}}$

  end

end

The position value for each player $i \in N$ is

$$\pi_i(N, v, g) = v(\{i\}) + \frac{1}{2} \sum_{j \in N \setminus \{i\}} \phi_{ij}\left(L, v^{Lg}\right),$$

where $\phi_{ij}\left(L, v^{Lg}\right)$ is the Shapley value of the link game $\left(L, v^{Lg}\right)$ evaluated by Harsanyi dividends.

$$\phi_{ij}\left(L, v^{Lg}\right) = \sum_{\{E \in \mathcal{F}^{Lg} : ij \in E\}} \frac{\Delta_E^{v^{Lg}}}{|E|}.$$

In the next table we can see the position values of the graphs in the cg-partition of the EP fuzzy communication structure.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	0.276504	0.289677	0.325992	0.351587	0.333333	0.	0.	0.
2	0.224578	0.224306	0.231151	0.249008	0.	0.	0.	0.
3	0.225688	0.232881	0.244444	0.222222	0.333333	0.	0.	0.
4	0.0826326	0.0874056	0.104167	0.116865	0.333333	0.	0.	0.
5	0.0651779	0.0637377	0.0422619	0.0109127	0.	0.	0.	0.
6	0.0329532	0.0181444	0.0121032	0.00892857	0.	0.	0.	0.
7	0.0393828	0.0418609	0.039881	0.0404762	0.	0.	0.	0.
8	0.053083	0.0419872	0.	0.	0.	0.	0.	0.

Table 3. Position values in the cg-partition of the EP fuzzy communication structure

Using the Choquet formulas we obtain the cg-values. In each of the next tables we compare the cg-value with the corresponding communication value and the corresponding classic value.

Players	Groups	Votes	$\phi(N, v)$	$\mu(N, v, g^\gamma)$	$M(N, v, \gamma)$
1	PPE	265	0.4214290	0.370238	0.253095
2	S&D	183	0.1785710	0.232143	0.125476
3	ADLE	84	0.1309520	0.175000	0.159048
4	CRE	55	0.0738095	0.0630952	0.0983333
5	Greens-ALE	55	0.0738095	0.0464286	0.0216667
6	GUE/NGL	35	0.0404762	0.0202381	0.0097619
7	EDF	29	0.0404762	0.0464286	0.0233333
8	NI	29	0.0404762	0.0464286	0.0092857

Table 4. cg-Myerson index in the European Parliament

Players	Groups	Votes	$\beta(N, v)$	$\eta(N, v, g^{\gamma})$	$B(N, v, \gamma)$
1	PPE	265	0.734375	0.632813	0.357813
2	S&D	183	0.265625	0.367188	0.192188
3	ADLE	84	0.234375	0.320313	0.201563
4	CRE	55	0.140625	0.117188	0.107812
5	Greens-ALE	55	0.140625	0.085938	0.039063
6	GUE/NGL	35	0.078125	0.039063	0.017188
7	EDF	29	0.078125	0.085938	0.042188
8	NI	29	0.078125	0.085938	0.017188

Table 5. cg-Banzhaf index in the European Parliament

Players	Groups	Votes	$\phi(N, v)$	$\pi(N, v, g^{\gamma})$	$P(N, v, \gamma)$
1	PPE	265	0.4214290	0.276504	0.226201
2	S&D	183	0.1785710	0.224578	0.117805
3	ADLE	84	0.1309520	0.225688	0.181412
4	CRE	55	0.0738095	0.0826326	0.11746
5	Greens-ALE	55	0.0738095	0.0651779	0.0193003
6	GUE/NGL	35	0.0404762	0.0329532	0.00810578
7	EDF	29	0.0404762	0.0393828	0.0202077
8	NI	29	0.0404762	0.053083	0.00950702

Table 6. cg-Position index in the European Parliament

We can observe that the aggregation of information (communication and then fuzzy communication) modified the results. Next table and figures compare the different cg-values studied.

Players	Groups	Votes	$P(N,v,\gamma)$	$M(N,v,\gamma)$	$B(N,v,\gamma)$
1	PPE	265	0.226201	0.2530950	0.357813
2	S&D	183	0.117805	0.1254760	0.192188
3	ADLE	84	0.181412	0.1590480	0.201563
4	CRE	55	0.11746	0.0983333	0.107813
5	Greens-ALE	55	0.0193003	0.0216667	0.0390625
6	GUE/NGL	35	0.0081057	0.0097619	0.0171875
7	EDF	29	0.0202077	0.0233333	0.0421875
8	NI	29	0.00950702	0.0092857	0.0171875

Table 7. cg-indices in the European Parliament

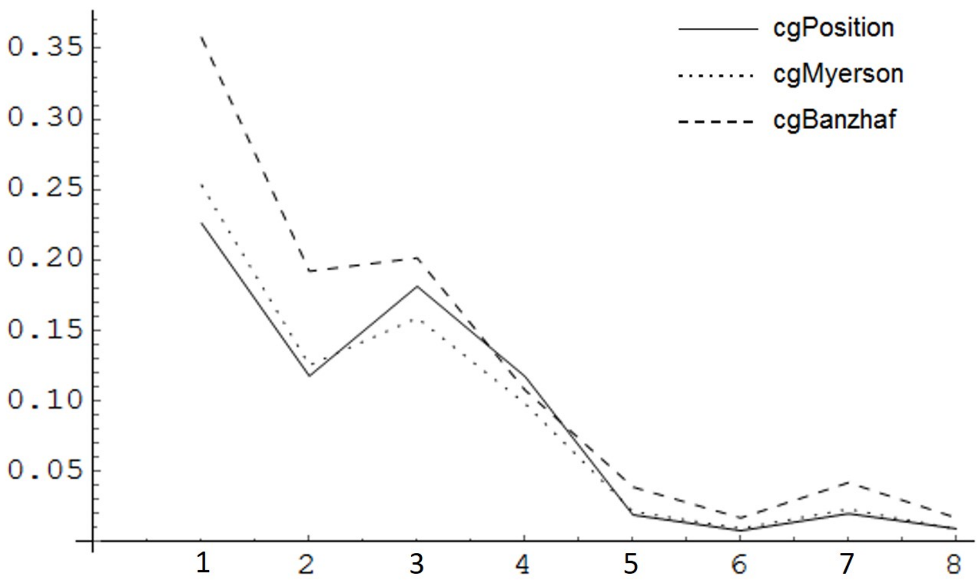


Figure 2: Comparative graph of the cg-indices of the EP-game (I)

We can see that the results of the cg-Myerson and cg-position values are different though both come from the Shapley value. The cg-position value stands out the situation of the groups in the structure more than the cg-Myerson value. We can see that in Table 7, moderate political groups (ADLE, CRE) with good relationships with the majority groups are benefited.

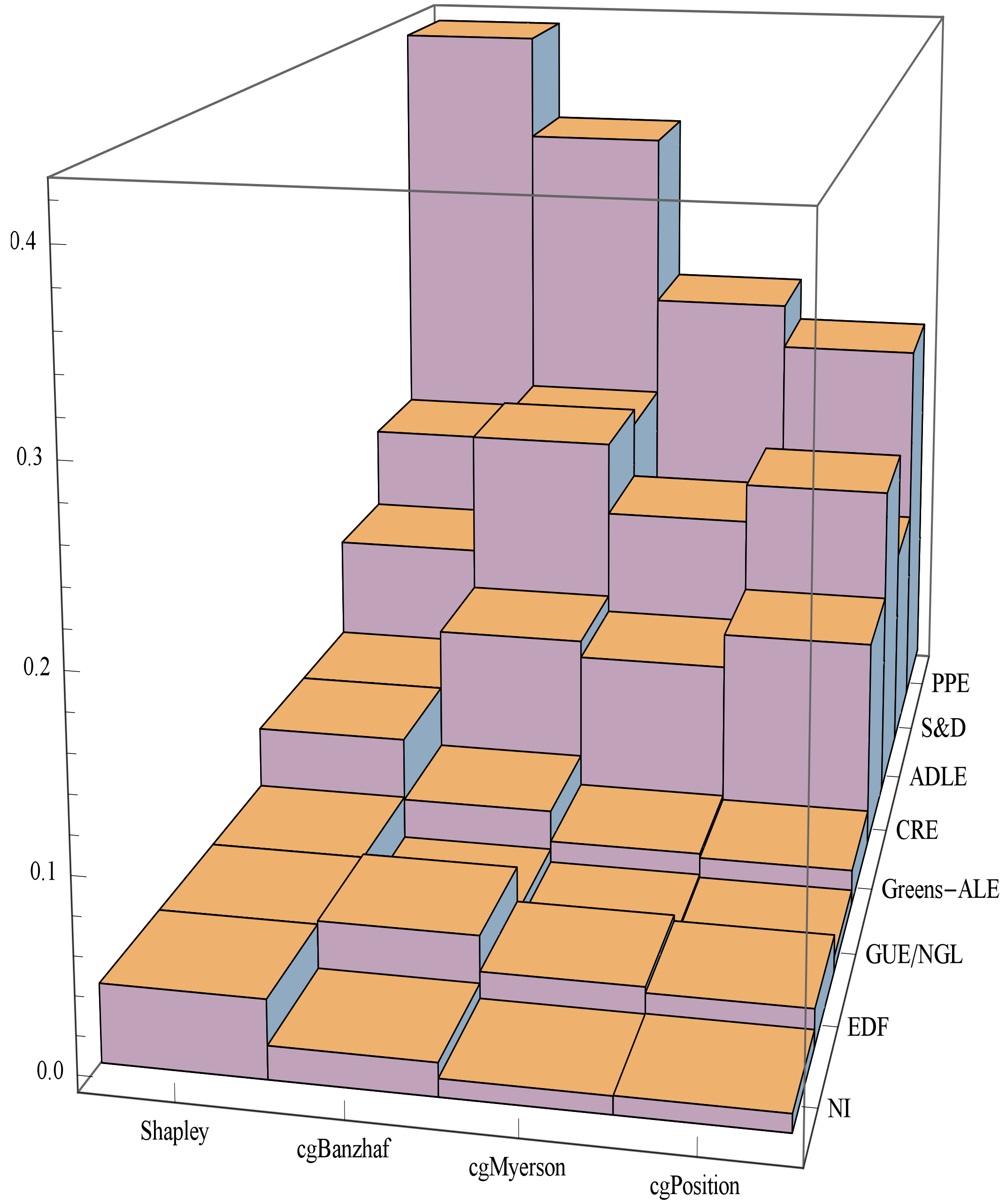


Figure 3: Comparative graph of the cg-indices of the EP-game (II)

Calculating prox-values

In order to determine the "ideological" component of the power we use the fuzzy graph of Figure 4. In this case the levels of the vertices are always $1,$ as it is expected for the ideological component of the power if the players are rational. The fuzzy graph is a proximity relation ρ over $N,$ where $\rho(ij)$ is interpreted as the total level of coincidence between groups i and j. It can be measured, for instance, by assigning a value in $[0,1]$ to each aspect of the ideology, for example, economy, immigration policies, etc., with the condition that the sum of the values of all issues considered is $1.$ Then, $\rho(ij) = 1$ if both groups have the same ideology in all issues.

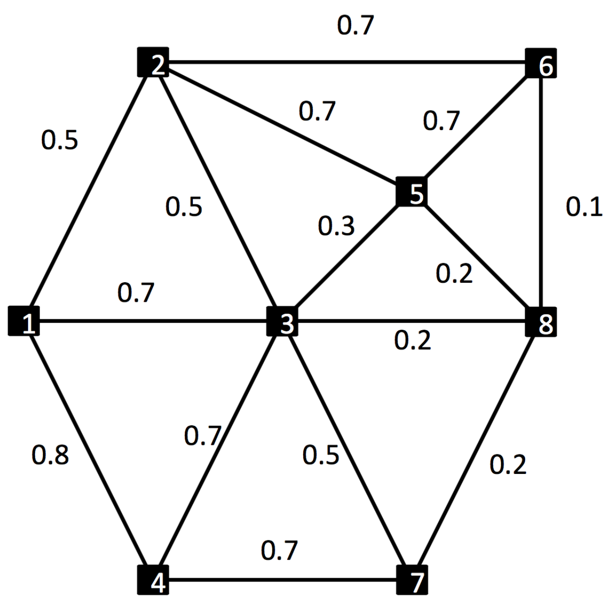


Figure 4: EP proximity relation

A proximity relation ρ is in particular a fuzzy graph, and then again we can store the fuzzy set of vertices and the fuzzy set of edges in an upper triangular matrix $\gamma = [\gamma(i,j)]_{N\times N}$ where $\gamma(i,i) = 1$ for every $i \in N$ and $\gamma(i,j) = \rho(ij)$ when $i < j$. So, in our case, the matrix representation of the EP proximity relation γ is:

$$\gamma = \begin{bmatrix} 1 & 0.5 & 0.7 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0.7 & 0.7 & 0 & 0 \\ 0 & 0 & 1 & 0.7 & 0.3 & 0 & 0.5 & 0.2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.7 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that depending on the specific situation, the numbers $\rho(ij)$ can increase or decrease in the "ideological" fuzzy graph with respect to the "national" fuzzy graph, or even remain the same. For the sake of simplicity, in the example we consider here, they remain the same.

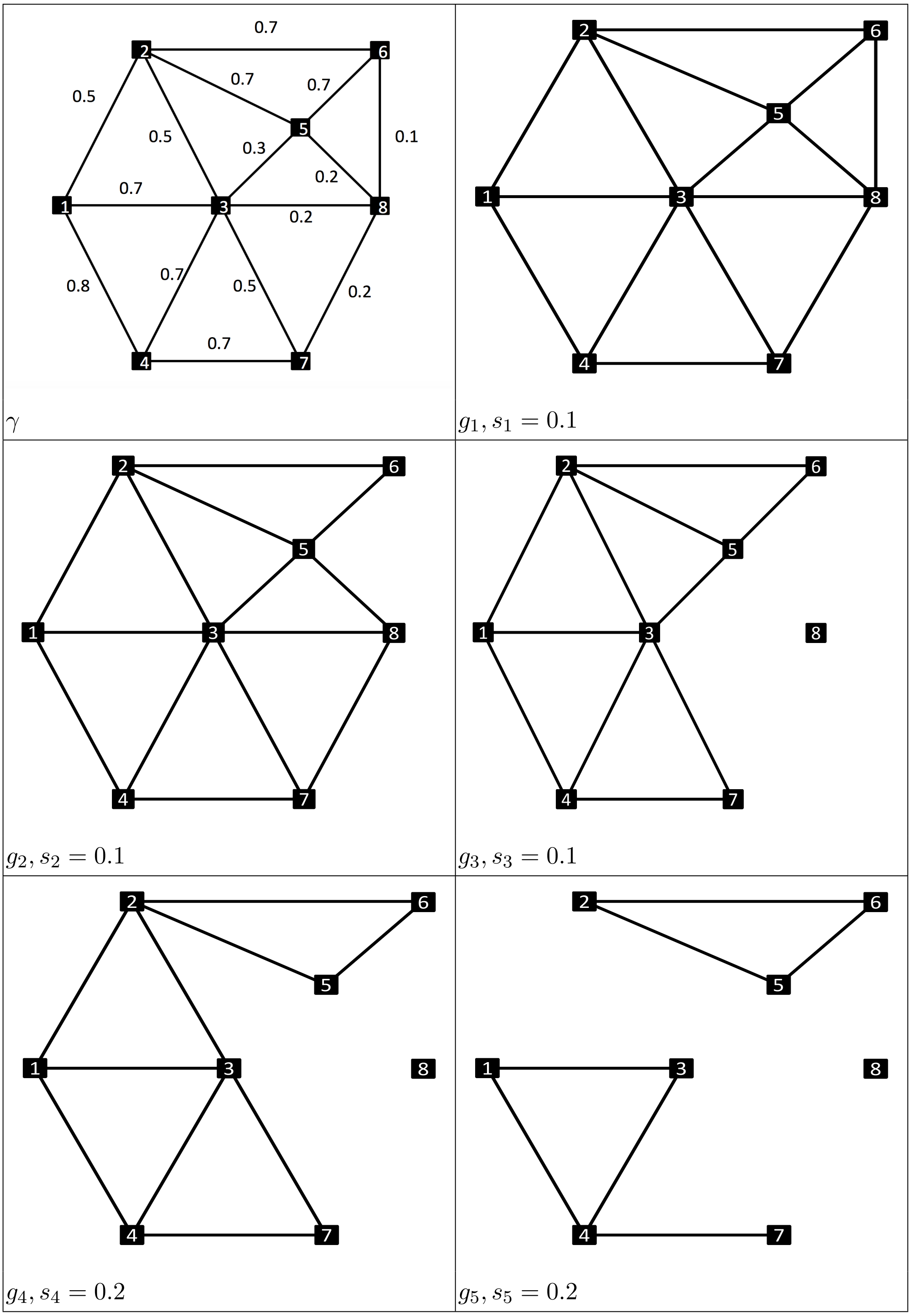
We are going to use proximity values to calculate the power indices. The proximity values studied (prox-Owen, prox-Banzhaf and prox-Banzhaf-Myerson) are defined in terms of the Choquet integral and a cooperation value: see Definitions 7.1, 7.6 and 7.12. The common procedure for all of them is the following.

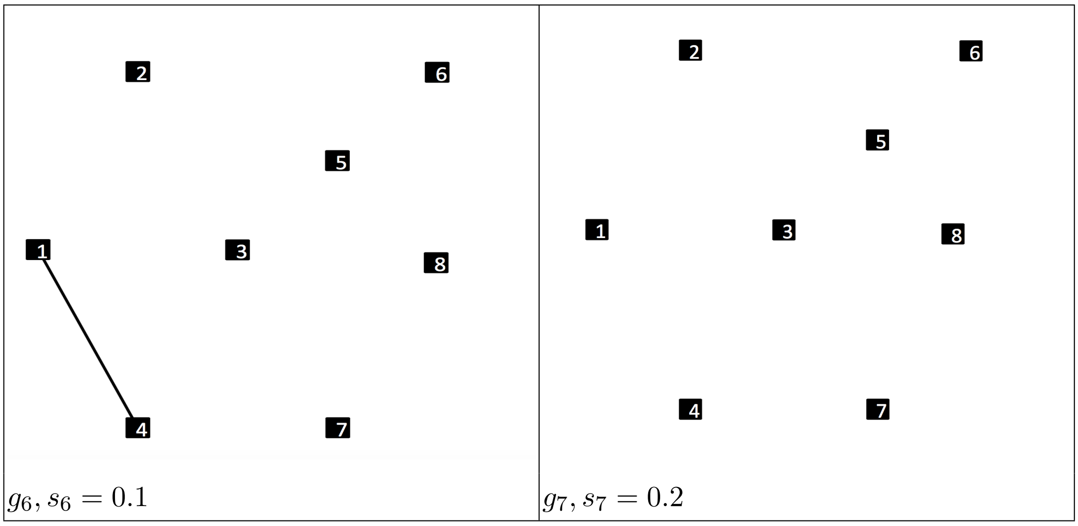
1. We get the cg-partition of the proximity relation.

2. We get the associated cooperation value for the corresponding graph in each level.

3. We calculate the prox-value using the Choquet integral.

The cg-partition of a fuzzy graph is obtained by the cg-algorithm of Section 2.3. Next figure shows the cg-algorithm applied to the EP proximity relation.


0.7
0.5
0.7
0.7
0.5
0.3
0.1
0.7
0.2
0.2
0.8
0.7
0.5
0.2
0.7
γ
$g_1, s_1 = 0.1$
$g_2, s_2 = 0.1$
$g_3, s_3 = 0.1$
$g_4, s_4 = 0.2$
$g_5, s_5 = 0.2$



Now we have to obtain for each graph in the partition the chosen cooperation value. The cooperation values (Myerson-Owen, coalitional graph Banzhaf and Banzhaf-Myerson) can be obtained by definition, namely getting first the games $v_{k(i)}$, $\forall i \in N$ (see (5.3)), and second calculating the graph Banzhaf or Myerson values using the Shapley or Banzhaf values of the graph games, but as happened in the previous section, the complexity order is too large if we compute all the characteristic functions. Even with the result in Owen [60], it involves a very large computational effort.

Then we use the algorithm of the previous section that calculates the Harsanyi dividends of the vertex games directly by using only connected coalitions and by (4) and (5) we get the Myerson and graph Banzhaf values. The only difference is that the game changes, in each step of the cg-algorithm now it is $v_{k(i)}$ instead of v. In order to compute the worth of a coalition in $v_{k(i)}$ we have to apply the Shapley or Banzhaf value to a quotient game (Definition 5.3) which in turn depends directly on that coalition, so we have a higher computational complexity. The total number of restricted games $v_{k(i)}^{L_{N_{k(i)}}}$ is $|N/g_1|+|N/g_2|+\cdots+|N/g_m|$, where m is the number of steps of the cg-partition. Despite these difficulties, we managed to obtain the prox-values of our example in a reasonable amount of time.

As an example we can see in Tables 8,9 and 10 the cooperation values of the graphs in the cg-partition of the EP proximity relation.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
1	0.370238	0.370238	0.4	0.391667	0.458333	0.459524	0.421
2	0.232143	0.232143	0.233333	0.241667	0.	0.104762	0.178
3	0.175	0.175	0.208333	0.2	0.319444	0.104762	0.130
4	0.0630952	0.0630952	0.0583333	0.0666667	0.180556	0.111905	0.073
5	0.0464286	0.0464286	0.0416667	0.0333333	0.	0.104762	0.073
6	0.0202381	0.0202381	0.0166667	0.0166667	0.	0.0380952	0.40
7	0.0464286	0.0464286	0.0416667	0.05	0.0416667	0.0380952	0.40
8	0.0464286	0.0464286	0.	0.	0.	0.0380952	0.40

Table 8. Myerson-Owen values of the graphs in the cg-partition of the EP proximity relation

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
1	0.632813	0.632813	0.640625	0.625	0.5	0.734375	0.734375
2	0.367188	0.367188	0.359375	0.375	0.	0.125	0.265625
3	0.320313	0.320313	0.328125	0.3125	0.3125	0.125	0.234375
4	0.117188	0.117188	0.109375	0.125	0.1875	0.140625	0.140625
5	0.0859375	0.859375	0.078125	0.0625	0.	0.125	0.140625
6	0.0390625	0.0390625	0.03125	0.03125	0.	0.0625	0.078125
7	0.0859375	0.0859375	0.078125	0.09375	0.0625	0.0625	0.078125
8	0.0859375	0.0859375	0.	0.	0.	0.0625	0.078125

Table 9. Coalitional graph Banzhaf values of the graphs in the cg-partition of the EP proximity relation

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
1	0.370238	0.370238	0.4	0.391667	0.5	0.734375	0.7343
2	0.232143	0.232143	0.233333	0.241667	0.	0.125	0.2656
3	0.175	0.175	0.208333	0.2	0.291667	0.125	0.2343
4	0.0630952	0.0630952	0.0583333	0.0666667	0.166667	0.140625	0.1406
5	0.0464286	0.0464286	0.0416667	0.0333333	0.	0.125	0.1406
6	0.0202381	0.0202381	0.0166667	0.0166667	0.	0.0625	0.0781
7	0.0464286	0.0464286	0.04166667	0.05	0.0416667	0.0625	0.0781
8	0.0464286	0.0464286	0.	0.	0.	0.0625	0.0781

Table 10. Banzhaf-Myerson values of the graphs in the cg-partition of the EP proximity relation

Using the Choquet integral we obtain the prox-values. In each of the next tables we compare the prox-value with the corresponding communication value and classic value.

Players	Groups	Votes	$\phi(N,v)$	$\mu(N,v,g^{\gamma})$	$W(N,v,\gamma)$
1	PPE	265	0.4214290	0.370238	0.414286
2	S&D	183	0.1785710	0.232143	0.164286
3	ADLE	84	0.1309520	0.175000	0.196389
4	CRE	55	0.0738095	0.0630952	0.0938492
5	Greens-ALE	55	0.0738095	0.0464286	0.0453571
6	GUE/NGL	35	0.0404762	0.0202381	0.0209524
7	EDF	29	0.0404762	0.0464286	0.0436905
8	NI	29	0.0404762	0.0464286	0.0211905

Table 11. Prox-Owen index in the European Parliament

Players	Groups	Votes	$\beta(N,v)$	$\eta(N,v,g^{\gamma})$	$D(N,v,\gamma)$
1	PPE	265	0.734375	0.632813	0.635938
2	S&D	183	0.265625	0.367188	0.25
3	ADLE	84	0.234375	0.320313	0.28125
4	CRE	55	0.140625	0.117188	0.139063
5	Greens-ALE	55	0.140625	0.085938	0.078125
6	GUE/NGL	35	0.078125	0.039063	0.0390625
7	EDF	29	0.078125	0.085938	0.078125
8	NI	29	0.078125	0.085938	0.0390625

Table 12. Prox-Banzhaf index in the European Parliament

Players	Groups	Votes	$\beta\,(N,v)$	$\mu(N,v,g^{\gamma})$	$Z(N,v,\gamma)$
1	PPE	265	0.734375	0.370238	0.512693
2	S&D	183	0.265625	0.232143	0.18372
3	ADLE	84	0.234375	0.175000	0.213542
4	CRE	55	0.140625	0.0630952	0.107307
5	Greens-ALE	55	0.140625	0.0464286	0.060744
6	GUE/NGL	35	0.078125	0.0202381	0.0309226
7	EDF	29	0.078125	0.0464286	0.0536607
8	NI	29	0.078125	0.0464286	0.0311607

Table 13. Prox-Banzhaf-Myerson index in the European Parliament

We can observe that the aggregation of information from the proximity relation modified the results. Next table and figures compare the different prox-values studied.

Players	Groups	Votes	$W(N,v,\gamma)$	$D(N,v,\gamma)$	$Z(N,v,\gamma)$
1	PPE	265	0.414286	0.635938	0.512693
2	S&D	183	0.164286	0.25	0.18372
3	ADLE	84	0.196389	0.28125	0.213542
4	CRE	55	0.0938492	0.139063	0.107307
5	Greens-ALE	55	0.0453571	0.078125	0.060744
6	GUE/NGL	35	0.0209524	0.0390625	0.0309226
7	EDF	29	0.0436905	0.078125	0.0536607
8	NI	29	0.0211905	0.0390625	0.0311607

Table 14. Prox-indices in the European Parliament

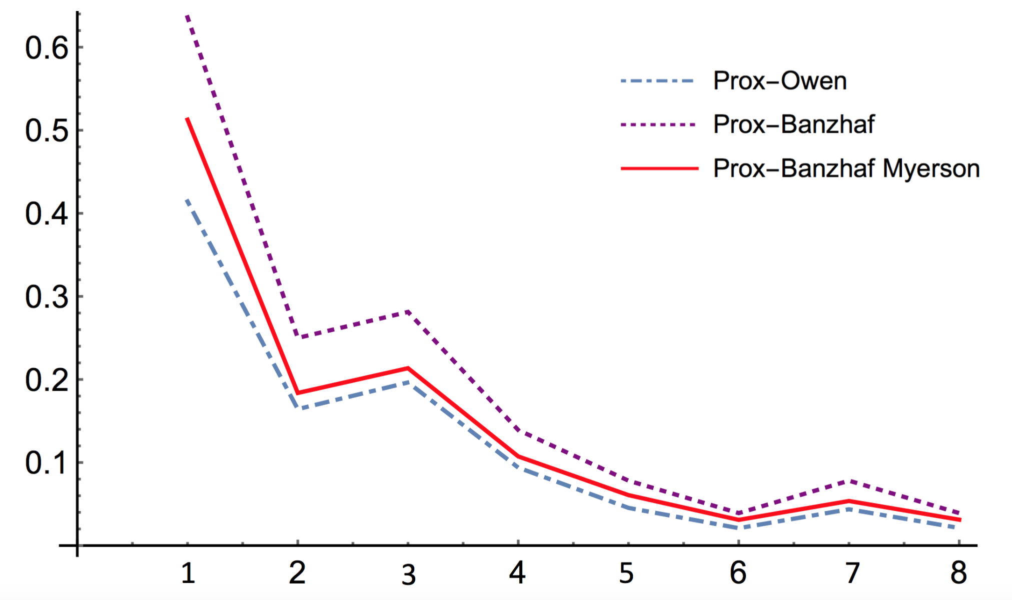


Figure 5: Comparative graph of the prox-indices of the EP-game (I)

For all players the prox-Banzhaf value is greater than the prox-Banzhaf-Myerson value and the latter is greater than the prox-Owen value. This fact is due to the definition of $v_{k(i)}$ as a Banzhaf value in the first two solutions.

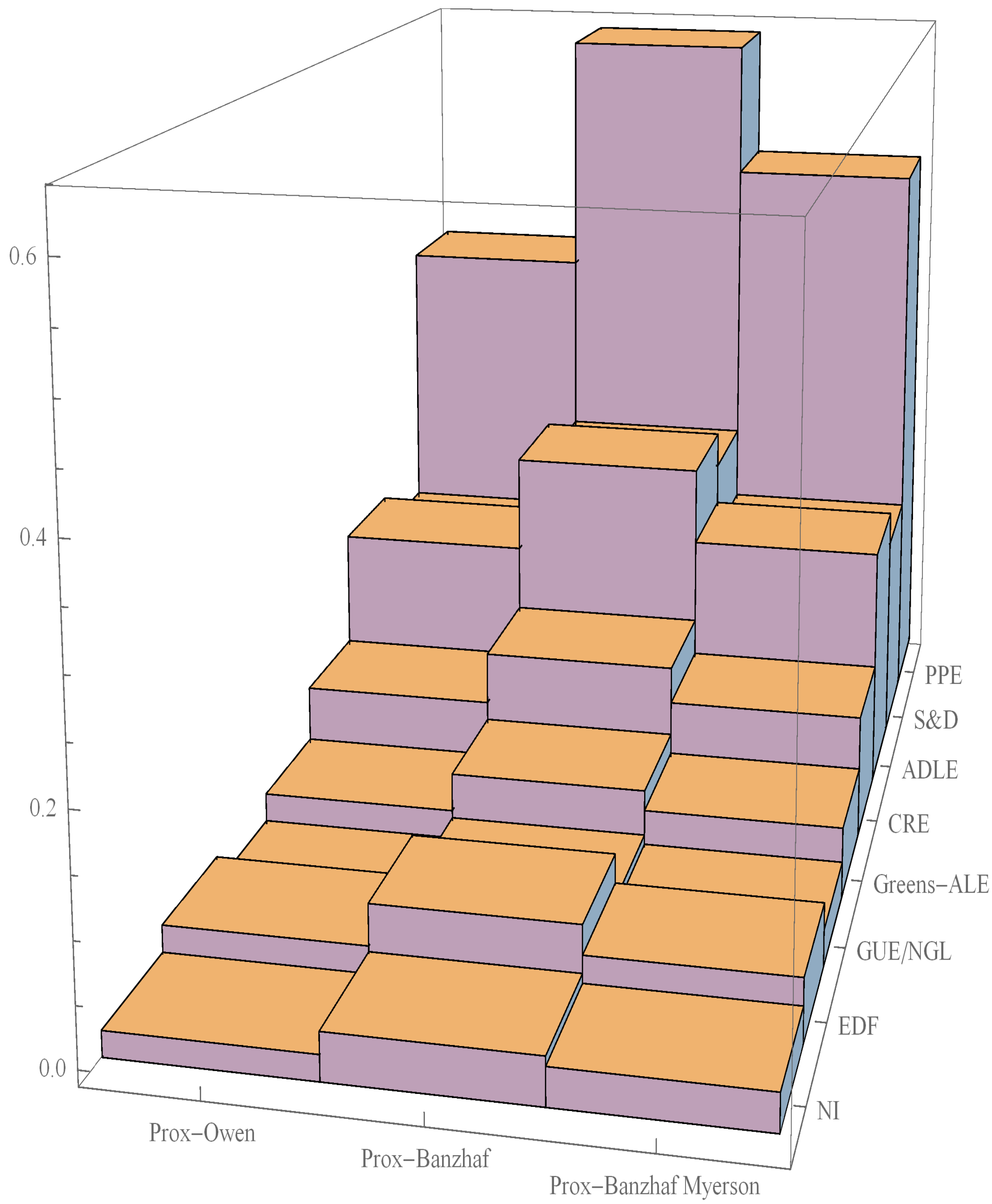


Figure 6: Comparative graph of the prox-indices of the EP-game (II)

Bibliography

[1] Albizuri, M.J. (2001) *An axiomatization of the modified Banzhaf Coleman index*. International Journal of Game Theory 30: 167–176.

[2] Alonso-Meijide, J.M. and Fiestras-Janeiro, M.G. (2002). *Modification of the Banzhaf Value for Games with a Coalition Structure*. Annals of Operations Research 109: 213-227.

[3] Alonso-Meijide, J.M. and Fiestras-Janeiro, M.G. (2006). *The Banzhaf value and communication situations*. Naval Research Logistics 53: 198-203.

[4] Alonso-Meijide, J.M., Carreras, F., Fiestras-Janeiro, M.G. and Owen, G. (2007). *A comparative axiomatic characterization of the Banzhaf-Owen coalitional value*. Decision Support Systems 43: 701-712.

[5] Amer, R., Carreras, F. and Giménez, J.M. (2001) *The modified Banzhaf value for games with coalition structure: an axiomatic characterization*. Mathematical Social Sciences 43: 45-54.

[6] Aumann, R.J. and Drèze, J.H. (1974) *Cooperative games with coalition structures*. International Journal of Game Theory 3: 217-237.

[7] Aubin, J.P. (1981). *Cooperative fuzzy games*. Mathematics of Operation Research 6: 1-13.

[8] Banzhaf, J.F. III (1965). *Weighted voting doesn't work: a mathematical analysis*. Rutgers Law Review 19: 317-343.

[9] Bergantiños, G., Carreras, F. and García-Jurado, I. (1993). *Cooperation when some players are incompatible*. Mathematical Methods of Operations Research 38: 187-201.

[10] Bilbao, J.M. (1998). *Axioms for the Shapley value on convex geometries*. European Journal of Operational Research 110: 368-376.

[11] Bilbao, J.M., Fernández, J.R., Jiménez-Losada, A. and López, J.J. (2000). *Generating functions for computing power indices efficiently*. TOP 8: 191-213.

[12] Bilbao, J.M., Fernández, J.R. and López, J.J. (2002). *On the complexity of computing values of restricted games*. International Journal of Foundations of Computer Science 13: 633-651.

[13] Borel, È. (1921). *La théorie du jeu et les équations intégrales á noyau symétrique*. Comptes Rendus de l'Académie des Sciences 173: 1304-1308.

[14] Borm, P., Owen, G. and Tijs, S. (1992). *On the position value for communication situations*. SIAM Journal of Discrete Mathematics 5: 305-320.

[15] Butnariu, D. (1980). *Stability and Shapley value for an n-persons fuzzy game*. Fuzzy Sets and Systems 4: 63-72.

[16] Carter, M. (1993) *Cooperative games*. In H. R. Varian, ed., Economic and Financial Modeling with Mathematica. Springer, Berlin (1993) 167-191.

[17] Casajus, A. (2007). *An efficient value for TU games with a cooperation structure*. Universitat Leipzig, Germany. Working paper.

[18] Casajus, A. (2007). *Beyond basic structures in game theory*. Ph.D. thesis. University of Leipzig, Germany.

[19] Chacón, C. (2008). *Interior operator games on antimatroids (in Spanish)*. Ph.D. Thesis. University of Seville, Spain.

[20] Chalkiadakis, G., Elkind, E. and Wooldrige, M. (2012). *Computational aspects of cooperative game theory*. Morgan and Claypool Publishers.

[21] Chetty, V. K., Dasgupta, D. and Raghavan, T. E. S. (1976). *Power and Distribution of Profits*. Tech. Rept, Indian Statistical Institute, N. Delhi.

[22] Choquet, G. (1954). *Théorie des capacités*. Annales de l'Institut Fourier 5: 131-295.

[23] Delgado, M., Verdegay, J.L. and Vila, M.A. (1985). *On fuzzy tree definition*. European Journal of Operational Research 22: 243-249.

[24] Deng, X. and Papadimitriou, C.H. (1994). *On the complexity of cooperative solutions concepts*. Mathematics of Operations Research 19: 257-266.

[25] Driessen, T.S.H. (1988). *Cooperative Games, Solutions and Applications*. Kluwer Academic Publishers, Dordrecht, The Netherlands.

[26] Dubey, P. and Shapley, L.S. (1979). *Mathematical Properties of the Banzhaf Power Index*. Mathematics of Operations Research 4: 99-131.

[27] Faigle, U. and Kern, W. (1992). *The Shapley value for cooperative games under precedence constraints*. International Journal of Game Theory 21: 249-266.

[28] Feltkamp, V. (1995). *Alternative Axiomatic Characterization of the Shapley and Banzhaf Values*. International Journal of Game Theory 24: 179-186.

[29] Fernández, J.R. (2000). *Complexity and algorithms in game theory (in Spanish)*. Ph.D. Thesis. University of Seville, Spain.

[30] Fernández, J.R., Algaba, E., Jiménez, A., Jiménez, N. and López, J.J. (2002). *Generating Functions for Computing the Myerson Value*. Annals of Operations Research 109: 143-158.

[31] Fernández, J.R., Gallego, I., Jiménez-Losada, A. and Ordóñez, M. (2016). *Cooperation among agents with a proximity relation*. European Journal of Operational Research 250: 555-565.

[32] Fernández, J.R., Gallego, I., Jiménez-Losada, A. and Ordóñez, M. (2016). *The cg-position value for games on fuzzy communication structures*. Preprint.

[33] Gallardo, J.M. (2015). *Values for games with authorization structure*. Ph.D. Thesis. University of Seville. Spain.

[34] Gallego, I., Fernández, J.R., Jiménez-Losada, A. and Ordóñez, M. (2014). *A Banzhaf value for games with fuzzy communication structure: computing the power of the political groups in the European Parliament*. Fuzzy Sets and Systems 255: 128-145.

[35] Gilles, R.P., Owen, G. and van den Brink, R. (1992). *Games with permission structures: the conjunctive approach*. International Journal of Game Theory 20: 277-293.

[36] Gould, H.W. (1956). *Some generalizations of Vandermonde's convolution*. The American Mathematical Monthly 63: 84-91.

[37] Haller, H. (1994). *Collusion properties of values*. International Journal of Game Theory 23: 261-281.

[38] Harsanyi, J.C. (1963). *A simplified bargaining model for the n-person cooperative game*. International Economic Review 4: 194-220.

[39] Herings, P.J.J., van der Laan, G. and Talman, A.J.J. (2010). *The Average Tree solution for Cycle-free Graph Games*. Games and Economic Behavior 62: 77-92.

[40] Herings, P.J.J., Talman, D. and van der Laan, G. (2010) *The Average Tree solution for games with communication structure*. Games and Economic Behavior 68: 626-633.

[41] Jiaquan, Z. and Zhang, Q. (2011). *Optimal fuzzy coalition structure and solution concepts of a class of fuzzy games*. Computer Science and Service System: 3971-3975.

[42] Jiménez, N. (1998). *Solution concepts for games on closure spaces*. Ph.D. Thesis. University of Seville, Spain.

[43] Jiménez-Losada, A. (1998). *Values for games on combinatorial structures (in Spanish)*. Ph.D. Thesis. University of Seville, Spain.

[44] Jiménez-Losada, A., Fernández, J.R., Ordóñez, M. and Grabisch, M. (2010). *Games on fuzzy communication structures with Choquet players*. European Journal of Operational Research 207: 836-847.

[45] Jiménez-Losada, A., Fernández, J.R. and Ordóñez, M. (2013). *Myerson values for games with fuzzy communication structure*. Fuzzy Sets and Systems 213: 74-90.

[46] Lehrer, E. (1988). *An axiomatization of the Banzhaf value*. International Journal of Game Theory 17: 89-99.

[47] Littlechild, S.C. and Owen, G. (1973). *A simple expression for the Shapley value in a special case*. Management Sciences 20: 370-372.

[48] López, J.J. (1996). *Partial cooperation in n-person games (in Spanish)*. Ph.D. Thesis. University of Seville, Spain.

[49] Meessen R. (1988). *Communication games (in Dutch)*. Master's thesis. University of Nijmegen. The Netherlands.

[50] Mordeson, J.N. and Nair, P.S. (2000). *Fuzzy graphs and fuzzy hypergraphs*. Studies in fuzziness and soft computing 46, Physica-Verlag Heidelberg.

[51] Myerson, R. (1977). *Graphs and cooperation in games*. Mathematics of Operations Research 2: 225-229.

[52] von Neumann, J. (1928). *Zur Theorie der Gesellschaftsspiele*. Mathematische Annalen 100: 295-320.

[53] von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, New Jersey.

[54] Nowak, A.S. (1997). *On an axiomatization of the Banzhaf value without the additivity axiom*. International Journal of Game Theory 26: 137-141.

[55] van den Nouweland, A. and Borm, P. (1991). *On the convexity of communication games*. International Journal of Game Theory 19: 421-430.

[56] O'Neill, B. (1982). *A problem of rights arbitration from the Talmud*. Mathematical Social Sciences 2: 345-371.

[57] Owen, G. (1975). *Multilinear extensions and the Banzhaf value*. Naval Research Logistics Quarterly 22: 741-750.

[58] Owen, G. (1977). *Values of games with a priori unions*. In Essays in Mathematical Economics and Game Theory 141: 76-88. R. Henn and O. Moeschlin eds. Springer-Verlag, Berlin.

[59] Owen, G. (1982). *Modification of the Banzhaf-Coleman index for games with a priori unions*. In: Holler MJ, editor. Power, voting and voting power. Wurzburg: Physica-Verlag; 1982. pp. 232-238.

[60] Owen, G. (1986). *Values of graph-restricted games*. SIAM Journal on Algebraic and Discrete Methods 7: 210-220.

[61] Rosenfeld, A. (1975). *Fuzzy graphs*. In Fuzzy Sets and Their Applications to Cognitive and Decision Processes, pp. 77-97. L.A. Zadeh, K.S. Fu, K. Tanaka and M. Shimura eds. Academic Press, New York, 1975.

[62] Shapley, L.S. (1953). *A Value for N-Person Games*. In Contributions to the Theory of Games 2: 307-317. H.W. Kuhn and A.W. Tucker eds. Princeton, New Jersey.

[63] Shapley, L.S. and Shubik, M. (1967). *Ownership and the production function*. The Quarterly Journal of Economics 81: 88-111.

[64] Shapley, L.S. (1969). *Utility comparison and theory of games*. In La Decision: Aggregation et Dynamique des Ordres de Preference 251-263. Editions du Centre National de la Recherche Scientifique. Paris.

[65] Shapley, L.S. (1971). *Cores of convex games*. International Journal of Game Theory 1: 11-26.

[66] Shmeidler, D. (1986). *Integral representation without additivity*. Proceedings of the American Mathematical Society 97: 255-261.

[67] Skiena, S. (1990) *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Addison-Wesley (1990), Reading.

[68] Slikker, M. (2005). *An axiomatization of the position value*. International Journal of Game Theory 33: 505-514.

[69] Tsurumi, M., Tanino, T. and Inuiguchi, M. (2001). *A Shapley function on a class of cooperative fuzzy games*. European Journal of Operational Research 129: 596-618.

[70] van Velzen, B., Hamers, H. and Solymosi, T. (2008). *Core stability in chain-component additive games*. Games and Economic Behavior 62: 116-139.

[71] De Waegenaere, A. and Wakker, P.P. (2000). *Nonmonotonic Choquet integrals*. Journal of Mathematical Economics 36: 45-60.