Coefficient fields and scalar extension in positive characteristic

M. Fernández-Lebrón, L. Narváez-Macarro
Department of Algebra, Faculty of Mathematics
University of Seville
c/ Tarfia s/n, 41012 Sevilla, Spain
E-mail: lebron@algebra.us.es, narvaez@algebra.us.es

Abstract

Let $k$ be a perfect field of positive characteristic, $k(t)_{per}$ the perfect closure of $k(t)$ and $A = k[[X_1, \ldots, X_n]]$. We show that for any maximal ideal $\mathfrak{n}$ of $A' = k(t)_{per} \otimes_k A$, the elements in $\mathcal{A}_\mathfrak{n}'$ which are annihilated by the “Taylor” Hasse-Schmidt derivations with respect to the $X_i$ form a coefficient field of $\mathcal{A}_\mathfrak{n}'$.

Keywords: Complete local ring; Coefficient field; Hasse-Schmidt derivation.


Introduction

Let $k$ be a perfect field, $k(\infty) = k(t)_{per}$ the perfect closure of $k(t)$ and $A = k[[X_1, \ldots, X_n]]$.

If $k$ is of characteristic 0, then $k(\infty) = k(t)$ and $A(t) = A \otimes_k k(t)$ is obviously noetherian. Actually, $A(t)$ is an $n$-dimensional regular non-local ring (see Example (2.3)) whose maximal ideals have the same height ($= n$). In [8] the second author proved that there is a uniform way to obtain a coefficient field in the completions $(\mathcal{A}(t))_n$, for all maximal ideals $\mathfrak{n}$ in $A(t)$. Namely, the elements in $(\mathcal{A}(t))_n$ which are annihilated by the partial derivatives $\frac{\partial}{\partial X_i}$ form a coefficient field of $(\mathcal{A}(t))_n$.

In this paper, we generalize the above result to the positive characteristic case.

At first sight, in positive characteristic it seems natural to consider Hasse-Schmidt derivations instead of usual derivations (see [4] Theorem 3.17), but Example [2.3] shows that the question is not so clear.

Consequently, in the characteristic $p > 0$ case we take the scalar extension $k \to k(\infty)$ instead of $k \to k(t)$, but a new problem appears: it is not obvious that the ring $A(\infty) = A \otimes_k k(\infty)$ is noetherian. We have proved that result in [3].

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The main result in this paper says that, for every maximal ideal \( \mathfrak{n} \) in \( A(\infty) \), the elements in \( (\hat{A(\infty)})_\mathfrak{n} \) which are annihilated by the “Taylor” Hasse-Schmidt derivations with respect to the \( X_i \) form a coefficient field of \( (\hat{A(\infty)})_\mathfrak{n} \).

Let us now comment on the content of this paper.

In Section 1 we introduce our basic notations and recall some results, mainly from [3].

In Section 2 we prove our main result and give the (counter)Example [2.3].

In the Appendix we give a complete proof of the Normalization Lemma for power series rings over perfect fields, which is an important ingredient in the proof of Theorem [2.1] and that we have not found in the literature. Our proof closely follows the proof in [1], but the latter works only for infinite perfect fields.

1 Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element. If \( B \) is a ring, we shall denote by \( \dim(B) \) its Krull dimension and by \( \Omega(B) \) the set of its maximal ideals. We shall use the letters \( K, L, k \) to denote fields and \( \mathbb{F}_p \) to denote the finite field of \( p \) elements, for a prime number \( p \). If \( p \in \text{Spec}(B) \), we shall denote by \( \text{ht}(p) \) the height of \( p \). Remember that a ring \( B \) is said to be biequidimensional if all its saturated chains of prime ideals have the same length.

If \( B \) is an integral domain, we denote by \( \text{Qt}(B) \) its quotient field.

If \( k \) is a ring and \( B \) is a \( k \)-algebra, the set of all derivations (resp. of all Hasse-Schmidt derivations) of \( B \) over \( k \) (cf. [5] and [6], §27) will be denoted by \( \text{Der}_k(B) \) (resp. \( \text{HS}_k(B) \)).

Now, we recall the notations and some results of [3] which are used in this paper.

For any \( \mathbb{F}_p \)-algebra \( B \), we denote \( B^\sharp := \bigcap_{e \geq 0} B^{pe} \).

Let \( k \) be a field of characteristic \( p > 0 \) and consider the field extension

\[ k_{(\infty)} := \bigcup_{m \geq 0} k \left( t^{\frac{1}{pm}} \right) \supset k(t). \]

If \( k \) is perfect, \( k_{(\infty)} \) coincides with the perfect closure of \( k(t) \).

For each \( k \)-algebra \( A \), we denote \( A(t) := k(t) \otimes_k A \). For the sake of brevity, we will write \( t_m = t^{\frac{1}{pm}} \) and denote

\[ A_{(m)} := A(t_m) := A \otimes_k k(t_m) = A(t) \otimes_k k(t) k(t_m), \quad A_{[m]} := A[t_m], \]

\[ A_{(\infty)} := A \otimes_k k_{(\infty)} = \bigcup_{m \geq 0} A_{(m)}, \quad A_{[\infty]} := \bigcup_{m \geq 0} A_{[m]}. \]

Each \( A_{(m)} \) (resp. \( A_{[m]} \)) is a free module over \( A(t) \) (resp. over \( A[t] \)) of rank \( p^m \).
For each prime ideal $N$ of $A_{(\infty)}$ we denote $N_{[\infty]} := N \cap A_{[\infty]}$, $N_{[m]} := N \cap A_{[m]}$ and $N_{(m)} := N \cap A_{(m)}$. Similarly, if $P$ is a prime ideal of $A_{[\infty]}$ we denote $P_{[m]} := P \cap A_{[m]}$.

(1.1) We have the following properties:

(i) $N = \bigcup_{m \geq 0} N_{(m)}$, $N_{[\infty]} = \bigcup_{m \geq 0} N_{[m]}$, (resp. $P = \bigcup_{m \geq 0} P_{[m]}$).

(ii) $N_{(n)} \cap A_{(m)} = N_{(m)}$ and $N_{[n]} \cap A_{[m]} = N_{[m]}$ for all $m \geq n$ (resp. $P_{[n]} \cap A_{[m]} = P_{[m]}$ for all $n \geq m$).

(iii) The following conditions are equivalent:

(a) $N$ is maximal (resp. $P$ is maximal).

(b) $N_{(m)}$ (resp. $P_{[m]}$) is maximal for some $m \geq 0$.

(c) $N_{(m)}$ (resp. $P_{[m]}$) is maximal for all $m \geq 0$.

(iv) ht($N$) = ht($N_{[\infty]}$) = ht($N_{(m)}$) = ht($N_{[m]}$) for all $m \geq 0$. Moreover, dim($A_{(\infty)}$) = dim($A_{(m)}$).

(v) Proposition (1.4) and Theorem (1.6)] Let us assume that $A$ is noetherian and that for every maximal ideal $m$ of $A$, the residue field $A/m$ is algebraic over $k$. Then for every $m \geq 0$ we have dim($A_{[\infty]}$) = dim($A_{(m)}$) = dim($A(t)$). Moreover, if $A$ is biequidimensional, universally catenarian of Krull dimension $n$, then every maximal ideal of $A_{(\infty)}$ (or of $A_{(m)}$) has height $n$.

(vi) Proposition 2.2] If $k$ is perfect and $B = k[[X_1, \ldots, X_n]]$, then $\text{Qt}(B)^{\sharp} = k$.

(vii) Proposition 3.4] If $k$ is perfect, $A$ is an integral $k$-algebra, $K = \text{Qt}(A)$ and $K^{\sharp}$ is algebraic over $k$, then any prime ideal $P \in \text{Spec}(A_{[\infty]})$ with $P \cap k[t] = 0$ and $P \cap A = 0$ is the extended ideal of some $P_{[m_0]}$, $m_0 \geq 0$.

(viii) Corollary 3.10] If $k$ is perfect, $A$ is noetherian and for every maximal ideal $m$ of $A$, the residue field $A/m$ is algebraic over $k$, then $A_{(\infty)}$ is also noetherian. In particular $k[[X_1, \ldots, X_n]]_{(\infty)}$ is noetherian.

(ix) Theorem 30.6] Let $(R, m)$ be an equicharacteristic $n$-dimensional regular local ring containing a quasi-coefficient field $k_0$, and $D_1, \ldots, D_n \in \text{Der}_{k_0}(R)$, $a_1, \ldots, a_n \in R$ such that $D_i(a_j) = \delta_{ij}$. Then, $\text{Der}_{k_0}(R)$ is a free $R$-module with basis $\{D_1, \ldots, D_n\}$.

(x) Theorem 3.17] Let $(R, m)$ be an equicharacteristic $n$-dimensional regular local ring containing a quasi-coefficient field $k_0$, and let $\tilde{D}^1, \ldots, \tilde{D}^n \in \text{HS}_{k_0}(R)$ such that their degree 1 components $\{D^1_1, \ldots, D^n_1\}$ form a basis of $\text{Der}_{k_0}(R)$. Let $\tilde{D}^1, \ldots, \tilde{D}^n$ be the extensions of $\tilde{D}^1, \ldots, \tilde{D}^n$ to $\tilde{R}$. Then, the set

$$\{a \in \tilde{R} \mid \tilde{D}^i_j(a) = 0 \quad \forall j = 1, \ldots, n, \ i \geq 1\}$$

is a coefficient field of $\tilde{R}$ (the only one containing $k_0$).
(1.2) Taylor expansions (cf. [7]).

Let \( n \geq 1 \) be an integer. We write \( X = (X_1, \ldots, X_n) \), \( T = (T_1, \ldots, T_n) \), \( X + T = (X_1 + T_1, \ldots, X_n + T_n) \) and, for \( \alpha \in \mathbb{N}^n \), \( X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \).

Let \( A \) be the formal power series ring \( k[[X]] \) (or the polynomial ring \( k[X] \)). For any \( f(X) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha X^\alpha \in A \) we define \( \Delta^{(\alpha)}(f(X)) \) by: \( f(X + T) = \sum_{\alpha \in \mathbb{N}^n} \Delta^{(\alpha)}(f(X)) T^\alpha \). One has

\[
\Delta^{(\alpha)}(f \cdot g) = \sum_{\beta + \sigma = \alpha} \Delta^{(\beta)}(f) \Delta^{(\sigma)}(g) \tag{1}
\]

and \( \alpha! \Delta^{(\alpha)} = \left( \frac{\partial}{\partial X_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial X_n} \right)^{\alpha_n} \). For \( i \in \mathbb{N} \), \( 1 \leq j \leq n \) and \( \alpha = (0, \ldots, \widehat{j}, \ldots, 0) \) we denote \( \Delta^j_i = \Delta^{(0, \ldots, i, \ldots, 0)} \). From (1) we obtain

\[
\Delta^j_i(f \cdot g) = \sum_{r+s=i} \Delta^r_i(f) \Delta^s_i(g),
\]
i.e. the sequences \( \Delta^j := (1_A, \Delta^1_j, \Delta^2_j, \ldots) \), \( 1 \leq j \leq n \), are Hasse-Schmidt derivations of \( A \) (over \( k \)) (cf. [7], §27).

Now, let us recall the following basic well known result (cf. [2] Propositions 5.5.3 and 5.5.6).

(1.3) Proposition. Let \( B \) be a noetherian ring, \( P \) be a prime ideal of \( B[t] \) and \( p = P \cap B \). Then, one of the following conditions holds:

(a) \( P = p[t] \), \( \text{ht}(P) = \text{ht}(p) \) and \( B[t]/P \cong (A/p)[t] \).

(b) \( P \supset p[t] \), \( \text{ht}(P) = \text{ht}(p) + 1 \) and \( B[t]/P \) is an algebraic extension of \( B/p \) (generated by \( t \) mod \( P \)).

2 Coefficients fields and the extension \( k \to k_{(\infty)} \).

Let \( k \) be a perfect field of characteristic \( p > 0 \) and \( A \) a \( k \)-algebra. For every Hasse-Schmidt derivation \( \hat{\Omega} \in \text{HS}_k(A) \), we also denote by \( \hat{\Omega} \in \text{HS}_{k_{(\infty)}}(A_{(\infty)}) \) the extended Hasse-Schmidt derivation. If \( n \subset A_{(\infty)} \) is a maximal ideal, we denote by \( \hat{\Omega}_n \) and \( \hat{\Delta}_n \) the extended Hasse-Schmidt derivations to \( (A_{(\infty)})_n \) and \( (A_{(\infty)})_n \), respectively.

The following theorem generalizes Theorem 2.3 of [8] to the positive characteristic case.

(2.1) Theorem. Let \( k \) be a perfect field of positive characteristic \( p > 0 \), \( A = k[[X_1, \ldots, X_n]] \) the power series ring and let us consider the Hasse-Schmidt derivations \( \Delta^j \in \text{HS}_k(A) \), \( j = 1, \ldots, n \), defined in (1.2). Then, for each maximal ideal \( n \subset A_{(\infty)} \) the set

\[
K_0 = \left\{ a \in (A_{(\infty)})_n \mid (\hat{\Delta}^j)_n(a) = 0 \quad \forall j = 1, \ldots, n; \quad \forall i \geq 1 \right\}
\]
is a coefficient field of the complete local ring \((\hat{A}_\infty)_{\mathfrak{n}}\).

**Proof.** We proceed in two steps, as in the proof of Theorem 2.3 of [3]: reduction to the case \(n = 1\) and treatment of this case.

**Step 1: the reduction.** Let us write \(P = \mathfrak{n} \cap A_{\infty}, \mathfrak{p} = \mathfrak{n} \cap A = P \cap A = P_{(m)} \cap A\).

From [1.1](iii), (iv) we know that the ideals \(n_{(m)}\) are maximal and \(\text{ht}(n_{(m)}) = \text{ht}(n)\) for all \(m \geq 0\). By Remark (1.8) of [3], there are only two possibilities for the prime ideal \(\mathfrak{p}:

(i) \(\text{ht}(\mathfrak{p}) = n\), and then \(\mathfrak{p} = (X_1, \ldots, X_n)\) and \(n = \mathfrak{p}^e\).

(ii) \(\text{ht}(\mathfrak{p}) = n - 1\).

In case (i), \(k_{(\infty)}\) is a coefficient field of \((A_{(\infty)})_{\mathfrak{n}}\) as well as of its completion, and \((\Delta^j_{\infty})_{\mathfrak{n}}(k_{(\infty)}) = 0\) for every \(j = 1, \ldots, n, i \geq 1\). The theorem is then a consequence of [1.1](ix), (x).

Let us suppose we are in case (ii). By Theorem [A.6] (Normalization Lemma) there exists a new set of variables \(X'_1, \ldots, X'_n\) in \(A\) such that

- \(\mathfrak{p} \cap k[[X'_1]] = (0),\)
- \(k[[X'_1]] \hookrightarrow A/\mathfrak{p}\) is a finite extension, and since \(A/\mathfrak{p}\) is finitely generated over \(k[[X'_1]], A/\mathfrak{p}\) is a finite \(k[[X'_1]]\)-module,
- \(k((X'_1)) \hookrightarrow \mathbb{Q}(A/\mathfrak{p})\) is a separable finite extension.

Since the Hasse-Schmidt derivations of \(A\) over \(k\) with respect to the variables \(X'_i\) can be expressed in terms of the \(\Delta^j\) ([2, Theorem 2.8]), we can suppose \(X'_i = X_i\).

Let us write \(K = A_{(\infty)}/\mathfrak{n} = \mathbb{Q}(A_{\infty}/P), R = A/\mathfrak{p}, A' = k[[X'_1]], n' = n \cap A'_{(\infty)}, P' = P \cap A'_{(\infty)} = n' \cap A'_{(\infty)}\) and \(K' = A'_{(\infty)}/n' = \mathbb{Q}(A'_{(\infty)}/P')\).

We have \(R_{(m)} = A_{(m)}/\mathfrak{p} A_{(m)}, R_{(\infty)} = A_{(\infty)}/\mathfrak{p} A_{(\infty)}\), \(K = \bigcup_{m \geq 0} A_{(m)}/n_{(m)}\) and \(K' = \bigcup_{m \geq 0} A'_{(m)}/n'_{(m)}\).

Let us consider the following commutative diagram of inclusion

\[
\begin{array}{ccc}
A'[t]/P'_{[0]} & \hookrightarrow & A'[t]/P_{[0]} \\
\downarrow & & \downarrow \\
A' & \hookrightarrow & A[t]/P_{[0],} \\
\downarrow & & \downarrow \\
R = A/\mathfrak{p} & \hookrightarrow & R_{[0]}.
\end{array}
\]

The bottom inclusions are algebraic (\(R\) is a finite \(A'\)-module and \(P_{[0]} \cap A = \mathfrak{p}\)), hence the top ones must be so. In particular \(A'[t]/P'_{[0]}\) is algebraic over \(A'\), which implies (Proposition [1.3]) that \(P'_{[0]} \neq 0\), then \(n'_{(0)} \neq 0\) and \(n' \neq 0\). Therefore \(n'\) is maximal since \(\text{dim}(A') = 1.\)
Let us show that the inclusion $K' \subset K$ is separable algebraic. For that, it is enough to prove that the extensions

$$\frac{A'(m)}{n'(m)} \subset \frac{A(m)}{n(m)}$$

are finite and separable.

Let us write $L' = \text{Qt}(A') = k((X_1))$, $L = \text{Qt}(A/p)$ and consider the following diagram of field extensions

$$
\begin{align*}
&L' = \text{Qt}(A') \subset \text{Qt}\left(\frac{A'_{[m]}}{P'_{[m]}}\right) = \frac{A'(m)}{n'(m)} \\
&L = \text{Qt}(R) \subset \text{Qt}\left(\frac{A_{[m]}}{P_{[m]}}\right) = \frac{A(m)}{n(m)}.
\end{align*}
$$

These extensions satisfy the following properties:

i) $L' \subset L$ is finite and separable. Hence, there is a primitive element $e$, $L = L'[e]$, whose minimal polynomial $f(X) \in L'[X]$ satisfies $f'(X) \neq 0$.

ii) By Proposition (1.3) the extensions $L \subset \text{Qt}\left(\frac{A_{[m]}}{P_{[m]}}\right)$, $L' \subset \text{Qt}\left(\frac{A'_{[m]}}{P'_{[m]}}\right)$ are finite and generated by the class $\overline{t}$ of $t$.

Therefore,

$$\frac{A(m)}{n(m)} = \text{Qt}\left(\frac{A_{[m]}}{P_{[m]}}\right) = L[\overline{t}] = L'[e][\overline{t}] = \left(\text{Qt}\left(\frac{A'_{[m]}}{P'_{[m]}}\right)\right)[e] = \left(\frac{A'(m)}{n'(m)}\right)[e]$$

and the extension

$$\frac{A'(m)}{n'(m)} \subset \frac{A(m)}{n(m)}$$

is finite and separable for all $m \geq 0$. Hence, $K' \subset K$ is separable algebraic.

Let us assume that the theorem is proved for $n = 1$. Then

$$K'_0 = \left\{ a \in (A'_{(\infty)})_{n'} \mid (\overline{\Delta_i})_{n'}(a) = 0 \quad \forall i \geq 1 \right\}$$

is a coefficient field of $(A'_{(\infty)})_{n'}$.

We can consider $K'_0$ as a subfield of $(A_{(\infty)})_n$ via the inclusion $(A'_{(\infty)})_{n'} \hookrightarrow (A_{(\infty)})_n$. Since $K'_0 \supseteq K'$ and $K' \subset K$ is separable algebraic, we deduce that $K'_0$ is a quasi-coefficient field of $(A_{(\infty)})_n$.

It is clear that for all $a \in K'_0$

$$\overline{(\Delta_i)_n}(a) = 0 \quad \forall j = 1, \ldots, n, \forall i \geq 1.$$
In particular, the $\hat{\Delta}_n$ are Hasse-Schmidt derivations over $K'_0$, and by (1.1) (ix), the $\{\Delta_1, \ldots, \Delta_n\}$ form a basis of $\text{Der}_{K'_0}((A_{(\infty)})_n)$.

Now, by applying (1.1) (x), we obtain that

$$\left\{ a \in (A_{(\infty)})_n \mid (\hat{\Delta}_j)_n(a) = 0 \quad \forall j = 1, \ldots, n, \forall i \geq 1 \right\}$$

is a coefficient field of $(A_{(\infty)})_n$ and the theorem is proved.

**Step 2: the case n=1.** Let us write $A = k[[X]]$, $L = \text{Qt}(A) = k((X))$ and let $n$ be a maximal ideal of $A_{(\infty)} = A \otimes_k k_{(\infty)}$. Let us denote $P = n \cap A_{[n]}$. By (1.1) (iv), we know that

$$\text{ht}(m) = \text{ht}(n_{(m)}) = \text{ht}(P_{[m]}) = \text{ht}(P) = 1.$$ 

As in the first step, we focus on the case $n \cap A = (0)$ (and then $P \cap A = (0)$). Since each $A_{[m]} = A[t_m]$ is a unique factorization domain and each $P_{[m]}$ is a prime ideal of $A_{[m]}$ of height 1, $P_{[m]}$ is generated by an irreducible polynomial $F_m(t_m) \in A[t_m]$ of degree $d \geq 1$ and with some non-constant coefficient, since $P_{[m]} \cap k[t_m] = (0)$. By irreducibility, at least one of the coefficients of $F_m(t_m)$ must be a unit, so we may assume that it is 1.

Let us write $K = \frac{A_{(\infty)}}{n}$ and $K_m = \frac{A_{(m)}}{n_{(m)}}$. Since $A_{(\infty)}$ and $A_{(m)}$ are localizations of $A_{[\infty]}$ and $A_{[m]}$ respectively, it follows that

$$K = \frac{A_{(\infty)}}{n} = \text{Qt} \left( \frac{A_{[\infty]}}{P} \right), \quad K_m = \frac{A_{(m)}}{n_{(m)}} = \text{Qt} \left( \frac{A_{[m]}}{P_{[m]}} \right).$$

The minimal polynomial of $\theta_m := (t_m \mod P_{[m]})$ over $L$ is $F_m(t_m)$. We have $K_m = L[\theta]$, $K = \bigcup_{m \geq 0} K_m = \bigcup_{m \geq 0} L[\theta_m] = L[\theta_0, \theta_1, \theta_2, \ldots]$, where $\theta_m = \theta_{m+1}^p$, and the inclusion $k_{(\infty)} \hookrightarrow K$ is a $k$-morphism which sends each $t_m$ onto $\theta_m$.

By (1.1) (vi), it follows that $L^2 = k((X))^2 = k$, and we can apply (1.1) (vii) to conclude that there exists $m_0 \geq 0$ such that $P$ is the extended ideal of $P_{[m_0]} = (F_{m_0}(t_{m_0}))$. Then, $P$ (resp. $n$) is the ideal of $A_{[\infty]}$ (resp. of $A_{(\infty)}$) generated by $\mu = F_{m_0}(t_{m_0})$. Moreover, for every $j \geq 1$, $P_{[m_0+j]}$ is the extended ideal of $P_{[m_0]}$ and some of the coefficients of $\mu$ is not a $p$-th power. Hence, we can take

$$F_{m_0+j}(t_{m_0+j}) = F_{m_0}(t_{m_0}) = F_{m_0}(t_{m_0+j}^p), \quad j \geq 1.$$ 

Since $k_{(\infty)}$ is perfect, the field extension $k_{(\infty)} \subset K$ is separable and, by Cohen structure theorem, there exists a $k_{(\infty)}$-isomorphism

$$\varphi : (A_{(\infty)})_n \overset{\sim}{\rightarrow} K[[s]] \quad (2)$$
which induces the identity on residue fields and sends the regular parameter $\mu$ of $(A_{(\infty)})_n$ onto $s$. One has:

\[
\varphi(\mu) = s \\
\varphi(t_m) = \theta_m \\
\varphi(X) = X + \xi \quad \text{with} \quad \xi \in (s).
\]

Let us denote by

\[
\Delta^X = (1, \Delta^1_1, \Delta^2_2, \ldots) \in \text{HS}_K(k[[X]])
\]

the Hasse-Schmidt derivation defined in (1.2) and let us assume, for the moment, that $\varphi$ satisfies the relation

\[
\varphi(a(X)) = a(X + \xi) \subseteq k[[X,\xi]] \subseteq K[[\xi]] \subseteq K[[s]] \tag{3}
\]

for all $a(X) \in A = k[[X]]$.

Then, writing $\mu = a_d(X)t_{m_0}^d + \cdots + a_0(X)$,

\[
s = \varphi(\mu) = \varphi \left( \sum_{r=0}^{d} a_r(X)t_{m_0}^r \right) = \sum_{r=0}^{d} \varphi(a_r(X)) \theta_{m_0}^r = \sum_{r=0}^{d} a_r(X + \xi)\theta_{m_0}^r \tag{1.2}
\]

\[
= \sum_{r=0}^{d} \left( \sum_{i=0}^{\infty} \Delta^X_i(a_r(X))\xi^i \right) \theta_{m_0}^r = \sum_{i=0}^{\infty} \sum_{r=0}^{d} \Delta^X_i(a_r(X))\theta_{m_0}^r \xi^i \in K[[\xi]],
\]

and $\xi$ must be of order one in $s$. Hence, $\xi$ is a new variable in $K[[s]]$ and $K[[s]] = K[[\xi]]$.

Let us denote by $\Delta'$ the unique extension of $\Delta^X$ to $K[[s]]$ through

\[
A \xrightarrow{\text{scalar ext.}} A \otimes_k k(\infty) \xrightarrow{\text{local}} (A_{(\infty)})_n \xrightarrow{\text{compl.}} (A_{(\infty)})_n \xrightarrow{\varphi} K[[s]],
\]

which belongs to $\text{HS}_{k(\infty)}(K[[s]])$, and let us denote by

\[
\Delta^\xi = (1, \Delta^1_1, \Delta^2_2, \ldots) \in \text{HS}_K(K[[\xi]]) = \text{HS}_K(K[[s]])
\]

the Hasse-Schmidt derivation defined in (1.2) this time with respect to the variable $\xi$.

We will show that relation (3) implies that $\Delta^\xi = \Delta'$, i.e.

\[
(\varphi \circ \Delta^X_i)(a) = (\Delta^\xi_i \circ \varphi)(a) \quad \forall i \geq 0, \forall a \in k[[X]], \tag{4}
\]

and then

\[
\varphi^{-1}(K) = \varphi^{-1} \left( \left\{ c \in K[[s]] \mid \Delta^\xi_i(c) = 0, \forall i > 0 \right\} \right) = \left\{ a \in (A_{(\infty)})_n \mid (\Delta^X_i)_n(a) = 0, \forall i > 0 \right\}
\]

is a coefficient field of $(A_{(\infty)})_n$ and the step 2 would be finished.

Let $\varphi_0 : A = k[[X]] \to k[[X,\xi]]$ be the local $k$-homomorphism defined by $\varphi_0(X) = X + \xi$. Relation (3) says that $\varphi(a(X)) = \varphi_0(a(X))$ for all $a(X) \in A$. 
Let $Y$ be a new variable and consider the local $k$-homomorphisms $\delta : k[[X]] \rightarrow k[[X,Y]]$, $\varepsilon : k[[X,\xi]] \rightarrow k[[X,\xi,Y]]$ and $\widetilde{\varphi}_0 : k[[X,Y]] \rightarrow k[[X,\xi,Y]]$ defined by:

$$\delta(X) = X + Y, \quad \varepsilon(X) = X, \quad \varepsilon(\xi) = \xi + Y, \quad \widetilde{\varphi}_0(Y) = Y, \quad \widetilde{\varphi}_0(X) = X + \xi.$$

Let us also consider the local $K$-homomorphism $\Theta : K[[\xi]] \rightarrow K[[\xi,Y]]$ defined by $\Theta(\xi) = \xi + Y$. Then, the following diagram

$$
\begin{array}{ccc}
k[[X]] & \xrightarrow{\varphi_0} & k[[X,\xi]] \\
\downarrow{\delta} & & \downarrow{\varepsilon} \\
k[[X,Y]] & \xrightarrow{\widetilde{\varphi}_0} & k[[X,\xi,Y]] \\
\end{array}
\xrightarrow{\subset} K[[\xi]] \quad \xrightarrow{\subset} K[[\xi,Y]]
$$

is commutative and we have

$$
\sum_{i=0}^{\infty} \Delta_\xi^i(\varphi(a))Y^i = \Theta(\varphi(a)) = \varepsilon(\varphi_0(a)) = \widetilde{\varphi}_0(\delta(a)) = \sum_{i=0}^{\infty} \varphi_0(\Delta_\xi^i(a))Y^i = \sum_{i=0}^{\infty} \varphi(\Delta_\xi^i(a))Y^i
$$

for all $a \in k[[X]]$. Therefore relation 4 is proved and $\Delta_\xi = \Delta'_\xi$.

The point now is to construct a $\varphi$ in (2) satisfying (3). We first find $\varphi(X) = X + \xi \in K[[s]]$, and for this we state and prove the following lemma which is a generalization of Lemma (2.3.3) of [S].

(2.2) Lemma. There exists a unique $\xi \in K[[s]]$ such that $\xi(0) = 0$ of order 1 satisfying

$$a_d(X + \xi)\theta_{m_0}^d + \cdots + a_0(X + \xi) = s.$$

PROOF. The lemma is a consequence of the implicit function theorem. Let $G(s,\sigma) = a_d(X + \sigma)\theta_{m_0}^d + \cdots + a_0(X + \sigma) - s \in K[[s,\sigma]]$, with

$$G(0,0) = a_d(X)\theta_{m_0}^d + \cdots + a_0(X) = F_{m_0}(\theta_{m_0}) = 0.$$

We have to check that

$$\left( \frac{\partial G}{\partial \sigma} \right)_{s=0} = a'_d(X)\theta_{m_0}^d + \cdots + a'_0(X) \neq 0 \quad \text{in} \quad K.$$

Assume the contrary: then $a'_d(X)\theta_{m_0}^d + \cdots + a'_0(X)$ should be a multiple of $F_{m_0}(t_{m_0})$ in $k((X))[t_{m_0}]$ and there would be an $\alpha \in k((X))$ such that

$$a'_r(X) = \alpha(X)a_r(X) \quad \text{for every} \quad r = 0,1,\ldots,d.$$
Since some of the coefficients $a_r$ is 1, we deduce that $\alpha(X) = 0$ and $a'_r(X) = 0$ for every $r = 0, 1, \ldots, d$, and then there are $b_r(X) \in k[[X]]$ such that $a_r(X) = b_r(X^p)$. Since $k$ is perfect we conclude that $a_r(X) = b_r(X)^p$, contradicting the fact that some of the coefficients of $\mu$ is not a $p$-th power.

So $(\frac{\partial G}{\partial \sigma})|_{s=\sigma=0} \neq 0$, and by the implicit function theorem, there is a unique $\xi \in K[[s]]$ such that $\xi(0) = 0$ and $G(s, \xi) = 0$. Then $\xi$ has order 1 since

$$
\left(\frac{\partial \xi}{\partial s}\right)(0) = \left[\left(\frac{\partial G}{\partial \sigma}\right)(0,0)\right]^{-1} \neq 0.
$$

Q.E.D.

Let us finish the proof of Theorem (2.1). Let $\xi \in K[[s]]$ be as in the Lemma (2.2) and let us consider the local $k$-homomorphism

$\varphi_0 : A = k[[X]] \to k[[X, \xi]]$

such that $\varphi_0(X) = X + \xi$. Let us call $\varphi : A \to K[[s]]$ the composition of $\varphi_0$ with the inclusion $k[[X, \xi]] \subset K[[\xi]] = K[[s]]$.

We extend $\varphi$ to $A_{(\infty)}$ by defining $\varphi(t_m) = \theta_m \in K_m \subseteq K$ and we obtain a $k_{(\infty)}$-homomorphism $\varphi : A_{(\infty)} \to K[[s]]$ satisfying (3) by construction and sending

$$
\mu = F_{m_0}(t_{m_0}) = a_d(X)t_{m_0}^d + \cdots + a_0(X)
$$

onto the element

$$
a_d(X + \xi)t_{m_0}^d + \cdots + a_0(X + \xi) = s.
$$

Therefore, the contraction of the maximal ideal $(s)$ by $\varphi$ must be $n = (\mu)$, and so we can extend $\varphi$, first to a local $k_{(\infty)}$-homomorphism $\varphi : (A_{(\infty)})_n \to K[[s]]$, and second, by completion, to $\varphi : (A_{(\infty)})_n \to K[[s]]$ satisfying (3) as desired. Q.E.D.

The following example shows that, in order to generalize Theorem (2.3) in [8] to the positive characteristic case, one has to consider the scalar extension $k \to k_{(\infty)}$ instead of $k \to k(t)$.

(2.3) Example. Let $k$ be a perfect field of characteristic $p > 0$, $A = k[[X]]$ and consider the maximal ideal $n = (X^p t - 1)$ in $A(t) = A \otimes_k k(t)$. Then, there is no coefficient field of $(\hat{A(t)})_n$ on which the $(\Delta^X_i)_n$, $i > 0$, vanish.
Assume the contrary, i.e. there exists a coefficient field \( K_0 \) of \( B := \widehat{\left( A(t) \right)}_n \) such that \( \left( \Delta^X_i \right)_n(K_0) = 0 \) for all \( i > 0 \), i.e. \( \left( \Delta^X_i \right)_n \in \text{HS}_{K_0}(B) \).

Since \( \left( \Delta^X_i \right)_n(X) = 1 \), \( \left( \Delta^X_i \right)_n \) would be a basis of \( \text{Der}_{K_0}(B) \) by Theorem 30.6 of [6], and by Theorem 3.17 of [4] we would have the equality

\[
K_0 = \left\{ a \in B \mid \left( \Delta^X_i \right)_n(a) = 0, \forall i > 0 \right\}.
\]

In particular \( k(t) \subset K_0 \).

The residue field of \( B \) is

\[
K = \frac{A(t)}{n} = \frac{\mathbb{Q}t}{(X^p t - 1)} = k[[X]][X^{-p}] = k((X)),
\]

where the inclusion \( k(t) \hookrightarrow K \) sends \( t \) to \( X^{-p} \). Let \( \tau : K_0 \sim K \) be the \( k(t) \)-isomorphism induced by the inclusion \( K_0 \subset B \).

By Cohen structure theorem, the inclusion \( K_0 \subset B \) would be extended to an isomorphism \( \psi : K_0[[s]] \sim B \) such that \( \psi(s) = X^p t - 1 \) (\( B \) is a one dimensional complete local noetherian local ring with parameter \( X^p t - 1 \)) and the diagram

\[
\begin{array}{ccc}
K_0[[s]] & \xrightarrow{\psi} & B = \widehat{\left( A(t) \right)}_n \\
\text{res.} & & \downarrow \text{res.} \\
K_0 & \xrightarrow{\tau} & K
\end{array}
\]

is commutative.

Since \( \tau^{-1}(X) \) is congruent to \( X \mod. \) the maximal ideal of \( B \), we deduce that \( \psi^{-1}(X) \) is congruent to \( \tau^{-1}(X) \mod. s \), i.e. \( \psi^{-1}(X) = \tau^{-1}(X) + \xi \), with \( \xi \in (s) \).

On the other hand,

\[
s = \psi^{-1}(X^p t - 1) = \psi^{-1}(X^p)\psi^{-1}(t) - 1 = \psi^{-1}(X)^p t - 1 = (\tau^{-1}(X) + \xi)^p t - 1 = (\tau^{-1}(X)^p + \xi^p) t - 1 = (t^{-1} + \xi^p) t - 1 = t\xi^p \in (s^p),
\]

which is a contradiction.

**Appendix: The Normalization Lemma for power series rings over perfect fields**

In this appendix we give a proof of the normalization lemma for power series rings over an arbitrary perfect field of positive characteristic. Our proof is an adaptation of Abhyankar’s proof [2], 23.7 and 24.5, which uses generic linear changes of coordinates and thus requires the field \( k \) to be infinite.
The following lemma is straightforward.

\textbf{(A.1) Lemma.} Let $L$ be a field of characteristic $p > 0$, and let $L \subset K = L[\alpha_1, \ldots, \alpha_n]$ a field extension with $\alpha_i^p \in L$ for $i = 1, \ldots, n$, and $[K : L] = p^e$. Then, there exist $\alpha_1, \ldots, \alpha_i$ such that $K = L[\alpha_1, \ldots, \alpha_i]$.

A series $f(X_1, \ldots, X_n) \in k[[X_1, \ldots, X_n]]$ is said to be $X_n$-distinguished if $f(0, \ldots, 0, X_n) \neq 0$.

The following combinatorial lemma is classical.

\textbf{(A.2) Lemma.} Let $\sigma = (\sigma_1, \ldots, \sigma_{n-1}) \in (\mathbb{N}^*)^{n-1}$ and $L_\sigma : \mathbb{N}^n \to \mathbb{N}$ defined by $L_\sigma(\alpha) = \sigma_1 \alpha_1 + \cdots + \sigma_{n-1} \alpha_{n-1} + \alpha_n$ for all $\alpha \in \mathbb{N}^n$. Then, for each finite subset $F \subset \mathbb{N}^n$, there exists a constant $C \geq 1$ such that the restriction $L_\sigma|_F$ is injective for all $\sigma$ with $\sigma_1 \geq \sigma_2 C$, $\sigma_2 \geq \sigma_3 C$, $\ldots$, $\sigma_{n-2} \geq \sigma_{n-1} C$, $\sigma_{n-1} \geq C$.

\textbf{Proof.} The proof is standard by a double induction on $n$ and $\sharp F$. Q.E.D.

\textbf{(A.3) Lemma.} Let $f(X_1, \ldots, X_n) \in k[[X]]$ be a non-zero and non-unit formal power series. Then for $\sigma_1 \gg \sigma_2 \gg \cdots \gg \sigma_{n-1} \gg 0$, the series $f(X_1 + X_n^{\sigma_1}, \ldots, X_{n-1} + X_n^{\sigma_{n-1}}, X_n)$ is $X_n$-distinguished.

\textbf{Proof.} Let us write $f(X_1, \ldots, X_n) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha X_1^{\alpha_1}, \ldots, X_n^{\alpha_n}$ and consider the Newton’s diagram

$$N(f) = \{ \alpha \in \mathbb{N}^n \mid f_\alpha \neq 0 \} \neq \emptyset, \quad \emptyset \notin N(f).$$

Let $F \subset N(f)$ be the finite set of minimal elements with respect to the usual partial ordering in $\mathbb{N}^n$. We have $N(f) \subset F + \mathbb{N}^n$.

By Lemma \textbf{(A.2)} we obtain that $L_\sigma|_F$ is injective for $\sigma_1 \gg \sigma_2 \gg \cdots \gg \sigma_{n-1} \gg 0$, and then the series

$$f(0 + X_n^{\sigma_1}, \ldots, 0 + X_n^{\sigma_{n-1}}, X_n) = \sum_{\alpha \in N(f)} f_\alpha X_n^{L_\sigma(\alpha)}$$

has order $\min_{\alpha \in F} L_\sigma(\alpha)$ and is non zero. Q.E.D.

\textbf{(A.4) Proposition.} Let $a \subset A = k[[X_1, \ldots, X_n]]$ be a proper ideal with $e = \dim (A/a)$. Then there exists a change of coordinates of the form

$$\begin{align*}
Y_1 &= X_1 + F_1(X_2^p, \ldots, X_n^p) \\
Y_2 &= X_2 + F_2(X_3^p, \ldots, X_n^p) \\
& \vdots \\
Y_{n-1} &= X_{n-1} + F_{n-1}(X_n^p) \\
Y_n &= X_n
\end{align*}$$

with $F_i \in \mathbb{F}_p[X_{i+1}, \ldots, X_n]$ for $i = 1, \ldots, n - 1$, such that $a \cap k[[Y_1, \ldots, Y_e]] = \{ 0 \}$ and the extension $k[[Y_1, \ldots, Y_e]] \hookrightarrow A/a$ is finite.
Proof. We proceed by induction on $n$. 
For $n = 1$: let $\mathfrak{a}$ a proper ideal of $A = k[[X_1]]$ of height 1. Then $\mathfrak{a} = (X_1^n)$ and 
\[
k \subset k[[X_1]]/\mathfrak{a} = k[X_1]
\]
is finite of rank $m$.

Suppose now the result is true for $n - 1$, and let $\mathfrak{a}$ be a proper ideal of $A = k[[X_1, \ldots, X_n]]$. Let us take a non-zero and non-unit formal power series $f(X_1, \ldots, X_n) \in \mathfrak{a}$.

By the change 
\[
\begin{aligned}
Y_j &= X_j - X_n^{\sigma_j}, \quad j = 1, \ldots, n - 1 \\
Y_n &= X_n,
\end{aligned}
\]
with $\sigma_j = \hat{p}$, $\sigma_1 \gg \sigma_2 \gg \cdots \gg \sigma_n-1 \gg 0$, and by Lemma (A.3) we deduce that the series 
\[
g(Y_1, \ldots, Y_{n-1}, Y_n) = f(Y_1 + Y_n^{\sigma_1}, \ldots, Y_{n-1} + Y_n^{\sigma_n-1}, Y_n) = f(X_1, \ldots, X_n)
\]
is $Y_n$-distinguished.

By Weierstrass preparation theorem we can write $g(Y_1, \ldots, Y_{n-1}, Y_n) = u \cdot H$, where $u$ is a unit and 
\[
H = Y_n^q + a_{q-1}(Y_1, \ldots, Y_{n-1})Y_n^{q-1} + \cdots + a_0(Y_1, \ldots, Y_{n-1}),
\]
where 
\[
q = \text{ord}_{X_n}(f(X_n^{\sigma_1}, \ldots, X_n^{\sigma_n-1}, X_n)) \geq 1 \text{ and } a_i(\mathbf{0}) = 0.
\]
Consequently $H \in \mathfrak{a}$ and the ring extension 
\[
\frac{k[[Y_1, \ldots, Y_{n-1}]]}{\mathfrak{a}^e} \subseteq \frac{k[[Y_1, \ldots, Y_n]]}{\mathfrak{a}} = k[[Y_1, \ldots, Y_{n-1}]][Y_n]
\]
is finite. The proposition follows by applying induction hypothesis to $\mathfrak{a}^e$.
Q.E.D.

From now on $k$ will be a perfect field of characteristic $p > 0$, $\mathfrak{p}$ a prime ideal in $A = k[[X_1, \ldots, X_n]]$, $R = A/\mathfrak{p}$, $L = \text{Qt}(A) = k((X_1, \ldots, X_n))$ and $K = \text{Qt}(R)$. Let us denote $e = \text{dim } R$ and $a \in R$ the class $a \mod \mathfrak{p}$ of any element $a \in A$.

The following proposition is an adaptation of (24.1) and (24.4) of [I], which uses Proposition (A.4) instead of (23.3) of loc. cit.

(A.5) Proposition. Under the above hypothesis, the relations

\[
K = K^p[X_1, \ldots, X_n], \quad [K : K^p] = p^e,
\]
hold and the set \( \{X_1^{\sigma_1}, \ldots, X_n^{\sigma_n} : 0 \leq \sigma_i < p, \ i = 1, \ldots, n \} \) is a system of generators of the extension $K^p \subset K$. Moreover, after a permutation of variables, we have $K = K^p[X_1, \ldots, X_e]$ and \( \{X_1^{\sigma_1}, \ldots, X_e^{\sigma_e} : 0 \leq \sigma_i < p, \ i = 1, \ldots, e \} \) is a basis of $K$ as $K^p$-vector space.

Proof. Since $k$ is perfect, one has $A = A^p[X_1, \ldots, X_n]$, $L = L^p[X_1, \ldots, X_n]$ and
\[
\{X_1^{\sigma_1}, \ldots, X_n^{\sigma_n} : 0 \leq \sigma_1 < p, \ldots, 0 \leq \sigma_n < p \}
\]
is basis of $L$ (resp. of $A$) as $L^p$-vector space (resp. as $A^p$-module). In particular $[L : L^p] = p^n$ and $A$ is a finite $A^p$-module.

Hence, $R = R^p[\overline{X}_1, \ldots, \overline{X}_n]$, $K = K^p[\overline{X}_1, \ldots, \overline{X}_n]$ and

$$\{\overline{X}^{\sigma_1}_1 \cdots \overline{X}^{\sigma_n}_n : 0 \leq \sigma_i < p, \ i = 1, \ldots, n\}$$

is a system of generators of the extension $K^p \subset K$.

By Proposition (A.4) we obtain a finite ring extension $B = k[[Y_1, \ldots, Y_e]] \subset R$ and then $L_1 = \text{Qt}(B) = k((Y_1, \ldots, Y_e)) \subset K$ is a finite field extension.

By using Frobenius morphism one proves that $[K : L_1] = [K^p : L_1^p]$, and from

$$[K : L_1][L_1 : L^p] = [K : L_1^p] = [K : K^p][K^p : L_1^p]$$

we deduce that $[K : K^p] = [L_1 : L^p] = p^e$.

Finally, by Lemma (A.1) we know that after a permutation of variables

$$\{\overline{X}^{\sigma_1}_1 \cdots \overline{X}^{\sigma_e}_e : 0 \leq \sigma_i < p, \ i = 1, \ldots, e\},$$

is a basis of $K$ as $K^p$-vector space.

Q.E.D.

(A.6) Theorem. (Normalization Lemma for power series ring over perfect fields in positive characteristics) In the situation of Proposition (A.5) there exists a new set of variables $Y_1, \ldots, Y_n \in A = k[[X_1, \ldots, X_n]]$ such that

1. $p \cap k[[Y_1, \ldots, Y_e]] = \{0\}$.
2. $B = k[[Y_1, \ldots, Y_e]] \hookrightarrow R = A/p$ is a finite ring extension.
3. $L_1 = \text{Qt}(B) \hookrightarrow K = \text{Qt}(R)$ is a separable finite extension.

Proof. In view of Proposition (A.5) after a permutation of variables $X_i$ we get $K = K^p[\overline{X}_1, \ldots, \overline{X}_e]$ and $\{\overline{X}^{\sigma_1}_1 \cdots \overline{X}^{\sigma_e}_e : 0 \leq \sigma_1 < p, \ldots, 0 \leq \sigma_e < p\}$ is basis of $K$ as $K^p$-vector space.

By Proposition (A.4) there is a new set of variables $Y_1, \ldots, Y_n$ in $k[[X_1, \ldots, X_n]]$ of the form

$$Y_j = X_j + F_j(X_{j+1}^p, \ldots, X_n^p), \quad 1 \leq j \leq n - 1$$

and $Y_n = X_n$, with $F_j \in F_p[X_{j+1}, \ldots, X_n]$, such that $p \cap k[[Y_1, \ldots, Y_e]] = \{0\}$ and the extension $B = k[[Y_1, \ldots, Y_e]] \hookrightarrow A/p$ is finite. Hence, $K$ is a finite field extension of $L_1 = \text{Qt}(B)$.

Since

$$\overline{X}^{\sigma_1}_1 \cdots \overline{X}^{\sigma_e}_e = (Y_1^p - F_1(\overline{X}_2^p, \ldots, \overline{X}_n^p))^{\sigma_1} \cdots (Y_e^p - F_e(\overline{X}^p_{e+1}, \ldots, \overline{X}_n^p))^{\sigma_e},$$

$Y_1, \ldots, Y_e \in L_1 = k((Y_1, \ldots, Y_e))$ and $F_j(\overline{X}_{j+1}^p, \ldots, \overline{X}_n^p) = F_j(\overline{X}_{j+1}, \ldots, \overline{X}_n)^p \in K^p$, we deduce that $\overline{X}^{\sigma_1}_1 \cdots \overline{X}^{\sigma_e}_e \in K^p(L_1)$ and $K = K^p(L_1)$. Therefore $K$ is a separable finite extension of $L_1$ (cf. [2], Theorem 8 on p. 69).

Q.E.D.
References


