

ON THE LINEARIZATION PROBLEM INVOLVING POCHHAMMER SYMBOLS AND THEIR q -ANALOGUES. ¹

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Abstract

In this paper we present a simple recurrent algorithm for solving the *Linearization Problem* involving some families of q -polynomials in the exponential lattice $x(s) = c_1q^s + c_3$. Some simple examples are worked out in details.

1 Introduction.

Given three families of polynomials, denoted by $P_n(x)$, $Q_m(x)$ and $R_j(x)$, of degree exactly equal to respectively n , m and j , the *Linearization Problem* asks to compute the so-called *linearization coefficients* L_{mjn} defined by the relation:

$$Q_m(x)R_j(x) = \sum_{n=0}^{m+j} L_{mjn}P_n(x).$$

The study of such a problem has known an increasing interest in the last few years. Special emphasis was given to the classical *continuous* (Hermite, Laguerre, Jacobi and Bessel) [8, 9, 11, 15, 19, 20, 21, 24, 26, 27, 28] and the discrete cases (Charlier, Meixner, Kravchuk and Hahn) [7, 9, 12, 14, 27]. The main aim of the present paper is to show that the ideas given in [14, 26] can be extended in a very easy way to the q -polynomials on the exponential lattice $x(s) = c_1q^s + c_3$. In fact, if $P_n(x)$, $Q_m(x)$ and $R_j(x)$ are polynomials in $x(s) = c_1q^s + c_3$, then it is possible to find a recurrence relation for the linearization coefficients L_{mjn} , which is an alternative approach to the one given in [5]. This approach requires the knowledge of the so-called structure relation and three-term recurrence relation

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for the polynomials $P_n(x)$ as well as the second order difference equation which satisfy the other two polynomials $Q_m(x)$ and $R_j(x)$.

In this paper we will present an algorithm for computing the above coefficients, showing, as an example, the case when the polynomials $P_n(x)$, $Q_m(x)$ and $R_j(x)$ are the three Pochhammer and q -Pochhammer symbols. In both cases the solutions are given explicitly. The obtained expression can be used for solving more “*complicated*” examples.

The structure of the paper is as follows. In Section 2, a general algorithm for finding a recurrence relation for the L_{mjn} , is presented. In Section 3, two special cases, of particular importance, are worked out, namely, the linearization of a product of two q -Pochhammer or q -Stirling polynomials in terms of single q -Pochhammer or q -Stirling polynomials, respectively. Finally, some more complicated examples involving the Charlier polynomials and their q -analogues in the lattice $x(s) = q^s$ are presented.

2 General algorithm for solving the linearization problem.

In this section we will present a general algorithm to find a recurrence relation for the linearization coefficients L_{mjn} in the expansion

$$Q_m(x(s))R_j(x(s)) = \sum_{n=0}^{m+j} L_{mjn}P_n(x(s)), \quad x(s) = c_1q^s + c_3, \quad (2.1)$$

where c_1 , c_3 and q are constants, $Q_m(x(s)) \equiv Q_m(s)_q$ and $R_j(x(s)) \equiv R_j(s)_q$ are polynomials which satisfy a second order difference equation of the form

$$a(s)Q_m(s+1)_q + b(s)Q_m(s)_q + c(s)Q_m(s-1)_q = 0, \quad (2.2)$$

and

$$\alpha(s)R_j(s+1)_q + \beta(s)R_j(s)_q + \gamma(s)R_j(s-1)_q = 0, \quad (2.3)$$

respectively. A special case of such polynomials are the q -polynomials of hypergeometric type [3, 22, 23], which satisfy the difference equation

$$\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \quad (2.4)$$

$$\nabla f(s) = f(s) - f(s-1), \quad \Delta f(s) = f(s+1) - f(s).$$

Obviously, the Eq. (2.4) is of the type (2.2) ($y \equiv Q_m$), with

$$a(s) = \sigma(s) + \tau(s) \Delta x(s - \frac{1}{2}), \quad c(s) = \sigma(s) \frac{\Delta(s)}{\nabla x(s)}, \quad (2.5)$$

$$b(s) = \lambda \Delta x(s - \frac{1}{2}) \Delta x(s) - a(s) - c(s).$$

Notice that when $x(s) = c_1q^s + c_3$, $\nabla x(s)$, $\Delta x(s)$ and $\Delta x(s - \frac{1}{2})$ are polynomials of first degree in $x(s)$, then, in this case, the functions a , b , c , α , β and γ in (2.2)-(2.3) are polynomials of 2th degree at most.

In the following, we will use the operators \mathcal{T} and \mathcal{I} defined as follows

$$\begin{aligned} \mathcal{T} : \mathbb{P} &\rightarrow \mathbb{P} & \mathcal{I} : \mathbb{P} &\rightarrow \mathbb{P} \\ \mathcal{T}p(s) &= p(s+1) & \mathcal{I}p(s) &= p(s) \end{aligned} .$$

Using the above operators, we can rewrite the Eqs. (2.2)-(2.3) in the form

$$a(s+1)\mathcal{T}^2Q_m(s)_q + b(s+1)\mathcal{T}Q_m(s)_q + c(s+1)\mathcal{I}Q_m(s)_q = 0, \quad (2.6)$$

and

$$\alpha(s+1)\mathcal{T}^2R_j(s)_q + \beta(s+1)\mathcal{T}R_j(s)_q + \gamma(s+1)\mathcal{I}R_j(s)_q = 0. \quad (2.7)$$

Other important examples of q -polynomials are the so-called q -Pochhammer symbols and the q -Stirling polynomials (see section 3 of the present work for more details) which satisfy a first order difference equation of type (2.6)- (2.7).

2.1 The fourth order difference equation for the polynomials $Q_m(s)_q R_j(s)_q$.

Using the two equations (2.6) and (2.7), it is easy to prove that the polynomials $u(s)_q \equiv Q_m(s)_q R_j(s)_q$, satisfy a fourth order difference equation of the form

$$\mathcal{L}_4 u(s) \equiv p_4(s)\mathcal{T}^4 u(s)_q + p_3(s)\mathcal{T}^3 u(s)_q + p_2(s)\mathcal{T}^2 u(s)_q + p_1(s)\mathcal{T} u(s)_q + p_0(s)\mathcal{I} u(s)_q. \quad (2.8)$$

To prove this, we will follow the works [13, 14]. The idea is the following. Since (2.6)-(2.7),

$$\begin{aligned} a(s+1)\alpha(s+1)\mathcal{T}^2 u(s) &= \\ &= [b(s+1)\mathcal{T}Q_m(s)_q + c(s+1)\mathcal{I}Q_m(s)_q] [\beta(s+1)\mathcal{T}R_j(s)_q + \gamma(s+1)\mathcal{I}R_j(s)_q], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \mathcal{L}_2 u(s) &\equiv a(s+1)\alpha(s+1)\mathcal{T}^2 u(s) - b(s+1)\beta(s+1)\mathcal{T} u(s) - c(s+1)\gamma(s+1)\mathcal{I} u(s) = \\ &= b(s+1)\gamma(s+1) [\mathcal{T}Q_m(s)_q \mathcal{I}R_j(s)_q] + c(s+1)\beta(s+1) [\mathcal{I}Q_m(s)_q \mathcal{T}R_j(s)_q] = \\ &= l_1(s) [\mathcal{T}Q_m(s)_q \mathcal{I}R_j(s)_q] + l_2(s) [\mathcal{I}Q_m(s)_q \mathcal{T}R_j(s)_q]. \end{aligned}$$

Next, we change in the last expression $s \rightarrow s+1$, and substitute in the right-hand side the expression $\mathcal{T}^2 Q_m(s)_q$ and $\mathcal{T}^2 R_j(s)_q$, using the Eqs. (2.6)-(2.7), respectively. This allows us to rewrite the resulting expression in the form

$$\mathcal{M}_3 u(s) = m_1(s) [\mathcal{T}Q_m(s)_q \mathcal{I}R_j(s)_q] + m_2(s) [\mathcal{I}Q_m(s)_q \mathcal{T}R_j(s)_q],$$

where \mathcal{M}_3 is a difference operator of third order (there is one term proportional to \mathcal{T}^3), m_1 and m_2 are known functions of s . Repeating the same procedure, but now starting from the above equation we obtain

$$\mathcal{N}_4 u(s) = n_1(s) [\mathcal{T}Q_m(s)_q \mathcal{I}R_j(s)_q] + n_2(s) [\mathcal{I}Q_m(s)_q \mathcal{T}R_j(s)_q].$$

Then

$$\begin{vmatrix} \mathcal{L}_2 u(s) & l_1(s) & l_2(s) \\ \mathcal{M}_3 u(s) & m_1(s) & m_2(s) \\ \mathcal{N}_4 u(s) & n_1(s) & n_2(s) \end{vmatrix} = 0. \quad (2.9)$$

Expanding the determinant from the first column, the Eq. (2.8) holds.

Remark: The above equation (2.9), and its proof, remains true for any lattice function $x(s)$ and not only for the exponential lattice $x(s) = c_1 q^s + c_3$.

2.2 The generalized linearization algorithm

As before, we will suppose that $Q_m(s)_q$ and $R_j(s)_q$ satisfy the equations (2.6) and (2.7), respectively, and that $P_n(s)_q$ satisfy the so-called structure relation in the exponential lattice $x(s) = c_1 q^s + c_3$ and a three-term recurrence relation. An example of the latest are, again, the q -hypergeometric orthogonal polynomials in the aforementioned lattice which satisfy the structure relation [4] (for the case $x(s) = q^s$ see [6])

$$[\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})] \frac{\Delta P_n(s)_q}{\Delta x(s)} = S_n P_{n+1}(s)_q + T_n P_n(s)_q + R_n P_{n-1}(s)_q, \quad (2.10)$$

or, written in its equivalent form

$$\Sigma(s) \mathcal{T} P_n(s)_q = \sum_{k=n-2}^{n+2} A_k(n) P_k(s)_q, \quad \Sigma(s) = \sigma(s) + \tau(s) \Delta x(s - \frac{1}{2}), \quad (2.11)$$

as well as the three-term recurrence relation (TTRR)

$$x(s) P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \quad P_{-1}(x) \equiv 0, \quad n \geq 0. \quad (2.12)$$

To obtain (2.11) from (2.10) we use the facts that $\Delta x(s)$ is a polynomial of first degree in $x(s)$ (which is not valid in general for any lattice $x(s)$), $\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})$ is a polynomial of degree two in $x(s)$, and the TTRR (2.12).

From the above expression (2.11), one easily obtains that

$$\begin{aligned} \Sigma(s) \Sigma(s+1) \mathcal{T}^2 P_n(s)_q &= \sum_{k=n-4}^{n+4} \tilde{A}_k(n) P_k(s)_q, \\ \Sigma(s) \Sigma(s+1) \Sigma(s+2) \mathcal{T}^3 P_n(s)_q &= \sum_{k=n-6}^{n+6} \hat{A}_k(n) P_k(s)_q, \\ \Sigma(s) \Sigma(s+1) \Sigma(s+2) \Sigma(s+3) \mathcal{T}^4 P_n(s)_q &= \sum_{k=n-8}^{n+8} \bar{A}_k(n) P_k(s)_q. \end{aligned} \quad (2.13)$$

To obtain a recurrence relation for the linearization coefficients we can do the following:

Since (2.8), $\mathcal{L}_4 Q_m(s)_q R_j(s)_q = 0$, then applying \mathcal{L}_4 to both sides of (2.1), we find

$$0 = \sum_{n=0}^{m+j} L_{mjn} \Sigma(s) \Sigma(s+1) \Sigma(s+2) \Sigma(s+3) \mathcal{L}_4 P_n(x(s)).$$

Taking into account that \mathcal{L}_4 is a fourth degree operator with polynomial coefficients, and

using the structure relation (2.11) as well as (2.13) we find

$$\begin{aligned}
0 = & \sum_{n=0}^{m+j} L_{mjn} \left\{ p_4(s) \sum_{k=n-8}^{n+8} \bar{A}_k(n) P_k(s)_q + p_3(s) \Sigma(s+3) \sum_{k=n-6}^{n+6} \hat{A}_k(n) P_k(s)_q + \right. \\
& + p_2(s) \Sigma(s+2) \Sigma(s+3) \sum_{k=n-4}^{n+4} \tilde{A}_k(n) P_k(s)_q + \\
& + p_1(s) \Sigma(s+1) \Sigma(s+2) \Sigma(s+3) \sum_{k=n-2}^{n+2} A_k(n) P_k(s)_q + \\
& \left. + \Sigma(s) \Sigma(s+1) \Sigma(s+2) \Sigma(s+3) p_0(s) P_n(s)_q \right\},
\end{aligned}$$

from where, and by taking into account that $\Sigma(s+k)$, $k = 0, 1, 2, 3$, is a polynomial of degree two in $x(s) = c_1 q^s + c_3$, as well as the TTRR (2.12) we obtain that the coefficients L_{mjn} satisfy a recurrence relation of the form

$$\sum_{k=0}^r c_k(i, j, n) L_{mj, n+k} = 0. \quad (2.14)$$

In general, the present algorithm may not give the minimal order recurrence for the linearization coefficients. To get the order r minimal it is necessary to use more specific properties of the families of polynomials involved in (2.1).

Remark: Notice that the present algorithm also works in the case when the product $Q_m(x(s))R_j(x(s))$ satisfy a k th-linear difference equation with polynomial coefficients (not necessary of order 4 as in (2.9)), so it can be used for solving more general linearization problems, e. g., linearization problems involving the product of three or more q -polynomials.

3 Examples

In this section we will work out some examples. For simplicity we will consider the case when the involved polynomials Q_m and R_j satisfy first order difference equations, i.e., equation of the form (2.6) and (2.7) with $a(s) = \alpha(s) \equiv 0$.

3.1 Linearization of a product of two q -Pochhammer symbols.

Let us define the quantities $(s)_q$ by

$$(s)_q = \frac{q^s - 1}{q - 1} = q^{\frac{s}{2}-1} [s]_q, \quad (3.1)$$

and let $[(s)_q]_n$, the q -Pochhammer symbol, be defined by

$$[(s)_q]_n = (s)_q (s+1)_q \cdots (s+n-1)_q = \prod_{k=0}^{n-1} \frac{q^{s+k} - 1}{q - 1}. \quad (3.2)$$

Notice that $[(s)_q]_n$ is a polynomial of degree exactly equal n in q^s . The polynomials $[(s)_q]_n$ satisfy the following difference equation

$$(s)_q[(s+1)_q]_n - (s+n)_q[(s)_q]_n = 0, \quad (3.3)$$

and a recurrence relation

$$(s)_q[(s)_q]_n - q^{-n}[(s)_q]_{n+1} + q^{-n}(n)_q[(s)_q]_n = 0. \quad (3.4)$$

Notice also that

$$[(s)_q]_n = \frac{(q^s; q)_n}{(1-q)^n}, \quad \text{where } (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k). \quad (3.5)$$

Obviously, in the limit $q \rightarrow 1$, the q -Pochhammer symbol $[(s)_q]_n$, becomes into the classical Pochhammer symbol $(s)_n = s(s+1) \cdots (s+n-1)$.

Since the product $[(s)_q]_i [(s)_q]_j$ is a polynomial in q^s , it can be represented as a linear combination of the single q -Pochhammer symbols $[(s_q)]_n$. In particular,

$$[(s)_q]_i [(s)_q]_j = \sum_{n=0}^{i+j} L_{ijn}(q) [(s_q)]_n. \quad (3.6)$$

In order to obtain the recurrence relation for the linearization coefficients L_{ijn} in (3.6) we apply the operator

$$(s)_q^2 \mathcal{T} - (s+i)_q (s+j)_q \mathcal{I} \quad (3.7)$$

to both sides of (3.6). Using formula (3.3) we obtain the following expression

$$0 = \sum_{n=0}^{i+j} L_{ijn} \left[\left(\frac{q^s - 1}{q - 1} \right)^2 \mathcal{T} [(s_q)]_n - \left(\frac{q^{s+i} - 1}{q - 1} \right) \left(\frac{q^{s+j} - 1}{q - 1} \right) [(s_q)]_n \right]. \quad (3.8)$$

Taking into account the Eq. (3.3) for the q -Pochhammer symbol, we find

$$\begin{aligned} 0 &= \sum_{n=0}^{i+j} L_{ijn} [(s_q)]_n \left[\left(\frac{q^s - 1}{q - 1} \right) \left(\frac{q^{s+n} - 1}{q - 1} \right) - \left(\frac{q^{s+i} - 1}{q - 1} \right) \left(\frac{q^{s+j} - 1}{q - 1} \right) \right] = \\ &= \sum_{n=0}^{i+j} L_{ijn} [(s_q)]_n [(s)_q (s+n)_q - (s+i)_q (s+j)_q]. \end{aligned}$$

Next, we will rewrite the expression inside the quadratic brackets using the identity

$$(s+n)_q = q^n (s)_q + (n)_q,$$

to obtain

$$0 = \sum_{n=0}^{i+j} L_{ijn} [(s_q)]_n \left\{ (s)_q^2 [q^n - q^{i+j}] + (s)_q [(n)_q - q^i (j)_q - q^j (i)_q] - (i)_q (j)_q \right\},$$

from where, using Eq. (3.4), we arrive to the expression

$$\begin{aligned}
& \sum_{n=0}^{i+j} L_{ijn} \left\{ q^{-2n-1} [q^n - q^{i+j}] [(s_q)_{n+2}] + \right. \\
& + [((n)_q - q^i(j)_q - q^j(i)_q) q^{-n} - (q^n - q^{i+j}) (q^{-2n-1}(n+1)_q + q^{-2n}(n)_q)] [(s_q)_{n+1}] + \\
& + \left. [(q^n - q^{i+j}) q^{-2n} (n)_q^2 - ((n)_q - q^i(j)_q - q^j(i)_q) q^{-n} - (i)_q(j)_q] [(s_q)_n] \right\} = \\
& = \sum_{n=0}^{i+j} L_{ijn} \left\{ q^{-2n-1} [q^n - q^{i+j}] [(s_q)_{n+2}] - \right. \\
& - \left\{ q^{-n-1}(n+1)_q + q^{i+j-1-2n} [1 + q^{n+j+1}(j)_q + q^{n+i+1}(i)_q - 2(n+1)_q] \right\} [(s_q)_{n+1}] - \\
& - \left. q^{i+j-2n} [(n)_q - q^{n+j}(j)_q] [(n)_q - q^{n+i}(i)_q] [(s_q)_n] \right\} = 0.
\end{aligned}$$

The above equation leads us to the following three-term recurrence relation for the linearization coefficients L_{ijn}

$$A_n L_{ijn-2} + B_n L_{ijn-1} + C_n L_{ijn} = 0, \quad (3.9)$$

where

$$\begin{aligned}
A_n &= q^{-2n+3} [q^{n-2} - q^{i+j}], \\
B_n &= -q^{-n}(n)_q - q^{i+j+1-n} [q^{-j}(j)_q + q^{-i}(i)_q - q^{-n}(n)_q - q^{-n+1}(n-1)_q], \\
C_n &= -q^{i+j} [q^{-n}(n)_q - q^{-j}(j)_q] [q^{-n}(n)_q - q^{-i}(i)_q],
\end{aligned} \quad (3.10)$$

with the initial conditions $L_{ij\ i+j+1} = 0$ and $L_{ij\ i+j} = q^{-ij}$.

To solve the above recurrence we apply the algorithm qHyper [1, 2, 25] which allows us to find an equivalent two-term recurrence relation for the linearization coefficients. Namely,

$$L_{ijn+1} = -\frac{q^{-k-1}(i+j-n)_q}{(i-n-1)_q(j-n-1)_q} L_{ijn}, \quad (3.11)$$

so that,

$$L_{ijn} = (-1)^{i+j-n} q^{\frac{i(i+1)+j(j+1)-n(n+1)}{2}} \frac{[(-j)_q]_{i+j-n} [(-i)_q]_{i+j-n}}{(i+j-n)_q!}, \quad (3.12)$$

for $n \geq \max(i, j)$ and vanishes otherwise.

Notice that, in the limit $q \rightarrow 1$, the above recurrence relations (3.9)-(3.11) transform into a two-term recurrence relations for the standard Pochhammer symbols $(s)_n$ of the form

$$(k-i-j-1)L_{ijn-1} - (k^2 - (i+j)k + ij)L_{ijn} = 0, \quad L_{ij\ i+j+1} = 0, \quad L_{ij\ i+j} = 1,$$

which solution

$$L_{ijn} = \begin{cases} \frac{(-1)^{i+j+n}(-j)_{i+j-n}(-i)_{i+j-n}}{(i+j-n)!} & n \geq \max(i, j) \\ 0 & \text{otherwise} \end{cases},$$

corresponds to (3.12) in the limit $q \rightarrow 1$.

3.2 Linearization of a product of two q -Stirling polynomials.

Let us define the q -Stirling polynomials or q -falling factorials $(s)_q^{[n]}$, by

$$(s)_q^{[n]} = (s)_q(s-1)_q \cdots (s-n+1)_q = \prod_{k=0}^{n-1} \frac{q^{s-k} - 1}{q - 1} \quad (3.13)$$

Also we will use the notation

$$(s)_q^{[n]} = \frac{(q^s; q)_q^{[n]}}{(1-q)^n}, \quad (a; q)_q^{[n]} = (1-a)(1-aq^{-1}) \cdots (1-aq^{-n+1}). \quad (3.14)$$

These quantities $(s)_q^{[n]}$ are closely related to the q -Stirling numbers $\tilde{S}_q(n, k)$, $s_q^*(n, k)$ [30] by formulas

$$(s)_q^n = \sum_{k=0}^n \tilde{S}_q(n, k)(s)_q^{[n]}, \quad (s)_q^{[n]} = \sum_{k=0}^n s_q^*(n, k)(s)_q^k, \quad (3.15)$$

and satisfy the following difference equation

$$(s)_q(s-1)_q^{[n]} - (s-n)_q(s)_q^{[n]} = 0, \quad (3.16)$$

as well as the recurrence relation

$$(s)_q(s)_q^{[n]} - q^n(s)_q^{[n+1]} - (n)_q[(s)_q]^{[n]} = 0. \quad (3.17)$$

Again, since the product $(s)_q^{[i]}(s)_q^{[j]}$ is a polynomial in $x(s) = q^s$, it can be represented as a linear combination of the q -falling factorials $(s)_q^{[n]}$. In particular,

$$(s)_q^{[i]}(s)_q^{[j]} = \sum_{n=0}^{i+j} \tilde{L}_{ijn}(q)(s)_q^{[n]}. \quad (3.18)$$

To obtain the recurrence relation for the linearization coefficients \tilde{L}_{ijn} in (3.18) we apply the operator

$$(s)_q^2 \mathcal{T}^{-1} - (s-i)_q(s-j)_q \mathcal{I} \quad (3.19)$$

to both sides of (3.18) and do similar calculations as before but now using the equations (3.16) and (3.17), respectively. This leads us to the expression

$$\begin{aligned} & \sum_{n=0}^{i+j} \tilde{L}_{ijn} \left\{ q^{2n+1} [q^{-n} - q^{-i-j}] (s)_q^{[n+2]} + \right. \\ & + \left\{ (n+1)_q + q^{n-i-j} [(j)_q + (i)_q - (n)_q - (n+1)_q] \right\} (s)_q^{[n+1]} - \\ & \left. - q^{-i-j} [(n)_q - (j)_q] [(n)_q - (i)_q] (s)_q^{[n]} \right\} = 0, \end{aligned}$$

which allows us to obtain the following three-term recurrence relation for the linearization coefficients \tilde{L}_{ijn}

$$\tilde{A}_n \tilde{L}_{ijn-2} + \tilde{B}_n \tilde{L}_{ijn-1} + \tilde{C}_n \tilde{L}_{ijn} = 0, \quad (3.20)$$

where

$$\begin{aligned} \tilde{A}_n &= q^{2n-3} [q^{-n+2} - q^{-i-j}], \\ \tilde{B}_n &= (n)_q + q^{n-i-j-1} [(j)_q + (i)_q - (n-1)_q - (n)_q], \\ \tilde{C}_n &= -q^{-i-j} [(n)_q - (j)_q] [(n)_q - (i)_q], \end{aligned} \quad (3.21)$$

with the initial conditions $\tilde{L}_{ij i+j+1} = 0$ and $\tilde{L}_{ij i+j} = q^{ij}$.

Again, applying to the above recurrence relation the algorithm qHyper [1, 2, 25] we find that the coefficients \tilde{L}_{ijn} satisfy an equivalent two-term recurrence relation

$$\tilde{L}_{ijn+1} = \frac{q(i+j-n)_q}{(i-n-1)_q (j-n-1)_q} \tilde{L}_{ijn}, \quad (3.22)$$

so that,

$$\tilde{L}_{ijn} = q^{i+j+ij-n} \frac{(-j)_q [i+j-n] (-i)_q [i+j-n]}{(i+j-n)_q!}, \quad \text{for } n \geq \max(i, j), \quad (3.23)$$

and vanishes otherwise.

We want to point out here that the above recurrence relation (3.20), as well as its solution (3.23), can be obtained from the previous equations (3.9) and (3.12). In order to do that we use the identity

$$(s)_q^{[n]} = (-1)^n q^{-n} [(-s)_{q^{-1}}]_n, \quad (3.24)$$

and substitute it in (3.6). Comparing the obtained expression with (3.18) we find

$$\tilde{L}_{ijn}(q) = (-1)^{i+j-n} q^{n-i-j} L_{ijn}(q^{-1}). \quad (3.25)$$

The above three-term recurrence relation (3.20) in the limit $q \rightarrow 1$, transforms into a two-term recurrence relation for the classical Stirling polynomials $(s)^{[n]} = s(s-1) \cdots (s-n+1)$, of the form

$$(j+i+1-k)\tilde{L}_{ijn-1} - (k^2 - (i+j)k + ij)\tilde{L}_{ijn} = 0, \quad L_{ij i+j+1} = 0, \quad L_{ij i+j} = 1, \quad (3.26)$$

which have the solution

$$\tilde{L}_{ijn} = \begin{cases} \frac{(-j)_{i+j-n} (-i)_{i+j-n}}{(i+j-n)!} & n \geq \max(i, j) \\ 0 & \text{otherwise} \end{cases}, \quad (3.27)$$

and that is in agreement with (3.23).

Remark: To conclude this section we want to remark that the method presented here can be used also for solving the linearization problem

$$\{[(s)_q]_i\}^m = \sum_{m=0}^{i+m} L_{in}^m(q)[(s_q)]_n,$$

or

$$[(s)_q^{[i]}]^m = \sum_{m=0}^{i+m} \tilde{L}_{in}^m(q)(s_q)^{[n]},$$

since the the m -power of a q -Pochhammer or a q -Stirling polynomials satisfy a first order difference equation of the form

$$(s)_q^m \mathcal{T}\{[(s)_q]_i\}^m - (s+i)_q^m \{[(s)_q]_i\}^m = 0, \quad (s)_q^m \mathcal{T}^{-1}[(s)_q^{[i]}]^m - (s-i)_q^m [(s)_q^{[i]}]^m,$$

respectively. Then, it is easy to see that the corresponding recurrence relations for the coefficients $L_{in}^m(q)$ and $\tilde{L}_{in}^m(q)$, are of degree at most $m+1$ (for the classical case see [18]).

3.3 Further examples

In this section we will show the usefulness of the obtained relations for solving some linerization formulas involving some classical hypergeometric polynomials and their q -analogues.

Classical Charlier polynomials.

Firstly, we will obtain the linearization of a product of two Stirling polynomials $x^{[m]}x^{[j]}$ in terms of the monic Charlier polynomials

$$x^{[m]}x^{[j]} = \sum_{n=0}^{m+j} c_{m,j,n} C_n^\mu(x), \quad (3.28)$$

where the Charlier polynomials C_n^μ are defined by

$$C_n^\mu(x) = (-\mu)^n {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{\mu}\right) = x^n + \dots \quad (3.29)$$

Here ${}_pF_q$ denotes the generalized hypergeometric function

$${}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}. \quad (3.30)$$

To obtain the explicit expression for the coefficients $c_{m,j,n}$ in (3.28), we will use the expression (3.27) as well as the well known inversion formula (see, for example, [7])

$$x^{[k]} = \sum_{n=0}^k c_{kn} C_n^\mu(x), \quad c_{kn} = \binom{k}{n} \mu^{k-n}. \quad (3.31)$$

Then, substituting (3.31) in

$$x^{[m]}x^{[j]} = \sum_{k=0}^{m+j} \tilde{L}_{mjk} x^{[k]},$$

and interchanging the sums we obtain

$$c_{m,j,n} = \sum_{k=0}^{m+j-n} \tilde{L}_{mj \ k+n} c_{k+n \ n}.$$

Substituting in the above expressions the Eqs. (3.27) and (3.31) for the coefficients $\tilde{L}_{mj \ k+n}$ and $c_{k+n \ n}$ and making some straightforward calculations, we finally obtain

$$c_{m,j,n} = \binom{m+j}{n} \mu^{m+j-n} {}_3F_1 \left(\begin{matrix} -m, & -j, & n-m-j \\ & -m-j & \end{matrix} \middle| \frac{1}{\mu} \right). \quad (3.32)$$

An analogous expression have been obtained in [7] by others means. Obviously, the same can be done for other hypergeometric polynomials (e.g. Hahn, Meixner and Kravchuk).

Notice also that, since

$$C_j^\gamma(x) = \sum_{k=0}^j a_{jk} x^{[k]}, \quad a_{jk} = \frac{(-1)^j (-j)_k \gamma^{j-k}}{k!},$$

then, multiplying (3.28) by a_{jk} and taking the sum over k , we obtain

$$x^{[m]} C_j^\gamma(x) = \sum_{n=0}^{m+j} l_{m,j,n} C_n^\mu(x), \quad (3.33)$$

where

$$l_{m,j,n} = (-\gamma)^j \mu^{m-n} \sum_{k=0}^j \binom{m+k}{n} \frac{(-j)_k}{k!} \left(\frac{\mu}{\gamma}\right)^k {}_3F_1 \left(\begin{matrix} -m, & -k, & n-m-k \\ & -m-k & \end{matrix} \middle| \frac{1}{\mu} \right).$$

A similar result was obtained in [7] by a different way.

Notice also the finiteness of the last sum as well as the termination character of the involved hypergeometric serie ${}_3F_1$.

***q*-Charlier polynomials.**

Next we will obtain some formulas for the linearization coefficients involving a *q*-analogue of the Charlier polynomials $c_n^\mu(s, q)$ in the exponential lattice $x(s) = q^s$ [3, 4].

These monic Charlier *q*-polynomials $c_n^\mu(s, q)$ are defined by

$$\begin{aligned} c_n^{(\mu)}(s, q) &= \mu^n (1-q)^n q^{n^2} {}_2\varphi_0 \left(\begin{matrix} q^{-n}, q^{-s} \\ - \end{matrix} ; q, -\frac{q^s}{(q-1)\mu} \right) = \\ &= \mu^n (1-q)^n q^{n^2} \sum_{k=0}^n \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{1}{\mu^k} (s)_q^{[k]}, \quad 1 < q < 1, \quad 0 < \mu < 1, \end{aligned} \quad (3.34)$$

where $(s)_q^{[k]}$ are the q -Stirling polynomials defined in (3.13), the symbols $(a; q)_k$ are given in (3.5), and the q -basic hypergeometric series is defined by [16]

$${}_r\hat{\varphi}_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)} \right]^{p-r+1}. \quad (3.35)$$

Also we need the q -analogue of the inversion formula (3.31) [5]

$$(s)_q^{[k]} = \sum_{n=0}^k c_{kn}^q c_n^\mu(s, q), \quad c_{kn}^q = \frac{q^{k-\frac{n}{2}(n+1)}}{(q-1)^n} \binom{k}{n}_q \mu^{k-n}, \quad (3.36)$$

where

$$\binom{k}{n}_q = \frac{(q; q)_k}{(q; q)_n (q; q)_{k-n}}.$$

Notice that the above expressions transform into the classical ones when $q \rightarrow 1$. Then, to solve the linearization problem

$$(s)_q^{[m]} (s)_q^{[j]} = \sum_{n=0}^{m+j} c_{m,j,n}^q c_n^\mu(s, q), \quad (3.37)$$

we apply the same procedure as before, i.e.,

$$c_{m,j,n}^q = \sum_{k=0}^{m+j-n} \tilde{L}_{mj \ k+n}(q) c_{k+n \ n}^q,$$

where $\tilde{L}_{mj \ k+n}(q)$ are given in (3.23) and $c_{k+n \ n}^q$ by (3.36). Again, some straightforward calculations (in which we use some identities involving the $(a; q)_n$ and $(a; q)^{[n]}$ symbols [16, 17]) lead us to the expression

$$c_{m,j,n}^q = \frac{q^{m+j+mj-\frac{n}{2}(n+1)} \mu^{m+j-n}}{(q-1)^n} \binom{j+m}{n}_q {}_3\hat{\varphi}_1 \left(\begin{matrix} q^{-m}, q^{-j} q^{n-m-j} \\ q^{-m-j} \end{matrix} ; q, \frac{1}{(1-q)q^{n+1}\mu} \right).$$

Here, we use the function ${}_r\hat{\varphi}_p$ defined by

$${}_r\hat{\varphi}_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k}. \quad (3.38)$$

Notice also that in the case when $q \rightarrow 1$, Eq. (3.37) transforms into the classical one (3.32).

Obviously, the same can be done with others q -polynomials (e.g. q -Hahn, q -Meixner, etc [3, 17]). Also, analogues of the problem (3.33) can be solved in a similar way.

Summary

In the present work we have developed a general recurrent method for solving the linearization problem for a product of two polynomials in the exponential lattice $x(s) = c_1 q^s + c_3$, obtaining the explicit solution in the case of the q -analogues of the Pochhammer symbol and

the Stirling polynomials. Finally, some linearization problems involving the q -analogues of the classical Charlier polynomials were solved.

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