LACK OF COMPACTNESS IN TWO-SCALE CONVERGENCE

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Abstract. This article deals with the links between compensated compactness and two-scale convergence. More precisely, we ask the following question: Is the div-curl compactness assumption sufficient to pass to the limit in a product of two sequences which two-scale converge with respect to the pair of variables \( (x, x/\epsilon) \)? We reply in the negative. Indeed, the div-curl assumption allows us to control oscillations which are faster than \(1/\epsilon\) but not the slower ones.

Key words. two-scale convergence, compensated compactness, counterexample

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1. Introduction. In order to study the asymptotic behavior of periodic problems arising in homogenization theory, Nguetseng introduced in [7] (see also Allaire [1]) the notion of two-scale convergence:

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^d \), \( Y := (-\frac{1}{2}, \frac{1}{2})^d \), and let \( M \) be a positive integer. A bounded sequence \( u_\varepsilon \in L^1_{\text{loc}}(\Omega)^M \) two-scale converges to a function \( \hat{u} \) in \( L^1_{\text{loc}}(\Omega \times \mathbb{R}^d)^M \) and \( Y \)-periodic with respect to the last variable if, for any \( \psi \in C_\infty(\Omega, C_\infty^#(Y))^M \), we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) \psi(x, \frac{x}{\varepsilon}) \, dx = \int_{\Omega} \int_{Y} \hat{u}(x, y) \psi(x, y) \, dx \, dy.
\]

A compactness theorem due to Nguetseng [7] establishes that if \( u_\varepsilon \) is bounded in \( L^p(\Omega)^M \), then there exists a subsequence of \( u_\varepsilon \) which two-scale converges to \( \hat{u} \in L^p(\Omega; L^p_{\text{loc}}(Y))^M \).

Taking in (1.1) \( \psi(x, y) \) independent of \( y \), we deduce that if \( u_\varepsilon \) two-scale converges to \( \hat{u} \), then it converges weakly in \( L^p(\Omega)^M \) to \( u := \int_Y \hat{u}(x, y) \, dy \). On the other hand, if \( u_\varepsilon \) strongly converges to \( u \) in \( L^1(\Omega)^M \), then it also two-scale converges to \( u \). Therefore two-scale convergence is stronger than weak convergence and weaker than the strong one. Moreover, it provides an expression of the limit of the product \( u_\varepsilon (x, \frac{x}{\varepsilon}) \) of (1.1) in which each term only weakly converges.

In the periodic homogenization we usually deal with a sequence \( u_\varepsilon \) which is not only bounded in \( L^p(\Omega)^M \) but whose some combinations of its derivatives are also bounded. In this context, let us recall that if \( u_\varepsilon \) converges weakly in \( W^{1,p}(\Omega)^M \), for \( 1 \leq p < +\infty \), to a function \( u \), then it converges strongly in \( L^p_{\text{loc}}(\Omega)^M \) (\( L^p(\Omega)^M \) if \( \Omega \) smooth) and so \( u_\varepsilon \) two-scale converges to \( u \). Then we can conjecture that the classical results of the compensated compactness theory due to Murat and Tartar (see, e.g., [6] and [8]), and in particular the div-curl theorem, still hold true when we replace the weak convergence in \( L^p(\Omega)^M \) with two-scale convergence. In fact we have the following result:

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**Proposition 1.1.** Let \((Y, Y_1, \ldots, Y_n)\) be \((n + 1)\) paralleloptes of \(\mathbb{R}^d\) of Lebesgue measure equal to 1, and let \(U, V\) be two vector-valued functions in \(L^2(\Omega; C_\#(Y \times Y_1 \times \cdots \times Y_n))^d\), where \(C_\#(Y \times Y_1 \times \cdots \times Y_n)\) denotes the set of the continuous functions on \((\mathbb{R}^d)^{n+1}\) which are \(Y\)-periodic with respect to the variable \(y\) and \(Y_k\)-periodic with respect to the variable \(y_k\) for any \(k = 1, \ldots, n\). Let \(\varepsilon_k = \varepsilon_k(\varepsilon)\) for \(k = 1, \ldots, n\) be \(n\) well-ordered scales such that

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon_1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \quad \text{for any } k = 1, \ldots, n-1.
\]

Consider the vector-valued sequences \(u_\varepsilon\) and \(v_\varepsilon\) defined by

\[
u_\varepsilon(x) := U \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n} \right) \quad \text{and} \quad v_\varepsilon(x) := V \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n} \right),
\]

and assume that

\[
div u_\varepsilon \text{ is compact in } H^{-1}(\Omega) \quad \text{and} \quad \text{curl} v_\varepsilon \text{ is compact in } H^{-1}(\Omega)^{d\times d}.
\]

Then the two-scale limits \(\hat{u}\) of \(u_\varepsilon\), \(\hat{v}\) of \(v_\varepsilon\), and \(\hat{w}\) of \(w_\varepsilon\) exist and satisfy

\[
\hat{w} = \hat{u} \cdot \hat{v}.
\]

Proposition 1.1 shows that the div-curl condition (1.4) implies some compactness in the two-scale convergence process (as in the classical case) when the oscillations of the sequences are faster than \(\frac{1}{\varepsilon}\). Unfortunately, this is not the case for general sequences, particularly when the oscillations are slower than \(\frac{1}{\varepsilon}\). This assertion follows from the following theorem, which is the main result of the present paper:

**Theorem 1.2.** Assume that \(d \geq 2\). Then there exist two functions \(U, V \in C_\#^\infty(2Y)^d\) such that the sequence \(u_\varepsilon(x) := U(\varepsilon x)\) is divergence-free, the sequence \(v_\varepsilon(x) := V(\varepsilon x)\) is curl-free, but the two-scale limits of \(u_\varepsilon\), \(v_\varepsilon\), and \(u_\varepsilon \cdot v_\varepsilon\) do not satisfy (1.5).

The key ingredient of this counterexample is that 2-periodic functions are considered although the test functions are 1-periodic.

In order to understand the lack of compactness in two-scale convergence, let us recall the equivalence between the two-scale convergence theory and the method introduced by Arbogast, Douglas, and Hornung [3] to study the oscillations of a sequence \(u_\varepsilon\) in \(L^1_{loc}(\mathbb{R}^d)^M\). Their method consists in introducing the function \(\bar{u}_\varepsilon : \mathbb{R}^d \times Y \to \mathbb{R}^M\) defined by

\[
\bar{u}_\varepsilon(x, y) = \sum_{k \in \mathbb{Z}^d} 1_{\varepsilon k + \varepsilon Y}(x) u_\varepsilon(\varepsilon k + \varepsilon y).
\]

The equivalence between the two approaches is then given by the following result (see, e.g., [5] and [4]):

**Theorem 1.3.** Assume that \(u_\varepsilon\) is bounded in \(L^p(\Omega)^M\), with \(1 < p < +\infty\). Then \(\bar{u}_\varepsilon\) converges weakly to \(\bar{u}\) in \(L^p(\Omega; L^p(Y))^M\) if and only if \(u_\varepsilon\) two-scale converges to \(\bar{u}\).

The functions \(\bar{u}_\varepsilon(x, y)\) are not continuous with respect to the variable \(x\). If a combination of derivatives of \(u_\varepsilon\) is bounded, we also get a bound for the same combination of derivatives with respect to the variable \(y\) of \(\bar{u}_\varepsilon\), but not with respect to the variable \(x\). This explains the lack of compactness in two-scale convergence.
2. Proof of the results. In this section we prove Proposition 1.1 and Theorem 1.2.

Proof of Proposition 1.1. We follow the multiscale procedure of [2]. Thanks to the separation of scales (1.2) the sequences \( u_\varepsilon, v_\varepsilon \) and \( u_\varepsilon \cdot v_\varepsilon \), respectively, two-scale converge to \( \hat{u} := \int_{Y_1} \cdots \int_{Y_n} U \), \( \hat{v} := \int_{Y_1} \cdots \int_{Y_n} V \), and \( \hat{w} := \int_{Y_1} \cdots \int_{Y_n} U \cdot V \). Putting test functions of type \( \varepsilon_k \Phi(x, \frac{x}{\varepsilon}, \frac{s}{\varepsilon^2}, \ldots, \frac{s}{\varepsilon^2}) \) from \( k = n \) to 1 in the div-curl assumption (1.4) implies that

\[
\text{div}_{y_k} \left( \int_{Y_{k+1}} \cdots \int_{Y_n} U \right) = 0 \quad \text{and} \quad \text{curl}_{y_k} \left( \int_{Y_{k+1}} \cdots \int_{Y_n} V \right) = 0 \quad \text{for} \quad k = 1, \ldots, n,
\]

whence, integrating by parts the product of \( \int_{Y_{k+1}} \cdots \int_{Y_n} U \) and \( \int_{Y_{k+1}} \cdots \int_{Y_n} V \) (which is equal to the gradient in \( y_k \) of a periodic function plus a function depending only on the other variables \( y_1, \ldots, y_{k-1} \)) successively from \( k = n \) to 1, yields

\[
\hat{w} = \int_{Y_1} \cdots \int_{Y_n} U \cdot V = \left( \int_{Y_1} \cdots \int_{Y_n} U \right) \cdot \left( \int_{Y_1} \cdots \int_{Y_n} V \right) = \hat{u} \cdot \hat{v},
\]

which implies the desired equality (1.5).

Proof of Theorem 1.2. Let us consider two vector-valued functions \( \Phi, \Psi \in C^\infty_c(Y)^d \) such that \( \text{div} \Phi = \text{curl} \Psi = 0 \), and \( \Phi \cdot \Psi \neq 0 \) (this is possible since \( d > 1 \)), which we extend to \( \mathbb{R}^d \) by \( Y \)-periodicity. Let \( \eta : \mathbb{R} \to \mathbb{R} \) be the 1-periodic function \( \eta := \sum_{i \in \mathbb{Z}} 1_{i - \frac{1}{4}, i + \frac{1}{4}} \) and let us define the following sequences

\[
u_\varepsilon(x) := \eta \left( \frac{x_1}{2\varepsilon} \right) \Phi \left( \frac{x}{\varepsilon} \right) \quad \text{and} \quad v_\varepsilon(x) := \eta \left( \frac{x_1}{\varepsilon} \right) \Psi \left( \frac{x}{\varepsilon} \right).
\]

Since in each cube \( \varepsilon k + \varepsilon Y \), for \( k \in \mathbb{Z}^d \), \( \eta \left( \frac{x}{2\varepsilon} \right) \) is constant, and \( \Phi \left( \frac{x}{\varepsilon} \right), \Psi \left( \frac{x}{\varepsilon} \right) \) vanish on the boundary of \( \varepsilon k + \varepsilon Y \), we have \( u_\varepsilon, v_\varepsilon \in C^\infty_c(\mathbb{R}^N) \), \( \text{div} u_\varepsilon = 0 \), and \( \text{curl} v_\varepsilon = 0 \) in \( \mathbb{R}^d \). Moreover, since \( \eta \left( \frac{x}{\varepsilon} \right) \) is constant in \( \varepsilon k + \varepsilon Y \) for any \( k \in \mathbb{Z}^d \), it is invariant by the transformation (1.6). We get

\[
\hat{u}_\varepsilon(x, y) = \eta \left( \frac{x_1}{2\varepsilon} \right) \Phi(y), \quad \hat{v}_\varepsilon(x, y) = \eta \left( \frac{x_1}{\varepsilon} \right) \Psi(y), \quad u_\varepsilon \cdot v_\varepsilon(x, y) = \eta^2 \left( \frac{x_1}{2\varepsilon} \right) \Phi(y) \cdot \Psi(y).
\]

By Theorem 1.3 the two-scale limits \( \hat{u} \) of \( u_\varepsilon \), \( \hat{v} \) of \( v_\varepsilon \), and \( \hat{w} \) of \( u_\varepsilon \cdot v_\varepsilon \) are thus given by

\[
\hat{u}(x, y) = \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta(s) \, ds \right) \Phi(y) = \frac{1}{2} \Phi(y), \quad \hat{v}(x, y) = \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta(s) \, ds \right) \Psi(y) = \frac{1}{2} \Psi(y),
\]

and

\[
\hat{w}(x, y) = \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta^2(s) \, ds \right) \Phi(y) \cdot \Psi(y) = \frac{1}{2} \Phi(y) \cdot \Psi(y),
\]

whence \( \hat{w} \neq \hat{u} \cdot \hat{v} \).

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