LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS SATISFYING AN IMPROVED EQUALITY INVOLVING $\delta(2,2)$

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ABSTRACT. It was proved in [8, 9] that every Lagrangian submanifold M of a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$ satisfies the following optimal inequality:

$$\delta(2,2) \leq \frac{25}{4}H^2 + 8c, \tag{A}$$

where H^2 is the squared mean curvature and $\delta(2,2)$ is a δ-invariant on M introduced by the first author. This optimal inequality improves a special case of an earlier inequality obtained in [B.-Y. Chen, Japan. J. Math. **26** (2000), 105–127].

The main purpose of this paper is to classify Lagrangian submanifolds of $\tilde{M}^5(4c)$ satisfying the equality case of the improved inequality (A).

1. Introduction

Let $\tilde{M}^n$ be a Kähler n-manifold with the complex structure J, a Kähler metric g and the Kähler 2-form ω. An isometric immersion $\psi : M \to \tilde{M}^n$ of a Riemannian n-manifold M into $\tilde{M}^n$ is called *Lagrangian* if $\psi^*\omega = 0$.

Let $\tilde{M}^n(4c)$ denote a Kähler n-manifold with constant holomorphic sectional curvature $4c$, called a *complex space form*. A complete simply-connected complex space form $\tilde{M}^n(4c)$ is holomorphically isometric to the complex Euclidean n-plane $\mathbf{C}^n$, the complex projective n-space $CP^n(4c)$, or a complex hyperbolic n-space $CH^n(4c)$ according to $c = 0$, $c > 0$ or $c < 0$, respectively.

B.-Y. Chen introduced in 1990s new Riemannian invariants $\delta(n_1, \ldots, n_k)$. For any n-dimensional submanifold M in a real space form $R^m(c)$ of constant

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curvature c, he proved the following sharp general inequality (see [5, 7] for details):

$$\begin{aligned}\delta(n_1,\dots,n_k) \le {} & \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}H^2 \\ & + \frac{1}{2}\Big(n(n-1)-\sum_{j=1}^{k} n_j(n_j-1)\Big)c.\end{aligned} \tag{1.1}$$

For Lagrangian submanifolds in a complex space form $\tilde{M}^n(4c)$, we have

Theorem A. *Let M be an n-dimensional Lagrangian submanifold in a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Then inequality (1.1) holds for each k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$.*

The following result from [6] extends a result in [10] on $\delta(2)$.

Theorem B. *Every Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ is minimal if it satisfies the equality case of (1.1) identically.*

Theorem B was improved recently in [8, 9] to the following inequality.

Theorem C. *Let M be an n-dimensional Lagrangian submanifold of $\tilde{M}^n(4c)$. Then, for an $(n_1,\dots,n_k)\in\mathcal{S}(n)$ with $\sum_{i=1}^k n_i < n$, we have*

$$\begin{aligned}\delta(n_1,\dots,n_k) \le {} & \frac{n^2\Big\{\Big(n-\sum_{i=1}^k n_i+3k-1\Big)-6\sum_{i=1}^k(2+n_i)^{-1}\Big\}}{2\Big\{\Big(n-\sum_{i=1}^k n_i+3k+2\Big)-6\sum_{i=1}^k(2+n_i)^{-1}\Big\}}H^2 \\ & + \frac{1}{2}\Big\{n(n-1)-\sum_{i=1}^k n_i(n_i-1)\Big\}c.\end{aligned} \tag{1.2}$$

The equality sign holds at a point $p\in M$ if and only if there is an orthonormal basis $\{e_1,\dots,e_n\}$ at p such that the second fundamental form h satisfies

$$\begin{aligned} & h(e_{\alpha_i},e_{\beta_i}) = \sum\nolimits_{\gamma_i} h^{\gamma_i}_{\alpha_i\beta_i}Je_{\gamma_i} + \frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda Je_{N+1},\ \ \sum\nolimits_{\alpha_i=1}^{n_i} h^{\gamma_i}_{\alpha_i\alpha_i}=0, \\ & h(e_{\alpha_i},e_{\alpha_j})=0,\ i\ne j;\ h(e_{\alpha_i},e_{N+1}) = \frac{3\lambda}{2+n_i}Je_{\alpha_i},\ h(e_{\alpha_i},e_u)=0, \\ & h(e_{N+1},e_{N+1}) = 3\lambda Je_{N+1},\ h(e_{N+1},e_u)=\lambda Je_u,\ N=n_1+\cdots+n_k, \\ & h(e_u,e_v)=\lambda\delta_{uv}Je_{N+1},\ i,j=1,\dots,k;\ u,v=N+2,\dots,n.\end{aligned} \tag{1.3}$$

For simplicity, we call a Lagrangian submanifold of a complex space form $\delta(n_1,\dots,n_k)$-*ideal* (resp., *improved* $\delta(n_1,\dots,n_k)$-*ideal*) if it satisfies the equality case of (1.1) (resp., the equality case of (1.2)) identically.

For $k=2$ and $n_1=n_2=2$, Theorem C reduces to the following.

Theorem D. *Let M be a Lagrangian submanifold in a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$. Then we have*

$$\delta(2,2) \leq \frac{25}{4}H^2 + 8c. \tag{1.4}$$

If the equality sign of (1.4) holds identically, then with respect some suitable orthonormal frame $\{e_1, \dots, e_5\}$ the second fundamental form h satisfies

$$\begin{aligned} &h(e_1,e_1) = \alpha Je_1 + \beta Je_2 + \mu Je_5,\ h(e_1,e_2) = \beta Je_1 - \alpha Je_2,\\ &h(e_2,e_2) = -\alpha Je_1 - \beta Je_2 + \mu Je_5,\\ &h(e_3,e_3) = \gamma Je_3 + \delta Je_4 + \mu Je_5,\ h(e_3,e_4) = \delta Je_3 - \gamma Je_4,\\ &h(e_4,e_4) = -\gamma Je_3 - \delta Je_4 + \mu Je_5,\ h(e_5,e_5) = 4\mu Je_5,\\ &h(e_i,e_5) = \mu Je_i,\ i \in \Delta;\ h(e_i,e_j) = 0,\ \ \textit{otherwise}, \end{aligned} \tag{1.5}$$

for some functions $\alpha, \beta, \gamma, \delta, \mu$, where $\Delta = \{1,2,3,4\}$.

The classification of $\delta(2,2)$-ideal Lagrangian submanifolds in complex space forms $\tilde{M}^5(4c)$ is done in [13]. In this paper we classify improved $\delta(2,2)$-ideal Lagrangian submanifolds in $\tilde{M}^5(4c)$. The main results of this paper are stated as Theorem 6.1, Theorem 7.1 and Theorem 8.1.

2. Preliminaries

2.1. **Basic formulas.** Let $\tilde{M}^n(4c)$ denote a complete simply-connected Kähler n-manifold with constant holomorphic sectional curvature $4c$. Then $\tilde{M}^n(4c)$ is holomorphically isometric to the complex Euclidean n-plane $\mathbf{C}^n$, the complex projective n-space $CP^n(4c)$, or a complex hyperbolic n-space $CH^n(-4c)$ according to $c = 0, c > 0$ or $c < 0$.

Let M be a Lagrangian submanifold of $\tilde{M}^n(4c)$. We denote the Levi-Civita connections of M and $\tilde{M}^n(4c)$ by ∇ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by (cf. [7])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \ \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.1}$$

for tangent vector fields X and Y and normal vector fields ξ, where h is the second fundamental form, A is the shape operator and D is the normal connection.

The second fundamental form and the shape operator are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The mean curvature vector $\overrightarrow{H}$ of M is defined by $\overrightarrow{H} = \frac{1}{n}\,\mathrm{trace}\, h$ and the *squared mean curvature* is given by $H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$.

For Lagrangian submanifolds, we have (cf. [7, 12])

$$D_X JY = J\nabla_X Y, \tag{2.2}$$

$$A_{JX}Y = -Jh(X,Y) = A_{JY}X. \tag{2.3}$$

Formula (2.3) implies that $\langle h(X,Y), JZ\rangle$ is totally symmetric.

The equations of Gauss and Codazzi are given respectively by

$$\langle R(X,Y)Z,W\rangle = \left\langle A_{h(Y,Z)}X,W\right\rangle - \left\langle A_{h(X,Z)}Y,W\right\rangle + c(\langle X,W\rangle \langle Y,Z\rangle - \langle X,Z\rangle \langle Y,W\rangle), \tag{2.4}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{2.5}$$

where R is the curvature tensor of M and ∇h is defined by

$$(\nabla_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.6}$$

For an orthonormal basis $\{e_1,\ldots,e_n\}$ of T_pM, we put

$$h^i_{jk} = \langle h(e_j,e_k), Je_i\rangle\,,\quad i,j,k = 1,\ldots,n.$$

It follows from (2.3) that $h^i_{jk} = h^j_{ik} = h^k_{ij}$.

2.2. **δ-invariants.** Let M be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1,\ldots,e_n$ of T_pM, the scalar curvature τ at p is $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$.

Let L be a r-subspace of T_pM with $r \geq 2$ and $\{e_1,\ldots,e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha,\beta \leq r. \tag{2.7}$$

For given integers $n \geq 3$, $k \geq 1$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of k-tuples $(n_1,\ldots,n_k)$ of integers satisfying $2 \leq n_1, \cdots, n_k < n$ and $\sum_{j=1}^k i \leq n$.

Put $\mathcal{S}(n) = \cup_{k\geq 1}\mathcal{S}(n,k)$. For each k-tuple $(n_1,\ldots,n_k) \in \mathcal{S}(n)$, the first author introduced in 1990s the Riemannian invariant $\delta(n_1,\ldots,n_k)$ by

$$\delta(n_1,\ldots,n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \;\; p \in M, \tag{2.8}$$

where $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1,\ldots,k$ (cf. [7] for details).

2.3. **Horizontal lift of Lagrangian submanifolds.** The following link between Legendrian submanifolds and Lagrangian submanifolds is due to [16] (see also [7, pp. 247–248]).

Case (i): $CP^n(4)$. Consider Hopf's fibration $\pi : S^{2n+1} \to CP^n(4)$. For a given point $u \in S^{2n+1}(1)$, the horizontal space at u is the orthogonal complement of $\imath u$, $\imath = \sqrt{-1}$, with respect to the metric on S^{2n+1} induced from the metric on $\mathbf{C}^{n+1}$. Let $\iota : N \to CP^n(4)$ be a Lagrangian isometric immersion. Then there is a covering map $\tau : \hat{N} \to N$ and a horizontal immersion $\hat{\iota} : \hat{N} \to S^{2n+1}$ such that $\iota \circ \tau = \pi \circ \hat{\iota}$. Thus each Lagrangian immersion can be lifted locally (or globally if N is simply-connected) to a Legendrian immersion of the same Riemannian manifold. In particular, a minimal Lagrangian submanifold of $CP^n(4)$ is lifted to a minimal Legendrian submanifold of the Sasakian $S^{2n+1}(1)$.

Conversely, suppose that $f : \hat{N} \to S^{2n+1}$ is a Legendrian isometric immersion. Then $\iota = \pi \circ f : N \to CP^n(4)$ is again a Lagrangian isometric immersion. Under this correspondence the second fundamental forms h^f and h^ι of f and ι satisfy $\pi_* h^f = h^\iota$. Moreover, h^f is horizontal with respect to π.

Case (ii): $CH^n(-4)$. We consider the complex number space $\mathbf{C}_1^{n+1}$ equipped with the pseudo-Euclidean metric: $g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j$.

Consider $H_1^{2n+1}(-1) = \{z \in \mathbf{C}_1^{2n+1} : \langle z, z\rangle = -1\}$ with the canonical Sasakian structure, where $\langle\ ,\ \rangle$ is the induced inner product.

Put $T'_z = \{u \in \mathbf{C}^{n+1} : \langle u, z\rangle = 0\}$, $H_1^1 = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$. Then there is an H_1^1-action on $H_1^{2n+1}(-1)$, $z \mapsto \lambda z$ and at each point $z \in H_1^{2n+1}(-1)$, the vector $\xi = -\imath z$ is tangent to the flow of the action. Since the metric g_0 is Hermitian, we have $\langle \xi, \xi\rangle = -1$. The quotient space $H_1^{2n+1}(-1)/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^n(-4)$ with constant holomorphic sectional curvature -4 whose complex structure J is induced from the complex structure J on $\mathbf{C}_1^{n+1}$ via Hopf's fibration $\pi : H_1^{2n+1}(-1) \to CH^n(4c)$.

Just like case (i), suppose that $\iota : N \to CH^n(-4)$ is a Lagrangian immersion, then there is an isometric covering map $\tau : \hat{N} \to N$ and a Legendrian immersion $f : \hat{N} \to H_1^{2n+1}(-1)$ such that $\iota \circ \tau = \pi \circ f$. Thus every Lagrangian immersion into $CH^n(-4)$ an be lifted locally (or globally if N is simply-connected) to a Legendrian immersion into $H_1^{2n+1}(-1)$. In particular, Lagrangian minimal submanifolds of $CH^n(-4)$ are lifted to Legendrian minimal submanifolds of $H_1^{2n+1}(-1)$. Conversely, if $f:\hat{N}\to H_1^{2n+1}(-1)$ is a Legendrian immersion, then $\iota=\pi\circ f:N\to CH^n(-4)$ is a Lagrangian immersion. Under this correspondence the second fundamental forms h^f and h^ι are related by $\pi_* h^f=h^\iota$. Also, h^f is horizontal with respect to π.

Let h be the second fundamental form of M in $S^{2n+1}(1)$ (or in $H_1^{2n+1}(-1)$). Since $S^{2n+1}(1)$ and $H_1^{2n+1}(-1)$ are totally umbilical with one as its mean curvature in $\mathbf{C}^{n+1}$ and in $\mathbf{C}_1^{n+1}$, respectively, we have

$$\hat{\nabla}_X Y=\nabla_X Y+h(X,Y)-\varepsilon L, \tag{2.9}$$

where $\varepsilon=1$ if the ambient space is $\mathbf{C}^{n+1}$; and $\varepsilon=-1$ if it is $\mathbf{C}_1^{n+1}$.

3. H-umbilical Lagrangian submanifolds and complex extensors

3.1. H-umbilical Lagrangian submanifolds.

Definition 3.1. A non-totally geodesic Lagrangian submanifold of a Kähler n-manifold is called *H-umbilical* if its second fundamental form satisfies

$$\begin{aligned} &h(e_j,e_j)=\mu Je_n,\;\; h(e_j,e_n)=\mu Je_j,\;\; j=1,\dots,n-1,\\ &h(e_n,e_n)=\varphi Je_n,\;\; h(e_j,e_k)=0,\;\; 1\le j\ne k\le n-1, \end{aligned} \tag{3.1}$$

for some functions μ,φ with respect to an orthonormal frame $\{e_1,\dots,e_n\}$. If the ratio of $\varphi:\mu$ is a constant r, the H-umbilical submanifold is said to be of ratio r.

If $G:N^{n-1}\to\mathbb{E}^n$ is a hypersurface of a Euclidean n-space $\mathbb{E}^n$ and $\gamma:I\to\mathbf{C}^*$ is a unit speed curve in $\mathbf{C}^*=\mathbf{C}-\{0\}$, then we may extend $G:N^{n-1}\to\mathbb{E}^n$ to an immersion $I\times N^{n-1}\to\mathbf{C}^n$ by $\gamma\otimes G:I\times N^{n-1}\to\mathbf{C}\otimes\mathbb{E}^n=\mathbf{C}^n$, where $(\gamma\otimes G)(s,p)=F(s)\otimes G(p)$ for $s\in I,\ p\in N^{n-1}$. This extension of G via tensor product $\otimes$ is called the *complex extensor* of G via the *generating curve* γ.

H-umbilical Lagrangian submanifolds in complex space forms were classified in a series of papers by the first author (cf. [2, 3, 4]). In particular, the following two results were proved in [2].

Theorem E. *Let $\iota:S^{n-1}\subset\mathbb{E}^n$ be the unit hypersphere in $\mathbb{E}^n$ centered at the origin. Then every complex extensor of ι via a unit speed curve*

$\gamma : I \to \mathbf{C}^*$ is an H-umbilical Lagrangian submanifold of $\mathbf{C}^n$ unless γ is contained in a line through the origin (which gives a totally geodesic Lagrangian submanifold).

Theorem F. *Let M be an H-umbilical Lagrangian submanifold of $\mathbf{C}^n$ with $n \geq 3$. Then M is either a flat space or congruent to an open part of a complex extensor of $\iota : S^{n-1} \subset \mathbb{E}^n$ via a curve $\gamma : I \to \mathbf{C}^*$.*

3.2. **Legendre curves.** A unit speed curve $z : I \to S^3(1) \subset \mathbf{C}^2$ (resp., $z : I \to H^3_1(-1) \subset \mathbf{C}^2_1$) is called *Legendre* if $\langle z', \mathrm{i}z\rangle = 0$. It was proved in [3] that a unit speed curve z in $S^3(1)$ (resp., in $H^3_1(-1)$) is Legendre if and only if it satisfies

$$z'' = \mathrm{i}\lambda z' - z \ \text{ (resp., } z'' = \mathrm{i}\lambda z' + z) \tag{3.2}$$

for a real-valued function λ. It is known in [3] that λ is the curvature function of z in $S^3(1)$ (resp., in $H^3_1(-1)$) (see also [1, Lemmas 3.1 and 3.2]).

3.3. H**-umbilical submanifolds with arbitrary ratio.** We provide a general method to construct H-umbilical Lagrangian submanifolds with any given ratio in $CP^n(4)$ via curves in $S^2(\frac{1}{2})$ (resp., in $CH^n(-4)$ via curves in $H^2(-\frac{1}{2})$).

Proposition 3.2. *For any real number r there exist H-umbilical Lagrangian submanifolds of ratio r in $CP^n(4)$ and in $CH^n(-4)$.*

Proof. If $r = 2$ this was done in [3, Theorems 5.1 and 6.1]. If $r \neq 2$, H-umbilical Lagrangian submanifolds of ratio r can be constructed as follows:

Case (a): $CP^n(4)$. Let $S^2(\frac{1}{2}) = \{\mathbf{x} \in \mathbb{E}^3; \langle \mathbf{x}, \mathbf{x}\rangle = \frac{1}{4}\}$. The Hopf fibration π from $S^3(1)$ onto $S^2(\frac{1}{2}) \equiv CP^1(4)$ is given by (cf. [1])

$$\pi(z_1, z_2) = \left(z_1\bar{z}_2, \tfrac{1}{2}(|z_1|^2 - |z_2|^2)\right), \ \ (z_1, z_2) \in S^3(1) \subset \mathbf{C}^2. \tag{3.3}$$

For a Legendre curve z in $S^3(1)$, the projection $\gamma_z = \pi \circ z$ is a curve in $S^2(\frac{1}{2})$. Conversely, each curve γ in $S^2(\frac{1}{2})$ gives rise to a horizontal lift $\tilde{\gamma}$ in $S^3(1)$ via π which is unique up to a factor $e^{i\theta}, \theta \in \mathbf{R}$. Notice that each horizontal lift of γ is a Legendre curve in $S^3(1)$. Moreover, since the Hopf fibration is a Riemannian submersion, each unit speed Legendre curve z in $S^3(1)$ is projected to a unit speed curve γ_z in $S^2(\frac{1}{2})$ with the same curvature.

It was known in [3, Lemma 7.2] that, for a given H-umbilical Lagrangian submanifold of ratio $r \neq 2$ in $\tilde{M}^n(4c)$, the function μ in (3.1) satisfies

$$\mu\mu'' - \left(\frac{r-3}{r-2}\right)\mu'^2 + (r-2)\mu^2((r-1)\mu^2 + c) = 0. \tag{3.4}$$

If μ is a non-trivial solution of (3.4) with $c = 1$, then there is a unit speed curve γ in $S^2(\frac{1}{2})$ whose curvature equals to $r\mu$. Let z be a horizontal lift of γ in $S^3(1)$. Then z is a unit speed Legendre curve satisfying $z''(x) = \mathrm{i}r\mu z'(x) - z(x)$ (cf. [3, Theorem 4.1] or [1, Lemma 3.1]).

Consider the map $\psi : M^5 \to S^{11}(1) \subset \mathbf{C}^6$ defined by

$$\psi(x, y_1, \ldots, y_5) = (z_1(x), z_2(x)y_1, \ldots, \ldots, z_2(x)y_5), \quad \sum_{j=1}^{5} y_j^2 = 1. \tag{3.5}$$

It follows from [3, Theorem 4.1 and Lemma 7.2] that $\pi \circ \psi$ is a H-umbilical Lagrangian submanifold of ratio r in $CP^n(4)$ such that

$$\begin{aligned} &h(e_j, e_j) = \mu J e_5, \quad h(e_j, e_n) = \mu J e_j, \\ &h(e_n, e_n) = r\mu J e_n, \ h(e_j, e_k) = 0, \quad 1 \le j \neq k \le n-1, \end{aligned} \tag{3.6}$$

with respect to suitable orthonormal frame $\{e_1, \ldots, e_5\}$.

Case (b): $CH^n(-4)$. For a non-trivial solution of (3.4) with $c = -1$, we can construct an H-umbilical Lagrangian submanifold of $CH^n(-4)$ via the Hopf fibration $\pi : H_1^3(-1) \to CH^1(-4) \equiv H^2(-\frac{1}{2})$ in a similar way as case (a), where

$$\pi(z_1, z_2) = \left(z_1\bar{z}_2, \tfrac{1}{2}(|z_1|^2 + |z_2|^2)\right), \quad (z_1, z_2) \in H_1^3(-1) \subset \mathbf{C}_1^2, \tag{3.7}$$

and $H^2(-\frac{1}{2}) = \{(x_1, x_2, x_3) \in \mathbb{E}_1^3 \, : \, x_1^2 - x_2^2 - x_3^2 = \frac{1}{4}, \, x_1 \ge \frac{1}{2}\}$ is the model of the real projective plane of curvature -4. □

3.4. **Classification of H-umbilical submanifolds of ratio 4.** The equation of Gauss and (3.1) imply that H-umbilical Lagrangian submanifolds of ratio $r \neq 4$ in complex space forms contain no open subsets of constant sectional curvature. Hence we conclude from [3, Theorems 4.1 and 7.1] and §3.3 the following results.

Lemma 3.3. *An H-umbilical Lagrangian submanifold M of ratio 4 in $CP^5(4)$ is congruent to an open portion of $\pi \circ \psi$, where $\pi : S^{11}(1) \to CP^5(4)$ is Hopf's fibration, $\psi : M \to S^{11}(1) \subset \mathbf{C}^6$ is given by*

$$\psi(t, y_1, \ldots, y_5) = (z_1(t), z_2(t)\mathbf{y}), \quad \{\mathbf{y} \in \mathbb{E}^5 : \langle \mathbf{y}, \mathbf{y} \rangle = 1\}, \tag{3.8}$$

and $z = (z_1, z_2) : I \to S^3(1) \subset \mathbf{C}^2$ *is a unit speed Legendre curve satisfying* $z'' = 4\mathrm{i}\mu z' - z$, *and* μ *is a nonzero solution of* $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2+1) = 0$.

Let M be an H-umbilical Lagrangian submanifold in $CH^5(-4)$ satisfying (3.1). We may assume that μ is defined on an open interval $I \ni 0$. Since H-umbilical submanifolds of ratio 4 in $CH^5(-4)$ contain no open subsets of constant curvature, Theorems 4.2 and 9.1 of [3] and results in §3.3 imply the following classification of H-umbilical submanifolds of ratio 4 in $CH^5(-4)$.

Lemma 3.4. *An* H*-umbilical Lagrangian submanifold* M *of ratio 4 in* $CH^5(-4)$ *is congruent to an open part of* $\pi \circ \psi$, *where* $\pi : H_1^{11}(-1) \to CH^5(-4)$ *is Hopf's fibration and* $\psi : M \to H_1^{11}(-1) \subset \mathbf{C}_1^6$ *is either one of*

$$\psi(t, y_1, \ldots, y_4) = (z_1(t), z_2(t)\mathbf{y}), \quad \{\mathbf{y} \in \mathbb{E}^5 : \langle \mathbf{y}, \mathbf{y} \rangle = 1\}, \tag{3.9}$$

$$\psi(t, y_1, \ldots, y_4) = (z_1(t)\mathbf{y}, z_2(t)), \quad \{\mathbf{y} \in \mathbb{E}_1^5 : \langle \mathbf{y}, \mathbf{y} \rangle = -1\}, \tag{3.10}$$

where z *is a unit speed Legendre curve in* $H_1^3(-1)$ *satisfying* $z'' = 4\mathrm{i}\mu z' + z$ *and* μ *is a non-trivial solution of* $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 - 1) = 0$; *or* ψ *is*

$$\begin{aligned}\psi(t, u_1, \ldots, u_4) = \sqrt{\mu} e^{\mathrm{i}\int_0^t \mu(t)dt} \Bigg(& 1 + \frac{1}{2}\sum_{j=1}^4 u_j^2 - \mathrm{i}t + \frac{1}{2\mu} - \frac{1}{2\mu(0)}, \\ & \left(i\mu(0) - \frac{\mu'(0)}{2\mu(0)}\right)\left(\frac{1}{2}\sum_{j=1}^4 u_j^2 - \mathrm{i}t + \frac{1}{2\mu} - \frac{1}{2\mu(0)}\right), u_1, \ldots, u_4\Bigg),\end{aligned} \tag{3.11}$$

where $z = (z_1, z_2) : I \to H_1^3(-1) \subset \mathbf{C}_1^2$ *is a unit speed Legendre curve and* μ *is a non-trivial solution of* $\mu'^2 = 4\mu^2(1 - \mu^2)$.

Example. It is easy to verify that $\mu = \operatorname{sech} 2t$ is a non-trivial solution of $\mu'^2 = 4\mu^2(1-\mu^2)$. Using $\mu = \operatorname{sech} 2t$, (3.11) reduces to

$$\begin{aligned}\psi(t, u_1, \ldots, u_4) = \frac{e^{\mathrm{i}\tan^{-1}(\tanh t)}}{\sqrt{\cosh 2t}} \Bigg(& \frac{1}{2} - \mathrm{i}t + \frac{1}{2}\sum_{j=1}^4 u_j^2 + \frac{\cosh 2t}{2}, \\ & t - \frac{\mathrm{i}}{2} + \frac{\mathrm{i}}{2}\sum_{j=1}^4 u_j^2 + \frac{\mathrm{i}\cosh 2t}{2}, u_1, \ldots, u_4\Bigg).\end{aligned} \tag{3.12}$$

It is direct to verify that (3.12) satisfies $\langle \psi, \psi \rangle = -1$ and the composition $\pi \circ \psi$ gives rise to an H-umbilical Lagrangian submanifold of ratio 4 in $CH^5(-4)$.

4. Some Lemmas

We need the following lemmas for the proof of the main theorems.

Lemma 4.1. *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold of $\tilde{M}^5(4c)$. Then with respect to some orthonormal frame $\{e_1,\ldots,e_5\}$ we have*

$$\begin{aligned}
&h(e_1,e_1)=aJe_1+\mu Je_5,\ h(e_1,e_2)=-aJe_2,\\
&h(e_2,e_2)=-aJe_1+\mu Je_5,\ h(e_3,e_3)=bJe_3+\mu Je_5,\\
&h(e_3,e_4)=-bJe_4,\ h(e_4,e_4)=-bJe_3+\mu Je_5,\\
&h(e_i,e_5)=\mu Je_i,\ i\in\Delta,\ h(e_5,e_5)=4\mu Je_5,\\
&h(e_i,e_j)=0,\ \ \textit{otherwise}.
\end{aligned}\tag{4.1}$$

Proof. Under the hypothesis, we have (1.5) with respect to an orthonormal frame $\{e_1,\ldots,e_5\}$. Thus, after applying [6, Lemma 1] to $V=\mathrm{Span}\,\{e_1,e_2\}$ and $V=\mathrm{Span}\,\{e_3,e_4\}$, we obtain (4.1). □

Let us put

$$\nabla_X e_i=\sum_{j=1}^{5}\omega_i^j(X)e_j,\ \ i=1,\ldots,5,\ \ X\in TM^5.\tag{4.2}$$

Then $\omega_i^j=-\omega_j^i$, $i,j=1,\ldots,5$.

If $\mu=0$, then M is a minimal Lagrangian submanifold according (4.1). Such submanifolds in complex space forms $\tilde{M}^5(4c)$ have been classified in [13].

If $a=b=0$ and $\mu\neq 0$, then M is an H-umbilical Lagrangian submanifold with ratio 4. Therefore, from now on we assume that $a,\mu\neq 0$.

Lemma 4.2. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies* (4.1) *with $a,b,\mu\neq 0$. Then we have*

$$\begin{aligned}
&\nabla_{e_1}e_1=\frac{e_2a}{3a}e_2-\nu e_5,\ \nabla_{e_1}e_2=-\frac{e_2a}{3a}e_1,\ \nabla_{e_2}e_1=-\frac{e_1a}{3a}e_2,\\
&\nabla_{e_2}e_2=\frac{e_1a}{3a}e_1-\nu e_5,\ \nabla_{e_3}e_3=\frac{e_4b}{3b}e_4-\nu e_5,\ \nabla_{e_3}e_4=-\frac{e_4b}{3b}e_3,\\
&\nabla_{e_4}e_3=-\frac{e_3b}{3b}e_4,\ \ \nabla_{e_4}e_4=\frac{e_3b}{3b}e_3-\nu e_5,\ \nabla_{e_i}e_5=\nu e_i,\ i\in\Delta,\\
&\nabla_{e_k}e_j=0,\ \ \textit{otherwise},
\end{aligned}\tag{4.3}$$

with $\nu = \frac{1}{2}e_5(\ln \mu) = -e_5(\ln a) = -e_5(\ln b)$*, where* $\Delta = \{1,2,3,4\}$*. Moreover, we have*

$$e_j\mu = 0, j \in \Delta, \ \ e_1 b = e_2 b = e_3 a = e_4 a = 0. \tag{4.4}$$

Proof. This lemma is obtained from Codazzi's equations via Lemma 4.1 and (4.2) and long computations. □

Lemma 4.3. *Under the hypothesis of Lemma 4.2, we have*

(a) T_0 *is a totally geodesic distribution, i.e.* T_0 *is integrable whose leaves are totally geodesic submanifolds;*
(b) $T_0 \oplus T_1$ *and* $T_0 \oplus T_2$ *are totally geodesic distributions;*
(c) T_1 *and* T_2 *are spherical distributions, i.e.* T_1*,* T_2 *are integrable distributions whose leaves are totally umbilical submanifolds with parallel mean curvature vector,*

where $T_0 = \mathrm{Span}\{e_5\}, T_1 = \mathrm{Span}\{e_1, e_2\}$ *and* $T_2 = \mathrm{Span}\{e_3, e_4\}$*.*

Proof. Since the distribution T_0 is of rank one, it is integrable. Moreover, since $\nabla_{e_5} e_5 = 0$ by Lemma 4.2, the integral curves of e_5 are geodesics in M. Thus we have statement (a). Statement (b) follows easily from (4.3).

To prove statement (c), first we observe that $[e_1, e_2] \in T_1$ and $[e_3, e_4] \in T_2$ follow from (4.3). Thus T_1, T_2 are integrable. Also, it follows from (4.3) that the second fundamental form h_1 of a leaf $\mathcal{L}_1$ of T_1 in M is given by

$$h_1(X, Y) = -\nu g_1(X_1, Y_1)e_5, \ \ X_1, Y_1 \in T\mathcal{L}_1, \tag{4.5}$$

where g_1 is the metric of $\mathcal{L}_1$. From (4.3) we obtain $\nabla_{e_i} e_5 = \nu e_i$, $i = 1, 2$. Thus $D^1_{e_1} e_5 = D^1_{e_2} e_5 = 0$, where D^1 is the normal connection of $\mathcal{L}_1$ in M. It follows from Gauss' equation and Lemma 4.1 that the curvature tensor R satisfies

$$\langle R(e_1, e_2)e_1, e_j\rangle = 0, \ \ j = 3, 4, 5. \tag{4.6}$$

Thus (4.6) and Lemma 4.2 imply that $0 \equiv R(e_1, e_2)e_1 \equiv (e_2\nu)e_5 \pmod{T_1}$. Hence $e_2\nu = 0$. Similarly, by considering $R(e_2, e_1)e_2$, we also have $e_1\alpha = 0$. After combining these with $D^1 e_5 = 0$, we conclude that $\mathcal{L}_1$ has parallel mean curvature vector in M. Hence T_1 is a spherical distribution. Similarly, T_2 is also a spherical distribution. Consequently, we obtain statement (c). □

Lemma 4.4. *Under the hypothesis of Lemma 4.2,* M *is locally a warped product* $I \times_{\rho_1(t)} M_1^2 \times_{\rho_2(t)} M_2^2$*, where* t *is function such that* $e_5 = \partial_t$ *(i.e.,* $e_5 =$

$\frac{\partial}{\partial t}$), ρ_1 and ρ_2 are two positive functions in t and M_1^2, M_2^2 are Riemannian 2-manifolds.

Proof. This lemma follows from Lemma 4.3 and a result of Hiepko [15] (see also [7, Theorem 4.4, p. 90]). □

Lemma 3.3 and (4.4) imply that μ depends only on t. Thus $\mu = \mu(t)$.

Lemma 4.5. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies* (4.1) *with $a, b, \mu \neq 0$. Then we have $c = -\nu^2 - \mu^2 < 0$. Thus μ satisfies $\mu'(t)^2 = -4\mu^2(t)(c + \mu^2(t))$.*

Proof. Under the hypothesis, it follows from Gauss' equation and Lemma 4.1 that $\langle R(e_1, e_3)e_3, e_1\rangle = c + \mu^2$. On the other hand, the definition of curvature tensor and Lemma 4.2 imply that $\langle R(e_1, e_3)e_3, e_1\rangle = -\nu^2$. Thus $c = -\nu^2 - \mu^2 < 0$. By combining this with the definition of ν, we obtain the lemma. □

5. More lemmas

Next, we consider the case $a, \mu \neq 0$ and $b = 0$.

Lemma 5.1. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies* (4.1) *with $a, \mu \neq 0$ and $b = 0$. Then we have*

$$\begin{aligned}
&\nabla_{e_1} e_1 = \frac{e_2 a}{3a} e_2 + \frac{e_3 a}{a} e_3 + \frac{e_4 a}{3a} e_4 - \nu e_5, \\
&\nabla_{e_1} e_2 = -\frac{e_2 a}{3a} e_1 - 3\omega_1^2(e_3) e_3 - 3\omega_1^2(e_4) e_4, \\
&\nabla_{e_1} e_3 = -\frac{e_3 a}{a} e_1 + 3\omega_1^2(e_3) e_2 + \omega_3^4(e_1) e_4, \\
&\nabla_{e_1} e_4 = -\frac{e_4 a}{a} e_1 + 3\omega_1^2(e_4) e_2 - \omega_3^4(e_1) e_3, \\
&\nabla_{e_2} e_1 = -\frac{e_1 a}{3a} e_2 + 3\omega_1^2(e_3) e_3 + \omega_1^4(e_2) e_4, \\
&\nabla_{e_2} e_2 = \frac{e_1 a}{3a} e_1 + \frac{e_3 a}{a} e_3 + \frac{e_4 a}{a} e_4 - \nu e_5, \\
&\nabla_{e_2} e_3 = -3\omega_1^2(e_3) e_1 - \frac{e_3 a}{a} e_2 + \omega_3^4(e_2) e_4, \\
&\nabla_{e_2} e_4 = -\omega_1^4(e_2) e_1 - \frac{e_4 a}{a} e_2 - \omega_3^4(e_2) e_3, \\
&\nabla_{e_3} e_1 = \omega_1^2(e_3) e_2, \quad \nabla_{e_3} e_2 = -\omega_1^2(e_3) e_1, \\
&\nabla_{e_3} e_3 = \omega_3^4(e_3) e_4 - \nu e_5, \ \nabla_{e_3} e_4 = -\omega_3^4(e_3) e_3, \\
&\nabla_{e_4} e_1 = \omega_1^2(e_4) e_2, \ \nabla_{e_4} e_2 = -\omega_1^2(e_4) e_1, \\
&\nabla_{e_4} e_3 = \omega_3^4(e_4) e_4, \ \nabla_{e_4} e_4 = -\omega_3^4(e_4) e_3 - \nu e_5, \\
&\nabla_{e_5} e_3 = \omega_3^4(e_5) e_4, \ \nabla_{e_5} e_4 = -\omega_3^4(e_5) e_5, \\
&\nabla_{e_i} e_5 = \nu e_i, \quad i \in \Delta, \ \nabla_{e_k} e_j = 0, \quad \textit{otherwise}.
\end{aligned} \tag{5.1}$$

with $\nu = \frac{1}{2} e_5(\ln \mu) = -e_5(\ln a)$. Moreover, we have

$$e_j \mu = 0, \quad j \in \Delta = \{1, 2, 3, 4\}. \tag{5.2}$$

Proof. Follows from Codazzi's equations via Lemma 4.1 and (4.2). □

Lemma 5.2. *Under the hypothesis of Lemma 5.1, we have*

(i) *T_0 is a totally geodesic distribution;*
(ii) *T_3 is a spherical distribution,*

where $T_0 = \mathrm{Span}\{e_5\}$ and $T_3 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$.

Proof. Clearly, T_0 is integrable. Moreover, since $\nabla_{e_5} e_5 = 0$ by Lemma 5.1, integral curves of e_5 are geodesics in M^5. Thus statement (i) follows. To prove statement (ii), we observe that the integrability of T_3 follows from (5.1). Also, (5.1) implies that the second fundamental form $\hat{h}$ of a leaf $\mathcal{L}$ of T_3 in M^5 is given by $\hat{h}(X, Y) = -\nu \hat{g}(X, Y) e_5$ for $X, Y \in T\mathcal{L}$, where $\hat{g}$ is the metric of $\mathcal{L}$. Since $[e_j, e_5]\mu = 0$ by (5.1) and $e_j\mu = 0$, for $j \in \Delta$, we find $e_ie_5\mu - e_5e_i\mu = 2e_1\nu = 0$. Therefore T_3 is a spherical distribution. □

Lemma 5.3. *Under the hypothesis of Lemma 5.1, M is locally a warped product $I \times_{\rho(t)} N^4$, where t is function such that $e_5 = \frac{\partial}{\partial t}$ and ρ is a positive function in t and N^4 is a Riemannian 4-manifold.*

Proof. Follows from Lemma 5.2 and Hiepko's theorem. □

It follows from (5.2) and the definition of ν that $\mu = \mu(t)$ and $\nu = \nu(t)$.

Lemma 5.4. *Under the hypothesis of Lemma 5.1, we have*

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2 - c, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{5.3}$$

Proof. From Gauss' equation and (5.1) we find $\langle R(e_1, e_5)e_5, e_1\rangle = 3\mu^2 + c$. On the other hand, (5.1) of Lemma 5.1 yields $\langle R(e_1, e_5)e_5, e_1\rangle = -e_5\nu - \nu^2$. Thus we find the first equation of (5.3). The second one follows immediately from the definition of ν given in Lemma 5.1. □

6. Improved $\delta(2, 2)$-ideal Lagrangian submanifolds of $\mathbf{C}^5$

Theorem 6.1. *Let M be an improved $\delta(2, 2)$-ideal Lagrangian submanifold in $\mathbf{C}^5$. Then it is one of the following Lagrangian submanifolds:*

(a) *a $\delta(2, 2)$-ideal Lagrangian minimal submanifold;*
(b) *an H-umbilical Lagrangian submanifold of ratio 4;*
(c) *a Lagrangian submanifold defined by*

$$L(\mu, u_2, \ldots, u_n) = \frac{e^{\frac{4}{3}\mathrm{i}\tan^{-1}\sqrt{\mu^3/(c^2-\mu^3)}}}{\sqrt{c^2\mu^{-1} - \mu^2} + \mathrm{i}\mu}\phi(u_2, \ldots, u_n), \tag{6.1}$$

where c is a positive real number and $\phi(u_2, \ldots, u_n)$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $CP^4(4)$.

Proof. Assume that M is an improved $\delta(2, 2)$-ideal Lagrangian submanifold in $\mathbf{C}^5$. Then there exists an orthonormal frame $\{e_1, \ldots, e_5\}$ such that (4.1) holds. If $\mu = 0$, then M is a minimal $\delta(2, 2)$-ideal Lagrangian submanifold. Thus, we obtain case (a). If $\mu \neq 0$ and $a = b = 0$, we obtain case (b).

Now, let us assume $a, \mu \neq 0$. Then Lemma 4.5 implies $b = 0$. So, by Lemmas 5.1 we have (5.1) and $e_j\mu = 0$, $j \in \Delta$. Further, by Lemma 5.3, M

is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$. Moreover, 4.1 shows that the second fundamental form satisfies

$$\begin{aligned} &h(e_1,e_1) = aJe_1 + \mu Je_5,\ h(e_1,e_2) = -aJe_2,\\ &h(e_2,e_2) = -aJe_1 + \mu Je_5,\\ &h(e_3,e_3) = h(e_4,e_4) = \mu Je_5,\\ &h(e_i,e_5) = \mu Je_i,\ i \in \Delta,\\ &h(e_5,e_5) = 4\mu Je_5,\ h(e_i,e_j) = 0,\ \ otherwise. \end{aligned} \tag{6.2}$$

From Lemma 5.4 we have the following differential system:

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2,\ \ \frac{d\mu}{dt} = 2\mu\nu. \tag{6.3}$$

Let $\varphi(t)$ be a function satisfying $\dfrac{d\varphi}{dt} = -4\mu$. Consider the map

$$\phi = e^{\mathrm{i}\varphi} e_5. \tag{6.4}$$

Then $\langle \phi, \phi \rangle = 1$. It follows from $\nabla_{e_5} e_5 = 0$, $\frac{d\varphi}{dt} = -4\mu$ and (6.2) that $\tilde\nabla_{e_5}\phi = 0$, where $\tilde\nabla$ is the Levi-Civita connection of $\mathbf{C}^5$. Thus ϕ is independent of t.

Let L denote the Lagrangian immersion of M in $\mathbf{C}^5$. Then (6.4) yields

$$e_5 = L_t = e^{-\mathrm{i}\varphi}\phi(u_1,\ldots,u_4), \tag{6.5}$$

where $u_1,\ldots,u_4$ are local coordinates of N^4. For each $j \in \Delta$, we obtain from $\nabla_{e_j} e_5 = \nu e_j$ of Lemma 5.1 and the first equation of (6.3) that

$$\phi_*(e_j) = \tilde\nabla_{e_j}\phi = e^{\mathrm{i}\varphi}\tilde\nabla_{e_j} e_5 = e^{\mathrm{i}\varphi}(\nu + \mathrm{i}\mu) e_j. \tag{6.6}$$

Thus

$$\tilde\nabla_{e_j}(\phi_*(e_i)) = e^{\mathrm{i}\varphi}(\nu + \mathrm{i}\mu)\tilde\nabla_{e_j} e_i. \tag{6.7}$$

In view of $\nabla_{e_j} e_5 = \nu e_j$ and (6.2), we may put

$$\tilde\nabla_{e_i} e_j = \Big(\sum_{k=1}^{4} \Gamma_{ij}^k + \mathrm{i} h_{ij}^k\Big) e_k - (\nu - \mathrm{i}\mu)\delta_{ij} e_5,\ \ i,j \in \Delta, \tag{6.8}$$

for some functions Γ_{ij}^k. Now, it follows from (6.4), (6.6), (6.7), and (6.8) that

$$\begin{aligned} \tilde\nabla_{e_j}(\phi_*(e_i)) &= \sum_{\gamma=2}^{n} \left(\Gamma_{ij}^k + \mathrm{i} h_{ij}^k\right) \phi_*(e_k) - \left(\mu^2 + \nu^2\right) \delta_{ij}\phi \\ &= \sum_{\gamma=2}^{n} \left(\Gamma_{ij}^k + \mathrm{i} h_{ij}^k\right) \phi_*(e_k) - \langle \phi_*(e_i), \phi_*(e_j)\rangle\, \phi. \end{aligned} \tag{6.9}$$

Since M is a Lagrangian submanifold in $\mathbf{C}^5$, (6.4) and (6.6) show that $\mathrm{i}\phi$ is perpendicular to each tangent space of M. Hence ϕ is a horizontal immersion in the unit hypersphere $S^9(1) \subset \mathbf{C}^5$. Moreover, it follows from (6.9) that the second fundamental form of ϕ is the original second fundamental form of M respect to to the second factor N^4 of the warped product $I \times_{\rho(t)} N^4$. Hence, ϕ is a minimal horizontal immersion in $S^9(1)$. Therefore, ϕ is a horizontal lift of a minimal Lagrangian immersion in $CP^4(4)$. Now, it follows from (6.2) that ϕ is a horizontal lift of a $\delta(2)$-ideal minimal Lagrangian submanifold of $CP^4(4)$.

By direct computation we find

$$\tilde{\nabla}_{e_\alpha}\left(L - \frac{e_5}{\nu + \mathrm{i}\mu}\right) = 0, \quad \alpha = 1, \ldots, 5. \tag{6.10}$$

Thus, by (6.4), up to translations the Lagrangian immersion L is

$$L = \frac{e^{-\mathrm{i}\varphi}}{\nu + \mathrm{i}\mu}\phi(u_1, \ldots, u_4), \tag{6.11}$$

where ϕ is a horizontal minimal immersion in $S^9(1)$ and ν, φ, μ satisfy

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2, \quad \frac{d\varphi}{dt} = -4\mu, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{6.12}$$

From (6.12) we find

$$\frac{d\nu}{d\mu} + \frac{\nu}{2\mu} = -\frac{3\mu}{2\nu}. \tag{6.13}$$

After solving (6.13) we get $\nu = \pm\sqrt{c^2\mu^{-1} - \mu^2}$ for some real number $c > 0$. Replacing e_5 by $-e_5$ if necessary, we have

$$\nu = \sqrt{c^2\mu^{-1} - \mu^2}. \tag{6.14}$$

It follows from (6.12) an (6.14) that $\varphi'(\mu) = -2/\sqrt{c^2\mu^{-1} - \mu^2}$. By solving the last equation we find $\varphi = -\frac{4}{3}\mathrm{i}\tan^{-1}\sqrt{\mu^3/(c^2 - \mu^3)} + c_0$ for some constant c_0. Therefore, we have the theorem after applying a suitable translation in μ. □

Remark 6.2. Minimal $\delta(2,2)$-ideal Lagrangian submanifolds in complex space forms $\mathbf{C}^5$, CP^5 and CH^5 are classified in [13]. Also $\delta(2)$-ideal minimal Lagrangian submanifolds in CP^4 and CH^4 have been classified recently in [14].

Let $\gamma(t)$ be a unit speed curve in $\mathbf{C}^*$. We put

$$\gamma(t)=r(t)e^{i\theta(t)},\;\; \gamma'(t)=e^{i\zeta(t)}. \tag{6.15}$$

The following result gives H-umbilical submanifolds of $\mathbf{C}^5$ with ratio 4.

Proposition 6.3. *If M is an H-umbilical Lagrangian submanifold of $\mathbf{C}^5$ of ratio 4, then M is an open part of a complex extensor $\gamma\otimes\iota$ of the unit hypersphere $\iota: S^4(1)\subset\mathbb{E}^5$ via a generating curve $\gamma: I\to\mathbf{C}^*$ whose curvature satisfies $\kappa=4\theta'$.*

Proof. If M is an H-umbilical Lagrangian submanifold of $\mathbf{C}^5$ with ratio 4, then the second fundamental form satisfies

$$\begin{aligned} &h(e_j,e_j)=\mu Je_5,\;\; h(e_j,e_5)=\mu Je_j,\;\; j\in\Delta,\\ &h(e_5,e_5)=4\mu Je_5,\; h(e_j,e_k)=0,\;\; 1\le j\ne k\le 4, \end{aligned}$$

for a nonzero function μ. Thus Gauss' equation yields $K(e_1\wedge e_5)=3\mu^2$. Hence M is non-flat. Therefore, according to Theorem F, M is an open part of a complex extensor of $\iota: S^{n-1}(1)\subset\mathbb{E}^n$ via a generating curve $\gamma: I\to\mathbf{C}^*$. It follows from [2] that the functions φ and μ in (4.1) are related with the two angle functions ζ and θ by $\varphi=\zeta'(t)=\kappa$ and $\mu=\theta'(t)$. Thus whenever γ is a unit speed curve satisfying $\kappa=4\theta'$, the complex extensor $\gamma\otimes\iota$ is an H-umbilical Lagrangian submanifold of ratio 4. Conversely, every H-umbilical Lagrangian submanifold of ratio 4 in $\mathbf{C}^n$ can be obtained in such way. ☐

7. Improved $\delta(2,2)$-ideal Lagrangian submanifolds of CP^5

Theorem 7.1. *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $CP^5(4)$. Then it is one of the following Lagrangian submanifolds:*

1. *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*
2. *an H-umbilical Lagrangian submanifold of ratio 4;*
3. *a Lagrangian submanifold defined by*

$$L(\mu,u_2,\ldots,u_4)=\frac{1}{c}\Big(\sqrt{\mu}e^{\mathrm{i}\theta}\phi,e^{3\mathrm{i}\theta}(\sqrt{c^2-\mu^3-\mu}-\mathrm{i}\mu^{\frac{3}{2}})\Big), \tag{7.1}$$

where c is a positive real number, $\phi: N^4\to S^9(1)\subset\mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $CP^4(4)$, and $\theta(\mu)$ satisfies

$$\frac{d\theta}{d\mu}=\frac{1}{2\sqrt{c^2\mu^{-1}-\mu^2-1}}. \tag{7.2}$$

Proof. Under the hypothesis there is an orthonormal frame $\{e_1,\dots,e_5\}$ such that (4.1) holds. If $\mu=0$, then M is a $\delta(2,2)$-ideal Lagrangian minimal submanifold. Thus we obtain case (1). If $\mu\neq 0$ and $a,b=0$, then M is an H-umbilical Lagrangian submanifold of ratio 4, which gives case (2).

Next, assume that $a,\mu\neq 0$. Then Lemma 4.5 implies $b=0$. So, by Lemmas 5.1 we obtain (5.1) and (5.2). Also, in this case M is locally a warped product $I\times_{\rho(t)}N^4$ with $e_5=\partial_t$ according to Lemma 5.3. From Lemma 4.1, we find

$$\begin{aligned} &h(e_1,e_1)=aJe_1+\mu Je_5,\ h(e_1,e_2)=-aJe_2,\\ &h(e_2,e_2)=-aJe_1+\mu Je_5,\\ &h(e_3,e_3)=h(e_4,e_4)=\mu Je_5,\ h(e_5,e_5)=4\mu Je_5,\\ &h(e_i,e_5)=\mu Je_i,\ i\in\Delta,\ \ h(e_i,e_j)=0,\ \textit{otherwise}. \end{aligned} \tag{7.3}$$

By Lemma 5.4 we have the following ODE system:

$$\frac{d\nu}{dt}=-1-\nu^2-3\mu^2,\quad \frac{d\mu}{dt}=2\mu\nu. \tag{7.4}$$

Let $\theta(t)$ be a function on M satisfying

$$\theta'(t)=\mu. \tag{7.5}$$

Let L denote the horizontal lift in $S^{11}(1)\subset \mathbf{C}^6$ of the Lagrangian immersion of M in $CP^5(4)$ via Hopf 's fibration. Consider the maps:

$$\xi=\frac{e^{-3\mathrm{i}\theta}\left(e_5-(\nu+\mathrm{i}\mu)\,L\right)}{\sqrt{1+\mu^2+\nu^2}},\ \ \phi=\frac{e^{-\mathrm{i}\theta}\left(L+(\nu-\mathrm{i}\mu)\,e_5\right)}{\sqrt{1+\mu^2+\nu^2}}. \tag{7.6}$$

Then $\langle\xi,\xi\rangle=\langle\phi,\phi\rangle=1$. From $\nabla_{e_j}e_5=\nu e_j$, $j\in\Delta$, and (7.4), we find $\tilde\nabla_{e_j}\xi=0$. Moreover, it follows from Lemma 5.1 and (7.3) that $\tilde\nabla_{e_5}e_5=4\mathrm{i}\mu e_5-L$. Thus we also hhve$\tilde\nabla_{e_5}\xi=0$. Hence ξ is a constant unit vector in $\mathbf{C}^6$. Similarly, we also have $\tilde\nabla_{e_5}\phi=0$. So ϕ is independent of t. Therefore, by combining (7.6) we find

$$L=\frac{e^{\mathrm{i}\theta}\phi-e^{3\mathrm{i}\theta}(\nu-\mathrm{i}\mu)\xi}{\sqrt{1+\mu^2+\nu^2}}. \tag{7.7}$$

Since ϕ is orthogonal to $\xi,\mathrm{i}\xi$, after choosing $\xi=(0,\dots,0,1)\in\mathbf{C}^6$ we obtain

$$L=\frac{1}{\sqrt{1+\mu^2+\nu^2}}\left(e^{\mathrm{i}\theta}\phi,e^{3\mathrm{i}\theta}(\nu-\mathrm{i}\mu)\right) \tag{7.8}$$

It follows from (7.4) and (7.5) that

$$\frac{d\nu}{d\mu} = -\frac{1+\nu^2+3\mu^2}{2\mu\nu}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu}. \tag{7.9}$$

Solving the first differential equation in (7.9) gives

$$\nu = \pm\sqrt{c^2\mu^{-1}-\mu^2-1}, \ \ c \in \mathbf{R}^+. \tag{7.10}$$

By replacing e_5 by $-e_5$ if necessary, we have $\nu = \sqrt{c^2\mu^{-1}-\mu^2-1}$. Consequently,

$$L = \frac{1}{c}\left(\sqrt{\mu}e^{\mathrm{i}\theta}\phi, e^{3\mathrm{i}\theta}(\sqrt{c^2-\mu^3-\mu} - \mathrm{i}\mu^{\frac{3}{2}})\right), \tag{7.11}$$

It follows from (5.1), (7.3) and the second formula in (7.6) that

$$\hat{\nabla}_{e_j}\phi = \frac{ce^{-\mathrm{i}\theta}}{\sqrt{\mu}}e_j, \ \ j \in \Delta. \tag{7.12}$$

Thus after applying (6.11) and (7.12) we derive that

$$\hat{\nabla}_{e_\beta}\hat{\nabla}_{e_\alpha}\phi = \sum_{\gamma=2}^{n}\left(\Gamma_{ij}^k + \mathrm{i}h_{ij}^k\right)\phi_*(e_k) - \langle\phi_*(e_i), \phi_*(e_j)\rangle\,\phi, \ i,j \in \Delta. \tag{7.13}$$

Hence ϕ is a horizontal immersion in $S^9(1)$. Moreover, it follows from (7.13) that the second fundamental form of ϕ is a scalar multiple of the original second fundamental form of M restricted to the second factor of the warped product $I \times_\rho N$. Consequently, ϕ is a minimal horizontal immersion in $S^9(1)$ of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal submanifold of $CP^4(4)$.

The converse is easy to verify. □

8. Improved $\delta(2,2)$-ideal Lagrangian submanifolds of CH^5

Theorem 8.1. *Let M be an improved $\delta(2,2)$-ideal Lagrangian submanifold in $CH^5(-4)$. Then M is one of the following Lagrangian submanifolds:*

- (i) *a $\delta(2,2)$-ideal Lagrangian minimal submanifold;*
- (ii) *an H-umbilical Lagrangian submanifold of ratio 4;*
- (iii) *a Lagrangian submanifold defined by*

$$L(\mu, u_1, \ldots, u_4) = \frac{1}{c}\left(\sqrt{\mu}e^{\mathrm{i}\theta}\phi(u_2,\ldots,u_4), e^{-\mathrm{i}\theta}(\sqrt{\mu-\mu^3-c^2} - \mathrm{i}\mu^{\frac{3}{2}})\right), \tag{8.1}$$

where c is a positive number, $\phi : N^4 \to H_1^9(-1) \subset \mathbf{C}_1^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CH^4(-4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1-\mu^2-c^2\mu^{-1}}$;

(iv) *a Lagrangian submanifold defined by*

$$L(\mu, u_1, \dots, u_4) = \frac{1}{c}\left(e^{-\mathrm{i}\theta}(\sqrt{\mu-\mu^3+c^2} - \mathrm{i}\mu^{\frac{3}{2}}), \sqrt{\mu}e^{\mathrm{i}\theta}\phi(u_2, \dots, u_4)\right), \tag{8.2}$$

where c is a positive number, $\phi : N^4 \to S^9(1) \subset \mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CP^4(4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1-\mu^2+c^2\mu^{-1}}$;

(v) *a Lagrangian submanifold defined by*

$$\begin{aligned} L(t, u_1, \dots, u_4) = \frac{1}{\cosh t - \mathrm{i}\sinh t}&\left(2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi,\psi\rangle - \frac{1}{4}\right),\right. \\ &\left.\psi,\, 2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi,\psi\rangle + \frac{1}{4}\right)\right), \end{aligned} \tag{8.3}$$

where $\psi(u_1, \dots, u_4)$ is a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $\mathbf{C}^4$ and up to a constant $w(u_1, \dots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2\left\langle \psi_{u_j}, \mathrm{i}\psi\right\rangle$, $j = 1, 2, 3, 4$;

(vi) *a Lagrangian submanifold defined by*

$$\begin{aligned} L(t, u_1, \dots, u_4) = \frac{1}{\cosh t - \mathrm{i}\sinh t}&\left(2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi,\psi\rangle - \frac{1}{4}\right),\right. \\ &\left.\psi_1, \psi_2,\, 2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi,\psi\rangle + \frac{1}{4}\right)\right), \end{aligned} \tag{8.4}$$

where $\psi = (\psi_1, \psi_2)$ is the direct product immersion of two non-totally geodesic Lagrangian minimal immersions $\psi_\alpha : N_\alpha^2 \to \mathbf{C}^2$, $\alpha = 1, 2$, and up to a constant $w(u_1, \dots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2\left\langle \psi_{u_j}, \mathrm{i}\psi\right\rangle$, $j = 1, 2, 3, 4$.

Proof. Under the hypothesis there exists an orthonormal frame $\{e_1, \dots, e_5\}$ such that (4.1) holds.

Case (1) $\mu = 0$. In this case, we obtain case (i) of the theorem.

Case (2): $\mu \neq 0$ *and* $a, b = 0$. In this case M is an H-umbilical Lagrangian submanifold with ratio 4, which gives case (ii).

Case (3): $\mu \neq 0$ *and at least one of* a, b *is nonzero.* Without loss of generality, we may assume $a \neq 0$ and $\mu > 0$. We divide this into two cases.

Case (3.a): $a, \mu \neq 0$ *and* $b = 0$. By Lemmas 5.1 we obtain (5.1) and (5.2). Also, M is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$ according to Lemma 5.3. From Lemma 4.1 we find

$$\begin{aligned} &h(e_1, e_1) = aJe_1 + \mu Je_5,\ h(e_1, e_2) = -aJe_2, \\ &h(e_2, e_2) = -aJe_1 + \mu Je_5, \\ &h(e_3, e_3) = h(e_4, e_4) = \mu Je_5,\ h(e_5, e_5) = 4\mu Je_5, \\ &h(e_i, e_5) = \mu Je_i,\ i \in \Delta,\ \ h(e_i, e_j) = 0,\ \textit{otherwise}. \end{aligned} \tag{8.5}$$

Let L be a horizontal immersion of M in $H_1^{11}(-1) \subset \mathbf{C}_1^6$ of the Lagrangian immersion of M in $CH^5(-4)$ via Hopf 's fibration and $\theta(t)$ a function satisfying

$$\frac{d\theta}{dt} = \mu. \tag{8.6}$$

From Lemma 5.4 we obtain the following ODE system:

$$\frac{d\nu}{dt} = 1 - 3\mu^2 - \nu^2, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{8.7}$$

It follows from (8.6) and (8.7) that

$$\frac{d\nu}{d\mu} = \frac{1 - 3\mu^2 - \nu^2}{2\mu\nu}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu}. \tag{8.8}$$

Solving the first differential equation in (8.8) gives $\nu = \pm\sqrt{1 - \mu^2 - k\mu^{-1}}$ for some real number k. By replacing e_5 by $-e_5$ if necessary, we find

$$\nu = \sqrt{1 - \mu^2 - k\mu^{-1}}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\sqrt{1 - \mu^2 - k\mu^{-1}}}. \tag{8.9}$$

It follows from (8.7) that $\frac{d}{dt}(1 - \mu^2 - \nu^2) = -2\nu(1 - \mu^2 - \nu^2)$. Since this equation for $y(t) = 1 - \mu^2 - \nu^2 = k\mu^{-1}$ has a unique solution for each given initial condition, each solution either vanishes identically or is nowhere zero.

Case (3.a.1): $\mu^2 + \nu^2 < 1$. In this case, (8.9) implies $k > 0$. Thus we may put $k = c^2$, $c > 0$. Consider the maps:

$$\eta = \frac{e^{-3\mathrm{i}\theta}(e_5 - (\nu + \mathrm{i}\mu)L)}{\sqrt{1 - \mu^2 - \nu^2}}, \quad \phi = \frac{e^{-\mathrm{i}\theta}\left((\nu - \mathrm{i}\mu)\, e_5 - L\right)}{\sqrt{1 - \mu^2 - \nu^2}}. \tag{8.10}$$

Then $\langle \eta, \eta \rangle = 1$ and $\langle \phi, \phi \rangle = -1$. From $\nabla_{e_j} e_5 = \nu e_j$, $j \in \Delta$, and (8.5), we obtain $\tilde{\nabla}_{e_j}\xi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\mathbf{C}_1^6$. Lemma 5.1 and (8.5) give $\tilde{\nabla}_{e_5} e_5 = 4\mathrm{i}\mu e_5 + L$. Thus we find $\tilde{\nabla}_{e_5}\xi = 0$. So η is a constant unit vector. Also, we find $\tilde{\nabla}_{e_5}\phi = 0$. Hence ϕ is independent of t. From (8.10) we get

$$L = -\frac{e^{\mathrm{i}\theta}\phi + e^{-\mathrm{i}\theta}(\nu - \mathrm{i}\mu)\eta}{\sqrt{1-\mu^2-\nu^2}}. \tag{8.11}$$

Since ϕ is orthogonal to $\eta, \mathrm{i}\eta$ and η is a constant unit space-like vector, we conclude from (8.9) and (8.11) that L is congruent to (8.1). Next, by applying the same method of the proof of Theorem 7.1, we conclude that ϕ is a horizontal immersion in $H^9_1(-1)$ whose second fundamental form is a scalar multiple of the original second fundamental form restricted to the second factor of $I \times_\rho N$. Consequently, ϕ is a minimal horizontal immersion in $H^9_1(-1)$ of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal submanifold of $CH^4(-4)$. This gives case (iii).

Case (3.a.2): $\mu^2 + \nu^2 > 1$. In this case (8.8) implies $k < 0$. Thus we may put $k = -c^2$, $c > 0$. Now, we consider the maps:

$$\eta = \frac{e^{-3\mathrm{i}\theta}(e_5 - (\nu + \mathrm{i}\mu)L)}{\sqrt{\mu^2+\nu^2-1}}, \quad \phi = \frac{e^{-\mathrm{i}\theta}\left((\nu - \mathrm{i}\mu)\, e_5 - L\right)}{\sqrt{\mu^2+\nu^2-1}} \tag{8.12}$$

instead. Then $\langle \phi, \phi \rangle = -\langle \eta, \eta \rangle = 1$. By applying similar arguments as case (3.a.1), we know that η is a constant time-like vector and ϕ is independent of t and orthogonal to $\eta, \mathrm{i}\eta$. Moreover, we may prove that ϕ is a minimal Legendre immersion in $S^9(1)$. Therefore we have case (iv) after choosing $\eta = (1, 0, \dots, 0)$.

Case (3.a.3): $\mu^2 + \nu^2 = 1$. In this case system (8.7) gives $\frac{d\nu}{dt} = 2(\nu^2 - 1)$ and $\mu = \pm\sqrt{1-\nu^2}$. Solving these and applying a suitable translations in t, we find

$$\mu = \operatorname{sech} 2t, \quad \nu = -\tanh 2t. \tag{8.13}$$

It follows from $\nabla_{e_5} e_5 = 0$, (8.5) and (8.13) that the horizontal lift L of the Lagrangian immersion of M in $CH^5(-4) \subset \mathbf{C}^6_1$ satisfies

$$L_{tt} - 4\mathrm{i}(\operatorname{sech} 2t)L_t - L = 0. \tag{8.14}$$

Solving this second order differential equation gives

$$L = \frac{\phi(u_1, \dots, u_4) + B(u_1, \dots, u_4)(2t + \mathrm{i}\cosh 2t)}{\cosh t - \mathrm{i}\sinh t}, \tag{8.15}$$

where $\phi(u_1, \dots, u_4)$ and $B(u_1, \dots, u_4)$ are $\mathbf{C}^6_1$-valued functions.

On the other hand, it follows from Lemma 5.1, (8.5) and (8.13) that

$$L_{tu_j} = (\mathrm{i}\,\mathrm{sech}\,2t - \tanh 2t)L_{u_j}, \quad j \in \Delta. \tag{8.16}$$

Substituting (8.15) into (8.16) shows that B is a constant vector ζ. Thus

$$L(t, u_1, \ldots, u_4) = \frac{\phi(u_1, \ldots, u_4)}{\cosh t - \mathrm{i}\sinh t} + \frac{(2t + \mathrm{i}\cosh 2t)}{\cosh t - \mathrm{i}\sinh t}\zeta, \tag{8.17}$$

Since $\langle L, L\rangle = -1$, (8.17) implies

$$-\cosh 2t = \langle \phi, \phi\rangle + \langle \phi, (4t + 2\mathrm{i}\cosh 2t)\zeta\rangle + (4t^2 + \cosh^2(2t))\langle \zeta, \zeta\rangle . \tag{8.18}$$

Since $\phi_t = 0$, by differentiating (8.18) with respect t we find

$$-\sinh 2t = 2t\langle \phi, \zeta\rangle + 2\sinh 2t\langle \phi, \mathrm{i}\zeta\rangle + (4t + \sinh 4t)\langle \zeta, \zeta\rangle . \tag{8.19}$$

We find from (8.19) at $t = 0$ that $\langle \phi, \zeta\rangle = 0$. Thus (8.19) gives

$$0 = \sinh 2t(1 + \langle \phi, \mathrm{i}\zeta\rangle) + (4t + \sinh 4t)\langle \zeta, \zeta\rangle . \tag{8.20}$$

Differentiating (8.20) gives $\langle \phi, \mathrm{i}\zeta\rangle = -\frac{1}{2} - 2\langle \zeta, \zeta\rangle$. Thus (8.17) yields $\langle \phi, \mathrm{i}\zeta\rangle = -\frac{1}{2}$ and $\langle \zeta, \zeta\rangle = 0$. Now, we find from (8.18) that $\langle \phi, \phi\rangle = 0$. Consequently we have

$$\langle \phi, \phi\rangle = \langle \zeta, \zeta\rangle = \langle \phi, \zeta\rangle = 0, \quad \langle \phi, \mathrm{i}\zeta\rangle = -\tfrac{1}{2}. \tag{8.21}$$

Since ζ is a constant light-like vector, we may put

$$\zeta = (1, 0, \ldots, 0, 1), \quad \phi = (a_1 + \mathrm{i}b_1, \ldots, a_6 + \mathrm{i}b_6). \tag{8.22}$$

It follows from (8.21) and (8.22) that $a_6 = a_1$ and $b_6 = b_1 + \frac{1}{2}$. Therefore

$$\phi = \left(a_1 + \mathrm{i}b_1, a_2 + \mathrm{i}b_2, \ldots, a_1 + \mathrm{i}(b_1 + \tfrac{1}{2})\right). \tag{8.23}$$

Now, by using $\langle \phi, \phi\rangle = 0$ and (8.23), we find $\psi = (a_2 + \mathrm{i}b_2, \ldots, a_5 + \mathrm{i}b_5)$ and $b_1 = -\frac{1}{4} - \langle \psi, \psi\rangle$. Combining these with (8.23) yields

$$\phi = \left(w - \mathrm{i}\langle \psi, \psi\rangle - \frac{\mathrm{i}}{4}, \psi, w - \mathrm{i}\langle \psi, \psi\rangle + \frac{\mathrm{i}}{4}\right) \tag{8.24}$$

with $w = a_1$. It follows from (8.22) and (8.24) that $\langle \phi_{u_j}, \zeta\rangle = \langle \phi_{u_j}, \mathrm{i}\zeta\rangle = 0$. Thus, by applying $\langle L_{u_j}, \mathrm{i}L\rangle = 0$, $j \in \Delta$, we find from (8.17) that $\langle \phi_{u_j}, \mathrm{i}\phi\rangle = 0$.

On the other hand, (8.24) implies that

$$\langle \phi_{u_j}, \mathrm{i}\phi\rangle = -\tfrac{1}{2}w_{u_j} + \langle \psi_{u_j}, \mathrm{i}\psi\rangle \tag{8.25}$$

with $w_{u_j} = \frac{\partial w}{\partial u_j}$. Therefore w satisfies the PDE system: $w_{u_j} = 2\langle \psi_{u_j}, \mathrm{i}\psi\rangle$.

Now, we derive from (8.17), (8.22) and (8.23) that

$$L = \frac{1}{\cosh t - \mathrm{i}\sinh t}\Bigg(2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi,\psi\rangle - \frac{1}{4}\right), \\ \psi, 2t + w + \mathrm{i}\left(\cosh 2t - \langle\psi,\psi\rangle + \frac{1}{4}\right)\Bigg). \tag{8.26}$$

It follows from (8.26) that

$$L_{u_j} = \frac{1}{\cosh t - \mathrm{i}\sinh t}\left(w_{u_j} - \mathrm{i}\,\langle\psi,\psi\rangle_{u_j},\, \psi_{u_j}, w_{u_j} - \mathrm{i}\,\langle\psi,\psi\rangle_{u_j}\right). \tag{8.27}$$

Thus we find $\left\langle\psi_{u_j},\psi_{u_k}\right\rangle = \cosh 2t\left\langle L_{u_j}, L_{u_k}\right\rangle$ which implies that ψ is an immersion in $\mathbf{C}^4$. Also, we find from (8.27) and $\left\langle L_{u_j}, \mathrm{i}L_{u_k}\right\rangle = 0$ that $\left\langle\psi_{u_j}, \mathrm{i}\psi_{u_k}\right\rangle = 0$. Thus ψ is a Lagrangian immersion. Now, by applying an argument similar to the last part of the proof of [11, Theorem 6.1], we conclude that

$$\psi_{u_j u_k} = \sum_{i=1}^{4}(\Gamma_{jk}^i + \mathrm{i}h_{jk}^i)\phi_{u_i}, \quad j,k \in \Delta.$$

Therefore, according to (8.5), ψ is a $\delta(2)$-ideal minimal Lagrangian immersion in $\mathbf{C}^4$. Consequently, we obtain case (v) of the theorem.

Case (3.b): $a, b, \mu \neq 0$. We obtain case (vi) of the theorem by applying the same argument as case (3.a.3). □

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