SIAM J. Math. Anal.
Vol. 37, No. 2, pp. 343–346

LACK OF COMPACTNESS IN TWO-SCALE CONVERGENCE*

MARC BRIANE† AND JUAN CASADO-DÍAZ‡

Abstract. This article deals with the links between compensated compactness and two-scale convergence. More precisely, we ask the following question: Is the div-curl compactness assumption sufficient to pass to the limit in a product of two sequences which two-scale converge with respect to the pair of variables $(x, x/\varepsilon)$? We reply in the negative. Indeed, the div-curl assumption allows us to control oscillations which are faster than $1/\varepsilon$ but not the slower ones.

Key words. two-scale convergence, compensated compactness, counterexample

AMS subject classifications. 35B27, 35B40

DOI. 10.1137/040621351

1. Introduction.

In order to study the asymptotic behavior of periodic problems arising in homogenization theory, Nguetseng introduced in [7] (see also Allaire [1]) the notion of two-scale convergence:

Let Ω be a bounded open subset of $\mathbb{R}^d$, $Y := (-\frac{1}{2}, \frac{1}{2})^d$, and let M be a positive integer. A bounded sequence u_ε in $L^1_{\text{loc}}(\Omega)^M$ two-scale converges to a function $\hat{u}$ in $L^1_{\text{loc}}(\Omega \times \mathbb{R}^d)^M$ and Y-periodic with respect to the last variable if, for any $\psi \in C_c^\infty(\Omega, C_\#^\infty(Y))^M$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y \hat{u}(x,y)\, \psi(x,y)\, dx\, dy. \tag{1.1}$$

A compactness theorem due to Nguetseng [7] establishes that if u_ε is bounded in $L^p(\Omega)^M$, then there exists a subsequence of u_ε which two-scale converges to $\hat{u} \in L^p(\Omega; L^p_\#(Y))^M$.

Taking in (1.1) $\psi(x,y)$ independent of y, we deduce that if u_ε two-scale converges to $\hat{u}$, then it converges weakly in $L^p(\Omega)^M$ to $u := \int_Y \hat{u}(x,y)\, dy$. On the other hand, if u_ε strongly converges to u in $L^1(\Omega)^M$, then it also two-scale converges to u. Therefore two-scale convergence is stronger than weak convergence and weaker than the strong one. Moreover, it provides an expression of the limit of the product $u_\varepsilon\, \psi(x, \frac{x}{\varepsilon})$ of (1.1) in which each term only weakly converges.

In the periodic homogenization we usually deal with a sequence u_ε which is not only bounded in $L^p(\Omega)^M$ but whose some combinations of its derivatives are also bounded. In this context, let us recall that if u_ε converges weakly in $W^{1,p}(\Omega)^M$, for $1 \le p < +\infty$, to a function u, then it converges strongly in $L^p_{\text{loc}}(\Omega)^M$ ($L^p(\Omega)^M$ if Ω smooth) and so u_ε two-scale converges to u. Then we can conjecture that the classical results of the compensated compactness theory due to Murat and Tartar (see, e.g., [6] and [8]), and in particular the div-curl theorem, still hold true when we replace the weak convergence in $L^p(\Omega)^M$ with two-scale convergence. In fact we have the following result:

*Received by the editors December 13, 2004; accepted for publication (in revised form) December 28, 2004; published electronically October 14, 2005.
http://www.siam.org/journals/sima/37-2/62135.html

†Centre de Mathématiques, I.N.S.A. de Rennes & I.R.M.A.R., 35043 Rennes cedex, France (mbriane@insa-rennes.fr).
‡Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, 41012 Sevilla, Spain (jcasadod@us.es).

PROPOSITION 1.1. *Let $(Y, Y_1, \ldots, Y_n)$ be $(n+1)$ parallelotops of $\mathbb{R}^d$ of Lebesgue measure equal to 1, and let U, V be two vector-valued functions in $L^2(\Omega; C_\#(Y \times Y_1 \times \cdots \times Y_n))^d$, where $C_\#(Y \times Y_1 \times \cdots \times Y_n)$ denotes the set of the continuous functions on $(\mathbb{R}^d)^{n+1}$ which are Y-periodic with respect to the variable y and Y_k-periodic with respect to the variable y_k for any $k = 1, \ldots, n$. Let $\varepsilon_k = \varepsilon_k(\varepsilon)$ for $k = 1, \ldots, n$ be n well-ordered scales such that*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \quad \textit{for any } k = 1, \ldots, n-1. \tag{1.2}$$

Consider the vector-valued sequences u_ε and v_ε defined by

$$u_\varepsilon(x) := U\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right) \quad \textit{and} \quad v_\varepsilon(x) := V\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right), \tag{1.3}$$

and assume that

$$\operatorname{div} u_\varepsilon \ \textit{is compact in } H^{-1}(\Omega) \quad \textit{and} \quad \operatorname{curl} v_\varepsilon \ \textit{is compact in } H^{-1}(\Omega)^{d\times d}. \tag{1.4}$$

Then the two-scale limits $\hat{u}$ of u_ε, $\hat{v}$ of v_ε, and $\hat{w}$ of $u_\varepsilon \cdot ? v_\varepsilon$ exist and satisfy

$$\hat{w} = \hat{u} \cdot \hat{v}. \tag{1.5}$$

Proposition 1.1 shows that the div-curl condition (1.4) implies some compactness in the two-scale convergence process (as in the classical case) when the oscillations of the sequences are faster than $\frac{1}{\varepsilon}$. Unfortunately, this is not the case for general sequences, particularly when the oscillations are slower than $\frac{1}{\varepsilon}$. This assertion follows from the following theorem, which is the main result of the present paper:

THEOREM 1.2. *Assume that $d \geq 2$. Then there exist two functions $U, V \in C^\infty_\#(2Y)^d$ such that the sequence $u_\varepsilon(x) := U(\frac{x}{\varepsilon})$ is divergence-free, the sequence $v_\varepsilon(x) := V(\frac{x}{\varepsilon})$ is curl-free, but the two-scale limits of u_ε, v_ε, and $u_\varepsilon \cdot v_\varepsilon$ do not satisfy* (1.5).

The key ingredient of this counterexample is that 2-periodic functions are considered although the test functions are 1-periodic.

In order to understand the lack of compactness in two-scale convergence, let us recall the equivalence between the two-scale convergence theory and the method introduced by Arbogast, Douglas, and Hornung [3] to study the oscillations of a sequence u_ε in $L^1_{\mathrm{loc}}(\mathbb{R}^d)^M$. Their method consists in introducing the function $\hat{u}_\varepsilon : \mathbb{R}^d \times Y \to \mathbb{R}^M$ defined by

$$\hat{u}_\varepsilon(x, y) = \sum_{k \in \mathbb{Z}^d} 1_{\varepsilon k + \varepsilon Y}(x)\, u_\varepsilon(\varepsilon k + \varepsilon y). \tag{1.6}$$

The equivalence between the two approaches is then given by the following result (see, e.g., [5] and [4]):

THEOREM 1.3. *Assume that u_ε is bounded in $L^p(\Omega)^M$, with $1 < p < +\infty$. Then $\hat{u}_\varepsilon$ converges weakly to $\hat{u}$ in $L^p(\Omega; L^p(Y))^M$ if and only if u_ε two-scale converges to $\hat{u}$.*

The functions $\hat{u}_\varepsilon(x, y)$ are not continuous with respect to the variable x. If a combination of derivatives of u_ε is bounded, we also get a bound for the same combination of derivatives with respect to the variable y of $\hat{u}_\varepsilon$ but not with respect to the variable x. This explains the lack of compactness in two-scale convergence.

2. Proof of the results. In this section we prove Proposition 1.1 and Theorem 1.2.

Proof of Proposition 1.1. We follow the multiscale procedure of [2]. Thanks to the separation of scales (1.2) the sequences u_ε, v_ε, and $u_\varepsilon \cdot v_\varepsilon$, respectively, two-scale converge to $\hat{u} := \int_{Y_1} \cdots \int_{Y_n} U$, $\hat{v} := \int_{Y_1} \cdots \int_{Y_n} V$, and $\hat{w} := \int_{Y_1} \cdots \int_{Y_n} U \cdot V$. Putting test functions of type $\varepsilon_k \, \Phi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_k})$ from $k = n$ to 1 in the div-curl assumption (1.4) implies that

$$\operatorname{div}_{y_k} \left(\int_{Y_{k+1}} \cdots \int_{Y_n} U \right) = 0 \text{ and } \operatorname{curl}_{y_k} \left(\int_{Y_{k+1}} \cdots \int_{Y_n} V \right) = 0 \quad \text{for } k = 1, \ldots, n,$$

whence, integrating by parts the product of $\int_{Y_{k+1}} \cdots \int_{Y_n} U$ and $\int_{Y_{k+1}} \cdots \int_{Y_n} V$ (which is equal to the gradient in y_k of a periodic function plus a function depending only on the other variables $y_1, \ldots, y_{k-1}$) successively from $k = n$ to 1, yields

$$\hat{w} = \int_{Y_1} \cdots \int_{Y_n} U \cdot V = \left(\int_{Y_1} \cdots \int_{Y_n} U \right) \cdot \left(\int_{Y_1} \cdots \int_{Y_n} V \right) = \hat{u} \cdot \hat{v},$$

which implies the desired equality (1.5).

Proof of Theorem 1.2. Let us consider two vector-valued functions $\Phi, \Psi \in C_c^\infty(Y)^d$ such that $\operatorname{div} \Phi = 0$, $\operatorname{curl} \Psi = 0$, and $\Phi \cdot \Psi \neq 0$ (this is possible since $d > 1$), which we extend to $\mathbb{R}^d$ by Y-periodicity. Let $\eta : \mathbb{R} \to \mathbb{R}$ be the 1-periodic function $\eta := \sum_{i \in \mathbb{Z}} 1_{(i - \frac{1}{4}, i + \frac{1}{4})}$ and let us define the following sequences

$$u_\varepsilon(x) := \eta \left(\frac{x_1}{2\,\varepsilon} \right) \Phi \left(\frac{x}{\varepsilon} \right) \quad \text{and} \quad v_\varepsilon(x) := \eta \left(\frac{x_1}{2\,\varepsilon} \right) \Psi \left(\frac{x}{\varepsilon} \right).$$

Since in each cube $\varepsilon k + \varepsilon Y$, for $k \in \mathbb{Z}^d$, $\eta(\frac{x_1}{2\,\varepsilon})$ is constant, and $\Phi(\frac{x}{\varepsilon})$, $\Psi(\frac{x}{\varepsilon})$ vanish on the boundary of $\varepsilon k + \varepsilon Y$, we have $u_\varepsilon, v_\varepsilon \in C^\infty(\mathbb{R}^N)$, $\operatorname{div} u_\varepsilon = 0$, and $\operatorname{curl} v_\varepsilon = 0$ in $\mathbb{R}^d$. Moreover, since $\eta(\frac{x_1}{2\,\varepsilon})$ is constant in $\varepsilon k + \varepsilon Y$ for any $k \in \mathbb{Z}^d$, it is invariant by the transformation (1.6). So we get

$$\hat{u}_\varepsilon(x,y) = \eta \left(\frac{x_1}{2\,\varepsilon} \right) \Phi(y), \quad \hat{v}_\varepsilon(x,y) = \eta \left(\frac{x_1}{2\,\varepsilon} \right) \Psi(y), \quad \widehat{u_\varepsilon \cdot v_\varepsilon}(x,y) = \eta^2 \left(\frac{x_1}{2\,\varepsilon} \right) \Phi(y) \cdot \Psi(y).$$

By Theorem 1.3 the two-scale limits $\hat{u}$ of u_ε, $\hat{v}$ of v_ε, and $\hat{w}$ of $u_\varepsilon \cdot v_\varepsilon$ are thus given by

$$\hat{u}(x,y) = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta(s)\, ds \right) \Phi(y) = \frac{1}{2}\, \Phi(y), \quad \hat{v}(x,y) = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta(s)\, ds \right) \Psi(y) = \frac{1}{2}\, \Psi(y),$$
$$\text{and} \quad \hat{w}(x,y) = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta^2(s)\, ds \right) \Phi(y) \cdot \Psi(y) = \frac{1}{2}\, \Phi(y) \cdot \Psi(y),$$

whence $\hat{w} \neq \hat{u} \cdot \hat{v}$.

REFERENCES

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., 23 (1992), pp. 1482–1518.
[2] G. Allaire and M. Briane, *Multiscale convergence and reiterated homogenization*, Proc. Roy. Soc. Edinburgh A, 126 (1996), pp. 297–342.

[3] T. Arbogast, J. Douglas, and U. Hornung, *Derivation of the double porosity model of single phase flow via homogenization theory*, SIAM J. Math. Anal., 21 (1990), pp. 823–836.

[4] D. Cioranescu, A. Damlamian, and G. Griso, *Periodic unfolding and homogenization*, C. R. Acad. Sci. Paris, 335 (2002), pp. 99–104.

[5] M. Lenczner, *Homogénéisation d'un circuit électrique*, C. R. Acad. Sci. Paris, 324 (1997), pp. 537–542.

[6] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 5 (1978), pp. 489–507.

[7] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989), pp. 608–623.

[8] L. Tartar, *Compensated compactness and applications to partial differential equations, nonlinear analysis and mechanics*, in Heriot-Watt Symposium IV, Pitman, San Francisco, 1979, pp. 136–212.