A $q$-analog of the Racah polynomials and the $q$-algebra $SU_q(2)^*$

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Abstract

We study some $q$-analogues of the Racah polynomials and some of their applications in the theory of representation of quantum algebras.

1 Introduction

In the paper [6] an orthogonal polynomial family that generalizes the Racah coefficients or $6j$-symbols was introduced: the so-called Racah and $q$-Racah polynomials. These polynomials were in the top of the so-called Askey Scheme (see e.g. [14]) that contains all classical families of hypergeometric orthogonal polynomials. Some years later the same authors [7] introduced the celebrated Askey-Wilson polynomials. One of the important properties of these polynomials is that from them one can obtain all known families of hypergeometric polynomials and $q$-polynomials as particular cases or as limit cases (for a review on this see the nice survey [14]). The main tool in these two works was the hypergeometric and basic series, respectively. On the other hand, the authors of [21] (see also [20, Russian Edition]) considered the $q$-polynomials as the solution of a second order difference equation of hypergeometric-type on the non-linear lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. In particular, they show that the solution of the hypergeometric-type equation can be expressed as certain basic series and, in such a way, they recovered the results by Askey & Wilson.

The interest of such polynomials increase after the appearance of the $q$-algebras and quantum groups [9, 10, 12, 14, 24]. However, from the first attends to built the $q$-analog of the Wigner-Racah formalism for the simplest quantum algebra $U_q(su(2))$ [13] (see also [11, 14, 17]) becomes clear that for obtaining the $q$-polynomials intimately connected with the $q$-analogues of the Racah and Clebsch Gordan coefficients, i.e., a $q$-analogue of the Racah polynomials $R_{\alpha,\beta}^\gamma(x(s), a, b)_q$ and the dual Hahn polynomials $\omega^{(\alpha,\beta)}(x(s), a, b)_q$, respectively, it is better to use a different lattice —in fact the $q$-Racah polynomials $R_{n,\gamma}^{\alpha,\beta}(x(s), N, \delta)_q$ introduced in [7] (see also [14]) were defined on the lattice $x(s) = q^{-s} + \delta q^{-N} q^s$ that depends not only of the variable $s$ but also on the parameters of the polynomials—, namely,

$$x(s) = [s]_q [s + 1]_q,$$

that only depends on $s$, where by $[s]_q$ we denote the $q$-numbers (in its symmetric form)

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad \forall s \in \mathbb{C}.$$
With this choice the $q$-Racah polynomials $u_n^\alpha,\beta(x(s), a, b)_q$ are proportional to the $q$-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$. A very nice and simple approach to $6j$-symbols has been recently developed in [23].

Moreover, this connection gives the possibility to a deeper study of the Wigner-Racah formalism (or the $q$-analogue of the quantum theory of angular momentum [25, 26, 27, 28] for the quantum algebras $U_q(su(2))$ and $U_q(su(1, 1))$ using the powerful and well-known theory of orthogonal polynomials on non-uniform lattices. On the other hand, using the $q$-analogue of the quantum theory of angular momentum [25, 26, 27, 28] we can obtain several results for the $q$-polynomials, some of which are non trivial from the point of view of the theory of orthogonal polynomials (see e.g. the nice surveys [13, 20]). In fact, in the present paper we present a detailed study of some $q$-analogues of the Racah polynomials on the lattice (1): the $u_n^\alpha,\beta(x(s), a, b)_q$ and the $\tilde{u}_n^\alpha,\beta(x(s), a, b)_q$ as well as their connection with the $q$-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$ in order to establish which properties of the polynomials correspond to the $6j$-symbols and vice versa.

The structure of the paper is as follows: In section 2 we present some general results from the theory of orthogonal polynomials on the non-uniform lattices taken from [21, 20]. In Section 2.1 a detailed discussion of the Racah polynomials $u_n^\alpha,\beta(x(s), a, b)_q$ is presented, whereas in Section 2.2 the $\tilde{u}_n^\alpha,\beta(x(s), a, b)_q$ are considered. In particular, a relation between these families is established. In Section 3 the comparative analysis of such families and the $6j$-symbols of the quantum algebra $U_q(su(2))$ is developed which gives, on one hand, some information about the Racah coefficients and, on the other hand, allow us to give a group-theoretical interpretation of the Racah polynomials on the lattice (1). Finally, some comments and remarks about $q$-Racah polynomials and the quantum algebra $U_q(su(3))$ are included.

## 2 Some general properties of $q$-polynomials

We will start with some general properties of orthogonal hypergeometric polynomials on the non-uniform lattices [20].

The hypergeometric polynomials are the polynomial solutions $P_n(x(s))_q$ of the second order linear difference equation of hypergeometric-type on the non-uniform lattice $x(s)$ (SODE)

$$\sigma(s) \frac{\Delta}{\Delta x(s-1/2)} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \quad x(s) = c_1[q^s + q^{-s-\mu}] + c_3, \quad q^\mu = \frac{c_1}{c_2},$$

(3)

or, equivalently

$$A_s y(s+1) + B_s y(s) + C_s y(s-1) + \lambda y(s) = 0,$$

(4)

where

$$A_s = \frac{\sigma(s) + \tau(s) \Delta x(s-1/2)}{\Delta x(s) \Delta x(s-1/2)}, \quad C_s = \frac{\sigma(s)}{\nabla x(s) \Delta x(s-1/2)}, \quad B_s = -(A_s + C_s).$$

Notice that $x(s) = x(-s-\mu)$.

In the following we will use the following notations$^1$ $P_n(s)_q := P_n(x(s))_q$ and $\sigma(-s-\mu) = \sigma(s) + \tau(s) \Delta x(s-1/2)$. With this notation the Eq. (3) becomes

$$\sigma(-s-\mu) \frac{\Delta P_n(s)_q}{\Delta x(s)} - \sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} + \lambda_n \Delta x(s-1/2) P_n(s)_q = 0.$$

(5)

The polynomial solutions $P_n(s)_q$ of [20] can be obtained by the following Rodrigues-type formula [20, 22]

$$P_n(s)_q = \frac{B_n}{\rho(s)} \nabla^{(n)} \rho_n(s), \quad \nabla^{(n)} := \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)},$$

where $x_m(s) = x(s + m/2)$,

$$\rho_n(s) = \rho(s+n) \prod_{m=1}^{n} \sigma(s+m),$$

(7)

$^1$In the exponential lattice $x(s) = c_1 q^{2s} + c_3$, so $\mu = \pm \infty$, therefore instead of using $\sigma(-s-\mu)$ one should use the equivalent function $\sigma(s) + \tau(s) \Delta x(s-1/2)$. 
and $\rho(s)$ is a solution of the Pearson-type equation $\Delta [\sigma(s)\rho(s)] = \tau(s)\rho(s)\Delta x(s - 1/2)$, or equivalently,

$$
\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\sigma(-s - \mu)}{\sigma(s+1)}.
$$

(8)

Let us point out that the function $\rho_n$ satisfy the equation $\Delta [\sigma(s)\rho_n(s)] = \tau_n(s)\rho_n(s)\Delta x_n(s-1/2)$, where $\tau_n(s)$ is given by

$$
\tau_n(s) = \frac{\sigma(s+n) + \tau(s+n)\Delta x(s+n - \frac{1}{2}) - \sigma(s)}{\Delta x_n(s - \frac{1}{2})} = \frac{\sigma(-s - n - \mu) - \sigma(s)}{\Delta x_n(s - \frac{1}{2})} = \tau'_n x_n(s) + \tau_n(0),
$$

being

$$
\tau'_n = -\frac{\lambda_{2n+1}}{[2n+1]_q}, \quad \tau_n(0) = \frac{\sigma(-s^* - n - \mu) - \sigma(s^*)}{x_n(s^*_n + \frac{1}{2}) - x_n(s^*_n - \frac{1}{2})},
$$

where $s^*_n$ is the zero of the function $x_n(s)$, i.e., $x_n(s^*_n) = 0$.

From [20] follows an explicit formula for the polynomials $P_n$ [20, Eq.(3.2.30)]

$$
P_n(s)_q = B_n \sum_{m=0}^{n} \frac{[n]_q!(-1)^{m+n}}{[m]_q!\lfloor n-m \rfloor_q!} \nabla x(s + m - \frac{n-1}{2}) \rho_n(s - n + m) \rho(s),
$$

(10)

where $[n]_q$ denotes the symmetric $q$-numbers [2] and the $q$-factorials are given by

$$
[0]_q! := 1, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q, \quad n \in \mathbb{N}.
$$

It can be shown [20] that the most general polynomial solution of the $q$-hypergeometric equation [20] corresponds to

$$
\sigma(s) = A \prod_{i=1}^{4} (s - s_i)_q = C q^{-2s} \prod_{i=1}^{4} (q^s - q^{s_i}), \quad A \cdot C \neq 0
$$

(11)

and has the form [22, Eq. (49a), page 240]

$$
P_n(s)_q = D_n \Phi_3 \left( q^{-n}, q^{2\mu + n - 1 + \sum_{i=1}^{r} s_i}, q^{s_1 + s_2 + \mu}, q^{s_1 + s_3 + \mu}, q^{s_1 + s_4 + \mu}; q, q \right),
$$

(12)

where the normalizing factor $D_n$ is given by $(x_q := q^{1/2} - q^{-1/2})$

$$
D_n = B_n \left( \frac{-A}{c_1 q^\mu x^2_q} \right)^n q^{-\frac{3}{2}(s_1 + s_2 + s_3 + s_4 + 2\mu) - 1} (q^{s_1 + s_2 + \mu}; q)_n (q^{s_1 + s_3 + \mu}; q)_n (q^{s_1 + s_4 + \mu}; q)_n.
$$

The basic hypergeometric series $\Phi_\mu$ are defined by [14]

$$
\Phi_\mu \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_p \\ ; q, z \end{array} \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k (q; q)_k} \frac{z^k}{(q^k q^\mu)^{p-r+1}},
$$

where $(a; q)_k = \prod_{m=0}^{k-1} (1 - a q^m)$, is the $q$-analogue of the Pochhammer symbol.

In this paper we will deal with orthogonal $q$-polynomials and functions. It can be proven [20], by using the difference equation of hypergeometric-type [8], that if the boundary conditions $\sigma(s)\rho(s)x^k(s - 1/2)|_{s=a,b} = 0$, for all $k \geq 0$, holds, then the polynomials $P_n(s)_q$ are orthogonal with respect to the weight function $\rho$, i.e.,

$$
\sum_{s=a}^{b-1} P_n(s)_q P_m(s)_q \rho(s) \Delta x(s - 1/2) = \delta_{nm} d^2_n, \quad s = a, a+1, \ldots, b - 1.
$$

(13)

The squared norm in [20] is [20, Eq. (3.7.15)]

$$
d^2_n = (-1)^n A_{n,n} B^2_n \sum_{s=a}^{b-1} \rho_n(s) \Delta x_n(s - 1/2),
$$

(14)
where [20 page 66]
\[ A_{n,k} = \frac{[n]_q}{[n-k]_q} \prod_{m=0}^{k-1} \left( -\frac{\lambda_{n+m}}{[n+m]_q} \right). \]  

A simple consequence of the orthogonality is the three-term recurrence relation (TTRR)
\[ x(s)P_n(s)q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \]  

where \( \alpha_n, \beta_n \) and \( \gamma_n \) are given by
\[ \alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{\lambda_n^2}. \]  

being \( a_n \) and \( b_n \) the first and second coefficients in the power expansion of \( P_n \), i.e., \( P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \cdots \). Substituting \( s = a \) in (16) we find
\[ \beta_n = \frac{x(a)P_n(a)_q - \alpha_n P_{n+1}(a)_q - \gamma_n P_{n-1}(a)_q}{P_n(a)_q}, \]  

which is an alternative way for finding the coefficient \( \beta_n \). Also we can use the expression [3 page 148]
\[ \beta_n = \frac{[n]_q \tau_{n-1}(0)}{\tau_{n-1}'} - \frac{[n+1]_q \tau_n(0)}{\tau_n'} + c_3([n]_q + 1 - [n+1]_q). \]  

To compute \( \alpha_n \) and \( \beta_n \) we need the following formulas (see e.g. [3 page 147])
\[ a_n = \frac{B_n A_{n,n}}{[n]_q^2}, \quad b_n = \frac{[n]_q \tau_{n-1}(0)}{\tau_{n-1}'} + c_3([n]_q - n). \]  

The explicit expression of \( \lambda_n \) is [22 Eq. (52) page 232]
\[ \lambda_n = -\frac{Aq^\mu}{c_1(q^{1/2} - q^{-1/2})^4} [1 + q^{s_1} + q^{s_2} + q^{s_3} + q^{s_4} + 2\mu + n - 1]_q 
\[ = -\frac{Cq^{-n+1/2}}{c_1(q^{1/2} - q^{-1/2})^2} (1 - q^n) (1 - q^{s_1+s_2+s_3+s_4+2\mu+n-1}), \]  

which can be obtained equating the largest powers of \( q^s \) in (21).

From the Rodrigues formula [19 §5.6] follows that
\[ \frac{\Delta P_n(s - \frac{1}{2})_q}{\Delta x(s - \frac{1}{2})} = -\frac{\lambda_n B_n}{B_{n-1}} P_{n-1}(s)_q, \]  

where \( P_{n-1} \) denotes the polynomial orthogonal with respect to the weight function \( \rho(s) = \rho_1(s - \frac{1}{2}) \). On the other hand, rewriting (3) as
\[ \left( \sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) \frac{\Delta}{\Delta x(s)} P_n(s)_q = -\lambda_n P_n(s)_q, \]  

it can be substituted by the following two first-order difference equations
\[ \frac{\Delta}{\Delta x(s)} P_n(s)_q = Q(s), \quad \left( \sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) Q(s) = -\lambda_n P_n(s)_q. \]  

Using the fact that \( \frac{\Delta}{\Delta x(s)} P_n(s)_q \) is a polynomial of degree \( n - 1 \) on \( x(s + 1/2) \) (see [20 §3.1]) it follows that
\[ \frac{\Delta}{\Delta x(s)} P_n(s)_q = C_n Q_{n-1}(s + \frac{1}{2}), \]  

where \( C_n \) is a normalizing constant. Comparison with [21] implies that \( Q(s) \) is the polynomial \( P_{n-1} \) orthogonal with respect to the function \( \rho_1(s - \frac{1}{2}) \) and \( C_n = -\lambda_n B_n / B_{n-1} \). Therefore, the second expression in (22) becomes
\[ P_n(s)_q = \frac{B_n}{B_{n-1}} \left( \sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) P_{n-1}(s + \frac{1}{2})_q. \]  

(23)
The $q$-polynomials satisfy the following differentiation-type formula \[ [19, 3] \text{§5.6.1]}
\[
\sigma(s) \frac{\Delta P_n(s)_q}{\Delta x(s)} = \frac{\lambda_n}{[n]_q q^n} \left[ \tau_n(s) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right]. \tag{24}
\]

Then, using the explicit expression for the coefficient $\alpha_n$, we find
\[
\sigma(s) \frac{\Delta P_n(s)_q}{\Delta x(s)} = \frac{\lambda_n}{[n]_q q^n} \left[ \tau_n(s) P_n(s)_q - \frac{\alpha_n \lambda_2 n}{[2n]_q} P_{n+1}(s)_q \right]. \tag{25}
\]

From the above equation using the identity $\Delta \frac{\Delta P_n(s)_q}{\Delta x(s)} = \Delta P_n(s)_q - \frac{\Delta P_n(s)_q}{\Delta x(s)}$ as well as the SODE 55 we find
\[
\sigma(-s - \mu) \frac{\Delta P_n(s)_q}{\Delta x(s)} = \frac{\lambda_n}{[n]_q q^n} \left[ \left( \tau_n(s) - [n]_q \tau_n \Delta x(s - \frac{1}{2}) \right) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right]. \tag{26}
\]

To conclude this section we will introduce the following notation by Nikiforov and Uvarov 20 22. First we define another $q$-analog of the Pochhammer symbols 20 Eq. (3.11.1)]
\[
(a|q)_k = \prod_{m=0}^{k-1} [a + m]_q = \frac{\bar{\Gamma}_q(a + k)}{\bar{\Gamma}_q(a)} = (-1)^k (q^a)_k (q^{1/2} - q^{-1/2})^{-k} q^{-\frac{k^2}{4}(k-1) - \frac{k}{4}}, \tag{27}
\]
where $\bar{\Gamma}_q(x)$ is the $q$-analog of the $\Gamma$ function introduced in 20 Eq. (3.2.24), and related to the classical $q$-Gamma function $\Gamma_q$ by formula
\[
\bar{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}} \Gamma_q(s) = q^{-\frac{(s-1)(s-2)}{4}} (1 - q)^{1-s} \frac{(q; q)_\infty}{(q^s; q)_\infty}, \quad 0 < q < 1.
\]

Next we define the $q$-hypergeometric function $\left._rF_p\right. (|q, z)
\[
\left._rF_p\right. \left( \begin{array}{c} a_1, \ldots, a_r \cr b_1, b_2, \ldots, b_p \end{array} \right| q, z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_p)_k} \frac{z^k}{k!} \left[ q^{-k} q^k (k-1) \right]^{p-r+1}, \tag{28}
\]
where, as before, $\varphi_q = q^{1/2} - q^{-1/2}$, and $(a|q)_k$ are given by 24. Notice that
\[
\lim_{q \to 1} \left._rF_p\right. \left( \begin{array}{c} a_1, a_2, \ldots, a_r \cr b_1, b_2, \ldots, b_p \end{array} \right| q, z \varphi_q^{p-r+1} = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_p)_k} \frac{z^k}{k!} = \left._{r+1}F_{p+1}\right. \left( \begin{array}{c} a_1, a_2, \ldots, a_r, a_{p+1} \cr b_1, b_2, \ldots, b_p \end{array} \right| q, z), \tag{29}
\]
and
\[
\left._{p+1}F_{p+1}\right. \left( \begin{array}{c} a_1, a_2, \ldots, a_r \cr b_1, b_2, \ldots, b_p \end{array} \right| q, t) \bigg|_{t = t_0} = \left._{p+1}F_{p+1}\right. \left( \begin{array}{c} a_1, a_2, \ldots, a_r \cr b_1, b_2, \ldots, b_p \end{array} \right| q, t), \tag{29}
\]
where $t_0 = z q^{\frac{1}{2}(\sum_{i=1}^{p+1} a_i - \sum_{i=1}^{r} b_i - 1)}$.

Using the above notation the polynomial solutions of 43 is 22 Eq. (49), page 232
\[
P_n(s)_q = B_n \left( \frac{A}{c_1 q^{\frac{s}{2} \varphi_q^2}} \right)^n (s_1 + s_2 + \mu|q)_n (s_1 + s_3 + \mu|q)_n \times
\]
\[
(s_1 + s_4 + \mu|q)_n \left._4F_3\right. \left( \begin{array}{c} -n, 2\mu + n - 1 + \sum_{i=1}^{4} s_i, s_1 - s, s_1 + s + \mu \cr s_1 + s_2 + \mu, s_1 + s_3 + s_4 + \mu, s_1 + s_4 + \mu \end{array} \right| q, 1). \tag{30}
\]

2.1 The $q$-Racah polynomials

Here we will consider the $q$-Racah polynomials $u_\alpha^{\alpha, \beta}(x(s), a, b)_q$ on the lattice $x(s) = [s]_q[s + 1]_q$ introduced in 2 17 20. For this lattice one has
\[
c_1 = q^{\frac{1}{2} \varphi_q^2}, \quad \mu = 1, \quad c_3 = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \varphi_q^{-2}. \tag{31}
\]
Let choose \( \sigma \) in (11) as
\[
\sigma(s) = -\frac{q^{-2s}}{\varphi_q^{\frac{1}{2}}(q^a - q^b)}(q^a - q^b)(q^a - q^{b+\alpha})(q^a - q^{b+\alpha}) = [s - a]_q [s + b]_q [s + a - \beta]_q [b + \alpha - s]_q,
\]
i.e., \( s_1 = a, s_2 = b, s_3 = \beta - a, s_4 = b + \alpha, C = -q^{-\frac{1}{4}(\alpha + \beta)} \varphi_q^{-\frac{1}{2}}, A = -1 \), and let \( B_n = (-1)^n / [n]_q \).
Here, as before, \( \varphi_q = q^{1/2} - q^{-1/2} \). Now from (20) we find
\[
\lambda_n = q^{-\frac{1}{4}(\alpha + \beta + 2n+1)} \varphi_q^{-2}(1 - q^n)(1 - q^{\alpha + \beta + n+1}) = [n]_q [n + \alpha + \beta + 1]_q.
\]
To obtain \( \tau_n(s) \) we use (9). In this case \( x_n(s) = [s + n/2]_q [s + n/2 + 1]_q \), then, choosing \( s_n^* = -n/2 \), we get
\[
\tau_n(s) = \tau'_n x_n(s) + \tau_n(0), \quad \tau'_n = -[2n + \alpha + \beta + 2]_q, \quad \tau_n(0) = \sigma(-n/2 - 1) - \sigma(-n/2).
\]
Taking into account that \( \tau(s) = \tau_0(s) \), we obtain the corresponding function \( \tau(s) \)
\[
\tau(s) = -[2 + \alpha + \beta] q x(s) + \sigma(-1) - \sigma(0).
\]

### 2.1.1 The orthogonality and the norm \( d_n^2 \)

A solution of the Pearson-type difference equation (8) is
\[
\rho_n(s) = \frac{\Gamma_q(s + n + 1) \Gamma_q(s + n - a + \beta + 1) \Gamma_q(s + n + a + b + 1) \Gamma_q(b + \alpha - s)}{\Gamma_q(s - a + 1) \Gamma_q(s + b + 1) \Gamma_q(s + a - \beta + 1) \Gamma_q(b - s - n)}.
\]

Since \( \sigma(a)\rho(a) = \sigma(b)\rho(b) = 0 \), then the \( q \)-Racah polynomials satisfy the orthogonality relation
\[
\sum_{s=a}^{b-1} u_n^{a,\beta}(x(s), a, b) q u_n^{a,\beta}(x(s), a, b) \rho(s)[2s + 1]_q = 0, \quad n \neq m,
\]
with the restrictions \( -\frac{1}{2} < a < b - 1, \alpha > -1, -1 < \beta < 2a + 1 \). Let us now compute the square of the norm \( d_n^2 \). From (10) and (15) follow
\[
\rho_n(s) = \frac{\Gamma_q(s + n + 1) \Gamma_q(s + n - a + \beta + 1) \Gamma_q(s + n + a + b + 1) \Gamma_q(b + \alpha - s)}{\Gamma_q(s - a + 1) \Gamma_q(s + b + 1) \Gamma_q(s + a - \beta + 1) \Gamma_q(b - s - n)}.
\]

\[
A_{n,n} = [n]_q (-1)^n \frac{\Gamma_q(a + \beta + 2n + 1)}{\Gamma_q(a + \beta + n + 1)} \quad \Rightarrow \quad A_n := (-1)^n A_{n,n} B_n^2 = \frac{\Gamma_q(a + \beta + 2n + 1)}{[n]_q [\Gamma_q(a + \beta + n + 1)}.
\]

Taking into account that \( \nabla x_{n+1}(s) = [2s + n + 1]_q \), using (14), and the identity
\[
\tilde{\Gamma}_q(A - s) = \tilde{\Gamma}_q(A)^{-1/x} (1 - A[q]_s),
\]
we have
\[
d_n^2 = \Lambda_n \sum_{s=a}^{b-n-1} \Gamma_q(s + n + a + 1) \Gamma_q(s + n - a + \beta + 1) \Gamma_q(s + n + a + b + 1) \Gamma_q(b + \alpha - s)
\]
\[
\Gamma_q(s - a + 1) \Gamma_q(s + b + 1) \Gamma_q(s + a - \beta + 1) \Gamma_q(b - s - n)[2s + n + 1]_q^{-1},
\]
\[
= \Lambda_n \sum_{s=0}^{b-a-n-1} \frac{\Gamma_q(s + n + 2a + 1) \Gamma_q(s + n + \beta + 1) \Gamma_q(s + n + a + b + a + 1) \Gamma_q(b - a + \alpha - s)}{\Gamma_q(s + 1) \Gamma_q(s + b + a + 1) \Gamma_q(s + 2a - \beta + 1) \Gamma_q(b - s - n)[2s + 2a + n + 1]_q^{-1}}
\]
\[
= \frac{\Gamma_q(a + \beta + 2n + 1) \Gamma_q(2a + n + 1) \Gamma_q(n + \beta + 1) \Gamma_q(a + b + n + a + 1) \Gamma_q(b + \alpha - a)}{[n]_q [\Gamma_q(a + \beta + n + 1) \Gamma_q(a + b + 1) \Gamma_q(2a - \beta + 1) \Gamma_q(b - a - n)}
\]
\[
\times \sum_{s=0}^{b-a-n-1} \frac{(n + 2a + 1, n + \beta + 1, n + a + a + b + b + 1, -b + a + n[n]_q)_s [2s + 2a + n + 1]_q}{(1, a + b + 1, 2a - \beta + 1, 1 - b + a - a[q]_s)[2s + 2a + n + 1]_q}.
\]
In the following we denote by $S_n$ the sum in the last expression. If we now use that $(a|q)_n = (-1)^n (q^a; q)_n q^{-\frac{1}{2}(n+2a-1)} x^{-n}$, as well as the identity

$$ [2s + 2a + n + 1]_q = q^{-s} [2a + n + 1]_q (q^{\alpha + \frac{n+1}{2}}; q)_s (-q^{\alpha + \frac{n+1}{2}}; q)_s, $$

we obtain

$$ S_n = \sum_{s=0}^{b-a-n-1} \left( \frac{(q^{2a+n+1}, q^{n+1}, q^{n+a+b+a+1}, q^{1-b+a+n}, q^{\Gamma + (2a+n+3)}, q^0)^2 (2a+n+1)}{(q, q^{a+b+1}, q^{2a+n+1}, q^{1-b+a+n}, q^{\Gamma + (2a+n+1)}; q_s)_{2a+n+1}} \frac{1}{q^{b-a-n-1}} \right) $$

$$ = [2a+n+1]_q \delta \varphi_5 \left( \left( \frac{2a+n+2}{2a+b-\alpha-\beta+1} \right) \frac{(q^{2a+n+1}, q^{n+1}, q^{n+a+\beta+1}, q^{1-b+a+n}, q^{\Gamma + (2a+n+3)}, q^0)^2 (2a+n+1)}{(q, q^{a+b+1}, q^{2a+n+1}, q^{1-b+a+n}, q^{\Gamma + (2a+n+1)}; q_s)_{2a+n+1}} \frac{1}{q^{\alpha-1-\beta-n}} \right). $$

But the above $\delta \varphi_5$ series is a very well-poised $\delta \varphi_5$ basic series and therefore by using the summation formula $\text{Li}$ (II.21) page 238

$$ \delta \varphi_5 \left( \begin{array}{c} a, q^{a+1/2}, -q^{a+1/2}, b, c, q^{-k} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq^{k+1} \end{array} \right) \begin{vmatrix} q, aq^{-k+1} \end{vmatrix} \begin{vmatrix} aq/b, aq/c, q \end{vmatrix} = \begin{vmatrix} (aq, aq/bc; q)_k \end{vmatrix} \begin{vmatrix} (aq/b, aq/c; q)_k \end{vmatrix} (aq/b, aq/c, q) $$

with $k = b - a - n - 1, a = q^{2a+n+1}, b = q^{n+\beta+1}, c = q^{n+a+b+1}$, we obtain

$$ S_n = [2a+n+1]_q \Gamma_q(a+b+1) \Gamma_q(b-a+\alpha+\beta+n+1) \Gamma_q(\alpha+n+1) \Gamma_q(2a-\beta+1) \Gamma_q(b-a+\beta-n) \Gamma_q(\alpha+\beta+2n+2) \Gamma_q(b-a+\alpha), $$

Thus

$$ \Gamma_q(\alpha+n+1) \Gamma_q(\beta+n+1) \Gamma_q(b-a+\alpha+\beta+n+1) \Gamma_q(\beta+n+1) \Gamma_q(b-a+\beta-n) \Gamma_q(a+b+\alpha+n+1) \Gamma_q(\alpha+\beta+2n+1) \Gamma_q(a+b-\beta-n) \Gamma_q(a+b+\alpha+n+1) \Gamma_q(\alpha+\beta+n+1) \Gamma_q(b-a+\beta-n) $$

Thus

$$ a^2_n = \Gamma_q(\alpha+\beta+2a+1) \Gamma_q(2a+n+1) \Gamma_q(n+\beta+1) \Gamma_q(a+b+n+\alpha+1) \Gamma_q(b+\alpha-a) \Gamma_q(2a-\beta+1) \Gamma_q(b-a) \Gamma_q(a+b+\alpha+1) \Gamma_q(a+b+\alpha) \Gamma_q(b-a+\beta-n) \Gamma_q(a+b-\beta-n) \Gamma_q(a+b+\alpha+n+1) \Gamma_q(\alpha+\beta+n+1) \Gamma_q(b-a+\beta-n) \Gamma_q(a+b+\alpha+n+1) \Gamma_q(\alpha+\beta+n+1) \Gamma_q(b-a+\beta-n) $$

\textbf{2.1.2 The hypergeometric representation}

From formula $\text{Li}$ and $\text{Li}$ the following two equivalent hypergeometric representations hold

$$ u^\alpha_\beta(x, s) = q^{-\frac{1}{2}(2a+\alpha+\beta+n+1)} (q^{a-b+1}; q)_n (q^{\beta+1}; q)_n (q^{a+b+\alpha+1}; q)_n \times \varphi_n(q^n, q)_n x^{-2n} \begin{vmatrix} q^{-n}, q^{a+\alpha+\beta+1}, q^{a-s}, q^{a+s+1} \\ q^{a+b+1}, q^{\beta+1}, q^{a+b}+1 \end{vmatrix}, $$

$$ 4 \varphi_3 \left( \begin{array}{c} q^{-n}, q^{a+\alpha+\beta+1}, q^{a-s}, q^{a+s+1} \\ q^{a+b+1}, q^{\beta+1}, q^{a+b}+1 \end{array} \begin{vmatrix} q, q \end{vmatrix} \right), $$

and

$$ u^\alpha_\beta(x, s) = \frac{(a-b+1)q_n(\beta+1)q_n(a+b+\alpha+1)q_n}{[n]_q} \begin{vmatrix} q^{-n}, q^{a+b+\alpha+1}, q^{a-s}, q^{a+s+1} \\ q^{a+b+1}, q^{\beta+1}, q^{a+b}+1 \end{vmatrix}, $$

$$ 4 \varphi_3 \left( \begin{array}{c} -n, a+b+n+1, a-s, a+s+1 \\ a+b+1, \beta+1, a+b+\alpha+1 \end{array} \begin{vmatrix} q, 1 \end{vmatrix} \right). $$

Using the Sears transformation formula $\text{Li}$ (III.15) we obtain the equivalent formulas

$$ u^\alpha_\beta(x, s) = q^{-\frac{1}{2}(-2a+\alpha+\beta+n+1)} (q^{a-b+1}; q)_n (q^{\alpha+1}; q)_n (q^{a-b+1}; q)_n \times \varphi_n(q^n, q)_n x^{-2n} \begin{vmatrix} q^{-n}, q^{a+\alpha+\beta+1}, q^{a-s}, q^{a+s+1} \\ q^{a+b+1}, q^{\alpha+1}, q^{a-b}+1 \end{vmatrix}, $$

$$ 4 \varphi_3 \left( \begin{array}{c} q^{-n}, q^{a+\alpha+\beta+1}, q^{a-s}, q^{a+s+1} \\ q^{a+b+1}, q^{\alpha+1}, q^{a-b}+1 \end{vmatrix} \begin{vmatrix} q, q \end{vmatrix} \right). $$

(34)

(35)

(36)
and

\[ u_n^{\alpha,\beta}(x(s), a, b)_q = \frac{(a - b + 1|q)_n}{[n]_q!} \times {}_4\phi_3 \left( \begin{array}{c} -n, \alpha + \beta + n + 1, -b - s, -b + s + 1 \\ a - b + 1, \alpha + 1, -a - b + \beta + 1 \end{array} \right) | q, 1 \right). \] (37)

**Remark:** From the above formulas follow that the polynomials \( u_n^{\alpha,\beta}(x(s), a, b)_q \) are multiples of the standard \( q \)-Racah polynomials \( R_n(\mu(q^{b+s}); q^\alpha, q^\beta, q^{a-b}, q^{a+b}) \).

From the above hypergeometric representations also follow the values

\[ u_n^{\alpha,\beta}(x(a), a, b)_q = \frac{(a - b + 1|q)_n}{[n]_q!} \times \frac{(q^{a-b+1}; q)_n(q^{\alpha+1}; q)_n(q^{a+b+1}; q)_n}{q^{2(a+\beta+n+1)}{}_{2}\phi_1(q; q)_n}, \] (38)

\[ u_n^{\alpha,\beta}(x(b - 1), a, b)_q = \frac{(a - b + 1|q)_n}{[n]_q!} \times \frac{(q^{a-b+1}; q)_n(q^{a+1}; q)_n(q^{\beta-a-b+1}; q)_n}{q^{2(2b+a+b+n+1)}{}_{2}\phi_1(q; q)_n}. \] (39)

The formula \( \text{[10]} \) leads to the following explicit formula\(^2\)

\[ u_n^{\alpha,\beta}(x(s), a, b)_q = \frac{\Gamma_s(s - a - 1)\Gamma_s(s + b + 1)\Gamma_s(b - s)}{\Gamma_s(s + a + 1)\Gamma_s(s - a + \beta + 1)\Gamma_s(b + a - s)} \times \sum_{k=0}^{n} (-1)^k [2s + 2k - n + 1] \binom{s}{k} \binom{s + k + a + 1}{k} \binom{s + k - n + 1}{k} \times \frac{\Gamma_s(s + k - a - \beta + 1)\Gamma_s(s + k + a - \beta + 1)\Gamma_s(b + a - s + n + k)}{\Gamma_s(s - n + k + b + 1)\Gamma_s(s - n + k + a - \beta + 1)\Gamma_s(b - s - k)}, \] (40)

from where follows

\[ u_n^{\alpha,\beta}(x(a), a, b)_q = \frac{(-1)^n \Gamma_s(b - a)\Gamma_s(\beta + n + 1)\Gamma_s(b + a + n + 1)}{[n]! \Gamma_s(b - a - n)\Gamma_s(\beta + 1)\Gamma_s(b + a + 1)}; \]
\[ u_n^{\alpha,\beta}(x(b - 1), a, b)_q = \frac{\Gamma_s(b - a)\Gamma_s(\alpha + n + 1)\Gamma_s(b - a - \beta)}{[n]! \Gamma_s(b - a - n)\Gamma_s(\alpha + 1)\Gamma_s(b - a - \beta - n)}; \] (41)

that coincide with the values \( \text{[38]} \) and \( \text{[39]} \) obtained before.

From the hypergeometric representation the following symmetry property follows

\[ u_n^{\alpha,\beta}(x(s), a, b)_q = u_n^{b-a+\beta,b+a+\alpha}(x(s), a, b)_q. \]

Finally, notice that from \( \text{[34]} \) (or \( \text{[36]} \)) follows that \( u_n^{\alpha,\beta}(x(s), a, b)_q \) is a polynomial of degree \( n \) on \( x(s) = [s]_q[s + 1]_q \). In fact,

\[ (q^{a-s}; q)_k(q^{a+s+1}; q)_k = (-1)^k q^{k(a+k+1)} \prod_{l=0}^{k-1} \left( \frac{x(s) - c_3}{c_1} - q^{-l} (q^{a+l} + q^{a-l}) \right), \]

where \( c_1 \) and \( c_3 \) are given in \( \text{[31]} \).

### 2.1.3 Three-term recurrence relation and differentiation formulas

To derive the coefficients of the TTRR \( \text{[10]} \) we use \( \text{[17]} \) and \( \text{[18]} \). Using \( \text{[15]} \) and \( \text{[17]} \), we obtain

\[ a_n = \frac{\Gamma_s(\alpha + \beta + 2n + 1)}{[n]_q! \Gamma_s(\alpha + \beta + n + 1)}, \quad \alpha_n = \frac{[n + 1]_q[\alpha + \beta + n + 1]}{[\alpha + \beta + 2n + 1]_q[\alpha + \beta + 2n + 2]_q}. \]

\(^2\)Obviously the formulas \( \text{[34]} \) and \( \text{[36]} \) also give equivalent explicit formulas.
To compute $\beta_n$, we use [18]

$$
\beta_n = \frac{\left[a + b + \alpha + n\right]_q[a + b - \alpha - n]_q[a + n]_q[a + \alpha + \beta + n]_q[a - a + \alpha + \beta + n]_q [a - b + \alpha + n + 1]_q [a - b - \alpha - n + 1]_q}{[a + \alpha + \beta + n + 1]_q[a - a + \alpha + \beta + n + 1]_q [a - b + \alpha + n + 1]_q [a - b - \alpha - n + 1]_q}
$$

The differentiation formulas [21] and [28] yield

$$
\frac{\Delta u_n^{\alpha,\beta}(x(s), a, b)_q}{\Delta x(s)} = [\alpha + \beta + n + 1]_q u_{n-1}^{\alpha,\beta}(x(s + \frac{1}{2}), a + \frac{1}{2}, b - \frac{1}{2})_q,
$$

(42)
Before starting let us mention that from the representation (34) and the identity 
\[ W \]  
we will follow [20, pages 38-39]. First of all, notice that the orthogonal relation (13) for the Racah polynomials can be written in the form
\[ u_{n}^{\alpha,\beta}(x(s), a, b) \] 
respectively. Finally, formulas (41) (or (46)) and (42) lead to the differentiation formulas
\[ \sigma(s) \frac{\nabla u_{n}^{\alpha,\beta}(x(s), a, b)_{q}}{[2s]_{q}} = -\frac{[\alpha + \beta + n + 1]_{q}}{[\alpha + n + 2]_{q}} \left[ \tau_{n}(s)u_{n}^{\alpha,\beta}(x(s), a, b) + [n + 1]_{q}u_{n+1}^{\alpha,\beta}(x(s), a, b)_{q} \right], \] 
\[ \sigma(-s - 1) \frac{\Delta u_{n}^{\alpha,\beta}(x(s), a, b)_{q}}{[2s+2]_{q}} = -\frac{[\alpha + \beta + n + 1]_{q}}{[\alpha + n + 1]_{q}} \times \left[ \tau_{n}(s) + [n]_{q}\left[ \alpha + n + 2 \right]_{q}\left[ 2s + 1 \right]_{q}u_{n}^{\alpha,\beta}(x(s), a, b) + [n + 1]_{q}u_{n+1}^{\alpha,\beta}(x(s), a, b)_{q} \right], \] 
where \( \tau_{n}(s) \) is given in (32).

### 2.1.4 The duality of the Racah polynomials

In this section we will discuss the duality property of the \( q \)-Racah polynomials \( u_{n}^{\alpha,\beta}(x(s), a, b)_{q} \).

We will follow [20] pages 38-39. First of all, notice that the orthogonal relation (13) for the Racah polynomials can be written in the form
\[ \sum_{t=0}^{N-1} C_{tn}C_{tm} = \delta_{n,m}, \quad C_{tn} = \frac{u_{n}^{\alpha,\beta}(x(t + a), a, b)_{q} \sqrt{\rho(t + a)\Delta x(t + a - 1/2)}}{d_{n}}, \quad N = b - a, \]
where \( \rho(s) \) and \( d_{n} \) are the weight function and the norm of the \( q \)-Racah polynomials \( u_{n}^{\alpha,\beta}(x(s), a, b)_{q} \), respectively. The above relation can be understood as the orthogonality property of the orthogonal matrix \( C = (C_{tn})_{t,n=0}^{N-1} \) by its first index. If we now use the orthogonality of \( C \) by the second index we get
\[ \sum_{n=0}^{N-1} C_{tn}C_{\nu n} = \delta_{t,\nu}, \quad N = b - a, \]
that leads to the dual orthogonality relation for the \( q \)-Racah polynomials
\[ \sum_{n=0}^{N-1} u_{n}^{\alpha,\beta}(x(s), a, b)_{q}u_{n}^{\alpha,\beta}(x(s'), a, b)_{q} \frac{1}{d_{n}^{2}} = \frac{1}{\rho(s)\Delta x(s - 1/2)}\delta_{s,s'}. \]

The next step is to identify the functions \( u_{n}^{\alpha,\beta}(x(s), a, b)_{q} \) as polynomials on some lattice \( x(n) \). Before starting we mention that from the representation (41) and the identity
\[ (q^{-n}; q)_{k}(q^{\alpha+\beta+n+1}; q)_{k} = \prod_{t=0}^{k-1} \left( 1 + q^{\alpha+\beta+2t+1} - q^{\alpha+\beta+1+2t} \left( x(t + a) + q^{\frac{1}{2}} + q^{-\frac{1}{2}} \right) \right), \]
where \( x(t) = \left[ t \right]_{q}[t + 1]_{q} = [n + \frac{\alpha+\beta}{2}]_{q}[n + \frac{\alpha+\beta}{2} + 1]_{q} \), follows that \( u_{n}^{\alpha,\beta}(x(s), a, b)_{q} \) also constitutes a polynomial of degree \( s - a \) (for \( s = a, a + 1, \ldots, b - a - 1 \)) on \( x(t) \) with \( t = n + \frac{\alpha+\beta}{2} \).

Let us now define the polynomials —compare with the definition of the Racah polynomials
\[ u_{k}^{\alpha',\beta'}(x(t), a', b')_{q} = \frac{(-1)^{k}\Gamma_{q}(b' - a')\Gamma_{q}(b' + k + 1)\Gamma_{q}(b' + a' + \alpha' + k + 1)}{[k]!\Gamma_{q}(b' - a' - k)\Gamma_{q}(b' + \alpha' + k + 1)} \times 4F_{3} \left( -k, a' + \beta' + k + 1, a' - t, a' + t + 1 \right| q, 1 \right) \]
where

\[ k = s - a, \quad t = n + \frac{\alpha + \beta}{2}, \quad a' = \frac{\alpha + \beta}{2}, \quad b' = b - a + \frac{\alpha + \beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \]  

(48)

Obviously they are polynomials of degree \( k = s - a \) on the lattice \( x(t) \) that satisfy the orthogonality property

\[ b' - 1 \sum_{t=a'}^{t=b'} u_k^{\alpha', \beta'} (x(t), a', b') \eta u_n^{\alpha, \beta} (x(t), a', b') \rho'(t) \Delta x(t - 1/2) = (d_k')^2 \delta_{km}, \]

(49)

where \( \rho'(t) \) and \( d_k' \) are the weight function \( \rho \) and the norm \( d_n \) given in table I, with the corresponding change \( a, b, \alpha, \beta, s, n, b' \), \( a', b', \alpha', \beta', t, k \).

Furthermore, with the above choice of the parameters of \( u_k^{\alpha', \beta'} (x(t), a', b')_q \), the hypergeometric function \( _4F_3 \) in (47) coincides with the function \( _4F_3 \) in (58) and therefore the following relation between the polynomials \( u_k^{\alpha', \beta'} (x(t), a', b') \) and \( u_n^{\alpha, \beta} (x(s), a, b)_q \) holds

\[ u_k^{\alpha', \beta'} (x(t), a', b')_q = A(\alpha, \beta, a, b, n, s) u_n^{\alpha, \beta} (x(s), a, b)_q, \]

(50)

where

\[ A(\alpha, \beta, a, b, n, s) = \frac{(-1)^{s-a+n} \Gamma_q(b - a - n) \Gamma_q(s - a + \beta + 1) \Gamma_q(b + \alpha + s + 1) \Gamma_q(n + 1)}{\Gamma_q(b - s) \Gamma_q(n + \beta + 1) \Gamma_q(b + a + \alpha + n + 1) \Gamma_q(s - a + 1)}, \]

If we now substitute (50) in (49) and make the change \( 48 \), i.e., the polynomial set \( u_k^{\alpha', \beta'} (x(t), a', b')_q \) defined by (47) (or (51)) is the dual set associated to the Racah polynomials \( u_k^{\alpha', \beta'} (x(s), a, b)_q \).

To conclude this study, let us show that the TTRR (13) of the polynomials \( u_k^{\alpha', \beta'} (x(t), a', b')_q \) is the SODE (11) of the polynomials \( u_n^{\alpha, \beta} (x(s), a, b)_q \) whereas the SODE (41) of the \( u_n^{\alpha', \beta'} (x(t), a', b')_q \) becomes into the TTRR (11) of \( u_n^{\alpha, \beta} (x(s), a, b)_q \) and vice versa.

Let denote by \( \varsigma(t) \) the \( \sigma \) function of the polynomial \( u_k^{\alpha', \beta'} \), then

\[ \varsigma(t) = [t - a']_q[t + b']_q[t + a' - \beta']_q[b' + \alpha' - t]_q = [n]_q[n + b - a + \alpha + \beta]_q[n + \alpha]_q[b + a - n - \beta]_q, \]

and therefore,

\[ \varsigma(-t - 1) = [\alpha + \beta + n + 1]_q[b + a + \alpha + n + 1]_q[b - a - n - 1]_q[n + \beta + 1]_q, \]

\[ \lambda_k = [k]_q[\alpha' + \beta' + k + 1]_q = [s - a]_q[s + a + 1]_q. \]

For the coefficients \( \alpha_k', \beta_k' \) and \( \gamma_k' \) of the TTRR for the polynomials \( u_k^{\alpha', \beta'} \) we have

\[ \alpha_k' = \frac{[k + 1]_q[\alpha' + \beta' + k + 1]_q}{[\alpha' + \beta' + 2k + 1]_q[\alpha' + \beta' + 2k + 2]_q} = \frac{[s - a + 1]_q[s + a + 1]_q}{[2s + 1]_q[2s + 2]_q}, \]

\[ \gamma_k' = \frac{[b + a + s]_q[b + a - s]_q[s + a - \beta]_q[s - a + \beta]_q[b + s]_q[b - s]_q}{[2s + 1]_q[2s + 2]_q}, \]

and

\[ \beta_k' = [n + \frac{\alpha + \beta}{2}]_q[n + \frac{\alpha + \beta}{2} + 1]_q + \frac{\sigma(-s - 1)}{[2s + 1]_q[2s + 2]_q} + \frac{\sigma(s)}{[2s + 1]_q[2s + 2]_q}. \]

Also we have \( \Delta x(t) = [2t + 2]_q = [2n + \alpha + \beta + 2]_q \) and \( x(s) = [s]_q[s + 1]_q = [k + a]_q[k + a + 1]_q. \)

Let show that the SODE of the Racah polynomials \( u_n^{\alpha, \beta} (x(s), a, b)_q \) is the TTRR of the \( u_k^{\alpha', \beta'} (x(t), a', b')_q \) polynomials. First, we substitute the relation (50) in the SODE (41) of the polynomials \( u_n^{\alpha, \beta} (x(s), a, b)_q \) and use that \( u_n^{\alpha, \beta} (x(s + 1), a, b)_q \) is proportional to \( u_{k+1}^{\alpha', \beta'} (x(t), a', b')_q \) (see (50)). After some simplification, and using the last formulas we obtain

\[ \alpha_k' u_{k+1}^{\alpha', \beta'} (x(t), a', b')_q + \left( \beta_k' - [n]_q[\alpha + \beta + n + 1]_q - \frac{[s + \beta]_q[s + \beta + 1]_q}{[2s + 1]_q[2s + 2]_q} \right) u_k^{\alpha', \beta'} (x(t), a', b')_q 
+ \gamma_k' u_{k-1}^{\alpha', \beta'} (x(t), a', b')_q = 0, \]

but

\[ [n]_q[\alpha + \beta + n + 1]_q + \frac{[s + \beta]_q[s + \beta + 1]_q}{[2s + 1]_q[2s + 2]_q} = [n + \frac{\alpha + \beta}{2}]_q[n + \frac{\alpha + \beta}{2} + 1]_q = x(t), \]
i.e., we obtain the TTRR for the polynomials \( u_k^{a',b'}(x(t), a', b')_q \).

If we now substitute (53) in the TTRR (10) for the Racah polynomials \( u_n^{\alpha,\beta}(x(s), a, b)_q \), and use that \( u_{n+1}^{\alpha,\beta}(x(s), a, b)_q \sim u_k^{a',b'}(x(t \pm 1), a', b')_q \), then we obtain the SODE

\[
\begin{align*}
\frac{(t-1)}{\Delta x(t)} & u_k^{\alpha,\beta'}(x(t), a', b')_q + \frac{(t)}{\Delta x(t)} u_k^{\alpha',\beta}(x(t-1), a', b')_q \\
- \left[ \frac{(t-1)}{\Delta x(t)} + \frac{t}{\Delta x(t)} \right] & + [a]_q[a+1]_q - [k+a]_q[k+a+1]_q \right] u_k^{\alpha',\beta'}(x(t), a', b')_q = 0.
\end{align*}
\]

That is the SODE (11) of the \( u_k^{a',b'}(x(t), a', b')_q \) since

\[
[a]_q[a+1]_q - [k+a]_q[k+a+1]_q = -[k]_q[k + 2a + 1]_q = -[k]_q[k + a' + \beta + 1]_q = -\lambda_k.
\]

### 2.2 The \( q \)-Racah polynomials \( \bar{u}_n^{\alpha,\beta}(x(s), a, b)_q \)

There is another possibility to define the \( q \)-Racah polynomials as it is suggested in (17) (20). It corresponds to the function

\[
\sigma(s) = [s - a]_q[s + b]_q[s - a + \beta]_q[b + \alpha + s]_q,
\]
i.e., \( A = 1, s_1 = a, s_2 = -b, s_3 = a - \beta, s_4 = -b - \alpha \). With this choice we obtain a new family of polynomials \( \bar{u}_n^{\alpha,\beta}(x(s), a, b)_q \) that is orthogonal with respect to the weight function

\[
\rho(s) = \frac{\Gamma_q(s + a + 1)\Gamma_q(s + a - \beta + 1)}{\Gamma_q(s + a + b + 1)\Gamma_q(s + a + \alpha - \beta)\Gamma_q(s + a - 1)\Gamma_q(s - a + 1)\Gamma_q(s - a + b + 1)\Gamma_q(s - a + \beta + 1)\Gamma_q(b - s)}.
\]

All their characteristics can be obtained exactly in the same way as before. Moreover, they can be also obtained from the corresponding characteristics of the polynomials \( u_n^{\alpha,\beta}(x(s), a, b)_q \) by changing \( \alpha \rightarrow -2b - \alpha, \beta \rightarrow 2a - \beta \) — and using the properties of the functions \( \Gamma_q(s), \Gamma_q(s), (a)_n \), and \((a; q)_n\). We will resume the main data of the polynomials \( \bar{u}_n^{\alpha,\beta}(x(s), a, b)_q \) in table 2.

#### 2.2.1 The hypergeometric representation

For the \( \bar{u}_n^{\alpha,\beta}(x(s), a, b)_q \) polynomials we have the following hypergeometric representation

\[
\begin{align*}
\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q = & \frac{q^{-2(4a-2b-a-\beta+n+1)}(q^{a-b+1}; q)_n(q^{2a-\beta+n+1}; q)_n(q^{a-b-\alpha+1}; q)_n}{x_2^2(q; q)_n} \\
& 4F_3 \left( \begin{array}{c}
q^{-n}, q^{2a-2b-a-\beta+n+1}, q^{a-s}, q^{a+b+1} \\
q^{a-b+1}, q^{2a-\beta+n+1}, q^{a-b-\alpha+1}
\end{array} \right) | q, q \right)
\end{align*}
\]

or, in terms of the \( q \)-hypergeometric series (28)

\[
\begin{align*}
\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q = & \frac{(a-b+1)|n(2a-b+1|n)(a-b-\alpha+1|n)}{[n]_q !} \\
& 4F_3 \left( \begin{array}{c}
-n, 2a-2b-\alpha-\beta+n+1, a-s, a+s+1 \\
a-b+1, 2a-\beta+n+1, a-b-\alpha+1
\end{array} \right) | q, 1 \right).
\end{align*}
\]

Using the Sears transformation formula (11 Eq. (III.15)) we obtain the equivalent representation formulas

\[
\begin{align*}
\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q = & \frac{q^{-2(2a-2b-a-\beta+n+1)}(q^{a-b+1}; q)_n(q^{2a+b+1}; q)_n(q^{2a-\beta+1}; q)_n}{x_2^2(q; q)_n} \\
& 4F_3 \left( \begin{array}{c}
q^{-n}, q^{a-2b-a-\beta+n+1}, q^{b-s}, q^{b+s+1} \\
q^{a+b+1}, q^{2a-\beta+1}, q^{a-\beta+1+n}
\end{array} \right) | q, q \right)
\end{align*}
\]

and

\[
\begin{align*}
\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q = & \frac{(a-b+1)|n(-2b-a+1|n)(a-b-\beta+1|n)}{[n]_q !} \\
& 4F_3 \left( \begin{array}{c}
-n, 2a-2b-\alpha-\beta+n+1, -b-s, -b+s+1 \\
a-b+1, -2b-\alpha+1, a-b-\beta+1
\end{array} \right) | q, 1 \right).
\end{align*}
\]
Table 2: Main data of the $q$-Racah polynomials $\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q$

<table>
<thead>
<tr>
<th>$P_n(s)$</th>
<th>$\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q$, $x(s) = [s]_q[s + 1]_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a, b)$</td>
<td>$[a, b - 1]$</td>
</tr>
</tbody>
</table>

\begin{align*}
\rho(s) & = \frac{\bar{\Gamma}_q(s + a + 1)\bar{\Gamma}_q(s + a - \beta + 1)}{\bar{\Gamma}_q(s + a + b + 1)\bar{\Gamma}_q(s + a - \beta + 1)\bar{\Gamma}_q(s + a - \beta + 1)\bar{\Gamma}_q(b - s)} \\
& \quad \times \begin{cases} 
\frac{1}{\bar{\Gamma}_q(s + a + b + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(b - s)} \\
\frac{1}{\bar{\Gamma}_q(s + a + b + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(b - s)} \\
\frac{1}{\bar{\Gamma}_q(s + a + b + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(b - s)} \\
\frac{1}{\bar{\Gamma}_q(s + a + b + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(s + a + b - n + 1)\bar{\Gamma}_q(b - s)} \\
\end{cases} \\
\tau(-s - 1) & = \frac{[2a - \beta + 1]_q[b + \alpha]_q - [2b + \alpha - 1]_q[a - \beta]_q - [2b + \alpha - 1]_q[2a - \beta + 1]_q}{[2a - \beta + 1]_q[b + \alpha]_q - [2b + \alpha - 1]_q[a - \beta]_q - [2b + \alpha - 1]_q[2a - \beta + 1]_q} \\
\tau(s) & = \frac{\Gamma_q(2a + n - \beta + 1)\Gamma_q(2b + a + \alpha + \beta - n)\Gamma_q(b - a + n)\Gamma_q(b - a + \alpha + \beta - n)}{\Gamma_q(n + 1)\Gamma_q(b - a + n)\Gamma_q(b - a + n)\Gamma_q(b - a + \alpha + \beta + 1)} \\
\tau_n(s) & = \frac{[2b - 2a + \alpha + \beta - 2n - 2]_q x(s + \frac{n}{2}) + [a + \frac{n}{2} + 1]_q[b - \frac{n}{2} - 1]_q[a + \frac{n}{2} + 1 - \beta]_q[b - \frac{n}{2} + \alpha + 1]_q}{[2b - 2a + \alpha + \beta - 2n + 2]_q x(s) + [a + \frac{n}{2} + 1]_q[b - \frac{n}{2} - 1]_q[a + \frac{n}{2} + 1 - \beta]_q[b - \frac{n}{2} + \alpha + 1]_q} \\
\lambda_n & = \frac{[n]_q[2b - 2a + \alpha + \beta - n - 1]_q}{[n]_q[2b - 2a + \alpha + \beta - n - 1]_q} \\
B_n & = \frac{1}{[n]_q!} \\
\rho_n(s) & = \frac{\bar{\Gamma}_q(s + a + n + 1)\bar{\Gamma}_q(s + a + n - 1)\bar{\Gamma}_q(s + a + n + 1)\bar{\Gamma}_q(b - s)}{\bar{\Gamma}_q(s + a + b + 1)\bar{\Gamma}_q(s + a + b - n)\bar{\Gamma}_q(s + a + b + 1)\bar{\Gamma}_q(b - s)} \\
\alpha_n & = \frac{(-1)^n\bar{\Gamma}_q[2b - 2a + \alpha + \beta - n]_q}{[n]_q\bar{\Gamma}_q[2b - 2a + \alpha + \beta - 2n]_q} \\
\beta_n & = \frac{[a]_q[a + 1]_q + [2b - 2a + \alpha + \beta - n - 1]_q[a - b - n + 1]_q[a - b - n + 1]_q[a - b - n + 1]_q}{[2b - 2a + \alpha + \beta - 2n - 2]_q[2b - 2a + \alpha + \beta - 2n - 2]_q[2b - 2a + \alpha + \beta - 2n - 2]_q[2b - 2a + \alpha + \beta - 2n - 2]_q} \\
\gamma_n & = \frac{[2a - \beta - n]_q[b - a - n]_q[b - a - n]_q[b - a - n + 1]_q[b - a - n + 1]_q[b - a - n + 1]_q[b - a - n + 1]_q}{[2b - 2a + \alpha + \beta - 2n + 1]_q[2b - 2a + \alpha + \beta - 2n + 1]_q[2b - 2a + \alpha + \beta - 2n + 1]_q[2b - 2a + \alpha + \beta - 2n + 1]_q} \\
\end{align*}

Remark: From the above formulas follow that the polynomials $\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q$ are multiples of the standard $q$-Racah polynomials $R_n(\mu(q^{1-n}); q^{a-b-\alpha}, q^{a-b-\beta}, q^{a-b}, q^{a+b})_q$.

Moreover, from the above hypergeometric representations follow the values

\begin{align*}
\bar{u}_n^{\alpha,\beta}(x(a), a, b)_q &= \frac{(a - b + 1)q_n(2a - \beta + 1)q_n(a - b - \alpha + 1)q_n}{[n]_q!} \\
&= \frac{q^{a-b+1}; q_n(q^{2a-\beta+1}; q_n)(q^{a-b-\alpha+1}; q_n)}{q^{2(a-2b-\alpha-\beta+n+1)}z_q^{2n}(q; q)_n}, \\
\bar{u}_n^{\alpha,\beta}(x(b-1), a, b)_q &= \frac{(a - b + 1)q_n(-2b - \alpha + 1)q_n(a - b - \beta + 1)q_n}{[n]_q!} \\
&= \frac{q^{a-b+1}; q_n(q^{-2b-\alpha+1}; q_n)(q^{-\beta+b-\alpha+1}; q_n)}{q^{2(a-2b-\alpha-\beta+n+1)}z_q^{2n}(q; q)_n}.
\end{align*}
Using (10) we obtain an explicit formula
\[ \widetilde{u}_n^{\alpha,\beta}(x(s), a, b) = \frac{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(s+\alpha+b+1)}{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+a+\beta+1)} \times \]
\[ \widetilde{\Gamma}_q(b+\alpha-s) \sum_{k=0}^{n} \frac{(-1)^{k+n}[2s+2k-n+1]!}{\Gamma_q(k+1)\Gamma_q(n-k+1)\Gamma_q(2s+k+2)\Gamma_q(s-n+k+1)\Gamma_q(b-s-k)} \times \]
\[ \frac{\widetilde{\Gamma}_q(s+k+n+1)\widetilde{\Gamma}_q(b+\alpha-n+1)\widetilde{\Gamma}_q(b-a+\alpha-n)}{[n]!\Gamma_q(b-a-n)\Gamma_q(2a-\beta+1)\Gamma_q(b-a+\alpha-n)} \times \]
\[ \frac{\widetilde{\Gamma}_q(2a-\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha+n)}{[n]!\Gamma_q(b-a-n)\Gamma_q(2b+a-\alpha-n)\Gamma_q(b-a+\beta-n)} \] \quad (57)

From this expression follows that
\[ \widetilde{u}_n^{\alpha,\beta}(x(a), a, b) = \frac{\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2a-\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha)}{[n]!\Gamma_q(b-a-n)\Gamma_q(2a-\beta+1)\Gamma_q(b-a+\alpha-n)} \times \]
\[ \widetilde{u}_n^{\alpha,\beta}(x(b-1), a, b) = \frac{(-1)^n\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2b+a)\widetilde{\Gamma}_q(b-a+\beta)}{[n]!\Gamma_q(b-a-n)\Gamma_q(2b+a-\alpha-n)\Gamma_q(b-a+\beta-n)} \] \quad (58)

that are in agreement with the values (55) and (56) obtained before.

From the hypergeometric representation follows the symmetry property
\[ \widetilde{u}_n^{\alpha,\beta}(x(s), a, b) = \widetilde{u}_n^{b-a+\beta, b-a+\alpha}(x(s), a, b) \] \quad (59)

### 2.2.2 The differentiation formulas

Next we use the differentiation formulas (21) and (24) to obtain
\[ \frac{\Delta \widetilde{u}_n^{\alpha,\beta}(x(s), a, b)_{q}}{\Delta x(s)} = -[2b-2a+\alpha+\beta-n-1]_q \widetilde{u}_n^{\alpha,\beta}(x(s+\frac{1}{2}), a+\frac{1}{2}, b-\frac{1}{2})_q, \] \quad (60)

\[ \sigma(-s-1)\widetilde{u}_n^{\alpha,\beta}(x(s+\frac{1}{2}), a+\frac{1}{2}, b-\frac{1}{2})_q \]

respectively. Finally, the formulas (21) (or (25)) and (26) lead to the following differentiation formulas
\[ \sigma(s) \frac{\sum \widetilde{u}_n^{\alpha,\beta}(x(s), a, b)_q}{[2s]_q} = -\frac{[2b-2a+\alpha+\beta-2n-2]_q}{[2b-2a+\alpha+\beta-2n]_q} \left[ \tau_n(s) \widetilde{u}_n^{\alpha,\beta}(x(s), a, b)_q - [n+1]_q \widetilde{u}_{n+1}^{\alpha,\beta}(x(s), a, b)_q \right], \] \quad (61)

\[ \frac{\Delta \widetilde{u}_n^{\alpha,\beta}(x(s), a, b)_{q}}{[2s+2]_q} = \frac{[2b-2a+\alpha+\beta-2n-2]_q}{[2b-2a+\alpha+\beta-2n]_q} \times \]

\[ \left[ \tau_n(s) + [n]_q[2b-2a+\alpha+\beta-2n-2]_q[2s+1]_q \widetilde{u}_n^{\alpha,\beta}(x(s), a, b)_q - [n+1]_q \widetilde{u}_{n+1}^{\alpha,\beta}(x(s), a, b)_q \right], \] \quad (62)

respectively, where \( \tau_n(s) \) is given in table (24).

### 2.3 The dual set to \( \widetilde{u}_n^{\alpha,\beta}(x(s), a, b) \)

To obtain the dual set to \( \widetilde{u}_n^{\alpha,\beta}(x(s), a, b) \) we use the same method as in the previous section. We start from the orthogonality relation (13) for the \( \widetilde{u}_n^{\alpha,\beta}(x(s), a, b) \) polynomials defined by (54) and write the dual relation
\[ \sum_{n=0}^{N-1} \widetilde{u}_n^{\alpha,\beta}(x(s), a, b)_q \widetilde{u}_n^{\alpha,\beta}(x(s'), a, b)_q \frac{1}{d^2_n} = \frac{1}{p(s)\Delta x(s-1/2)} \delta_{s,s'}, \quad N = b-a, \] \quad (63)

\(^3\)Obviously the formulas (51) (52) also give two equivalent explicit formulas.
where $\rho$ and $d_n^2$ are the weight function and the norm of the $\tilde{u}_{n}^{\alpha, \beta}(x(s), a, b)_q$ given in table 2. Furthermore, from (54) follows that the functions $\tilde{u}_{n}^{\alpha, \beta}(x(s), a, b)_q$ are polynomials of degree $k = b - s - 1$ on the lattice $x(t) = [t]_q[t + 1]_q$ where $t = b - a + n + \frac{\alpha + \beta}{2} - 1$ (the proof is similar to the one presented in section 2.1.4 and we will omit it here). To identify the dual set let us define the new set

$$
\tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')_q = \frac{(-1)^k b^k \Gamma_q(b' - a') \Gamma_q(b' - a' + \beta') \Gamma_q(2b' + \alpha')}{[k! \Gamma_q(b' - a' - k) \Gamma_q(b' - a' + \beta' - k) \Gamma_q(2b' + \alpha' - k)]}
$$

$$
\times 4F_3 \left( \begin{array}{c}
-k, 2a' - 2b' - \alpha' - \beta' + k + 1, -b' - t, -b' + t + 1 \\
\alpha' - b' + 1, -2b' - \alpha' + 1, a' - b' - \beta' + 1
\end{array} \right| q, 1),
$$

where

$$
k = b - s - 1, \quad t = b - a + n + \frac{\alpha + \beta}{2} - 1, \quad a' = \frac{\alpha + \beta}{2}, \quad b' = b - a + \frac{\alpha + \beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta.
$$

Obviously they satisfy the following orthogonality relation

$$
\sum_{t=a'}^{b'-1} \tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')_q \tilde{u}_{n}^{\alpha', \beta'}(x(t), a', b')_q \rho'(t) \Delta x(t - 1/2) = (d_k')^2 \delta_{k, m},
$$

where $\rho'(t)$ and $d_k'$ are the weight function $\rho$ and the norm $d_n$, respectively, given in table 2 with the corresponding change of the parameters $a, b, \alpha, \beta, n, s$ by $a', b', \alpha', \beta', k, t$.

Furthermore, with the above definition (64) for the parameters of $\tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')_q$, the hypergeometric function $4F_3$ in (64) coincides with the function $4F_3$ in (55) and therefore the following relation between the polynomials $\tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')$ and $\tilde{u}_{n}^{\alpha, \beta}(x(s), a, b)_q$ holds

$$
\tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')_q = \tilde{A}(\alpha, \beta, a, b, n, s) \tilde{u}_{n}^{\alpha, \beta}(x(s), a, b)_q,
$$

where

$$
\tilde{A}(\alpha, \beta, a, b, n, s) = \frac{(-1)^{b - s - 1 - n} \Gamma_q(b - a - n) \Gamma_q(2b + \alpha - n) \Gamma_q(s + b + \alpha + 1)}{\Gamma_q(s + b + \alpha + 1) \Gamma_q(s - a + \beta + 1) \Gamma_q(s + b + \alpha + 1) \Gamma_q(s - a + 1)}
$$

To prove that the polynomials $\tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')_q$ are the dual set to $\tilde{u}_{n}^{\alpha, \beta}(x(s), a, b)_q$ it is sufficient to substitute (57) in (56) and do the change (54) that transforms (66) into (67).

Let also mention that, as in the case of the $q$-Racah polynomials, the TTRR (16) of the polynomials $\tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')_q$ is the SODE (4) of the polynomials $\tilde{u}_{n}^{\alpha, \beta}(x(s), a, b)_q$ whereas the SODE (4) of the $\tilde{u}_{k}^{\alpha', \beta'}(x(t), a', b')_q$ becomes into the TTRR (16) of $\tilde{u}_{n}^{\alpha, \beta}(x(s), a, b)_q$ and vice versa.

To conclude this section let us point out that there exist a simple relation connecting both polynomials $u_{n}^{\alpha, \beta}(x(s), a, b)_q$ and $\tilde{u}_{k}^{\alpha, \beta}(x(s), a, b)_q$ (see (57) from below). We will establish it at the end of the next section.

### 3 Connection with the 6j-symbols of the q-algebra $SU_q(2)$

#### 3.1 6j-symbols of the quantum algebra $SU_q(2)$

It is known (see e.g. [20] and references therein) that the Racah coefficients $U_q(j_1 j_2 j_3; j_1 j_2 j_3)$ are used for the transition from the coupling scheme of three angular momenta $j_1, j_2, j_3$ to the following ones

$$
| j_1, j_2, j_3 : jm \rangle = \sum_{m_1, m_2, m_3, m_12} \langle j_1 m_1 j_2 m_2 | j_1 m_1 | j_2 m_2 | j_3 m_3 | j_1 m_1 | j_2 m_2 | j_3 m_3 \rangle
$$

$$
\times | j_1 m_1 j_2 m_2 j_1 m_1 j_2 m_2 j_3 m_3 | jm \rangle | j_1 m_1 | j_2 m_2 | j_3 m_3 \rangle,
$$

to the following ones

$$
| j_1 j_2 j_3 : jm \rangle = \sum_{m_1, m_2, m_3, m_12} \langle j_2 m_2 j_3 m_3 | j_2 m_2 | j_3 m_3 | j_1 m_1 | j_2 m_2 | j_3 m_3 \rangle.
$$
where $\langle j_\alpha m_\alpha j_\beta m_\beta | j_\gamma m_\gamma \rangle$ denotes the Clebsch-Gordan Coefficients of the quantum algebra $su_q(2)$. In fact we have that the recoupling is given by

$$|j_1 j_2 (j_12), j_3 : jm \rangle = \sum_{j_{23}} U_q(j_1 j_2 j j_3; j_{12} j_{23}) |j_1 j_2 j_3 (j_{23}) : jm \rangle.$$ 

The Racah coefficients $U$ define an unitary matrix, i.e., they satisfy the orthogonality relations

$$\sum_{j_{23}} U_q(j_1 j_2 j j_3; j_{12} j_{23}) U_q(j_1 j_2 j j_3; j'_{12} j_{23}) = \delta_{j_{12}, j'_{12}},$$

$$\sum_{j_{12}} U_q(j_1 j_2 j j_3; j_{12} j_{23}) U_q(j_1 j_2 j j_3; j_{12} j'_{23}) = \delta_{j_{23}, j'_{23}}.$$ 

Usually instead of the Racah coefficients is more convenient to use the $6j$-symbols defined by

$$U_q(j_1 j_2 j j_3; j_{12} j_{23}) = (-1)^{j_1 + j_2 + j_3 + j} \sqrt{[2j_1 + 1]_q [2j_2 + 1]_q [2j_3 + 1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j & 3 & j_{23} \end{array} \right\}_q.$$ 

The $6j$-symbols have the following symmetry property

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j' & j & j_{23} \end{array} \right\}_q = \left\{ \begin{array}{ccc} j_3 & j_2 & j_1 \\ j' & j & j_{23} \end{array} \right\}_q.$$ 

(70)

Here and without lost of generality we will suppose that $j_1 \geq j_2$ and $j_3 \geq j_2$, then for the moments $j_{23}$ and $j_{12}$ we have the intervals

$$j_3 - j_2 \leq j_{23} \leq j_2 + j_3, \quad j_1 - j_2 \leq j_{12} \leq j_1 + j_2,$$ 

respectively. Now, in order to avoid any other restrictions on these two momenta (caused by the so called triangle inequalities for the $6j$-symbols) we will assume that the following restrictions hold

$$|j - j_3| \leq \min(j_{12}) = j_1 - j_2, \quad |j - j_1| \leq \min(j_{23}) = j_3 - j_2.$$ 

### 3.2 6j-symbols and the $q$-Racah polynomials $u_n^{a,\beta}(x(s), a, b)_q$

Now we are ready to establish the connection of $6j$-symbols with the $q$-Racah polynomials. We fix the variable $s$ as $s = j_{23}$ that runs on the interval $a \leq s \leq b - 1$ where $a = j_3 - j_2, b = j_2 + j_3 + 1$. Let us put

$$(-1)^{j_1 + j_2 + j_3} \sqrt{[2j_{12} + 1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}} \ u_n^{a,\beta}(x(s), a, b)_q,$$ 

(71)

where $\rho(s)$ and $d_n$ are the weight function and the norm, respectively, of the $q$-Racah polynomials on the lattice $\mathbb{U} u_n^{a,\beta}(x(s), a, b)_q$, and $n = j_{12} - j_1 + j_2, a = j_1 - j_2 - j_3 + j \geq 0, \beta = j_1 - j_2 + j_3 - j \geq 0$.4

To verify the above relation we use the recurrence relation [24, Eq. (5.17)]

$$[2]_q [2j_{23} + 2]_q A_q^{-} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} - 1 \end{array} \right\}_q =$$

$$\left( [2j_{23}]_q [2j_1 + 2]_q - [2]_q [j - j_{23} + j_1 + 1]_q [j + j_{23} - j_1]_q \right) \times$$

$$\left( [2j_2]_q [2j_{23} + 2]_q - [2]_q [j_3 - j_2 + j_{23} + 1]_q [j_3 + j_2 - j_{23}]_q \right)$$

$$\left( [2j_2]_q [j_1 + 2]_q - [2]_q [j_1 j_2 - j_1 + 1]_q [j_1 + j_2 - j_1]_q [2j_{23} + 2]_q [2j_{23}]_q \right) \times$$

$$[2j_{23} + 1]_q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q + [2]_q [2j_{23}]_q A_q^{+} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{array} \right\}_q = 0,$$

(72)

4Notice that this is equivalent to the following setting

$$j_1 = (b - a - 1 + \alpha + \beta)/2, \quad j_2 = (b - a - 1)/2, \quad j_3 = (a + b - 1)/2,$$

$$j_{12} = (2n + \alpha + \beta)/2, \quad j_{23} = s, \quad j = (a + b - 1 + \alpha - \beta)/2.$$
where

\[ A_q^- = \sqrt{[j + j_{23} + j_1 + 1]_q [j + j_{23} - j_1]_q [j - j_{23} + j_1 + 1]_q [j_{23} - j + j_1]_q \times [j_2 + j_3 + j_{23} + 1]_q [j_2 + j_3 - j_{23} + 1]_q [j_3 - j_2 + j_{23}]_q [j_2 - j_3 + j_{23}]_q}, \]
\[ A_q^+ = \sqrt{[j + j_{23} + j_1 + 2]_q [j + j_{23} - j_1 + 1]_q [j - j_{23} + j_1]_q [j_{23} - j + j_1 + 1]_q \times \sqrt{[j_2 + j_3 + j_{23} + 2]_q [j_2 + j_3 - j_{23} + 1]_q [j_3 - j_2 + j_{23} + 1]_q [j_2 - j_3 + j_{23} + 1]_q}}. \]

Notice that

\[ A_q^- = \sqrt{\sigma(j_{23}) \sigma(-j_{23})}, \quad A_q^+ = \sqrt{\sigma(j_{23} + 1) \sigma(-j_{23} - 1)}, \]

where

\[ \sigma(j_{23}) = [j_{23} - j_3 + j_2]_q [j_{23} + j_2 + j_3 + 1]_q [j_{23} - j_1 + j]_q [j + j_1 - j_{23} + 1]_q, \]
\[ \sigma(-j_{23} - 1) = [j_{23} + j_3 - j_2 + 1]_q [j_2 + j_3 - j_{23}]_q [j_{23} + j_1 - j + 1]_q [j + j_1 + j_{23} + 2]_q. \]

Substituting (71) in (72) and simplifying the obtained expression we get

\[ [2s]_q \sigma(-s - 1) u_0^\alpha \beta(x(s + 1), a, b)_q + [2s + 2]_q \sigma(s) u_0^\alpha \beta(x(s - 1), a, b)_q + \left( \lambda_2 [2s]_q [2s + 1]_q [2s + 2]_q - [2s]_q \sigma(-s - 1) [2s + 2]_q \sigma(s) \right) u_0^\alpha \beta(x(s), a, b)_q = 0, \]

which is the difference equation for the $q$-Racah polynomials. Since $u_0^\alpha \beta(x(s), a, b)_q = 1$, (71) leads to

\[ (-1)^{j_1 + j_{23} + j} \sqrt{[2j_1 - 2j_2 + 1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_1 - j_2 \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{\rho(s)}{d_0^2}} \Rightarrow \]

\[ (-1)^{j_1 + j_1 + s} \sqrt{[j_1 + j + s + 1]_q [j_1 + j - s]_q [j_1 - j + s]_q [j_1 + j - s]_q [j_{23} - j_2 + s]_q [j_{23} + j_2 - s]_q [j_2 + j_3 + s]_q [j_2 + j_3 - s]_q [j_2 + j_3 + s + 1]_q} \times \]

\[ \left[ \frac{[2j_1 - 2j_2]_q [2j_2]_q [2j_2]_q [j_2 + j_3 + j - j_1]_q [j_1 + j_3 - j_2 + j]_q [j_1 + j_3 - j_2 + j + 1]_q}{[2j_1 + 1]_q [j_1 + j_3 - j_2 + j]_q [j_1 + j_3 - j_2 + j + 1]_q} \right]. \]

Furthermore, substituting the values $s = a$ and $s = b - 1$ in (74) and using (41) we find

\[ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\
 j_3 & j & j_{3 - j_2} \end{array} \right\}_q = (-1)^{j_1 + j_3 + j} \times \]

\[ \frac{[j_{12} + j_3 - j]_q [2]_q [j_{12} + j_3 + j + 1]_q [2j_3 - 2j_2]_q [j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12}]_q [j_1 + j_3 - j_2 + j + 1]_q}{[j_1 - j_2 + j_3 - j]_q [2j_3 + 1]_q [2j_3 - j_1 - j_2 + j]_q [j_1 + j_3 + j]_q [j_1 + j_3 + j + 1]_q}, \]

and

\[ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\
 j_3 & j & j_{2 + j_3} \end{array} \right\}_q = (-1)^{j_1 + j_3 + j} \times \]

\[ \frac{[2]_q [j_{12} + j_3 + j]_q [j_2 - j_1 + j_3 + j]_q [2j_3]_q [j_1 + j_2 + j_3]_q [j_1 + j_2 + j_3 - j]_q [j_1 + j_2 + j_3 + j]_q [j_1 + j_2 + j_3 + j + 1]_q}{[j_1 + j_2 - j_3]_q [j_1 - j_2 + j_3 + j]_q [j_1 + j_2 + j_3 + j + 1]_q}, \]

that are in agreement with the results in [26].
The relation (71) allows us to obtain several recurrence relations for the 6j-symbols of the quantum algebra $SU_q(2)$ by using the properties of the $q$-Racah polynomials. So, the TTRR (110) gives

$$[2j_{12}]_q A_q^+ \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} \end{array} \right\}_q \{ \begin{array}{c} j_2 \\ j \end{array} \}_q \{ \begin{array}{c} j_{12} + 1 \\ j_{23} \end{array} \}_q + [2j_{12} + 2]_q A_q^- \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} \end{array} \right\}_q \{ \begin{array}{c} j_2 \\ j \end{array} \}_q \{ \begin{array}{c} j_{12} - 1 \\ j_{23} \end{array} \}_q$$

$$- \left( [2j_{12}]_q [2j_{12} + 1]_q [2j_{12} + 2]_q (j_{23})_q [j_{23} + 1]_q - [j_3 - j_2]_q [j_3 - j - 1]_q [j_3 + j + 2]_q - [2j_{12} + 2]_q [j_1 - j_2 + j_3 + j + 1]_q [j_1 + j_2 + j + 1]_q \right) = 0,$$

where

$$A_q^- = \sqrt{[j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12}]_q [j_{12} - j_3 + j]_q [j_{12} + j_3 - j]_q [j_1 + j_2 + j_{12} + 1]_q}$$

$$A_q^+ = \sqrt{[j_2 + j_3 + j + 1]_q [j_1 + j_2 - j_{12} + j + 1]_q}$$

(76)

The expressions (12) and (48) yield

$$\sqrt{\sigma(j_{23} + 1)} \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} + 1 \end{array} \right\}_q + \sqrt{\sigma(-j_{23} - 1)} \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} \end{array} \right\}_q$$

$$= [2j_{23} + 2]_q \sqrt{[j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12} + 1]_q} \left\{ \begin{array}{c} j_1 + \frac{1}{2} \\ j_3 \\ j - \frac{1}{2} \end{array} \right\}_q \{ \begin{array}{c} j_{12} \\ j \end{array} \}_q \{ \begin{array}{c} j_{23} + \frac{1}{2} \end{array} \}_q$$

(78)

and

$$\sqrt{\sigma(-j_{23} - 1)} \left\{ \begin{array}{c} j_1 + \frac{1}{2} \\ j_3 \\ j - \frac{1}{2} \end{array} \right\}_q \{ \begin{array}{c} j_{12} \\ j \end{array} \}_q \{ \begin{array}{c} j_{23} + 1 \end{array} \}_q$$

$$= [2j_{23} + 1]_q \sqrt{[j_{12} - j_1 + j_{12}]_q [j_{12} + j_1 - j_2 + 1]_q} \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \end{array} \right\}_q \{ \begin{array}{c} j_{12} \\ j \end{array} \}_q \{ \begin{array}{c} j_{23} - \frac{1}{2} \end{array} \}_q$$

(79)

respectively, whereas the differentiation formulas (84) - (85) give

$$[2j_{12} + 2]_q A_q^+ \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} - 1 \end{array} \right\}_q + [2j_{12}]_q A_q^- \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} + 1 \end{array} \right\}_q$$

$$+ \left( \sigma(j_{23}) [2j_{12} + 2]_q + [j_1 - j_2 + j_{12} + 1]_q [2j_{23}]_q A(j_{12}, j_{23}, j_1, j_2) \right) \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} \end{array} \right\}_q = 0$$

(80)

and

$$[2j_{12} + 2]_q A_q^+ \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} + 1 \end{array} \right\}_q - [2j_{23} + 2]_q A_q^- \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} \end{array} \right\}_q$$

$$+ \left( [2j_{12} + 2]_q \sigma(-j_{23} - 1) - [2j_{23} + 2]_q [j_1 - j_2 + j_{12} + 1]_q \right) A(j_{12}, j_{23}, j_1, j_2) +$$

$$[j_{12} - j_1 + j_2]_q [2j_{12} + 2]_q [2j_{23} + 1]_q \left\{ \begin{array}{c} j_1 \\ j_3 \\ j \\ j_{23} \end{array} \right\}_q \{ \begin{array}{c} j_{12} \\ j \end{array} \}_q \{ \begin{array}{c} j_{23} + j_1 - j_2 + 1 \end{array} \}_q = 0,$$

(81)

respectively, where $A_q^-$ are given by (38), $A_q^+$ by (77) and

$$A(j_{12}, j_{23}, j_1, j_2) = \sigma \left( \frac{-j_{12} + j_1 - j_2}{2} - 1 \right) - \sigma \left( \frac{-j_{12} + j_1 + j_2}{2} \right) -$$

$$[2j_{12} + 2]_q \left[ j_{23} + \frac{j_{12} - j_1 + j_2}{2} \right]_q \left[ j_{23} + \frac{j_{12} - j_1 + j_2 + 1}{2} \right]_q.$$
Using the hypergeometric representations \(53\) and \(54\), we obtain the representation of the 6\(j\)-symbols in terms of the \(q\)-hypergeometric function\(^5\) \(52\):

\[
\begin{align*}
\left\{ j_1, j_2, j_{12} \atop j_3, j \right\}_q & = (-1)^{j_1+j_2+j_{12}+j} \frac{[2j_{12}]_q!}{[j_1-j_2+j_3+j+1]_q!} \\
\left[ j_1+j_2+1 \right]_q! [j_1+j_2+3]_q! & \left[ [j_1+j_2+1]_q! [j_1+j_2+3]_q! [j_3-j_2+1]_q! \right] \\
\left[ j_1-j_2+j_3+j+1 \right]_q! & \left[ j_1-j_2+j_3+j+1 \right]_q! \times \\
\left[ j_2-j_3+j \right]_q! & \left[ j_2-j_3+j \right]_q! \times \\
\sum_{k=0}^{j_{12}-j_2+j_3} (-1)^{k+j_1+j_2+j+3+j} [k]_q! [k+j_3-k]_q! [k+j_3-j]_q! [k+j_3-j]_q! & [k+j_3-j]_q! [k+j_3-j]_q! [k+j_3-j]_q! [k+j_3-j]_q! \\
\left[ k+j_2+j_3 \right]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! & [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! \\
\end{align*}
\]

Notice that from the above representations the values \(74\) and \(75\) immediately follows. Notice also that the above formulas give two alternative explicit formulas for computing the 6\(j\)-symbols.

A third explicit formula follows from \(70\):

\[
\begin{align*}
\left\{ j_1, j_2, j_{12} \atop j_3, j \right\}_q & = \sqrt{\frac{j_2+j_3+1}{[j_1-j_2+j_3+j+1]_q!}} \\
\left[ j_2+j_3+1 \right]_q! [j_2+j_3+3]_q! & \left[ [j_2+j_3+1]_q! [j_2+j_3+3]_q! [j_3-j_2+1]_q! \right] \\
\left[ j_2+j_3+1 \right]_q! & \left[ j_2+j_3+1 \right]_q! \times \\
\left[ j_2+j_3+1 \right]_q! & \left[ j_2+j_3+1 \right]_q! \times \\
\left[ j_2+j_3+1 \right]_q! & \left[ j_2+j_3+1 \right]_q! \times \\
\sum_{k=0}^{j_{12}-j_2+j_3} (-1)^{k+j_1+j_2+j+3+j} [k]_q! [k+j_3-k]_q! [k+j_3-j]_q! [k+j_3-j]_q! & [k+j_3-j]_q! [k+j_3-j]_q! [k+j_3-j]_q! [k+j_3-j]_q! \\
\left[ k+j_2+j_3 \right]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! & [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! [k+j_2+j_3+1]_q! \\
\end{align*}
\]

To conclude this section let us point out that the orthogonality relations \(58\) and \(59\) lead to the orthogonality relations for the Racah polynomials \(u_{n}^{\alpha,\beta}(x(s), a, b)_q\) \(55\) and their duals \(u_{n}^{\alpha,\beta}(x(t), a', b')_q\), respectively, and also that the relation \(50\) between \(q\)-Racah and dual \(q\)-Racah corresponds to the symmetry property \(70\).

### 3.3 6\(j\)-symbols and the alternative \(q\)-Racah polynomials \(\tilde{u}_{n}^{\alpha,\beta}(x(s), a, b)_q\)

In this section we will make a comparison analysis but for the alternative \(q\)-Racah polynomials \(\tilde{u}_{n}^{\alpha,\beta}(x(s), a, b)_q\). We again choose \(s = j_{23}\) that runs on the interval \([a, b-1]\), \(a = j_3-j_2\), \(b = j_2+j_3+1\). In this case the connection is given by formula

\[
(-1)^{j_{12}+j_2+j_3} \sqrt{[2j_{12}+1]_q} \left\{ j_1, j_2, j_{12} \atop j_3, j \right\}_q = \sqrt{\frac{\rho(s)}{d^2}} \tilde{u}_{n}^{\alpha,\beta}(x(s), a, b)_q,
\]

where \(\rho(s)\) and \(d^2\) are the weight function and the norm, respectively, of the alternative \(q\)-Racah polynomials \(\tilde{u}_{n}^{\alpha,\beta}(x(s), a, b)_q\) (see Section 2.2) on the lattice \(1\), and \(n = j_1 + j_2 - j_{12}\), \(\alpha = j_1 - j_2 - j_3 + j \geq 0\), \(\beta = j_1 - j_2 + j_3 - j \geq 0\).

\(^5\)To obtain the representation in terms of the basic hypergeometric series it is sufficient to use the relation \(29\).
respectively, where

\[ \varsigma(j_{23}) = j_{23} - j_3 + j_2 \] 
\[ \varsigma(-j_{23} - 1) = j_{23} + j_3 - j_2 + 1 \] 

The differentiation formulas (81)–(84) give

\[ [2j_{12}]_q A^q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} - 1 \end{array} \right\}_q - [2j_{23}]_q A^q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} - 1 \\ j_3 & j & j_{23} \end{array} \right\}_q \]

\[ \left( \varsigma(j_{23})[2j_{12}]_q + [j_1 + j_2 + j_{12} + 1]_q [2j_{23}]_q \Lambda(j_{12}, j_{23}, j_1, j_2) \right) \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = 0 \] (85)

and

\[ [2j_{12}]_q A^q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{array} \right\}_q + [2j_{23} + 2]_q A^q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} - 1 \\ j_3 & j & j_{23} \end{array} \right\}_q \]

\[ \left( [2j_{12}]_q \varsigma(-j_{23} - 1) - [2j_{23} + 2]_q [j_1 + j_2 + j_{12} + 1]_q \left( \Lambda(j_{12}, j_{23}, j_1, j_2) \right) \right) \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = 0, \] (86)

respectively, where \( A^q \) are given by (83), \( \Lambda^q \) by (77) and

\[ \Lambda(j_{12}, j_{23}, j_1, j_2) = \varsigma \left( \frac{j_{12} - j_1 - j_2}{2} - 1 \right) - \varsigma \left( \frac{j_{12} - j_1 - j_2}{2} \right) - \]

\[ [2j_{12}]_q \left[ j_{23} + \frac{j_1 + j_2 - j_{12}}{2} \right]_q \left[ j_{23} + \frac{j_1 + j_2 - j_{12}}{2} + 1 \right]_q. \]
If we now use the hypergeometric representations (52) and (54) we obtain two new representations of the $6j$-symbols in terms of the $q$-hypergeometric function (28).

\[
\begin{cases}
\begin{array}{c}
\{ j_1, j_2, j_12 \} = (-1)^{j_1+j_2+j_3} \frac{[2j_2q]_l [j_1+j_2-j_3+j]_q!}{[j_3+j_2-j_1+j]_q!} \\
-j_1+j_2+j_3 \sqrt{[j_1+j_2-2j_2]_q [j_1+j_2-j_3-1]_q [j_1+j_2-j_3+1]_q [j_1+j_2-j_3+2]_q} \\
\end{array}
\end{cases}
\]

and

\[
\begin{cases}
\begin{array}{c}
\{ j_1, j_2, j_12 \} = (-1)^{j_1+j_2+j_3} \frac{[2j_2q]_l [j_1+j_2+j_3-j]_q!}{[j_3+j_2-j_1+j]_q!} \\
-j_1+j_2+j_3 \sqrt{[j_1+j_2-2j_2]_q [j_1+j_2+j_3+1]_q [j_1+j_2+j_3+2]_q} \\
\end{array}
\end{cases}
\]

Notice that from the above representations the values (71) and (74) also follows. Obviously the above formulas give another two alternative explicit formulas for computing the $6j$-symbols. Finally, from (87)

\[
\begin{cases}
\begin{array}{c}
\{ j_1, j_2, j_12 \} = \sqrt{[j_1+j_2+j_3-j]_q [j_1+j_2-2j_2]_q [j_1+j_2+j_3+1]_q [j_1+j_2+j_3+2]_q} \\
\end{array}
\end{cases}
\]

To conclude this section, let us point out that the orthogonality relations (88) and (89) lead to the orthogonality relations for the alternative Racah polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ and their duals $\tilde{u}_n^{\alpha,\beta}(x(t), a', b')_q$ (64), respectively, as well as the relation (71) between $q$-Racah and dual $q$-Racah corresponds to the symmetry property (70).

### 3.4 Connection between $\tilde{u}_k^{\alpha,\beta}(x(s), a, b)_q$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$

Let us obtain a formula connecting the two families $\tilde{u}_k^{\alpha,\beta}(x(s), a, b)_q$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$. In fact, Eqs. (71) and (72) suggest the following relation between both Racah polynomials $\tilde{u}_k^{\alpha,\beta}(x(s), a, b)_q$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$.

\[
\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q = (-1)^{s-a-n} \times \frac{\tilde{\Gamma}_q(\beta + 1) \tilde{\Gamma}_q(\alpha + 1 + s) \tilde{\Gamma}_q(a + b - \beta - n) \tilde{\Gamma}_q(a + b + \alpha + n) \tilde{\Gamma}_q(a + b - \beta + n) \tilde{\Gamma}_q(a + b + \alpha + s) \tilde{\Gamma}_q(\alpha + 1 + n) \tilde{\Gamma}_q(\beta + 1 + n)}{\tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1) \tilde{\Gamma}_q(s - a + \beta + 1)}.
\]

To prove it is sufficient to substitute the above formula in the difference equation (4) of the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ polynomials. After some straightforward computations the resulting difference equation becomes into the corresponding difference equation for the polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$.
Notice that from this relation follows that
\[
\begin{align*}
4F_3 \left( \begin{array}{c}
 a - b + n + 1, a - b - \alpha - \beta - n, a - s, a + s + 1 \\
 a - b + 1, 2a - \beta + 1, a - b - \alpha + 1
\end{array} \right| q, 1 \\
\right) = \frac{(\beta + 1)(q)_{s-a}(b + \alpha + a + 1)(q)_{s-a}}{(2a - \beta + 1)(q)_{s-a}(a - \beta - \alpha + 1)(q)_{s-a}} \left( \begin{array}{c}
 -n, \alpha + \beta + n + 1, a - s, a + s + 1 \\
 a - b + 1, \beta + 1, a + b + \alpha + 1
\end{array} \right| q, 1
\end{align*}
\]

This yield to the following identity for terminating $4\phi_3$ basic series, $n, N - n - 1, k = 0, 1, 2, \ldots$,
\[
4\varphi_3 \left( \begin{array}{c}
 q^{-N+1}, q^{-n-N+1}A^{-1}B^{-1}, q^{-k}, q^{-k}D \\
 q^{1-N}, q^{-2k}DB^{-1}, q^{1-N}A^{-1}
\end{array} \right| q, q
\right) = \frac{q^{-kN}}{A^k B^k (q^{-2k}DB^{-1}; q)_k, (q^{1-N}A^{-1}; q)_k} 4\varphi_3 \left( \begin{array}{c}
 q^{-n}, ABq^{n-k}, q^{-k}D \\
 q^{1-N}, qB, q^{-2k}DA
\end{array} \right| q, q
\right).
\]

## 4 Conclusions

Here we have provided a detailed study of two kind of Racah $q$-polynomials on the lattice $x(s) = [s]_q[s + 1]_q$ and also their comparative analysis with the Racah coefficients or $6j$-symbols of the quantum algebra $U_q(su(2))$.

To conclude this paper we will briefly discuss the relation of these $q$-Racah polynomials with the representation theory of the quantum algebra $U_q(su(3))$. In [20] §5.5.3 was shown that the transformation between two different bases ($\lambda, \mu$) of the irreducible representation of the classical (not $q$) algebra $su(3)$ corresponding to the reductions $su(3) \supset su(2) \times u(1)$ and $su(3) \supset u(1) \times su(2)$ of the $su(3)$ algebra in two different subalgebras $su(2)$ is given in terms of the Weyl coefficients that are, up to a sign (phase), the Racah coefficients of the algebra $su(2)$. The same statement can be done in the case of the quantum algebra $su_q(3)$ [5, 18]: The Weyl coefficients of the transformation between two bases of the irreducible representation ($\lambda, \mu$) corresponding to the reductions $su_q(3) \supset su_q(2) \times u_q(1)$ and $su_q(3) \supset u_q(1) \times su_q(2)$ of the quantum algebra $su_q(3)$ in two different quantum subalgebras $su_q(2)$ coincide (up to a sign) with the $q$-Racah coefficients of the $su_q(2)$.

In fact, the Weyl coefficients satisfy certain difference equations that are equivalent to the differentiation formulas for the $q$-Racah polynomials $u^{(n,3)}_\alpha(x(s), a, b)_q$ and $w^{(n,3)}_\alpha(x, a, b)_q$ so, following the idea in [20] §5.5.3 we can assure that the main properties of the $q$-Racah polynomials are closely related with the representations of the quantum algebra $U_q(su(3))$. Finally, let us point out that the same assertion can be done but with the non-compact quantum algebra $U_q(su(2,1))$. This will be carefully done in a forthcoming paper.

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