LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS SATISFYING AN IMPROVED EQUALITY INVOLVING $\delta(2, 2)$

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Abstract. It was proved in [8, 9] that every Lagrangian submanifold $M$ of a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$ satisfies the following optimal inequality:

$$\delta(2, 2) \leq \frac{25}{4} H^2 + 8c,$$

where $H^2$ is the squared mean curvature and $\delta(2, 2)$ is a $\delta$-invariant on $M$ introduced by the first author. This optimal inequality improves a special case of an earlier inequality obtained in [B.-Y. Chen, Japan. J. Math. 26 (2000), 105–127].

The main purpose of this paper is to classify Lagrangian submanifolds of $\tilde{M}^n(4c)$ satisfying the equality case of the improved inequality (A).

1. Introduction

Let $\tilde{M}^n$ be a Kähler $n$-manifold with the complex structure $J$, a Kähler metric $g$ and the Kähler 2-form $\omega$. An isometric immersion $\psi : M \to \tilde{M}^n$ of a Riemannian $n$-manifold $M$ into $\tilde{M}^n$ is called Lagrangian if $\psi^* \omega = 0$.

Let $\tilde{M}^n(4c)$ denote a Kähler $n$-manifold with constant holomorphic sectional curvature $4c$, called a complex space form. A complete simply-connected complex space form $\tilde{M}^n(4c)$ is holomorphically isometric to the complex Euclidean $n$-plane $\mathbb{C}^n$, the complex projective $n$-space $CP^n(4c)$, or a complex hyperbolic $n$-space $CH^n(4c)$ according to $c = 0$, $c > 0$ or $c < 0$, respectively.

B.-Y. Chen introduced in 1990s new Riemannian invariants $\delta(n_1, \ldots, n_k)$. For any $n$-dimensional submanifold $M$ in a real space form $R^m(c)$ of constant curvature $c$.
curvature $c$, he proved the following sharp general inequality (see [5], [7] for details):

$$
\delta(n_1, \ldots, n_k) \leq \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)} H^2 
+ \frac{1}{2} \left(n(n - 1) - \sum_{j=1}^k n_j(n_j - 1)\right) c. \tag{1.1}
$$

For Lagrangian submanifolds in a complex space form $\tilde{M}^n(4c)$, we have

**Theorem A.** Let $M$ be an $n$-dimensional Lagrangian submanifold in a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Then inequality (1.1) holds for each $k$-tuple $(n_1, \ldots, n_k) \in S(n)$.

The following result from [6] extends a result in [10] on $\delta(2)$.

**Theorem B.** Every Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ is minimal if it satisfies the equality case of (1.1) identically.

Theorem B was improved recently in [8], [9] to the following inequality.

**Theorem C.** Let $M$ be an $n$-dimensional Lagrangian submanifold of $\tilde{M}^n(4c)$. Then, for an $(n_1, \ldots, n_k) \in S(n)$ with $\sum_{i=1}^k n_i < n$, we have

$$
\delta(n_1, \ldots, n_k) \leq n^2 \left\{ \left(n - \sum_{i=1}^k n_i + 3k - 1\right) - 6 \sum_{i=1}^k (2 + n_i) - 1 \right\} H^2 
+ \frac{1}{2} \left(n(n - 1) - \sum_{i=1}^k n_i(n_i - 1)\right) c. \tag{1.2}
$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, \ldots, e_n\}$ at $p$ such that the second fundamental form $h$ satisfies

$$
\begin{align*}
h(e_{\alpha_i}, e_{\beta_i}) &= \sum_{i=1}^n h_{\alpha_i, \beta_i}^\gamma J e_\gamma + \frac{3 \delta_{\alpha_i, \beta_i}}{2 + n_i} \lambda J e_{N+1}, \quad \sum_{i=1}^n h_{\alpha_i, \alpha_i}^\gamma = 0, \\
h(e_{\alpha_i}, e_{\alpha_j}) &= 0, \quad i \neq j; \quad h(e_{\alpha_i}, e_{N+1}) = \frac{3 \lambda}{2 + n_i} J e_{\alpha_i}, \quad h(e_{\alpha_i}, e_u) = 0, \quad (1.3) \\
h(e_{N+1}, e_{N+1}) &= 3 \lambda J e_{N+1}, \quad h(e_{N+1}, e_u) = \lambda J e_u, \quad N = n_1 + \cdots + n_k, \\
h(e_u, e_v) &= \lambda \delta_{uv} J e_{N+1}, \quad i, j = 1, \ldots, k; \quad u, v = N + 2, \ldots, n.
\end{align*}
$$

For simplicity, we call a Lagrangian submanifold of a complex space form $\delta(n_1, \ldots, n_k)$-ideal (resp., improved $\delta(n_1, \ldots, n_k)$-ideal) if it satisfies the equality case of (1.1) (resp., the equality case of (1.2)) identically.

For $k = 2$ and $n_1 = n_2 = 2$, Theorem C reduces to the following.
Theorem D. Let $M$ be a Lagrangian submanifold in a complex space form $	ilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$. Then we have

$$\delta(2,2) \leq \frac{25}{4}\mathcal{H}^2 + 8c. \quad (1.4)$$

If the equality sign of (1.4) holds identically, then with respect some suitable orthonormal frame $\{e_1, \ldots, e_5\}$ the second fundamental form $h$ satisfies

$$h(e_1, e_1) = \alpha J e_1 + \beta J e_2 + \mu J e_5,$$
$$h(e_2, e_2) = -\alpha J e_1 - \beta J e_2 + \mu J e_5,$$
$$h(e_3, e_3) = \gamma J e_3 + \delta J e_4 + \mu J e_5,$$
$$h(e_4, e_4) = -\gamma J e_3 - \delta J e_4 + \mu J e_5,$$
$$h(e_5, e_5) = 4\mu J e_5,$$
$$h(e_i, e_5) = \mu J e_i, \ i \in \Delta; \ h(e_i, e_j) = 0, \ otherwise,$$

(1.5)

for some functions $\alpha, \beta, \gamma, \delta, \mu$, where $\Delta = \{1, 2, 3, 4\}$.

The classification of $\delta(2,2)$-ideal Lagrangian submanifolds in complex space forms $\tilde{M}^5(4c)$ is done in [13]. In this paper we classify improved $\delta(2,2)$-ideal Lagrangian submanifolds in $\tilde{M}^5(4c)$. The main results of this paper are stated as Theorem 6.1, Theorem 7.1 and Theorem 8.1.

2. Preliminaries

2.1. Basic formulas. Let $\tilde{M}^n(4c)$ denote a complete simply-connected Kähler $n$-manifold with constant holomorphic sectional curvature $4c$. Then $\tilde{M}^n(4c)$ is holomorphically isometric to the complex Euclidean $n$-plane $\mathbb{C}^n$, the complex projective $n$-space $CP^n(4c)$, or a complex hyperbolic $n$-space $CH^n(-4c)$ according to $c = 0, c > 0$ or $c < 0$.

Let $M$ be a Lagrangian submanifold of $\tilde{M}^n(4c)$. We denote the Levi-Civita connections of $M$ and $\tilde{M}^n(4c)$ by $\nabla$ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by (cf. [7])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

(2.1)

for tangent vector fields $X$ and $Y$ and normal vector fields $\xi$, where $h$ is the second fundamental form, $A$ is the shape operator and $D$ is the normal connection.

The second fundamental form and the shape operator are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$
The mean curvature vector $\vec{H}$ of $M$ is defined by $\vec{H} = \frac{1}{n} \text{trace } h$ and the squared mean curvature is given by $H^2 = \langle \vec{H}, \vec{H} \rangle$.

For Lagrangian submanifolds, we have (cf. [7, 12])

$$D_X JY = J \nabla_X Y, \quad (2.2)$$
$$AJX Y = -J h(X,Y) = AJY X. \quad (2.3)$$

Formula (2.3) implies that $\langle h(X,Y), JZ \rangle$ is totally symmetric.

The equations of Gauss and Codazzi are given respectively by

$$\langle R(X,Y)Z, W \rangle = \langle A_h(Y,Z)X, W \rangle - \langle A_h(X,Z)Y, W \rangle \quad (2.4)$$
$$+ c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle),$$

$$\nabla X h(Y, Z) = \nabla Y h(X, Z), \quad (2.5)$$

where $R$ is the curvature tensor of $M$ and $\nabla h$ is defined by

$$\nabla X h(Y, Z) = DX h(Y, Z) - h(\nabla X Y, Z) - h(Y, \nabla X Z). \quad (2.6)$$

For an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p M$, we put

$$h^i_{jk} = \langle h(e_j, e_k), Je_i \rangle, \quad i, j, k = 1, \ldots, n.$$  

It follows from (2.3) that $h^i_{jk} = h^j_{ik} = h^k_{ij}$.

2.2. $\delta$-invariants. Let $M$ be a Riemannian $n$-manifold. Denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_p M$, $p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of $T_p M$, the scalar curvature $\tau$ at $p$ is $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$.

Let $L$ be a $r$-subspace of $T_p M$ with $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $L$. The scalar curvature $\tau(L)$ of $L$ is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \quad (2.7)$$

For given integers $n \geq 3$, $k \geq 1$, we denote by $S(n, k)$ the finite set consisting of $k$-tuples $(n_1, \ldots, n_k)$ of integers satisfying $2 \leq n_1, \ldots, n_k < n$ and $\sum_{j=1}^k i_j \leq n$.

Put $S(n) = \cup_{k \geq 1} S(n, k)$. For each $k$-tuple $(n_1, \ldots, n_k) \in S(n)$, the first author introduced in 1990s the Riemannian invariant $\delta(n_1, \ldots, n_k)$ by

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf \{\tau(L_1) + \cdots + \tau(L_k)\}, \quad p \in M, \quad (2.8)$$

where $L_1, \ldots, L_k$ run over all $k$ mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j = 1, \ldots, k$ (cf. [7] for details).
2.3. **Horizontal lift of Lagrangian submanifolds.** The following link between Legendrian submanifolds and Lagrangian submanifolds is due to [16] (see also [7, pp. 247–248]).

**Case (i):** $CP^n(4)$. Consider Hopf’s fibration $\pi : S^{2n+1} \to CP^n(4)$. For a given point $u \in S^{2n+1}(1)$, the horizontal space at $u$ is the orthogonal complement of $i_u$, $i = \sqrt{-1}$, with respect to the metric on $S^{2n+1}$ induced from the metric on $C^{n+1}$. Let $\iota : N \to CP^n(4)$ be a Lagrangian isometric immersion. Then there is a covering map $\tau : \hat{N} \to N$ and a horizontal immersion $\hat{\iota} : \hat{N} \to S^{2n+1}$ such that $\iota \circ \tau = \pi \circ \hat{\iota}$. Thus each Lagrangian immersion can be lifted locally (or globally if $N$ is simply-connected) to a Legendrian immersion of the same Riemannian manifold. In particular, a minimal Lagrangian submanifold of $CP^n(4)$ is lifted to a minimal Legendrian submanifold of the Sasakian $S^{2n+1}(1)$.

Conversely, suppose that $f : \hat{N} \to S^{2n+1}$ is a Legendrian isometric immersion. Then $\iota = \pi \circ f : N \to CP^n(4)$ is again a Lagrangian isometric immersion. Under this correspondence the second fundamental forms $h_f$ and $h_\iota$ of $f$ and $\iota$ satisfy $\pi^* h_f = h_\iota$. Moreover, $h_f$ is horizontal with respect to $\pi$.

**Case (ii):** $CH^n(-4)$. We consider the complex number space $C_1^{n+1}$ equipped with the pseudo-Euclidean metric: $g_0 = -dz_1d\bar{z}_1 + \sum_{j=2}^{n+1} dz_jd\bar{z}_j$.

Consider $H_2^{2n+1}(-1) = \{z \in C_1^{2n+1} : \langle z, z \rangle = -1\}$ with the canonical Sasakian structure, where $\langle \ , \ \rangle$ is the induced inner product.

Put $T'_z = \{u \in C^{n+1} : \langle u, z \rangle = 0\}$, $H_1^1 = \{\lambda \in C : \lambda\bar{\lambda} = 1\}$. Then there is an $H_1^1$-action on $H_1^{2n+1}(-1)$, $z \mapsto \lambda z$ and at each point $z \in H_1^{2n+1}(-1)$, the vector $\xi = -iz$ is tangent to the flow of the action. Since the metric $g_0$ is Hermitian, we have $\langle \xi, \xi \rangle = -1$. The quotient space $H_1^{2n+1}(-1) / \sim$, under the identification induced from the action, is the complex hyperbolic space $CH^n(-4)$ with constant holomorphic sectional curvature $-4$ whose complex structure $J$ is induced from the complex structure $J$ on $C_1^{n+1}$ via Hopf’s fibration $\pi : H_1^{2n+1}(-1) \to CH^n(4c)$.

Just like case (i), suppose that $\iota : N \to CH^n(-4)$ is a Lagrangian immersion, then there is an isometric covering map $\tau : \hat{N} \to N$ and a Legendrian immersion $f : \hat{N} \to H_1^{2n+1}(-1)$ such that $\iota \circ \tau = \pi \circ f$. Thus every Lagrangian immersion into $CH^n(-4)$ an be lifted locally (or globally if $N$ is
simply-connected) to a Legendrian immersion into $H^{2n+1}(1)$. In particular, Lagrangian minimal submanifolds of $CH^n(-4)$ are lifted to Legendrian minimal submanifolds of $H^{2n+1}_1(1)$. Conversely, if $f: \tilde{N} \to H^{2n+1}_1(1)$ is a Legendrian immersion, then $\iota = \pi \circ f : N \to CH^n(-4)$ is a Lagrangian immersion. Under this correspondence the second fundamental forms $h_f$ and $h_\iota$ are related by $\pi^* h_f = h_\iota$. Also, $h_f$ is horizontal with respect to $\pi$.

Let $h$ be the second fundamental form of $M$ in $S^{2n+1}(1)$ (or in $H^{2n+1}_1(1)$). Since $S^{2n+1}(1)$ and $H^{2n+1}_1(1)$ are totally umbilical with one as its mean curvature in $C^{n+1}$ and in $C^{n+1}_1$, respectively, we have

\[ \hat{\nabla}_X Y = \nabla_X Y + h(X, Y) - \varepsilon L, \quad (2.9) \]

where $\varepsilon = 1$ if the ambient space is $C^{n+1}$; and $\varepsilon = -1$ if it is $C^{n+1}_1$.

3. $H$-umbilical Lagrangian submanifolds and complex extensors


Definition 3.1. A non-totally geodesic Lagrangian submanifold of a Kähler $n$-manifold is called $H$-umbilical if its second fundamental form satisfies

\[ h(e_j, e_j) = \mu Je_n, \quad h(e_j, e_n) = \mu Je_j, \quad j = 1, \ldots, n - 1, \]

\[ h(e_n, e_n) = \varphi Je_n, \quad h(e_j, e_k) = 0, \quad 1 \leq j \neq k \leq n - 1, \quad (3.1) \]

for some functions $\mu, \varphi$ with respect to an orthonormal frame $\{e_1, \ldots, e_n\}$. If the ratio of $\varphi : \mu$ is a constant $r$, the $H$-umbilical submanifold is said to be of ratio $r$.

If $G : N^{n-1} \to \mathbb{E}^n$ is a hypersurface of a Euclidean $n$-space $\mathbb{E}^n$ and $\gamma : I \to \mathbb{C}^*$ is a unit speed curve in $\mathbb{C}^* = \mathbb{C} - \{0\}$, then we may extend $G : N^{n-1} \to \mathbb{E}^n$ to an immersion $I \times N^{n-1} \to \mathbb{C}^n$ by $\gamma \otimes G : I \times N^{n-1} \to \mathbb{C}^n \otimes \mathbb{E}^n = \mathbb{C}^n$, where $(\gamma \otimes G)(s, p) = F(s) \otimes G(p)$ for $s \in I$, $p \in N^{n-1}$. This extension of $G$ via tensor product $\otimes$ is called the complex extensor of $G$ via the generating curve $\gamma$.

$H$-umbilical Lagrangian submanifolds in complex space forms were classified in a series of papers by the first author (cf. [2], [3], [4]). In particular, the following two results were proved in [2].

**Theorem E.** Let $\iota : S^{n-1} \subset \mathbb{E}^n$ be the unit hypersphere in $\mathbb{E}^n$ centered at the origin. Then every complex extensor of $\iota$ via a unit speed curve
\[ \gamma: I \to \mathbb{C}^* \] is an H-umbilical Lagrangian submanifold of \( \mathbb{C}^n \) unless \( \gamma \) is contained in a line through the origin (which gives a totally geodesic Lagrangian submanifold).

**Theorem F.** Let \( M \) be an H-umbilical Lagrangian submanifold of \( \mathbb{C}^n \) with \( n \geq 3 \). Then \( M \) is either a flat space or congruent to an open part of a complex extensor of \( i: S^{n-1} \subset \mathbb{E}^n \) via a curve \( \gamma: I \to \mathbb{C}^* \).

### 3.2. Legendre curves

A unit speed curve \( z: I \to S^3(1) \subset \mathbb{C}^2 \) (resp., \( z: I \to H^3_1(-1) \subset \mathbb{C}^2 \)) is called Legendre if \( \langle z', iz \rangle = 0 \). It was proved in [3] that a unit speed curve \( z \) in \( S^3(1) \) (resp., in \( H^3_1(-1) \)) is Legendre if and only if it satisfies

\[ z'' = i \lambda z' - z \quad \text{ (resp., } z'' = i \lambda z' + z) \quad (3.2) \]

for a real-valued function \( \lambda \). It is known in [3] that \( \lambda \) is the curvature function of \( z \) in \( S^3(1) \) (resp., in \( H^3_1(-1) \)) (see also [1, Lemmas 3.1 and 3.2]).

### 3.3. H-umbilical submanifolds with arbitrary ratio

We provide a general method to construct H-umbilical Lagrangian submanifolds with any given ratio in \( CP^n(4) \) via curves in \( S^2(\frac{1}{2}) \) (resp., in \( CH^n(-4) \) via curves in \( H^2(-\frac{1}{2}) \)).

**Proposition 3.2.** For any real number \( r \) there exist H-umbilical Lagrangian submanifolds of ratio \( r \) in \( CP^n(4) \) and in \( CH^n(-4) \).

**Proof.** If \( r = 2 \) this was done in [3, Theorems 5.1 and 6.1]. If \( r \neq 2 \), H-umbilical Lagrangian submanifolds of ratio \( r \) can be constructed as follows:

**Case (a):** \( CP^n(4) \). Let \( S^2(\frac{1}{2}) = \{ x \in \mathbb{E}^3; \langle x, x \rangle = \frac{1}{4} \} \). The Hopf fibration \( \pi \) from \( S^3(1) \) onto \( S^2(\frac{1}{2}) \equiv CP^1(4) \) is given by (cf. [1])

\[ \pi(z_1, z_2) = (z_1 \bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)) \], \( (z_1, z_2) \in S^3(1) \subset \mathbb{C}^2 \). \quad (3.3)

For a Legendre curve \( z \) in \( S^3(1) \), the projection \( \gamma \) is a curve in \( S^2(\frac{1}{2}) \). Conversely, each curve \( \gamma \) in \( S^2(\frac{1}{2}) \) gives rise to a horizontal lift \( \hat{\gamma} \) in \( S^3(1) \) via \( \pi \) which is unique up to a factor \( e^{i\theta}, \theta \in \mathbb{R} \). Notice that each horizontal lift of \( \gamma \) is a Legendre curve in \( S^3(1) \). Moreover, since the Hopf fibration is a Riemannian submersion, each unit speed Legendre curve \( z \) in \( S^3(1) \) is projected to a unit speed curve \( \gamma \) in \( S^2(\frac{1}{2}) \) with the same curvature.
It was known in [3] Lemma 7.2 that, for a given $H$-umbilical Lagrangian submanifold of ratio $r \neq 2$ in $M^n(4c)$, the function $\mu$ in (3.1) satisfies

$$\mu \mu'' - \left(\frac{r-3}{r-2}\right) \mu'^2 + (r-2)\mu^2((r-1)\mu^2 + c) = 0. \quad (3.4)$$

If $\mu$ is a non-trivial solution of (3.4) with $c = 1$, then there is a unit speed curve $\gamma$ in $S^2(\frac{1}{r})$ whose curvature equals to $r\mu$. Let $z$ be a horizontal lift of $\gamma$ in $S^3(1)$. Then $z$ is a unit speed Legendre curve satisfying $z''(x) = ir\mu z'(x) - z(x)$ (cf. [3] Theorem 4.1 or [1] Lemma 3.1).

Consider the map $\psi : M^5 \to S^{11}(1) \subset \mathbb{C}^6$ defined by

$$\psi(x, y_1, \ldots, y_5) = (z_1(x), z_2(x)y_1, \ldots, z_2(x)y_5), \quad \sum_{j=1}^5 y_j^2 = 1. \quad (3.5)$$

It follows from [3] Theorem 4.1 and Lemma 7.2 that $\pi \circ \psi$ is a $H$-umbilical Lagrangian submanifold of ratio $r$ in $CP^n(4)$ such that

$$h(e_j, e_j) = \mu Je_5, \quad h(e_j, e_n) = \mu Je_j, \quad h(e_n, e_n) = r\mu Je_n, \quad h(e_j, e_k) = 0, \quad 1 \leq j \neq k \leq n-1, \quad (3.6)$$

with respect to suitable orthonormal frame $\{e_1, \ldots, e_5\}$.

**Case (b): $CH^n(-4)$.** For a non-trivial solution of (3.4) with $c = -1$, we can construct an $H$-umbilical Lagrangian submanifold of $CH^n(-4)$ via the Hopf fibration $\pi : H^3_1(-1) \to CH^1(-4) \equiv H^2(-\frac{1}{2})$ in a similar way as case (a), where

$$\pi(z_1, z_2) = (z_1 \bar{z}_2, \frac{1}{2}(|z_1|^2 + |z_2|^2)), \quad (z_1, z_2) \in H^3_1(-1) \subset \mathbb{C}^2, \quad (3.7)$$

and $H^2(-\frac{1}{2}) = \{(x_1, x_2, x_3) \in \mathbb{E}^3_1 : x_1^2 - x_2^2 - x_3^2 = \frac{1}{4}, x_1 \geq \frac{1}{2}\}$ is the model of the real projective plane of curvature $-\frac{1}{4}$.

### 3.4. Classification of $H$-umbilical submanifolds of ratio 4

The equation of Gauss and (3.1) imply that $H$-umbilical Lagrangian submanifolds of ratio $r \neq 4$ in complex space forms contain no open subsets of constant sectional curvature. Hence we conclude from [3] Theorems 4.1 and 7.1 and §3.3 the following results.

**Lemma 3.3.** An $H$-umbilical Lagrangian submanifold $M$ of ratio 4 in $CP^5(4)$ is congruent to an open portion of $\pi \circ \psi$, where $\pi : S^{11}(1) \to CP^5(4)$ is Hopf’s fibration, $\psi : M \to S^{11}(1) \subset \mathbb{C}^6$ is given by

$$\psi(t, y_1, \ldots, y_5) = (z_1(t), z_2(t)y), \quad \{y \in \mathbb{E}^5 : \langle y, y \rangle = 1\}, \quad (3.8)$$
and \( z = (z_1, z_2) : I \to S^3(1) \subset \mathbb{C}^2 \) is a unit speed Legendre curve satisfying
\[ z'' = 4i\mu z' - z, \text{ and } \mu \text{ is a nonzero solution of } 2\mu'' - \mu^2 + 4\mu^2(3\mu^2 + 1) = 0. \]

Let \( M \) be an \( H \)-umbilical Lagrangian submanifold in \( CH^5(-4) \) satisfying \( §3.1 \). We may assume that \( \mu \) is defined on an open interval \( I \ni 0 \). Since \( H \)-umbilical submanifolds of ratio 4 in \( CH^5(-4) \) contain no open subsets of constant curvature, Theorems 4.2 and 9.1 of [2] and results in §3.3 imply the following classification of \( H \)-umbilical submanifolds of ratio 4 in \( CH^5(-4) \).

**Lemma 3.4.** An \( H \)-umbilical Lagrangian submanifold \( M \) of ratio 4 in \( CH^5(-4) \) is congruent to an open part of \( \pi \circ \psi \), where \( \pi : H^6_1(-1) \to CH^5(-4) \) is Hopf’s fibration and \( \psi : M \to H^6_1(-1) \subset \mathbb{C}^6 \) is either one of

\[
\psi(t, y_1, \ldots, y_4) = (z_1(t), z_2(t)y), \quad \{ y \in E^5 : \langle y, y \rangle = 1 \}, \quad (3.9)
\]

\[
\psi(t, y_1, \ldots, y_4) = (z_1(t)y, z_2(t)), \quad \{ y \in E^5_1 : \langle y, y \rangle = -1 \}, \quad (3.10)
\]

where \( z \) is a unit speed Legendre curve in \( H^6_1(-1) \) satisfying \( z'' = 4i\mu z' + z \) and \( \mu \) is a non-trivial solution of \( 2\mu'' - \mu^2 + 4\mu^2(3\mu^2 - 1) = 0 \); or \( \psi \) is

\[
\psi(t, u_1, \ldots, u_4) = \sqrt{\mu} e^{i \int_0^t \mu(\tau) dt} \left( 1 + \frac{1}{2} \sum_{j=1}^4 u_j^2 - it + \frac{1}{2\mu} - \frac{1}{2\mu(0)} \right) \left( i\mu(0) - \frac{\mu'(0)}{2\mu(0)} \right) \left( \frac{1}{2} \sum_{j=1}^4 u_j^2 - it + \frac{1}{2\mu} - \frac{1}{2\mu(0)} \right), \quad u_1, \ldots, u_4
\]

\[
(3.11)
\]

where \( z = (z_1, z_2) : I \to H^6_1(-1) \subset \mathbb{C}^6 \) is a unit speed Legendre curve and \( \mu \) is a non-trivial solution of \( \mu'^2 = 4\mu^2(1 - \mu^2) \).

**Example.** It is easy to verify that \( \mu = \text{sech} \, 2t \) is a non-trivial solution of \( \mu'^2 = 4\mu^2(1 - \mu^2) \). Using \( \mu = \text{sech} \, 2t \), \( (3.11) \) reduces to

\[
\psi(t, u_1, \ldots, u_4) = \frac{e^{i \tan^{-1}(\tanh t)}}{\sqrt{\cosh 2t}} \left( \frac{1}{2} - it + \frac{1}{2} \sum_{j=1}^4 u_j^2 + \frac{\cosh 2t}{2}, \right) \left( t - \frac{i}{2} + \frac{i}{2} \sum_{j=1}^4 u_j^2 + \frac{i \cosh 2t}{2}, u_1, \ldots, u_4 \right).
\]

\[
(3.12)
\]

It is direct to verify that \( (3.12) \) satisfies \( \langle \psi, \psi \rangle = -1 \) and the composition \( \pi \circ \psi \) gives rise to an \( H \)-umbilical Lagrangian submanifold of ratio 4 in \( CH^5(-4) \).
4. Some Lemmas

We need the following lemmas for the proof of the main theorems.

Lemma 4.1. Let $M$ be an improved $\delta(2,2)$-ideal Lagrangian submanifold of $\tilde{M}^5(4c)$. Then with respect to some orthonormal frame $\{e_1, \ldots, e_5\}$ we have

\[
\begin{align*}
    h(e_1, e_1) &= a Je_1 + \mu Je_5, \quad h(e_1, e_2) = -a Je_2, \\
    h(e_2, e_2) &= -a Je_1 + \mu Je_5, \quad h(e_3, e_3) = b Je_3 + \mu Je_5, \\
    h(e_3, e_4) &= -b Je_4, \quad h(e_4, e_4) = -b Je_3 + \mu Je_5, \\
    h(e_i, e_5) &= \mu Je_i, \quad i \in \Delta, \quad h(e_5, e_5) = 4\mu Je_5, \\
    h(e_i, e_j) &= 0, \quad \text{otherwise},
\end{align*}
\]

(4.1)

Proof. Under the hypothesis, we have (1.5) with respect to an orthonormal frame $\{e_1, \ldots, e_5\}$. Thus, after applying [6, Lemma 1] to $V = \text{Span} \{e_1, e_2\}$ and $V = \text{Span} \{e_3, e_4\}$, we obtain (4.1). □

Let us put

\[
\nabla_X e_i = \sum_{j=1}^{5} \omega^j_i(X)e_j, \quad i = 1, \ldots, 5, \quad X \in TM^5.
\]

(4.2)

Then $\omega^i_j = -\omega^j_i, \quad i, j = 1, \ldots, 5$.

If $\mu = 0$, then $M$ is a minimal Lagrangian submanifold according [13]. Such submanifolds in complex space forms $\tilde{M}^5(4c)$ have been classified in [13].

If $a = b = 0$ and $\mu \neq 0$, then $M$ is an $H$-umbilical Lagrangian submanifold with ratio 4. Therefore, from now on we assume that $a, \mu \neq 0$.

Lemma 4.2. Let $M$ be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies (4.1) with $a, b, \mu \neq 0$. Then we have

\[
\begin{align*}
    \nabla_{e_1} e_1 &= \frac{e_2}{3a}e_2 - \nu e_5, \quad \nabla_{e_1} e_2 = -\frac{e_2}{3a}e_1, \quad \nabla_{e_2} e_1 = -\frac{e_1}{3a}e_2, \\
    \nabla_{e_2} e_2 &= \frac{e_3}{3a}e_1 - \nu e_5, \quad \nabla_{e_3} e_3 = \frac{e_4}{3b}e_4 - \nu e_5, \quad \nabla_{e_4} e_4 = -\frac{e_4}{3b}e_3, \\
    \nabla_{e_3} e_3 &= -\frac{e_4}{3b}e_4, \quad \nabla_{e_4} e_4 = \frac{e_3}{3b}e_3 - \nu e_5, \quad \nabla_{e_i} e_5 = \nu e_i, \quad i \in \Delta, \\
    \nabla_{e_i} e_j &= 0, \quad \text{otherwise},
\end{align*}
\]

(4.3)
with \( \nu = \frac{1}{2} e_5 (\ln \mu) = - e_5 (\ln a) = - e_5 (\ln b) \), where \( \Delta = \{1, 2, 3, 4\} \). Moreover, we have

\[
e_j \mu = 0, \quad j \in \Delta, \quad e_1 b = e_2 b = e_3 a = e_4 a = 0.
\] (4.4)

**Proof.** This lemma is obtained from Codazzi’s equations via Lemma 4.1 and (4.2) and long computations. \( \square \)

**Lemma 4.3.** Under the hypothesis of Lemma 4.2, we have

(a) \( T_0 \) is a totally geodesic distribution, i.e. \( T_0 \) is integrable whose leaves are totally geodesic submanifolds;

(b) \( T_0 \oplus T_1 \) and \( T_0 \oplus T_2 \) are totally geodesic distributions;

(c) \( T_1 \) and \( T_2 \) are spherical distributions, i.e. \( T_1, T_2 \) are integrable distributions whose leaves are totally umbilical submanifolds with parallel mean curvature vector,

where \( T_0 = \text{Span}\{e_5\}, T_1 = \text{Span}\{e_1, e_2\} \) and \( T_2 = \text{Span}\{e_3, e_4\} \).

**Proof.** Since the distribution \( T_0 \) is of rank one, it is integrable. Moreover, since \( \nabla_{e_5} e_5 = 0 \) by Lemma 4.2, the integral curves of \( e_5 \) are geodesics in \( M \).

Thus we have statement (a). Statement (b) follows easily from (4.3).

To prove statement (c), first we observe that \([e_1, e_2] \in T_1\) and \([e_3, e_4] \in T_2\) follow from (4.3). Thus \( T_1, T_2 \) are integrable. Also, it follows from (4.3) that the second fundamental form \( h_1 \) of a leaf \( L_1 \) of \( T_1 \) in \( M \) is given by

\[
h_1(X, Y) = - \nu g_1(X_1, Y_1) e_5, \quad X_1, Y_1 \in T L_1,
\] (4.5)

where \( g_1 \) is the metric of \( L_1 \). From (4.3) we obtain \( \nabla_{e_i} e_5 = \nu e_i, \quad i = 1, 2 \). Thus \( D^1 e_5 = D^1 e_5 = 0 \), where \( D^1 \) is the normal connection of \( L_1 \) in \( M \). It follows from Gauss’ equation and Lemma 4.1 that the curvature tensor \( R \) satisfies

\[
\langle R(e_1, e_2) e_1, e_j \rangle = 0, \quad j = 3, 4, 5.
\] (4.6)

Thus (4.6) and Lemma 4.2 imply that \( 0 \equiv R(e_1, e_2) e_1 \equiv (e_2 \nu) e_5 \) (mod \( T_1 \)). Hence \( e_2 \nu = 0 \). Similarly, by considering \( R(e_2, e_1) e_2 \), we also have \( e_1 \alpha = 0 \). After combining these with \( D^1 e_5 = 0 \), we conclude that \( L_1 \) has parallel mean curvature vector in \( M \). Hence \( T_1 \) is a spherical distribution. Similarly, \( T_2 \) is also a spherical distribution. Consequently, we obtain statement (c). \( \square \)

**Lemma 4.4.** Under the hypothesis of Lemma 4.2, \( M \) is locally a warped product \( I \times \rho_1(t) M_1^2 \times \rho_2(t) M_2^2 \), where \( t \) is function such that \( e_5 = \partial_t \) (i.e., \( e_5 =
\[ \frac{\partial}{\partial t}, \rho_1 \text{ and } \rho_2 \text{ are two positive functions in } t \text{ and } M^2_1, M^2_2 \text{ are Riemannian 2-manifolds.} \]

**Proof.** This lemma follows from Lemma 4.3 and a result of Hiepko [15] (see also [7, Theorem 4.4, p. 90]). \[ \square \]

Lemma 3.3 and (4.4) imply that \( \mu \) depends only on \( t \). Thus \( \mu = \mu(t) \).

**Lemma 4.5.** Let \( M \) be a Lagrangian submanifold of \( \tilde{M}^5(4c) \) whose second fundamental form satisfies (4.1) with \( a, b, \mu \neq 0 \). Then we have \( c = -\nu^2 - \mu^2 < 0 \). Thus \( \mu \) satisfies \( \mu'(t)^2 = -4\mu^2(t)(c + \mu^2(t)) \).

**Proof.** Under the hypothesis, it follows from Gauss’ equation and Lemma 4.1 that \( \langle R(e_1, e_3)e_3, e_1 \rangle = c + \mu^2 \). On the other hand, the definition of curvature tensor and Lemma 4.2 imply that \( \langle R(e_1, e_3)e_3, e_1 \rangle = -\nu^2 \). Thus \( c = -\nu^2 - \mu^2 < 0 \). By combining this with the definition of \( \nu \), we obtain the lemma. \[ \square \]

5. MORE LEMMAS

Next, we consider the case \( a, \mu \neq 0 \) and \( b = 0 \).
Lemma 5.1. Let $M$ be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies (4.1) with $a, \mu \neq 0$ and $b = 0$. Then we have
\[
\nabla_{e_i} e_1 = e_2a/3a e_2 + e_3a/3a e_3 + e_4a/3a e_4 - \nu e_5,
\]
\[
\nabla_{e_i} e_2 = -e_2a/3a e_1 - 3\omega_1^2(e_3)e_3 - 3\omega_1^2(e_4)e_4,
\]
\[
\nabla_{e_i} e_3 = -e_3a/3a e_1 + 3\omega_1^2(e_3)e_2 + \omega_1^3(e_1)e_3,
\]
\[
\nabla_{e_i} e_4 = -e_4a/3a e_1 + 3\omega_1^2(e_4)e_2 - \omega_1^3(e_1)e_3,
\]
\[
\nabla_{e_i} e_5 = 3\omega_1^2(e_3)e_1 - e_3a/3a e_2 + \omega_1^3(e_3)e_4,
\]
\[
(5.1)
\]
\[
\nabla_{e_i} e_6 = -\omega_1^3(e_2)e_1 - e_4a/3a e_2 - \omega_1^3(e_2)e_3,
\]
\[
\nabla_{e_i} e_7 = \omega_1^3(e_3)e_2, \quad \nabla_{e_i} e_8 = -\omega_1^3(e_3)e_1,
\]
\[
\nabla_{e_i} e_9 = \omega_1^3(e_4)e_2, \quad \nabla_{e_i} e_{10} = -\omega_1^3(e_4)e_1,
\]
\[
\nabla_{e_i} e_{11} = \omega_1^3(e_5)e_2, \quad \nabla_{e_i} e_{12} = -\omega_1^3(e_5)e_1,
\]
\[
\\nabla_{e_i} e_{13} = \nu e_i, \quad i \in \Delta, \quad \nabla_{e_i} e_{14} = 0, \quad \text{otherwise}.
\]

with $\nu = 1/5 e_5(\ln \mu) = -e_5(\ln a)$. Moreover, we have
\[
e_j \mu = 0, \quad j \in \Delta = \{1, 2, 3, 4\}. \quad (5.2)
\]

Proof. Follows from Codazzi’s equations via Lemma 4.1 and 4.2. 

Lemma 5.2. Under the hypothesis of Lemma 5.1, we have

(i) $T_0$ is a totally geodesic distribution;

(ii) $T_3$ is a spherical distribution,

where $T_0 = \text{Span}\{e_5\}$ and $T_3 = \text{Span}\{e_1, e_2, e_3, e_4\}$.

Proof. Clearly, $T_0$ is integrable. Moreover, since $\nabla_{e_5} e_5 = 0$ by Lemma 5.1, integral curves of $e_5$ are geodesics in $M^5$. Thus statement (i) follows. To prove statement (ii), we observe that the integrability of $T_3$ follows from (5.1). Also, (5.1) implies that the second fundamental form $\hat{h}$ of a leaf $L$ of $T_3$ in $M^5$ is given by $\hat{h}(X, Y) = -\nu \hat{g}(X, Y) e_5$ for $X, Y \in TL$, where $\hat{g}$ is the
metric of \( \mathcal{L} \). Since \([e_j, e_5]\mu = 0 \) by (5.1) and \( e_j\mu = 0 \), for \( j \in \Delta \), we find \( e_i e_5\mu - e_5 e_i\mu = 2e_1\nu = 0 \). Therefore \( T_3 \) is a spherical distribution. \( \square \)

**Lemma 5.3.** Under the hypothesis of Lemma 5.1, \( M \) is locally a warped product \( I \times \rho(t) N^4 \), where \( t \) is function such that \( e_5 = \frac{\partial}{\partial t} \) and \( \rho \) is a positive function in \( t \) and \( N^4 \) is a Riemannian 4-manifold.

**Proof.** Follows from Lemma 5.2 and Hiepko’s theorem. \( \square \)

It follows from (5.2) and the definition of \( \nu \) that \( \mu = \mu(t) \) and \( \nu = \nu(t) \).

**Lemma 5.4.** Under the hypothesis of Lemma 5.1, we have
\[
\frac{d\nu}{dt} = -3\mu^2 - \nu^2 - c, \quad \frac{d\mu}{dt} = 2\mu\nu.
\] \( (5.3) \)

**Proof.** From Gauss’ equation and (5.1) we find \( \langle R(e_1, e_5) e_5, e_1 \rangle = 3\mu^2 + c \). On the other hand, (5.1) of Lemma 5.1 yields \( \langle R(e_1, e_5) e_5, e_1 \rangle = -e_5\nu - \nu^2 \). Thus we find the first equation of (5.3). The second one follows immediately from the definition of \( \nu \) given in Lemma 5.1. \( \square \)

6. **Improved \( \delta(2, 2) \)-ideal Lagrangian submanifolds of \( \mathbb{C}^5 \)**

**Theorem 6.1.** Let \( M \) be an improved \( \delta(2, 2) \)-ideal Lagrangian submanifold in \( \mathbb{C}^5 \). Then it is one of the following Lagrangian submanifolds:

(a) a \( \delta(2, 2) \)-ideal Lagrangian minimal submanifold;

(b) an \( H \)-umbilical Lagrangian submanifold of ratio 4;

(c) a Lagrangian submanifold defined by
\[
L(\mu, u_2, \ldots, u_n) = \frac{e^{\frac{4i}{\mu} \tan^{-1}\sqrt{\mu^2/(c^2 - \mu^2)}}}{\sqrt{c^2\mu^{-1} - \mu^2 + i\mu}} \phi(u_2, \ldots, u_n),
\] \( (6.1) \)

where \( c \) is a positive real number and \( \phi(u_2, \ldots, u_n) \) is a horizontal lift of a non-totally geodesic \( \delta(2) \)-ideal Lagrangian minimal immersion in \( CP^4(4) \).

**Proof.** Assume that \( M \) is an improved \( \delta(2, 2) \)-ideal Lagrangian submanifold in \( \mathbb{C}^5 \). Then there exists an orthonormal frame \( \{e_1, \ldots, e_5\} \) such that (4.1) holds. If \( \mu = 0 \), then \( M \) is a minimal \( \delta(2, 2) \)-ideal Lagrangian submanifold. Thus, we obtain case (a). If \( \mu \neq 0 \) and \( a = b = 0 \), we obtain case (b).

Now, let us assume \( a, \mu \neq 0 \). Then Lemma 4.5 implies \( b = 0 \). So, by Lemmas 5.1 we have (5.1) and \( e_j\mu = 0, j \in \Delta \). Further, by Lemma 5.3, \( M \)
is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$. Moreover, (4.1) shows that the second fundamental form satisfies
\begin{align*}
h(e_1, e_1) &= aJe_1 + \mu Je_5, \quad h(e_1, e_2) = -aJe_2, \\
h(e_2, e_2) &= aJe_1 + \mu Je_5, \\
h(e_3, e_3) &= h(e_4, e_4) = \mu Je_5, \\
h(e_i, e_5) &= \mu Je_i, \quad i \in \Delta, \\
h(e_5, e_5) &= 4\mu Je_5, \quad h(e_i, e_j) = 0, \text{ otherwise.}
\end{align*}

(6.2)

From Lemma 5.4 we have the following differential system:
\begin{align*}
\frac{d\nu}{dt} &= -3\mu^2 - \nu^2, \quad \frac{d\mu}{dt} = 2\mu\nu.
\end{align*}

(6.3)

Let $\varphi(t)$ be a function satisfying $\frac{d\varphi}{dt} = -4\mu$. Consider the map
\begin{align*}
\phi = e^{i\varphi}e_5.
\end{align*}

(6.4)

Then $\langle \phi, \phi \rangle = 1$. It follows from $\nabla_{e_5}e_5 = 0, \frac{d\varphi}{dt} = -4\mu$ and (6.2) that $\tilde{\nabla}_{e_5}\phi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\mathbb{C}^5$. Thus $\phi$ is independent of $t$.

Let $L$ denote the Lagrangian immersion of $M$ in $\mathbb{C}^5$. Then (6.4) yields
\begin{align*}
e_5 = L_t = e^{-i\varphi}\phi(u_1, \ldots, u_4),
\end{align*}

(6.5)

where $u_1, \ldots, u_4$ are local coordinates of $N^4$. For each $j \in \Delta$, we obtain from $\nabla_{e_j}e_5 = \nu e_j$ of Lemma 5.4 and the first equation of (6.3) that
\begin{align*}
\phi\ast(e_j) = \tilde{\nabla}_{e_j}\phi = e^{i\varphi}\nabla_{e_j}e_5 = e^{i\varphi}(\nu + i\mu)e_j.
\end{align*}

(6.6)

Thus
\begin{align*}
\tilde{\nabla}_{e_j}(\phi\ast(e_i)) &= e^{i\varphi}(\nu + i\mu)\tilde{\nabla}_{e_j}e_i.
\end{align*}

(6.7)

In view of $\nabla_{e_j}e_5 = \nu e_j$ and (6.2), we may put
\begin{align*}
\tilde{\nabla}_{e_i}e_j &= \left(\sum_{k=1}^{4} \Gamma^k_{ij} + ih^k_{ij}\right)e_k - (\nu - i\mu)\delta_{ij}e_5, \quad i, j \in \Delta,
\end{align*}

(6.8)

for some functions $\Gamma^k_{ij}$. Now, it follows from (6.4), (6.6), (6.7), and (6.8) that
\begin{align*}
\tilde{\nabla}_{e_j}(\phi\ast(e_i)) &= \sum_{\gamma=2}^{n} \left(\Gamma^k_{ij} + ih^k_{ij}\right)\phi\ast(e_k) - (\mu^2 + \nu^2)\delta_{ij}\phi \\
&= \sum_{\gamma=2}^{n} \left(\Gamma^k_{ij} + ih^k_{ij}\right)\phi\ast(e_k) - \langle \phi\ast(e_i), \phi\ast(e_j) \rangle \phi.
\end{align*}

(6.9)
Since $M$ is a Lagrangian submanifold in $\mathbb{C}^5$, (6.4) and (6.6) show that $i\phi$ is perpendicular to each tangent space of $M$. Hence $\phi$ is a horizontal immersion in the unit hypersphere $S^9(1) \subset \mathbb{C}^5$. Moreover, it follows from (6.9) that the second fundamental form of $\phi$ is the original second fundamental form of $M$ respect to the second factor $N^4$ of the warped product $I \times_{\rho(t)} N^4$. Hence, $\phi$ is a minimal horizontal immersion in $S^9(1)$. Therefore, $\phi$ is a horizontal lift of a minimal Lagrangian immersion in $\mathbb{C}P^4(4)$. Now, it follows from (6.2) that $\phi$ is a horizontal lift of a $\delta(2)$-ideal minimal Lagrangian submanifold of $\mathbb{C}P^4(4)$.

By direct computation we find
\[
\tilde{\nabla}_{e\alpha}(L - \frac{e_5}{\nu + i\mu}) = 0, \quad \alpha = 1, \ldots, 5.
\] (6.10)
Thus, by (6.4), up to translations the Lagrangian immersion $L$ is
\[
L = \frac{e^{-i\varphi}}{\nu + i\mu} \phi(u_1, \ldots, u_4),
\] (6.11)
where $\phi$ is a horizontal minimal immersion in $S^9(1)$ and $\nu, \varphi, \mu$ satisfy
\[
\frac{d\nu}{dt} = -3\mu^2 - \nu^2, \quad \frac{d\varphi}{dt} = -4\mu, \quad \frac{d\mu}{dt} = 2\mu\nu.
\] (6.12)

From (6.12) we find
\[
\frac{d\nu}{d\mu} + \frac{\nu}{2\mu} = -\frac{3\mu}{2\nu}.
\] (6.13)
After solving (6.13) we get $\nu = \pm \sqrt{c^2\mu^{-1} - \mu^2}$ for some real number $c > 0$. Replacing $e_5$ by $-e_5$ if necessary, we have
\[
\nu = \sqrt{c^2\mu^{-1} - \mu^2}.
\] (6.14)
It follows from (6.12) and (6.14) that $\varphi'(\mu) = -\frac{4}{3}i \tan^{-1}\sqrt{\mu^3/(c^2 - \mu^3)} + c_0$ for some constant $c_0$. Therefore, we have the theorem after applying a suitable translation in $\mu$. \hfill \Box

Remark 6.2. Minimal $\delta(2, 2)$-ideal Lagrangian submanifolds in complex space forms $\mathbb{C}^5$, $\mathbb{C}P^5$ and $\mathbb{C}H^5$ are classified in [13]. Also $\delta(2)$-ideal minimal Lagrangian submanifolds in $\mathbb{C}P^4$ and $\mathbb{C}H^4$ have been classified recently in [14].
Let \( \gamma(t) \) be a unit speed curve in \( \mathbb{C}^* \). We put
\[
\gamma(t) = r(t)e^{i\theta(t)}, \quad \gamma'(t) = e^{i\zeta(t)}.
\]
(6.15)

The following result gives \( H \)-umbilical submanifolds of \( \mathbb{C}^5 \) with ratio 4.

**Proposition 6.3.** If \( M \) is an \( H \)-umbilical Lagrangian submanifold of \( \mathbb{C}^5 \) of ratio 4, then \( M \) is an open part of a complex extensor \( \gamma \otimes \iota \) of the unit hyper sphere \( \iota : S^4(1) \subset E^5 \) via a generating curve \( \gamma : I \to \mathbb{C}^* \) whose curvature satisfies \( \kappa = 4\theta' \).

**Proof.** If \( M \) is an \( H \)-umbilical Lagrangian submanifold of \( \mathbb{C}^5 \) with ratio 4, then the second fundamental form satisfies
\[
\begin{align*}
  h(e_j,e_j) &= \mu Je_5, \quad h(e_j,e_5) = \mu Je_j, \quad j \in \Delta, \\
  h(e_5,e_5) &= 4\mu Je_5, \quad h(e_j,e_k) = 0, \quad 1 \leq j \neq k \leq 4,
\end{align*}
\]
for a nonzero function \( \mu \). Thus Gauss’ equation yields \( K(e_1 \wedge e_5) = 3\mu^2 \). Hence \( M \) is non-flat. Therefore, according to Theorem F, \( M \) is an open part of a complex extensor of \( \iota : S^{n-1}(1) \subset E^n \) via a generating curve \( \gamma : I \to \mathbb{C}^* \).

It follows from [2] that the functions \( \varphi \) and \( \mu \) in (4.1) are related with the two angle functions \( \zeta \) and \( \theta \) by \( \varphi = \zeta'(t) = \kappa \) and \( \mu = \theta'(t) \). Thus whenever \( \gamma \) is a unit speed curve satisfying \( \kappa = 4\theta' \), the complex extensor \( \gamma \otimes \iota \) is an \( H \)-umbilical Lagrangian submanifold of ratio 4. Conversely, every \( H \)-umbilical Lagrangian submanifold of ratio 4 in \( \mathbb{C}^n \) can be obtained in such way. \( \square \)

### 7. Improved \( \delta(2,2) \)-ideal Lagrangian submanifolds of \( \mathbb{C}P^5 \)

**Theorem 7.1.** Let \( M \) be an improved \( \delta(2,2) \)-ideal Lagrangian submanifold in \( \mathbb{C}P^5(4) \). Then it is one of the following Lagrangian submanifolds:

1. a \( \delta(2,2) \)-ideal Lagrangian minimal submanifold;
2. an \( H \)-umbilical Lagrangian submanifold of ratio 4;
3. a Lagrangian submanifold defined by
\[
L(\mu,u_2,\ldots,u_4) = \frac{1}{c} \left( \sqrt{i\mu} e^{i\theta} \rho, e^{3i\theta} \left( \sqrt{c^2 - \mu^2 - \mu - i\mu^2} \right) \right),
\]
where \( c \) is a positive real number, \( \rho : N^4 \to S^9(1) \subset \mathbb{C}^5 \) is a horizontal lift of a non-totally geodesic \( \delta(2) \)-ideal Lagrangian minimal immersion in \( \mathbb{C}P^4(4) \), and \( \theta(\mu) \) satisfies
\[
\frac{d\theta}{d\mu} = \frac{1}{2\sqrt{c^2 \mu - \mu^2 - 1}}.
\]
(7.1)
Proof. Under the hypothesis there is an orthonormal frame \( \{ e_1, \ldots, e_5 \} \) such that (4.1) holds. If \( \mu = 0 \), then \( M \) is a \( \delta(2,2) \)-ideal Lagrangian minimal submanifold. Thus we obtain case (1). If \( \mu \neq 0 \) and \( a, b = 0 \), then \( M \) is an \( H \)-umbilical Lagrangian submanifold of ratio 4, which gives case (2).

Next, assume that \( a, \mu \neq 0 \). Then Lemma 4.5 implies \( b = 0 \). So, by Lemmas 5.1 we obtain (5.1) and (5.2). Also, in this case \( M \) is locally a warped product \( I \times \rho(t) N^4 \) with \( e_5 = \partial_t \) according to Lemma 5.3. From Lemma 4.1, we find

\[
\begin{align*}
    h(e_1, e_1) &= aJe_1 + \mu Je_5, \\
    h(e_2, e_2) &= -aJe_1 + \mu Je_5, \\
    h(e_3, e_3) &= h(e_4, e_4) = \mu Je_5, \\
    h(e_5, e_5) &= 4\mu Je_5, \\
    h(e_i, e_5) &= \mu Je_i, \quad i \in \Delta, \\
    h(e_i, e_j) &= 0, \quad \text{otherwise.}
\end{align*}
\]  

(7.3)

By Lemma 5.4 we have the following ODE system:

\[
\begin{align*}
    \frac{d\nu}{dt} &= -1 - \nu^2 - 3\mu^2, \\
    \frac{d\mu}{dt} &= 2\mu\nu.
\end{align*}
\]  

(7.4)

Let \( \theta(t) \) be a function on \( M \) satisfying

\[
\theta'(t) = \mu.
\]  

(7.5)

Let \( L \) denote the horizontal lift in \( S^{11}(1) \subset \mathbb{C}^6 \) of the Lagrangian immersion of \( M \) in \( CP^5(4) \) via Hopf’s fibration. Consider the maps:

\[
\xi = \frac{e^{-3i\theta}(e_5 - \nu + i\mu)L}{\sqrt{1 + \mu^2 + \nu^2}}, \quad \phi = \frac{e^{-i\theta}(L + (\nu - i\mu)e_5)}{\sqrt{1 + \mu^2 + \nu^2}}.
\]  

(7.6)

Then \( \langle \xi, \xi \rangle = \langle \phi, \phi \rangle = 1 \). From \( \nabla_{e_j} e_5 = \nu e_j, \quad j \in \Delta, \) and (7.4), we find \( \nabla_{e_j} \xi = 0 \). Moreover, it follows from Lemma 5.1 and (7.3) that \( \nabla_{e_5} e_5 = 4i\mu e_5 - L \). Thus we also have \( \nabla_{e_5} \xi = 0 \). Hence \( \xi \) is a constant unit vector in \( \mathbb{C}^6 \). Similarly, we also have \( \nabla_{e_5} \phi = 0 \). So \( \phi \) is independent of \( t \). Therefore, by combining (7.6) we find

\[
L = \frac{e^{i\theta}\phi - e^{3i\theta}(\nu - i\mu)\xi}{\sqrt{1 + \mu^2 + \nu^2}}.
\]  

(7.7)

Since \( \phi \) is orthogonal to \( \xi, i\xi \), after choosing \( \xi = (0, \ldots, 0, 1) \in \mathbb{C}^6 \) we obtain

\[
L = \frac{1}{\sqrt{1 + \mu^2 + \nu^2}} \left( e^{i\theta}\phi, e^{3i\theta}(\nu - i\mu) \right)
\]  

(7.8)
It follows from (7.4) and (7.5) that
\[
\frac{d\nu}{d\mu} = -\frac{1 + \nu^2 + 3\mu^2}{2\nu} \quad \text{and} \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu},
\] (7.9)
Solving the first differential equation in (7.9) gives
\[
\nu = \pm \sqrt{c^2\mu - 1 - \mu^2}, \quad c \in \mathbb{R}^+.
\] (7.10)
By replacing \(e_5\) by \(-e_5\) if necessary, we have \(\nu = \sqrt{c^2\mu - 1 - \mu^2 - 1}\). Consequently,
\[
L = \frac{1}{c} \left( \sqrt{\mu e^{i\theta}} e^{3i\theta} (\sqrt{c^2 - \mu^2} - \mu - i\mu^2) \right),
\] (7.11)
It follows from (5.1), (7.3) and the second formula in (7.6) that
\[
\hat{\nabla}_{e_j} \phi = \frac{ce^{-i\theta}}{\sqrt{\mu}} e_j, \quad j \in \Delta.
\] (7.12)
Thus after applying (6.11) and (7.12) we derive that
\[
\hat{\nabla}_{e_i} \hat{\nabla}_{e_\alpha} \phi = \sum_{\gamma=2}^{n} \left( \Gamma_{ij}^k + ih_{ij}^k \right) \phi_k(e_{\gamma}) - \langle \phi_k(e_i), \phi_k(e_j) \rangle \phi, \quad i, j \in \Delta.
\] (7.13)
Hence \(\phi\) is a horizontal immersion in \(S^9(1)\). Moreover, it follows from (7.13) that the second fundamental form of \(\phi\) is a scalar multiple of the original second fundamental form of \(M\) restricted to the second factor of the warped product \(I \times_\rho N\). Consequently, \(\phi\) is a minimal horizontal immersion in \(S^9(1)\) of a non-totally geodesic \(\delta(2)\)-ideal Lagrangian minimal submanifold of \(CP^4(4)\).

The converse is easy to verify. \(\square\)

8. IMPROVED \(\delta(2, 2)\)-IDEAL LAGRANGIAN SUBMANIFOLDS OF \(CH^5\)

**Theorem 8.1.** Let \(M\) be an improved \(\delta(2, 2)\)-ideal Lagrangian submanifold in \(CH^5(-4)\). Then \(M\) is one of the following Lagrangian submanifolds:

(i) a \(\delta(2, 2)\)-ideal Lagrangian minimal submanifold;
(ii) an \(H\)-umbilical Lagrangian submanifold of ratio 4;
(iii) a Lagrangian submanifold defined by
\[
L(\mu, u_1, \ldots, u_4) = \frac{1}{c} \left( \sqrt{\mu e^{i\theta}} \phi(u_2, \ldots, u_4), e^{-i\theta} (\sqrt{\mu - \mu^3 - c^2} - i\mu^2) \right),
\] (8.1)
where $c$ is a positive number, $\phi : N^4 \to H_1^1(-1) \subset C_1^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CH^4(-4)$, and $\theta(t)$ satisfies $\frac{d\theta}{dt} = \frac{1}{2}\sqrt{1 - \mu^2 - c^2\mu^{-1}}$;

(iv) a Lagrangian submanifold defined by
$$L(\mu, u_1, \ldots, u_4) = \frac{1}{c} \left( e^{-i\theta}(\sqrt{\mu - \mu^3 + c^2} - i\mu^2), \sqrt{\mu}e^{i\theta}\phi(u_2, \ldots, u_4) \right),$$

(8.2)

where $c$ is a positive number, $\phi : N^4 \to S^9(1) \subset C^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$-ideal minimal Lagrangian immersion in $CP^4(4)$, and $\theta(t)$ satisfies $\frac{d\theta}{dt} = \frac{1}{2}\sqrt{1 - \mu^2 + c^2\mu^{-1}}$;

(v) a Lagrangian submanifold defined by
$$L(t, u_1, \ldots, u_4) = \frac{1}{\cosh t - i \sinh t} \left( 2t + w + i \left( \cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right) \right),$$
$$\psi_1, \psi_2, 2t + w + i \left( \cosh 2t - \langle \psi, \psi \rangle + \frac{1}{4} \right) \right),$$

(8.3)

where $\psi(u_1, \ldots, u_4)$ is a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal immersion in $C^4$ and up to a constant $w(u_1, \ldots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$, $j = 1, 2, 3, 4$;

(vi) a Lagrangian submanifold defined by
$$L(t, u_1, \ldots, u_4) = \frac{1}{\cosh t - i \sinh t} \left( 2t + w + i \left( \cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right) \right),$$
$$\psi_1, \psi_2, 2t + w + i \left( \cosh 2t - \langle \psi, \psi \rangle + \frac{1}{4} \right) \right),$$

(8.4)

where $\psi = (\psi_1, \psi_2)$ is the direct product immersion of two non-totally geodesic Lagrangian minimal immersions $\psi_\alpha : N^2_\alpha \to C^2$, $\alpha = 1, 2$, and up to a constant $w(u_1, \ldots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$, $j = 1, 2, 3, 4$.

Proof. Under the hypothesis there exists an orthonormal frame $\{e_1, \ldots, e_5\}$ such that (4.11) holds.

Case (1) $\mu = 0$. In this case, we obtain case (i) of the theorem.

Case (2): $\mu \neq 0$ and $a, b = 0$. In this case $M$ is an $H$-umbilical Lagrangian submanifold with ratio 4, which gives case (ii).

Case (3): $\mu \neq 0$ and at least one of $a, b$ is nonzero. Without loss of generality, we may assume $a \neq 0$ and $\mu > 0$. We divide this into two cases.
**Case (3.a):** \(a, \mu \neq 0 \) and \(b = 0\). By Lemmas 5.1 we obtain (5.1) and (5.2). Also, \(M\) is locally a warped product \(I \times \mu(t) N^4\) with \(e_5 = \partial_t\), according to Lemma 5.3. From Lemma 4.1 we find

\[
h(e_1, e_1) = aJe_1 + \mu Je_5, \quad h(e_1, e_2) = -aJe_2, \\
\]

\[
h(e_2, e_2) = -aJe_1 + \mu Je_5, \\
\]

\[
h(e_3, e_3) = h(e_4, e_4) = \mu Je_5, \quad h(e_5, e_5) = 4\mu Je_5, \\
\]

\[
h(e_i, e_5) = \mu Je_i, \quad i \in \Delta, \quad h(e_i, e_j) = 0, \text{ otherwise.} \tag{8.5}
\]

Let \(L\) be a horizontal immersion of \(M\) in \(H^{11}_{11}(-1) \subset C^6_1\) of the Lagrangian immersion of \(M\) in \(CH^5(-4)\) via Hopf's fibration and \(\theta(t)\) a function satisfying

\[
d\theta dt = \mu. \tag{8.6}
\]

From Lemma 5.4 we obtain the following ODE system:

\[
\frac{d\nu}{dt} = 1 - 3\mu^2 - \nu^2, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{8.7}
\]

It follows from (8.6) and (8.7) that

\[
\frac{d\nu}{d\mu} = \frac{1 - 3\mu^2 - \nu^2}{2\mu\nu}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu}. \tag{8.8}
\]

Solving the first differential equation in (8.8) gives \(\nu = \pm \sqrt{1 - \mu^2 - k\mu^{-1}}\) for some real number \(k\). By replacing \(e_5\) by \(-e_5\) if necessary, we find

\[
\nu = \sqrt{1 - \mu^2 - k\mu^{-1}}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\sqrt{1 - \mu^2 - k\mu^{-1}}}. \tag{8.9}
\]

It follows from (8.7) that \(\frac{d}{dt}(1 - \mu^2 - \nu^2) = -2\nu(1 - \mu^2 - \nu^2)\). Since this equation for \(y(t) = 1 - \mu^2 - \nu^2 = k\mu^{-1}\) has a unique solution for each given initial condition, each solution either vanishes identically or is nowhere zero.

**Case (3.a.1):** \(\mu^2 + \nu^2 < 1\). In this case, (8.9) implies \(k > 0\). Thus we may put \(k = c^2\), \(c > 0\). Consider the maps:

\[
\eta = \frac{e^{-3i\theta}(e_5 - (\nu + i\mu)L)}{\sqrt{1 - \mu^2 - \nu^2}}, \quad \phi = \frac{e^{-i\theta}((\nu - i\mu)e_5 - L)}{\sqrt{1 - \mu^2 - \nu^2}}. \tag{8.10}
\]

Then \(\langle \eta, \eta \rangle = 1\) and \(\langle \phi, \phi \rangle = -1\). From \(\nabla_{e_j} e_5 = \nu e_j, j \in \Delta\), and (8.5), we obtain \(\nabla e_5 \xi = 0\), where \(\nabla\) is the Levi-Civita connection of \(C^6_1\). Lemma 5.1 and (8.5) give \(\nabla e_5 e_5 = 4i\mu e_5 + L\). Thus we find \(\nabla e_5 \xi = 0\). So \(\eta\) is a constant.
unit vector. Also, we find $\tilde{\nabla}_e_5 \phi = 0$. Hence $\phi$ is independent of $t$. From (8.10) we get

$$L = -\frac{e^{i\theta} \phi + e^{-i\theta} (\nu - i\mu) \eta}{\sqrt{1 - \mu^2 - \nu^2}}. \quad (8.11)$$

Since $\phi$ is orthogonal to $\eta, i\eta$ and $\eta$ is a constant unit space-like vector, we conclude from (8.9) and (8.11) that $L$ is congruent to (8.1). Next, by applying the same method of the proof of Theorem 7.1, we conclude that $\phi$ is a horizontal immersion in $H_1^1(-1)$ whose second fundamental form is a scalar multiple of the original second fundamental form restricted to the second factor of $I \times \rho N$. Consequently, $\phi$ is a minimal horizontal immersion in $H_1^1(-1)$ of a non-totally geodesic $\delta(2)$-ideal Lagrangian minimal submanifold of $CH^4(-4)$. This gives case (iii).

**Case (3.a.2):** $\mu^2 + \nu^2 > 1$. In this case (8.8) implies $k < 0$. Thus we may put $k = -c^2, c > 0$. Now, we consider the maps:

$$\eta = e^{-i3\theta} (e_5 - (\nu + i\mu)L), \quad \phi = e^{-i\theta} ((\nu - i\mu)e_5 - L) \sqrt{\mu^2 + \nu^2 - 1} \quad (8.12)$$

instead. Then $\langle \phi, \phi \rangle = -\langle \eta, \eta \rangle = 1$. By applying similar arguments as case (3.a.1), we know that $\eta$ is a constant time-like vector and $\phi$ is independent of $t$ and orthogonal to $\eta, i\eta$. Moreover, we may prove that $\phi$ is a minimal Legendre immersion in $S^0(1)$. Therefore we have case (iv) after choosing $\eta = (1,0,\ldots,0)$.

**Case (3.a.3):** $\mu^2 + \nu^2 = 1$. In this case system (8.7) gives $\frac{d\mu}{dt} = 2(\nu^2 - 1)$ and $\mu = \pm \sqrt{1 - \nu^2}$. Solving these and applying a suitable translations in $t$, we find

$$\mu = \text{sech} \, 2t, \quad \nu = -\tanh \, 2t. \quad (8.13)$$

It follows from $\nabla_{e_5} e_5 = 0$, (8.5) and (8.13) that the horizontal lift $L$ of the Lagrangian immersion of $M$ in $CH^5(-4) \subset C_1^6$ satisfies

$$L_{tt} - 4i (\text{sech} \, 2t) L_t - L = 0. \quad (8.14)$$

Solving this second order differential equation gives

$$L = \phi(u_1, \ldots, u_4) + B(u_1, \ldots, u_4)(2t + i \cosh \, 2t) \cosh t - i \sinh t, \quad (8.15)$$

where $\phi(u_1, \ldots, u_4)$ and $B(u_1, \ldots, u_4)$ are $C_1^6$-valued functions.
On the other hand, it follows from Lemma 5.1, (8.5) and (8.13) that

$$L_{tu_j} = (i \, \text{sech}^2 t - \tanh^2 t)L_{u_j}, \quad j \in \Delta.$$  \hfill (8.16)

Substituting (8.15) into (8.16) shows that $B$ is a constant vector $\zeta$. Thus

$$L(t, u_1, \ldots, u_4) = \frac{\phi(u_1, \ldots, u_4)}{\cosh t - i \sinh t} + \frac{(2t + i \cosh 2t)}{\cosh t - i \sinh t} \zeta,$$  \hfill (8.17)

since $\langle L, L \rangle = -1$, (8.17) implies

$$- \cosh 2t = \langle \phi, \phi \rangle + \langle \phi, (4t + 2i \cosh 2t) \zeta \rangle + (4t^2 + \cosh^2(2t)) \langle \zeta, \zeta \rangle.$$  \hfill (8.18)

Since $\phi_t = 0$, by differentiating (8.18) with respect $t$ we find

$$- \sinh 2t = 2t \langle \phi, \zeta \rangle + 2 \sinh 2t \langle \phi, i\zeta \rangle + (4t + \sinh 4t) \langle \zeta, \zeta \rangle.$$  \hfill (8.19)

We find from (8.19) at $t = 0$ that $\langle \phi, \zeta \rangle = 0$. Thus (8.19) gives

$$0 = \sinh 2t(1 + \langle \phi, i\zeta \rangle) + (4t + \sinh 4t) \langle \zeta, \zeta \rangle.$$  \hfill (8.20)

Differentiating (8.20) gives $\langle \phi, i\zeta \rangle = -\frac{1}{2} - 2 \langle \zeta, \zeta \rangle$. Thus (8.17) yields $\langle \phi, i\zeta \rangle = -\frac{1}{2}$ and $\langle \zeta, \zeta \rangle = 0$. Now, we find from (8.18) that $\langle \phi, \phi \rangle = 0$. Consequently we have

$$\langle \phi, \phi \rangle = \langle \zeta, \zeta \rangle = \langle \phi, \zeta \rangle = 0, \quad \langle \phi, i\zeta \rangle = -\frac{1}{2}.$$  \hfill (8.21)

Since $\zeta$ is a constant light-like vector, we may put

$$\zeta = (1, 0, \ldots, 0, 1), \quad \phi = (a_1 + ib_1, \ldots, a_6 + ib_6).$$  \hfill (8.22)

It follows from (8.21) and (8.22) that $a_6 = a_1$ and $b_6 = b_1 + \frac{1}{2}$. Therefore

$$\phi = (a_1 + ib_1, a_2 + ib_2, \ldots, a_1 + i(b_1 + \frac{1}{2})).$$  \hfill (8.23)

Now, by using $\langle \phi, \phi \rangle = 0$ and (8.23), we find $\psi = (a_2 + ib_2, \ldots, a_5 + ib_5)$ and $b_1 = -\frac{1}{4} - \langle \psi, \psi \rangle$. Combining these with (8.23) yields

$$\phi = \left( w - i \langle \psi, \psi \rangle - \frac{i}{4}, \psi, w - i \langle \psi, \psi \rangle + \frac{i}{4} \right)$$  \hfill (8.24)

with $w = a_1$. It follows from (8.22) and (8.24) that $\langle \phi_{u_j}, \zeta \rangle = \langle \phi_{u_j}, i\zeta \rangle = 0$. Thus, by applying $\langle L_{u_j}, iL \rangle = 0, \quad j \in \Delta$, we find from (8.17) that $\langle \phi_{u_j}, i\phi \rangle = 0$.

On the other hand, (8.24) implies that

$$\langle \phi_{u_j}, i\phi \rangle = -\frac{1}{2} w_{u_j} + \langle \psi_{u_j}, i\psi \rangle$$  \hfill (8.25)

with $w_{u_j} = \frac{\partial w}{\partial u_j}$. Therefore $w$ satisfies the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$.  \hfill
Now, we derive from (8.17), (8.22) and (8.23) that
\[
L = \frac{1}{\cosh t - i \sinh t} \left( 2t + w + i \left( \cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right) \right),
\]
(8.26)

It follows from (8.26) that
\[
L_{u_j} = \frac{1}{\cosh t - i \sinh t} \left( w_{u_j} - i \langle \psi, \psi \rangle_{u_j}, \psi_{u_j}, w_{u_j} - i \langle \psi, \psi \rangle_{u_j} \right).
\]
(8.27)

Thus we find \( \langle \psi_{u_j}, \psi_{u_k} \rangle = \cosh 2t \langle L_{u_j}, L_{u_k} \rangle \) which implies that \( \psi \) is an immersion in \( \mathbb{C}^4 \). Also, we find from (8.27) and \( \langle L_{u_j}, iL_{u_k} \rangle = 0 \) that \( \langle \psi_{u_j}, i\psi_{u_k} \rangle = 0 \). Thus \( \psi \) is a Lagrangian immersion. Now, by applying an argument similar to the last part of the proof of [11, Theorem 6.1], we conclude that
\[
\psi_{u_j u_k} = \sum_{i=1}^4 (\Gamma^i_{jk} + i h^i_{jk}) \phi_{u_i}, \quad j, k \in \Delta.
\]

Therefore, according to (8.5), \( \psi \) is a \( \delta(2) \)-ideal minimal Lagrangian immersion in \( \mathbb{C}^4 \). Consequently, we obtain case (v) of the theorem.

**Case (3.b):** \( a, b, \mu \neq 0 \). We obtain case (vi) of the theorem by applying the same argument as case (3.a.3).

**Acknowledgement.** The authors thank the referee and Dr. Luc Vrancken for pointing out an error in the original version of this paper.

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