GROTHENDIECK LOCALLY CONVEX SPACES
OF CONTINUOUS VECTOR VALUED FUNCTIONS

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Let \( \mathcal{C}(X, E) \) be the space of continuous functions from the completely regular Hausdorff space \( X \) into the Hausdorff locally convex space \( E \), endowed with the compact-open topology. Our aim is to characterize the \( \mathcal{C}(X, E) \) spaces which have the following property: weak-star and weak sequential convergences coincide in the equicontinuous subsets of \( \mathcal{C}(X, E)' \). These spaces are here called Grothendieck spaces. It is shown that in the equicontinuous subsets of \( E' \) the \( \sigma(E', E) \)- and \( \beta(E', E) \)-sequential convergences coincide, if \( \mathcal{C}(X, E) \) is a Grothendieck space and \( X \) contains an infinite compact subset. Conversely, if \( X \) is a \( G \)-space and \( E \) is a strict inductive limit of Fréchet-Montel spaces \( \mathcal{C}(X, E) \) is a Grothendieck space. Therefore, it is proved that if \( E \) is a separable Fréchet space, then \( E \) is a Montel space if and only if there is an infinite compact Hausdorff \( X \) such that \( \mathcal{C}(X, E) \) is a Grothendieck space.

1. Introduction. In this paper \( X \) will always denote a completely regular Hausdorff topological space, \( E \) a Hausdorff locally convex space, and \( \mathcal{C}(X, E) \) the space of continuous functions from \( X \) into \( E \), endowed with the compact-open topology. When \( E \) is the scalar field of reals or complex numbers, we write \( \mathcal{C}(X) \) instead \( \mathcal{C}(X, E) \).

It is well known that \( \mathcal{C}(X, E) \) is a Montel space whenever \( \mathcal{C}(X) \) and \( E \) so are, hence, if and only if \( X \) is discrete and \( E \) is a Montel space (see [5], [16]).

We study what happens when \( X \) has the following weaker property: the compact subsets of \( X \) are \( G \)-spaces (see below for definitions).

We obtain in Theorem 4.4 that if \( E \) is a Fréchet-Montel space and \( X \) has that property, then \( \mathcal{C}(X, E) \) is a Grothendieck locally convex space. The key in the proof is the following fact: every countable equicontinuous subset of \( \mathcal{C}(X, E)' \) lies, via a Radon-Nikodým theorem, in a suitable \( L^1(\tau, E'_\beta) \). As a consequence of a theorem of Mújica [10], the same result is true when \( E \) is a strict inductive limit of Fréchet-Montel spaces.

In §3 we study the converse of 4.4. In Corollary 3.3 it is proved that if \( X \) contains an infinite compact subset, \( E \) is a Fréchet separable space and \( \mathcal{C}(X, E) \) is a Grothendieck space, then \( E \) is a Montel space. This property characterizes the Montel spaces among the Fréchet separable spaces.
Finally, in §5 we study the Grothendieck property in $\mathcal{B}(\Sigma, E)$, the space of $\Sigma$-totally measurable functions, by using the results for $\mathcal{C}(X, E)$.

2. Generalities. A compact Hausdorff topological space $K$ is called a $G$-space whenever $\mathcal{C}(K)$ is a Grothendieck Banach space, i.e. the weak-star and weak sequential convergences coincide in $\mathcal{C}(K)'$ [6].

We extend here this concept to completely regular spaces.

2.1. Definition. $X$ is a $G$-space if every compact subset $K$ of $X$ is a $G$-space.

If $X$ is compact, both definitions coincide [6]. Let us remark that there exist non-compact non-discrete $G$-spaces. Indeed, the topological subspace of the Stone-Čech compactification of a countable discrete set obtained removing a cluster point, is such a space.

We introduce a new definition of Grothendieck locally convex space, so that $\mathcal{C}(X)$ is a Grothendieck space if and only if $X$ is a $G$-space.

2.2. Definition. $E$ is a Grothendieck space whenever the $\sigma(E', E)$- and $\sigma(E', E'')$-sequential convergences coincide in the equicontinuous subsets of $E'$.

In [17] the $TG$-spaces are defined as those spaces $E$ in which the $\sigma(E', E)$- and $\sigma(E', E'')$-sequential convergences coincide. When one deals with $\mathcal{C}(X)$ spaces, our definition seems to be more reasonable than that of [17] (see 2.4 and 2.5).

The following permanence properties of the class of Grothendieck locally convex spaces are easy to see, thus we state them without proof.

2.3. Proposition. (a) $E$ is a Grothendieck space if and only if every, or some, dense subspace of $E$ so is.

(b) Let $T: E \to F$ be a linear continuous operator such that for every bounded subset $B$ of $F$ there is a bounded subset $C$ of $E$ so that $B$ is contained in the closure of $T(C)$. Then $F$ is a Grothendieck space if $E$ so is.

(c) If $E$ is the inductive limit of the sequence $(E_n)$ of Grothendieck spaces, and if every bounded subset of $E$ is contained in some $E_n$, then $E$ is a Grothendieck space.

2.4. Theorem. $\mathcal{C}(X, E)$ is a Grothendieck space if and only if $\mathcal{C}(K, E)$ so is for every compact subset $K$ of $X$. In particular, $X$ is a $G$-space if and only if $\mathcal{C}(X)$ is a Grothendieck space.
Proof. Let us recall that, if $K$ is a compact subset of $X$, the restriction map $T$ is a continuous linear operator from $\mathcal{C}(X, E)$ into $\mathcal{C}(X, E)$.

If $B \subset \mathcal{C}(K, E)$ is bounded, then the bounded subset $C$ of $\mathcal{C}(X, E)$, whose elements $g$ can be written $g = \sum_{n \leq m} f_n(\cdot) e_n$ with $f_n \in \mathcal{C}(X)$, $0 \leq f_n \leq 1$, $\sum_{n \leq m} f_n \leq 1$, and $e_n \in \bigcup \{ h(K) : h \in B \}$, satisfies $T(C) \supset B$ (see [14, I.5.3]).

If $\mathcal{C}(X, E)$ is a Grothendieck space, $\mathcal{C}(K, E)$ so is by 2.3(b).

Conversely, let $(g'_n)$ be an equicontinuous and $\sigma(\mathcal{C}(X, E)', \mathcal{C}(X, E))$-null sequence. By [14, III.3 and III.4], there exist a compact subset $K$ of $X$ and an equicontinuous sequence $(h'_n)$ in $\mathcal{C}(K, E)'$ such that $g'_n = h'_n \circ T$ for all $n \in \mathbb{N}$. Since $(h'_n)$ is $\sigma(\mathcal{C}(K, E)', T(\mathcal{C}(K, E)))$-null and equicontinuous, it is also $\sigma(\mathcal{C}(K, E)', \mathcal{C}(K, E)''')$-null if $\mathcal{C}(K, E)$ is a Grothendieck space. It follows that $(g'_n)$ is $\sigma(\mathcal{C}(X, E)', \mathcal{C}(X, E)'''')$-null.

2.5. Remark. We use an example of Haydon [4] to show that, while in the class of barreled spaces the $TG$-spaces and the Grothendieck spaces do coincide, this is not true in general.

Choose, for each infinite sequence in $\mathbb{N}$, a cluster point in the Stone-Čech compactification of $\mathbb{N}$, and let $X$ be the topological subspace of that compactification, formed by $\mathbb{N}$ and these cluster points. Then every compact subset of $X$ is finite, $\mathcal{C}(X)$ is infrabarrelled and every $f \in \mathcal{C}(X)$ is bounded. By Theorem 2.4, $X$ is a $G$-space. Let $f'_n(f) = n^{-1}f(n)$ for all $f \in \mathcal{C}(X)$ and $n \in \mathbb{N}$. Then $(f'_n)$ is a $\sigma(\mathcal{C}(X)'', \mathcal{C}(X))$-null sequence in $\mathcal{C}(X)'$, that is not $\sigma(\mathcal{C}(X)'', \mathcal{C}(X)''')$-null because it is not equicontinuous.

3. Necessary conditions for $\mathcal{C}(X, E)$ to be a Grothendieck space. It is well known, and easy to see, that $\mathcal{C}(X)$ and $E$ are topologically isomorphic to complemented subspaces of $\mathcal{C}(X, E)$. By 2.3(b), $\mathcal{C}(X)$ and $E$ must be Grothendieck spaces if $\mathcal{C}(X, E)$ is such a space.

However, unless $X$ is pseudofinite, i.e. their compact subsets are finite (hence $\mathcal{C}(X, E)$ is a Grothendieck space if and only if $E$ so is, by Theorem 2.4), $E$ has a stronger property if $\mathcal{C}(X, E)$ is a Grothendieck space, as we prove in the next theorem. To prove it we recall the following result of [2]:

Theorem A. Let $E$ and $F$ be Hausdorff locally convex spaces, and suppose that $F$ contains a subspace topologically isomorphic to the subspace of $e_0$ whose elements have only finitely many non-zero coordinates.

If the injective tensor product $F \otimes E$ is a Grothendieck space, then the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide in the equicontinuous subsets of $E'$. 

As was noted in [2], if $X$ is not pseudofinite, then $\mathcal{C}(X)$ contains a subspace topologically isomorphic to the above mentioned subspace of $c_0$. Moreover, the injective tensor product $\mathcal{C}(X) \otimes E$ can be linear and topologically identified with a dense subspace of $\mathcal{C}(X, E)$, namely, the subspace of all finite dimensional valued elements of $\mathcal{C}(X, E)$. Thus we obtain from Theorem A and Proposition 2.3 (a):

3.1. **Theorem.** If $\mathcal{C}(X, E)$ is a Grothendieck space and $X$ contains an infinite compact subset, then the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide in the equicontinuous subsets of $E'$.

3.2. **Remark.** By Theorem 2.4, if $X$ is pseudofinite and $E$ is a Grothendieck Banach space, $\mathcal{C}(X, E)$ is a Grothendieck space. However, if $E$ is infinite dimensional, the conclusion of Theorem 3.1 does not hold [11].

Using Theorem 3.1 and [7, 11.6.2], we obtain the following corollary, converse of Theorem 4.4:

3.3. **Corollary.** If $E$ is a Fréchet separable space, $X$ is not pseudofinite and $\mathcal{C}(X, E)$ is a Grothendieck space, then $E$ is a Montel space.

3.4. **Remark.** It is unknown for us if Corollary 3.3 is true without the separability assumption on $E$. This is related with the following question raised in [7, pg. 247]: is a Fréchet space $E$ already a Montel space if every $\sigma(E', E)$-convergent sequence in $E'$ converges for $\beta(E', E)$?

4. **Sufficient conditions for $\mathcal{C}(X, E)$ to be a Grothendieck space.**

We shall need some facts about vector integration, many of those can be found in [1] and [15].

Let $(X, \Sigma, \tau)$ be a complete measure space with $\tau(X) \leq 1$. We denote by $\mathcal{S}(\Sigma, E)$ (resp. $\mathcal{B}(\Sigma, E)$, $L^1(\tau, E)$, $L^\infty(\tau, E)$) the vector space of $\Sigma$-simple (resp. $\Sigma$-totally measurable, $\tau$-integrable, $\tau$-essentially bounded) $E$-valued (classes of) functions. Recall that $\mathcal{S}(\Sigma, E)$ and $\mathcal{B}(\Sigma, E)$ are endowed with the uniform convergence topology, and that the topology of $L^1(\tau, E)$ is defined by the seminorms $u \rightarrow \int p(u(x))\,d\tau(x)$, where $p$ runs over the set of all continuous seminorms in $E$ (unless contrary specification, all integrals will be extended to $X$).

The following Radon-Nikodym theorem is proved in [1]:

**Theorem B.** If $E$ is a quasi-complete (CM)-space, $\mu: \Sigma \to E$ is a countably additive vector measure, of bounded variation and $\tau$-absolutely continuous, then there exists $u \in L^1(\tau, E)$ such that $\mu(A) = \int_A u(x)\,d\tau(x)$ for every $A \in \Sigma$. 
Let us recall that $E$ is a quasi-complete (CM)-space, if, for instance, it is either a Fréchet-Montel space or a (DF)-Montel space [1].

Firstly we extend the classical duality theorem $L^1 - L^\infty$ to $L^1(\tau, E'_p)$, where $E$ is a Fréchet-Montel space.

The following lemma can be easily proved. As usual, $p_L$ will denote the gauge of the absolutely convex set $L$ in its linear span.

4.1. LEMMA. If $u \in \mathcal{S}(\Sigma, E')$, namely, $u = \sum_{i \leq m} \chi_{A_i} e'_i$ with $(A_i)_{i \leq m}$ disjoint in $\Sigma$, then

$$\int p_{B^o}(u(x)) \, d\tau(x) \leq \tau \left( \bigcup_{i \leq m} A_i \right) \sup_{i \leq m} p_{B^o}(e'_i)$$

for every bounded subset $B$ of $E$.

4.2. THEOREM. Let $E$ be a Fréchet-Montel space. The relation

$$u'(u) = \int u(x)(\psi(x)) \, d\tau(x) \quad \text{for all } u \in L^1(\tau, E'_p)$$

defined for $u' \in L^1(\tau, E'_p)'$ and $\psi \in L^\infty(\tau, E)$, is an algebraic isomorphism between $L^1(\tau, E'_p)'$ and $L^\infty(\tau, E)$.

Proof. Let $\psi \in L^\infty(\tau, E)$. The map $x \to u(x)(\psi(x))$ is measurable for every $u \in L^1(\tau, E'_p)$, because $\psi$ is strongly measurable and the assertion is clearly true when $\psi \in \mathcal{S}(\Sigma, E)$.

Furthermore, if $Z \in \Sigma$ is a $\tau$-null set such that $B = \psi(S \setminus Z)$ is bounded, then we have

$$|u(x)(\psi(x))| \leq p_{B^o}(u(x))$$

for every $x \in X \setminus Z$.

Hence $x \to u(x)(\psi(x))$ is $\tau$-integrable, and we can define a linear form $u'$ on $L^1(\tau, E'_p)$ by (1). Moreover, it follows from (2) that $u'$ is continuous.

Conversely, fix $u' \in L^1(\tau, E'_p)'$. There exists a bounded subset $B$ of $E$ such that

$$\int p_{B^o}(u(x)) \, d\tau(x) \leq 1 \quad \text{implies } |u'(u)| \leq 1$$

for every $u \in L^1(\tau, E'_p)$.

We define a map $\mu: \Sigma \to E''$ by

$$\mu(A)(e') = u'(\chi_A e')$$

for every $A \in \Sigma$ and $e' \in E'$ (it follows easily from Lemma 4.1 and (3) that $\mu(A) \in E''$). Since $E$ is reflexive we can suppose that $\mu(A) \in E$. 
Clearly, $\mu : \Sigma \to E$ is a finitely additive vector measure. We shall show that $\mu$ is countably additive: let $A$ be the union of the disjoint sequence $(A_n)$ in $\Sigma$. Given an absolutely convex zero-neighborhood $U$ in $E$ and $\varepsilon > 0$, we choose $\lambda$ with $0 < \lambda < \infty$ such that $B \subset \lambda U$, and $m_0 \in \mathbb{N}$ such that $\lambda \tau(\bigcup_{n \geq m} A_n) \leq \varepsilon$ for every $m \geq m_0$. Since
\[
e'(\mu(A)) - \sum_{n \leq m} e'(\mu(A_n)) = u'(\chi_{\bigcup_{n \geq m} A_n} e')
\]
it follows from Lemma 4.1 and (3) that
\[
\left| e'(\mu(A)) - \sum_{n \leq m} e'(\mu(A_n)) \right| \leq \varepsilon
\]
for every $m \geq m_0$ and $e' \in U^0$, as desired.

Furthermore, if $A = \bigcup_{n \leq m} A_n$ where $(A_n)_{n \leq m}$ is disjoint in $\Sigma$, and if $\varepsilon > 0$, there exists $(e'_n)_{n \leq m}$ in $U^0$ such that
\[
\sum_{n \leq m} p_U(\mu(A_n)) \leq \sum_{n \leq m} e'_n(\mu(A_n)) + \varepsilon = u'(\sum_{n \leq m} \chi_{A_n} e'_n) + \varepsilon.
\]
Hence the $p_U$-variation of $\mu$ satisfies the inequality $V_{p_U} \mu(A) \leq \lambda \tau(A)$, from Lemma 4.1 and (3) again.

Thus $\mu$ is $\tau$-absolutely continuous and has bounded variation. By Theorem B, there exists $v \in L^1(\tau, E)$ such that
\[
(5) \quad \mu(A) = \int_A v(x) \, d\tau(x) \quad \text{for every } A \in \Sigma.
\]

We claim that $v$ is $\tau$-essentially bounded and satisfies (1). Indeed, let $(U_j)_j$ be a countable basis in $E$ of absolutely convex zero-neighborhoods. Choose, for each $j \in \mathbb{N}$, $\lambda_j$ such that $0 < \lambda_j < \infty$ and $B \subset \lambda_j U_j$.

By Lemma 4.1, (3), (4) and (5), we have
\[
(6) \quad \left| \int_A e'(v(x)) \, d\tau(x) \right| \leq \lambda_j \tau(A)
\]
for all $e' \in U^0_j, A \in \Sigma$ and $j \in \mathbb{N}$.

Let $(e'_{j,k})_k$ be a sequence in $U^0_j$ such that $p_{U_j}(e) = \sup_k |e'_{j,k}(e)|$ for every $e \in E$.

By (6), there exists $Z \in \Sigma$ with $\tau(Z) = 0$ such that $|e'_{j,k}(v(x))| \leq \lambda_j$ for all $x \in X \setminus Z$ and all $j, k \in \mathbb{N}$. Hence $v(X \setminus Z)$ is bounded in $E$.

Finally, it follows from (4) that (1) is true for all $u \in \mathcal{F}(\Sigma, E')$, and, by density, for every $u \in L^1(\tau, E'_0)$. This concludes the proof.

Assume that $X$ is compact Hausdorff and $\Sigma$ contains the Borel subsets of $X$. For each $u \in L^1(\tau, E'_0)$, denote by $\mu_u$ the vector measure of density $u$ with respect to $\tau$. If $p$ is a continuous seminorm in $E$, the subset
$F$ of $L^1(\tau, E'_\mu)$ defined by the condition $V_p\nu_u(X) < \infty$, is a linear subspace. If $u \in F$ then $\nu_u$ has bounded semivariation, thus it defines a continuous linear form on $\mathcal{S}(\Sigma, E)$, which extends by continuity to the whole space $\mathcal{B}(\Sigma, E)$ [15]. Let $Tu \in \mathcal{C}(X, E)'$ be the restriction to $\mathcal{C}(X, E)$ of this linear form, i.e.

$$ (Tu)(g) = \int g(x) \, d\nu_u(x) $$

for every $g \in \mathcal{C}(X, E)$.

4.3. Lemma. The map $T: F \to \mathcal{C}(X, E)'$ defined by (7) is a linear continuous operator, when $\mathcal{C}(X, E)'$ is endowed with the strong topology with respect to $\mathcal{C}(X, E)$.

Proof. We have, for each $u \in F$,

$$ (Tu)(g) = \int u(x)(g(x)) \, d\tau(x) $$

for every $g \in \mathcal{C}(X, E)$. Indeed, the dominated convergence theorem and a standard density argument show that it suffices to see (8) when $g$ belongs to $\mathcal{S}(\Sigma, E)$, that is trivially true.

Let $H$ be a bounded subset of $\mathcal{C}(X, E)$. Then $B = \bigcup \{ g(X): g \in H \}$ is a bounded subset of $E$. Hence, by (8), $|(Tu)(g)| \leq \int p_B^\circ(u(x)) \, d\tau(x)$ and the lemma follows.

We are now ready to prove the sufficient condition:

4.4. Theorem. Let $X$ be a completely regular Hausdorff $G$-space and $E$ a Fréchet-Montel space. Then $\mathcal{C}(X, E)$ is a Grothendieck space.

Proof. By 2.4 we can suppose, without loss of generality, that $X$ is compact.

Let $(g'_n)_n$ be an equicontinuous sequence in $\mathcal{C}(X, E)'$. By [14, III.4.5] there exists a continuous seminorm $p$ in $E$ such that $V_p\mu_n(X) \leq 1$, for every $n \in \mathbb{N}$, where $\mu_n$ is the representing measure of $g'_n$ [14, III].

Let $\tau = \sum_n 2^{-n}V_p\mu_n$. $\tau$ is a countably additive $[0, 1]$-valued Borel measure, by [14, III.2.5]. Let $\Sigma$ be the completed $\sigma$-field of the Borel field of $X$ with respect to $\tau$. We shall denote also by $\tau$ and $\mu_n$ the natural extensions of the earlier measures to $\Sigma$. 
Since $E$ is a Montel space, the measure $\mu_n: \Sigma \to E'_B$ is countably additive. Clearly $V_p \mu_n \leq 2^n$, thus $\mu_n$ has bounded variation and is $\tau$-absolutely continuous (when it is considered as an $E'_B$-valued measure).

We apply Theorem B, obtaining, for each $n \in \mathbb{N}$, a function $u_n \in L^1(\tau, E'_B)$ such that $\mu_n$ is the vector measure of density $u_n$ with respect to $\tau$.

Clearly $u_n \in F$ and $Tu_n = g'_n$, for every $n \in \mathbb{N}$.

Fix $g'' \in \mathscr{C}(X, E)'$. By Lemma 4.3 and Theorem 4.2, there exists $v \in L^\infty(\tau, E)$ such that $g''(g'_n) = \int u_n(x)(v(x)) \, d\tau(x)$ for every $n \in \mathbb{N}$.

Let $Z$ be a set in $\Sigma$ with $\tau(Z) = 0$ and $v(X \setminus Z)$ bounded. The function $v_1 = \chi_{X \setminus Z} v$ is totally measurable, because $E$ is Montel and metrizable.

Given $\varepsilon > 0$, we can choose $v_2 \in \mathscr{S}(\Sigma, E)$ such that $p(v_3(x)) \leq \varepsilon/2$, for every $x \in X$, if $v_3 = v_1 - v_2$. Hence,

$$\int u_n(x)(v_3(x)) \, d\tau(x) = \varepsilon/2$$

for every $n \in \mathbb{N}$, because $V_p \mu_n(X) \leq 1$.

On the other hand, if $(g'_n)$ is $\sigma(\mathscr{C}(X, E)'$, $\mathscr{C}(X, E))$-null, then $(\mu_n(A)(e))$ is a null sequence, for every $e \in E$ and $A \in \Sigma$. Indeed, since $X$ is a $G$-space, for each $e \in E$, the weak-star null sequence $(\mu_n(\cdot)(e))$ in $\mathscr{C}(X)'$, is also weak null, hence $(\mu_n(A)(e))$ is null for every Borel subset $A$ of $X$, and so for every $A \in \Sigma$.

Since $v_2$ is simple, it follows that

$$\lim_{n \to \infty} \int u_n(x)(v_2(x)) \, d\tau(x) = 0.$$ 

By (9) and (10), $(g''(g'_n))$ is a null sequence, and we have shown that $(g'_n)$ is $\sigma(\mathscr{C}(X, E)'$, $\mathscr{C}(X, E)'')$-null.

4.5. COROLLARY. Let $X$ be a completely regular Hausdorff $G$-space and $E$ the inductive limit of the sequence $(E_n)$ of Fréchet-Montel spaces, such that every bounded subset of $E$ is localized in some $E_n$. Then $\mathscr{C}(X, E)$ is a Grothendieck space.

Proof. We can again suppose $X$ compact. By [10], the inductive limit of the sequence $(\mathscr{C}(X, E_n))$ is a dense topological subspace of $\mathscr{C}(X, E)$. By Proposition 2.3 (a) and (c), and Theorem 4.4, it follows that $\mathscr{C}(X, E)$ is a Grothendieck space.
4.6. Corollary. Let $E$ be a Fréchet separable space. The following conditions are equivalent:

(a) $E$ is a Montel space.

(b) There exists a non-pseudofinite completely regular Hausdorff space $X$ such that $\mathcal{C}(X, E)$ is a Grothendieck space.

(c) For every completely regular Hausdorff $G$-space $X$, $\mathcal{C}(X, E)$ is a Grothendieck space.

Proof. Use 4.4 and 3.3.

5. Application to spaces of totally measurable functions. Let $X$ be a nonempty set and $\Sigma$ a field of subsets of $X$. We will say that a subset $B$ of $X$ is open if for every $x \in B$ there is $A \in \Sigma$ with $x \in A$ and $A \subset B$. Endowed $X$ with this topology, let $X^*$ be the Hausdorff space associated to $X$, $\pi: X \to X^*$ the quotient map, and $\Sigma^* = \{ \pi(A): A \in \Sigma \}$.

The following lemma is easily established:

5.1. Lemma (a) $X^*$ is a completely regular Hausdorff zero-dimensional topological space.

(b) The map $A \in \Sigma \to \pi(A) \in \Sigma^*$ is a Boolean isomorphism.

(c) The map $g \in \mathcal{B}(\Sigma^*, E) \to g \circ \pi \in \mathcal{B}(\Sigma, E)$ is a topological isomorphism, and its restriction to $\mathcal{S}(\Sigma^*, E)$ so is onto $\mathcal{S}(\Sigma, E)$.

(d) The map $x^* \in X^* \to \{ B^* \in \Sigma^*: x^* \in B^* \} \in \mathcal{P}(\Sigma^*)$ is one-to-one.

By using 5.1, when one studies the linear topological properties of $\mathcal{B}(\Sigma, E)$, it can be supposed that $X$ is a dense subspace of a Hausdorff compact zero-dimensional topological space $K$ (namely, the Stone space of the Boolean algebra $\Sigma$), and $\Sigma$ is the trace in $X$ of the Boolean algebra of open and closed subsets of $K$. In this context we have the following theorem:

5.2. Theorem. There exists a subspace of $\mathcal{B}(\Sigma, E)$, containing $\mathcal{S}(\Sigma, E)$, that is topologically isomorphic to $\mathcal{C}(K, E)$.

Proof. It is easy to check that the set of restrictions to $X$ of all elements of $\mathcal{C}(K, E)$ is such a subspace.

By Proposition 2.3 (a), it follows that $\mathcal{B}(\Sigma, E)$ is a Grothendieck space if and only if $\mathcal{C}(K, E)$ so is. Hence we can apply to $\mathcal{B}(\Sigma, E)$ the results of §§3 and 4.
5.3. Remark. The question of when \( \mathcal{B}(\Sigma) \) (equivalently, \( \mathcal{C}(K) \)) is a Grothendieck space is related to the validity of the Vitali-Hahn-Saks theorem for finitely additive scalar measures on \( \Sigma \), of bounded variation. For instance, if \( \Sigma \) is \( \sigma \)-complete, or more generally, \( \Sigma \) has the subsequential interpolation property, then \( \mathcal{B}(\Sigma) \) is a Grothendieck space (see [13] and [3]).

Finally, we show that the following result of Mendoza [8], can be easily deduced from their earlier results in [9] and our Theorem 5.2.

5.4. Theorem. Suppose \( \Sigma \) infinite. Then \( \mathcal{B}(\Sigma, E) \) is infrabarrelled (resp. barrelled) if and only if \( E'_{\beta} \) has property \( (B) \) of Pietsch [12, 1.5.8], and \( E \) is infrabarrelled (resp. barrelled).

Proof. Let us observe that \( \mathcal{S}(\Sigma, E) \) is a large dense subspace of \( \mathcal{B}(\Sigma, E) \). Indeed, if \( H \) is a bounded subset of \( \mathcal{B}(\Sigma, E) \), then the set of all \( g \) in \( \mathcal{S}(\Sigma, E) \) for which there exists \( h \in H \) with \( g(X) \subseteq h(X) \), is a bounded subset of \( \mathcal{S}(\Sigma, E) \) whose closure in \( \mathcal{B}(\Sigma, E) \) contains \( H \).

Thus Theorem 5.2 implies that \( \mathcal{B}(\Sigma, E) \) is infrabarrelled whenever \( \mathcal{C}(K, E) \) so is, hence we have the first equivalence of the theorem, by [9].

If \( \mathcal{B}(\Sigma, E) \) is barrelled, then \( E \) is barrelled and \( \mathcal{B}(\Sigma, E) \) is infrabarrelled, so \( E'_{\beta} \) has property \( (B) \). The converse follows easily because \( \mathcal{C}(K, E) \) is topologically isomorphic to a dense subspace of \( \mathcal{B}(\Sigma, E) \), by 5.2.

5.5. Remark. We have also shown in 5.4 that, if \( \Sigma \) is infinite, \( \mathcal{S}(\Sigma, E) \) is infrabarrelled if and only if \( E'_{\beta} \) has property \( (B) \) and \( E \) is infrabarrelled, a result of Mendoza [8]. In [2] we prove that \( \mathcal{S}(\Sigma, E) \) is barrelled if and only if \( \mathcal{S}(\Sigma) \) and \( E \) so are, and \( E \) is nuclear.

References


[9] ———, *Necessary and sufficient conditions for $C(X, E)$ to be barrelled or infrabarrelled*, Simon Stevin, 57 (1983), 103–123.


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