Purity of exponential sums on $\mathbb{A}^n$, II

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Abstract

We give a purity result for exponential sums of the type $\sum_{x\in k^n}\psi(f(x))$, where k is a finite field of characteristic p, $\psi : k \to \mathbb{C}^\star$ is a non-trivial additive character and $f \in k[x_1, \ldots, x_n]$ is a polynomial whose highest degree homogeneous form splits as a product of factors defining a divisor with normal crossings in $\mathbb{P}^{n-1}$.

1 Introduction

Let k be a finite field of characteristic p and cardinality q, let $f \in k[x_1, \ldots, x_n]$ be a polynomial of degree d and $\psi : k \to \mathbb{C}^\star$ a non-trivial additive character. We are interested in the sum the $\sum_{x\in k^n}\psi(f(x))$. Under various different regularity conditions on f (cf. [4] Théorème 8.4, [10] Theorem 0.4, [1] Theorem 1.4, [20] Theorems 2 and 4) the sum is known to be pure of weight n and rank $\leq (d-1)^n$. In particular, we have the estimate

$$\left|\sum_{x\in k^n}\psi(f(x))\right| \leq (d-1)^n \cdot q^{n/2}.$$

In this article we consider the case where the highest degree homogeneous form of f is reducible, and the hypersurface defined in $\mathbb{P}^{n-1}$ by the product of its distinct factors is a divisor with normal crossings.

More precisely, let $f = f_d + f_{d'} + f'$, with f_i homogeneous of degree i and f' of degree $< d'$, and suppose that $f_d = g_1^{\alpha_1} \cdots g_r^{\alpha_r}$ in $\bar{k}[x_1, \ldots, x_n]$, where $g_{i_1} = \ldots = g_{i_k} = 0$ and $g_{i_1} = \ldots = g_{i_k} = f_{d'} = 0$ define smooth subvarieties of codimension k and $k+1$ respectively in $\mathbb{P}^{n-1}_{\bar{k}}$ for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$. These hypotheses appear for the first time in [2], where Adolphson and Sperber show that for d, d', $\alpha_1, \ldots, \alpha_r$ prime to p the L-function associated to the exponential sum $\sum_{x\in k^n}\psi(f(x))$ is a polynomial or the reciprocal of a polynomial, depending on the parity of n. This suggest that the cohomology of the sum is

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concentrated in degree n. Our main result (Theorem 1) will show that, under these hypotheses, the sum is indeed cohomologically pure of weight n.

The proof consists of two parts. First we will show that, for a fixed k, the property of being pure and the rank of the sum do not depend on the particular f chosen, but only on the combinatorial data: d, d', the α_i's and the degrees of the g_i's. For this we use an argument similar to the one in ([5], 3.7.3). In order to finish the proof it would then be enough to find a particular f for which the result is true. In *loc.cit.*, Deligne uses the diagonal polynomial $\sum_{i=1}^n x_i^d$. Unfortunately, in general we will not have such an explicit example for all possible choices of combinatorial data. Instead, we use the theory of perverse sheaves to show that the sum is pure for a certain subfamily of the family of all polynomials of a given type.

This will prove the purity of the sum, but still will not give the value of the rank. In Section 3 we will give some bounds for this rank that will allow us to effectively estimate the absolute value of the sum (cf. Corollary 9). Then in Section 4 we will give en explicit formula for it in terms of the well known formulas for the Euler characteristic of complete intersections in projective space. In particular, this will show that the rank is also independent of the base field k.

In the last section we will give some examples of sums that can be bounded using these results. We will find that several previously known estimates, like the ones for non-singular polynomials and for Kloosterman sums, arise as particular cases of this more general purity theorem.

Notice that, in this case, the general results in [15] about exponential sums where the highest degree form of f is singular do not give good estimates for the sum, due to the high dimension of the singular locus of the hypersurface $f_d = 0$ ($n-3$ if $\alpha_i = 1$ for all $i = 1, \ldots, r$ and $n-2$ otherwise).

2 Purity of the sums

Throughout this section we will fix a prime p, a finite field k of characteristic p and cardinality q and an integer $n \geq 1$. An *admissible type* will consist of the following data:

- a positive integer d prime to p
- a positive integer $d' < d$ prime to p
- a positive integer r
- two r-tuples of positive integers $\mathbf{e} = (e_1, \ldots, e_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that α_i is prime to p for all i and $d = \sum_i e_i \alpha_i$.

Given an admissible type $(d, d', r, \mathbf{e}, \alpha)$, we will say that a polynomial $f \in k[x_1, \ldots, x_n]$ is *admissible of type* $(d, d', r, \mathbf{e}, \alpha)$ if we can write $f = f_d + f_{d'} + f'$, where f_d (respectively $f_{d'}$) is homogeneous of degree d (resp. d'), f' has degree less than d' and:

- f_d factors in $\bar{k}[x_1, \ldots, x_n]$ as $g_1^{\alpha_1} \cdots g_r^{\alpha_r}$, where g_i is homogeneous of degree e_i.

- For every subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$, the subscheme of $\mathbb{P}^{n-1}_{\bar{k}}$ defined by the homogeneous ideal $(g_{i_1}, \ldots, g_{i_k})$ is smooth of codimension k (empty if $k \geq n$).

- For every subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$, the subscheme of $\mathbb{P}^{n-1}_{\bar{k}}$ defined by the homogeneous ideal $(f_{d'}, g_{i_1}, \ldots, g_{i_k})$ is smooth of codimension $k+1$ (empty if $k \geq n-1$).

Our main result is the following:

Theorem 1 *Let $(d, d', r, \mathbf{e}, \alpha)$ be an admissible type. There is a constant $M = M(p, n, d, d', r, \mathbf{e}, \alpha)$ such that, for every admissible polynomial $f \in k[x_1, \ldots, x_n]$ of type $(d, d', r, \mathbf{e}, \alpha)$ and every non-trivial additive character $\psi : k \to \mathbb{C}^\star$, the exponential sum*

$$\sum_{x \in k^n} \psi(f(x))$$

is pure of weight n and rank M. In particular we have the estimate

$$\left| \sum_{x \in k^n} \psi(f(x)) \right| \leq M \cdot q^{n/2}.$$

We will translate the result to cohomological language. Fix a prime $\ell \neq p$ and an isomorphism $\iota : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$. Consider the smooth Artin-Schreier $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1_k$ associated to the non-trivial additive character $\psi : k \to \mathbb{C}^\star \xrightarrow{\sim} \bar{\mathbb{Q}}_\ell^\star$ (cf. [6], 1.7) and, for every $f \in k[x_1, \ldots, x_n]$, let $\mathcal{L}_{\psi(f)}$ be its pull-back to $\mathbb{A}^n_k$. The cohomology groups with compact support $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ are endowed with an action of the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ and, in particular, of the geometric Frobenius element $F \in \mathrm{Gal}(\bar{k}/k)$. By the Grothendieck trace formula we have

$$\sum_{x \in k^n} \psi(f(x)) = \sum_{i=0}^{2n} (-1)^i \,\mathrm{Trace}(F | \mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})).$$

Then Theorem 1 is a consequence of the following cohomological result:

Theorem 2 *Let $(d, d', r, \mathbf{e}, \alpha)$ be an admissible type. There is a constant $M = M(p, n, d, d', r, \mathbf{e}, \alpha)$ such that, for every admissible polynomial $f \in k[x_1, \ldots, x_n]$ of type $(d, d', r, \mathbf{e}, \alpha)$ and every non-trivial additive character $\psi : k \to \mathbb{C}^\star$, the cohomology groups $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ vanish for $i \neq n$, and $\mathrm{H}^n_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ has dimension M and is pure of weight n.*

The remainder of this section will be devoted to the proof of Theorem 2. For every positive integer m, let S_m (respectively S^h_m) be the affine k-space parameterizing all polynomials (resp. all homogeneous polynomials) of degree $\leq m$ (resp. of degree m) in n variables. Given an admissible type $(d, d', r, \mathbf{e}, \alpha)$, let $U \subset S^h_{e_1} \times \cdots \times S^h_{e_r} \times S_{d'}$ be the dense open set parameterizing all $(r+1)$-tuples $(g_1, \ldots, g_r, h)$ such that $g_1^{\alpha_1} \cdots g_r^{\alpha_r} + h$ is admissible of type $(d, d', r, \mathbf{e}, \alpha)$ (Notice that this is *not* a parametrization of the set of admissible polynomials of type $(d, d', r, \mathbf{e}, \alpha)$ if there are $i \neq j$ such that $e_i = e_j$ and $\alpha_i = \alpha_j$, since in that case the same polynomial can appear more than once).

On the affine n-space $X = \mathbb{A}^n_U$ over U we have the universal polynomial $F \in \Gamma(X, \mathcal{O}_X)$ such that, for every $u = (g_1, \dots, g_r, h) \in U(k)$, the restriction of F to the fiber $X_u \cong \mathbb{A}^n_k$ is $f_u := g_1^{\alpha_1} \cdots g_r^{\alpha_r} + h$, and the corresponding Artin-Schreier sheaf $\mathcal{L}_{\psi(F)}$ whose restriction to X_u is $\mathcal{L}_{\psi(f_u)}$. Write $F = F_d + F_{d'} + F'$ where F_d (respectively $F_{d'}$) is the homogeneous component of degree d (resp. of degree d') and $F' = F - F_d - F_{d'}$.

Let $\bar{X} = \mathbb{P}^n_U$ be the projective n-space over U, $j : X \hookrightarrow \bar{X}$ the inclusion, $H = \bar{X} - X$, $\pi : X \to U$ and $\bar{\pi} : \bar{X} \to U$ the projections and $\mathcal{F} = j_! \mathcal{L}_{\psi(F)}$. By the proper base change theorem, the fiber of $\mathrm{R}\pi_! \mathcal{L}_{\psi(F)} = \mathrm{R}\bar{\pi}_\star \mathcal{F}$ at a geometric point $\bar{u}$ over $u \in U(k)$ is $\mathrm{R}\Gamma_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_u)})$. Since U is connected, Theorem 2 will then be a consequence of the following

Proposition 3 *The sheaves* $\mathrm{R}^i \pi_! \mathcal{L}_{\psi(F)}$ *vanish on* U *for* $i \neq n$, *and* $\mathrm{R}^n \pi_! \mathcal{L}_{\psi(F)}$ *is smooth and punctually pure of weight* n *on* U.

We will first show that the $\mathrm{R}^i \pi_! \mathcal{L}_{\psi(F)}$ are all smooth on U. Recall (cf. [7], 2.12) that a morphism $g : Z \to S$ is said to be *locally acyclic* for a constructible sheaf $\mathcal{G}$ on Z if, for every geometric point z of Z and every geometric point s of $S_{g(z)}$ (the henselization of S at $g(z)$), the induced map $\mathrm{R}\Gamma(Z_z, \mathcal{G}) \to \mathrm{R}\Gamma((Z_z)_s, \mathcal{G})$ is an isomorphism, where Z_z is the henselization of Z at z and $(Z_z)_s$ the fiber at s of the canonical map $Z_z \to S_{g(z)}$.

Lemma 4 *The projection* $\bar{\pi} : \bar{X} \to U$ *is locally acyclic for* $\mathcal{F}$.

Proof. Following ([5], 3.7.3) we will show that, locally for the étale topology on $\bar{X}$, the pair $(\bar{X}, \mathcal{F})$ is U-isomorphic to the product of U and a scheme endowed with a $\bar{\mathbb{Q}}_\ell$-sheaf. Let $x \in \bar{X}$ be a closed point, we have to distinguish several cases:

a) If $x \in X$, then $\mathcal{F}$ is smooth in a neighborhood of x, and in fact it is trivialized by the étale neighborhood (Y, y) of x, where Y is defined by the equation $t^d - t - F = 0$ and $y \in Y$ is any point mapping to x, so the assertion is clear in this case.

b) Let $x \in H$ such that $F_d(x) \neq 0$. Let F^h be the homogenization of F with respect to the variable x_0, where $x_0 = 0$ defines the hyperplane at infinity H. Then F is given by F^h / x_0^d. Pick $j \in \{1, \dots, n\}$ such that the j-th coordinate of x is non-zero, then $a = F^h / x_j^d$ and $z = x_0 / x_j$ are well defined regular functions around x, and $a(x) \neq 0$. Take the étale neighborhood (Y, y) of x defined by the equation $v^d = a$, $y \in Y$ mapping to x. Pick a system of parameters $t_1, \dots, t_n$ at y such that $t_1 = z/v$, they define an étale U-map ϕ from a Zariski neighborhood of y in Y to $\mathbb{A}^n \times U$ with $F_{|Y} = \hat{F} \circ \phi$, where $\hat{F}(t_1, \dots, t_n, u) = t_1^{-d}$ is independent of u.

c) Let $x \in H$ such that $F_d(x) = 0$ but $F_{d'}(x) \neq 0$. For simplicity, we can assume without loss of generality that there is some $k \in \{1, \dots, r\}$ such that $g_i(x) = 0$ for $i \leq k$ and $g_i(x) \neq 0$ for $i > k$. Let j and z be as in the previous case, $F^{h'} = (F^h - F_d)/x_0^{d-d'}$ the homogenization of $F_{d'} + F'$ with respect to

x_0, $a = F^{h'}/x_j^{d'}$ and $b_i = g_i/x_j^{e_i}$ for $i = 1, \ldots, r$. Then a and the b_i's are regular in a neighborhood of x, $a(x) \neq 0$, $b_i(x) \neq 0$ for $i > k$ and $b_1, \ldots, b_k, z$ are part of a system of parameters at x. Let (Y, y) be the étale neighborhood of x defined by the equations $v^{d'} = a$ and $w^{\alpha_k} = v^{-d} b_{k+1}^{\alpha_{k+1}} \cdots b_r^{\alpha_r}$, and $y \in Y$ mapping to x. Pick a system of parameters $t_1, \ldots, t_n$ at y such that $t_n = z/v$, $t_i = b_i$ for $i = 1, \ldots, k-1$ and $t_k = b_k w$. They define an étale U-map ϕ from a Zariski neighborhood of y in Y to $\mathbb{A}^n \times U$ with $F_{|Y} = \hat{F} \circ \phi$, where $\hat{F}(t_1, \ldots, t_n, u) = t_n^{-d} t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k} + t_n^{-d'}$ is independent of u.

d) Finally, let $x \in H$ such that $F_d(x) = F_{d'}(x) = 0$ and assume, as in case (c), that there is some $k \in \{1, \ldots, r\}$ such that $g_i(x) = 0$ for $i \leq k$ and $g_i(x) \neq 0$ for $i > k$. Let j, z, $F^{h'}$, a and b_i be as above. Then a and the b_i are regular in a neighborhood of x, $b_i(x) \neq 0$ for $i > k$ and $b_1, \ldots, b_k, a, z$ are part of a system of parameters at x. Let (Y, y) be the étale neighborhood of x defined by the equation $v^{\alpha_k} = b_{k+1}^{\alpha_{k+1}} \cdots b_r^{\alpha_r}$ and $y \in Y$ mapping to x. Pick a system of parameters $t_1, \ldots, t_n$ at y such that $t_n = z$, $t_{n-1} = a$, $t_i = b_i$ for $i = 1, \ldots, k-1$ and $t_k = b_k v$. They define an étale U-map ϕ from a Zariski neighborhood of y in Y to $\mathbb{A}^n \times U$ with $F_{|Y} = \hat{F} \circ \phi$, where $\hat{F}(t_1, \ldots, t_n, u) = t_n^{-d} t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k} + t_n^{-d'} t_{n-1}$ is independent of u.

So in either case we find that $\mathcal{F}$ is locally isomorphic, for the étale topology, to a sheaf of the form $\mathcal{L}_{\psi(g)}$ on $\mathbb{A}^n \times U$, where $g \in \Gamma(U, \mathcal{O}_U)(t_1, \ldots, t_n)$ (and hence $\mathcal{L}_{\psi(g)}$) does not depend on U, therefore being the product of U and a scheme endowed with a sheaf. $\Box$

Corollary 5 *The sheaves* $\mathrm{R}^i \pi_! \mathcal{L}_{\psi(F)}$ *are smooth on* U.

Proof. By ([7], A.2.2), for every geometric point u of U the specialization map $(\mathrm{R}\bar{\pi}_\star \mathcal{F})_u = \mathrm{R}\Gamma(\bar{X}_u, \mathcal{F}) \to \mathrm{R}\Gamma(\bar{X}_{\bar{\eta}}, \mathcal{F}) = (\mathrm{R}\bar{\pi}_\star \mathcal{F})_{\bar{\eta}}$ is a quasi-isomorphism, where $\bar{\eta}$ is a geometric generic point of U. Therefore the cohomology sheaves $\mathrm{R}^i \pi_! \mathcal{L}_{\psi(F)} = \mathrm{R}^i \bar{\pi}_\star \mathcal{F}$ of $\mathrm{R}\bar{\pi}_\star \mathcal{F}$ are smooth. $\Box$

In particular, it suffices to prove Theorem 2 for *one* particular admissible polynomial of type $(d, d', r, \mathbf{e}, \alpha)$. This will show that the stalk of $\mathrm{R}^i \pi_! \mathcal{L}_{\psi(F)}$ at one geometric point of U vanishes for $i \neq n$ and is pure of weight n for $i = n$. Since the sheaves are smooth on U, the same will be true at any other geometric point of U.

Replacing k by a finite extension if necessary, assume that $U(k) \neq \emptyset$, and pick any $u = (g_1, \ldots, g_r, h) \in U(k)$. Let $V \subset \hat{\mathbb{A}}^n$ be the dense open set of all linear forms l such that $(g_1, \ldots, g_r, h + l) \in U$ (which is the entire $\hat{\mathbb{A}}^n$ if $d' > 1$). Then V is a closed subset of U via the embedding $l \mapsto (g_1, \ldots, g_r, h + l)$.

Proposition 6 *The object* $K = \mathrm{R}\pi_! \mathcal{L}_{\psi(F)}[2n] \in \mathcal{D}_c^b(V, \bar{\mathbb{Q}}_\ell)$ *is perverse and pure of weight* $2n$.

Proof. This is a particular case of the results in [18] and [14]. The object K is just the restriction to V of the Fourier transform of $\mathcal{L}_{\psi(f_u)}[n]$ with respect to the character ψ (cf. [18], 2.1). Since $\mathcal{L}_{\psi(f_u)}$ is smooth and punctually pure of weight 0 and $\mathbb{A}^n$ is smooth, $\mathcal{L}_{\psi(f_u)}[n]$ is perverse and pure of weight n (cf. *loc.cit.*, 1.2.2(iii) and 1.3.2(iii)). But the Fourier transform preserves perversity and purity and shifts weights by n (cf. *loc.cit.*, 2.1.5(iii) and 2.2.1), therefore K is perverse and pure of weight $2n$. □

We can now finish the proof of Proposition 3. The object K is perverse by Proposition 6 and has smooth cohomology sheaves by Corollary 5. Therefore all of them must vanish except for $\mathcal{H}^{-n}(K) = \mathrm{R}^n\pi_!\mathcal{L}_{\psi(F)}$, which is smooth and punctually pure of weight n, since K is pure of weight $2n$ (cf. [18], 1.4.1). This proves the proposition (and consequently Theorems 1 and 2) for a particular ψ, but it is clear that M does not depend on the additive character chosen, since any other non-trivial character would be given by $t \to \psi(at)$ for some $a \in k^\star$, and multiplying an admissible polynomial by a non-zero constant does not change its type.

Remark. The following examples show that it is essential to consider the components of lower degree of f in the purity theorem, and therefore it is not possible to give a similar result depending only on the highest degree homogeneous component of f.

First, suppose that n is prime to p and let $f(x) = \prod_{x=1}^n x_i = x_1 \cdots x_n$, which clearly defines a divisor with normal crossings in $\mathbb{P}^{n-1}$. Then

$$\sum_{x\in k^n} \psi(f(x)) = \sum_{a\in k} \sum_{x_1\cdots x_{n-1}=a} \sum_{x_n\in k} \psi(ax_n).$$

But $\sum_{x_n\in k} \psi(ax_n) = 0$ for $a \neq 0$ and q for $a = 0$, so

$$\sum_{x\in k^n} \psi(f(x)) = \sum_{x_1\cdots x_{n-1}=0} q = q \cdot \#\{(x_1, \ldots, x_{n-1}) \in k^{n-1} : x_1 \cdots x_{n-1} = 0\} =$$

$$= q(q^{n-1} - (q-1)^{n-1}) = O(q^{n-1}).$$

As a second example, take $f(x) = x_1^d$, with d prime to p. Then, on $\mathbb{A}^n = \mathbb{A}^1 \times \mathbb{A}^{n-1}$ we have

$$\mathcal{L}_{\psi(f)} = \mathcal{L}_{\psi(x^d)} \boxtimes \bar{\mathbb{Q}}_\ell$$

and therefore, since $\mathrm{H}^i_c(\mathbb{A}^{n-1}_{\bar{k}}, \bar{\mathbb{Q}}_\ell) = 0$ for $i \neq 2n-2$, we deduce that

$$\mathrm{H}^{2n-1}_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = \mathrm{H}^1_c(\mathbb{A}^1_{\bar{k}}, \mathcal{L}_{\psi(x^d)}) \otimes \mathrm{H}^{2n-2}_c(\mathbb{A}^{n-1}_{\bar{k}}, \bar{\mathbb{Q}}_\ell) =$$

$$= \mathrm{H}^1_c(\mathbb{A}^1_{\bar{k}}, \mathcal{L}_{\psi(x^d)}) \otimes \bar{\mathbb{Q}}_\ell(1-n)$$

has dimension $d-1$ and is pure of weight $2n-1$, since $\mathrm{H}^1_c(\mathbb{A}^1_{\bar{k}}, \mathcal{L}_{\psi(x^d)})$ has dimension $d-1$ and is pure of weight 1 (cf. [4], 8.11). So in this case we get

$$\left| \sum_{x\in k^n} \psi(f(x)) \right| = O(q^{n-\frac{1}{2}}).$$

3 First estimates for the rank

In the next section we will prove a closed formula for M in terms of the type $(d, d', r, \mathbf{e}, \alpha)$. For now, we will give some easier bounds for it using the following "degeneration principle": Suppose that we have a family of polynomials of degree d in n variables whose coefficients are parameterized by the points of an irreducible variety, and suppose that the exponential sum associated to a generic element of the family is pure (in the sense of Theorem 2). Then, for every polynomial in the family whose associated exponential sum is pure, the rank of this sum is bounded above by the rank of the sum corresponding to a generic element of the family. This assertion is justified by the following theorem of Katz:

Theorem 7 (cf. [16], Proposition 12) *Let $\mathcal{G}$ be a $\bar{\mathbb{Q}}_\ell$-sheaf on a smooth variety S of dimension r, and suppose that there is a perverse object $K \in \mathcal{D}^b_c(S, \bar{\mathbb{Q}}_\ell)$ such that $\mathcal{G} = \mathcal{H}^{-r}(K)$. Then the integer valued function defined by $s \mapsto$ rank $\mathcal{G}_{\bar{s}}$ on S (where $\bar{s}$ is a geometric point over s) is lower semicontinuous. In other words, the rank of $\mathcal{G}$ can never increase under specialization.*

We will apply this result to the following situation: S is the affine k-space parameterizing all polynomials of degree $\leq d$ in n variables, $\pi : S \times \mathbb{A}^n_k \to S$ is the projection, $F \in \Gamma(S \times \mathbb{A}^n_k, \mathcal{O}_{S\times\mathbb{A}^n_k})$ is the universal polynomial, $K = \mathrm{R}\pi_! \mathcal{L}_{\psi(F)}[n + \dim S]$ and $\mathcal{G} = \mathrm{R}^n\pi_!\mathcal{L}_{\psi(F)}$. By ([17], Part (1) of Theorem 3.1.2) K is perverse, so the theorem applies to this case and we conclude that the rank of $\mathrm{R}^n\pi_!\mathcal{L}_{\psi(F)}$ is a lower semicontinuous function on S. As a first application of this, we can show

Proposition 8 *For every admissible type $(d, d', r, \mathbf{e}, \alpha)$ we have the upper bound*

$$M(p, n, d, d', r, \mathbf{e}, \alpha) \leq (d-1)^n.$$

Proof. It is well known (cf. [4], Lemme 8.5) that the generic rank of $\mathrm{R}^n\pi_!\mathcal{L}_{\psi(F)}$ on S is $(d-1)^n$. Therefore this is a direct consequence of the semicontinuity of the rank. □

Corollary 9 *For every admissible polynomial $f \in k[x_1, \ldots, x_n]$ of degree d and every non-trivial additive character $\psi : k \to \mathbb{C}^\star$ we have the estimate*

$$\left| \sum_{x\in k^n} \psi(f(x)) \right| \leq (d-1)^n \cdot q^{n/2}.$$

We will now give a few inequalities that can be deduced from the degeneration principle. In all cases, the proof is the same: the subvariety of S parameterizing all admissible polynomials of a given type is a subset of the closure of the subvariety of S parameterizing the admissible polynomials of another type. From the degeneration principle we conclude that the rank corresponding to the former type is bounded above by the rank corresponding to the latter.

Notice that applying the same permutation to the e_i's and the α_i's does not change M. Therefore in the following examples we will always modify the first factors of f_d, but the same inequalities will hold if we modify any of the other factors the same way.

1. Suppose that $e_1 = \sum_{j=1}^{s} h_j\beta_j$ with all β_j's prime to p. Then
$$M(p, n, d, d', r, (e_1, e_2, \ldots, e_r), (\alpha_1, \alpha_2, \ldots, \alpha_r)) \geq$$
$$\geq M(p, n, d, d', r{+}s{-}1, (h_1, \ldots, h_s, e_2, \ldots, e_r), (\alpha_1\beta_1, \ldots, \alpha_1\beta_s, \alpha_2, \ldots, \alpha_r))$$
(g_1 "becomes reducible").

2. If $r \geq 2$ and $e_1 = e_2$, then
$$M(p, n, d, d', r, (e_2, e_2, e_3, \ldots, e_r), (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)) \geq$$
$$\geq M(p, n, d, d', r-1, (e_2, \ldots, e_r), (\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_r))$$
(g_1 and g_2 "become equal").

3. If $d > d' > d'' \geq 1$ are all prime to p, then
$$M(p, n, d, d', r, (e_1, e_2, \ldots, e_r), (\alpha_1, \alpha_2, \ldots, \alpha_r)) \geq$$
$$\geq M(p, n, d, d'', r, (e_1, e_2, \ldots, e_r), (\alpha_1, \alpha_2, \ldots, \alpha_r))$$
($f_{d'}$ "degenerates to 0").

We can use these inequalities to give a lower bound for M. First of all, let us compute an easy case:

Lemma 10 $M(p, n, d, d', 1, (1), (d)) = (d-1)(d'-1)^{n-1}$.

Proof. We will use the polynomial $f(x_1, \ldots, x_n) = x_1^d + \sum_{i=1}^{n} x_i^{d'}$, which is clearly admissible of type $(d, d', 1, (1), (d))$. Since
$$\psi(f(x)) = \psi(x_1^d + x_1^{d'})\psi(x_2^{d'}) \cdots \psi(x_n^{d'}),$$
we have, as sheaves on $\mathbb{A}_k^n$,
$$\mathcal{L}_{\psi(f)} = \mathcal{L}_{\psi(x^d+x^{d'})} \boxtimes \mathcal{L}_{\psi(x^{d'})} \boxtimes \cdots \boxtimes \mathcal{L}_{\psi(x^{d'})}.$$
But $\mathcal{L}_{\psi(x^d+x^{d'})}$ (respectively $\mathcal{L}_{\psi(x^{d'})}$) has rank $d-1$ (resp. $d'-1$) (cf. [4], 8.11), so $\mathcal{L}_{\psi(f)}$ has rank $(d-1)(d'-1)^{n-1}$. □

Corollary 11 *For any admissible type* $(d, d', r, \mathbf{e}, \alpha)$ *we have the lower bound*
$$M(p, n, d, d', r, \mathbf{e}, \alpha) \geq (d-1)(d'-1)^{n-1}.$$

Proof. By repeated application of the first two inequalities, one shows that
$$M(p, n, d, d', r, (e_1, \ldots, e_r), (\alpha_1, \ldots, \alpha_r)) \geq M(p, n, d, d', 1, (1), (d)).$$
□

4 An explicit formula for the rank

This section will be entirely devoted to the proof of the following

Theorem 12 *For all positive integers $r \leq n$ and $d_1, \ldots, d_r$ let $\chi(n; d_1, \ldots, d_r)$ denote the Euler characteristic of a non-singular complete intersection of multidegree $(d_1, \ldots, d_r)$ in $\mathbb{P}^n$. Then for every prime p and every admissible type $(d, d', r, \mathbf{e}, \alpha)$ we have*

$$M(p, n, d, d', r, \mathbf{e}, \alpha) = (-1)^n + d\frac{(d'-1)^n - (-1)^n}{d'} + (-1)^n(d-d')\chi \qquad (1)$$

where

$$\chi := \sum_{\substack{I \subseteq \{1,\ldots,r\} \\ 1 \leq |I| \leq n-1}} (-1)^{|I|-1}\chi(n-1; e_I) - \sum_{\substack{I \subseteq \{1,\ldots,r\} \\ 1 \leq |I| \leq n-2}} (-1)^{|I|-1}\chi(n-1; d', e_I)$$

and e_I stands for $e_{i_1}, \ldots, e_{i_j}$ if $I = \{i_1, \ldots, i_j\}$.

Corollary 13 *$M(p, n, d, d', r, \mathbf{e}, \alpha)$ is independent of p and of α.*

We will denote this number by $M(n, d, d', r, \mathbf{e})$. Of course, given d and $\mathbf{e}$, there are only a finite number of possible choices for α, namely the solutions in positive integers $(x_1, \ldots, x_n)$ of $e_1 x_1 + \ldots + e_n x_n = d$.

Fix a finite field k of characteristic p and let $f \in k[x_1, \ldots, x_n]$ be an admissible polynomial of type $(d, d', r, \mathbf{e}, \alpha)$. We already know that the rank does not depend on the polynomial chosen, so we can assume that f does not have any terms of degree $< d'$. Therefore $f = f_d + f_{d'}$, where $f_d = g_1^{\alpha_1} \cdots g_r^{\alpha_r}$, g_i is homogeneous of degree e_i, $f_{d'}$ is homogeneous of degree d' and $g_1 \cdots g_r f_{d'} = 0$ defines a divisor with normal crossings in $\mathbb{P}^{n-1}_{\bar{k}}$.

By Theorem 2, the cohomology sheaves $\mathrm{H}^i_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)})$ vanish for $i \neq n$, so

$$M(p, n, d, d', r, \mathbf{e}, \alpha) = (-1)^n \chi_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}).$$

We need to compute this Euler characteristic. The idea is to reduce this to computing the Euler characteristics of some sheaves in $\mathbb{A}^1_{\bar{k}}$, which can be done by studying their local properties. Consider the object $\mathrm{R}f_{d!}\mathcal{L}_{\psi(f)}$ in $\mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$, we have $\chi_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = \chi_c(\mathbb{A}^1_{\bar{k}}, \mathrm{R}f_{d!}\mathcal{L}_{\psi(f)})$. By the projection formula,

$$\mathrm{R}f_{d!}\mathcal{L}_{\psi(f)} = \mathrm{R}f_{d!}\mathcal{L}_{\psi(f_d + f_{d'})} = \mathrm{R}f_{d!}(\mathcal{L}_{\psi(f_d)} \otimes \mathcal{L}_{\psi(f_{d'})}) = \mathcal{L}_\psi \otimes \mathrm{R}f_{d!}\mathcal{L}_{\psi(f_{d'})}$$

We will now study the object $\mathrm{R}f_{d!}\mathcal{L}_{\psi(f_{d'})}$ in detail. By abuse of language, we will say that an object $K \in \mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ is smooth (resp. unramified, tamely ramified, totally wild) at a point $t \in \mathbb{A}^1_{\bar{k}}$ (resp. $t \in \mathbb{P}^1_{\bar{k}}$) if so are all its cohomology sheaves. The dimension of the fiber of K at a point $t \in \mathbb{A}^1_{\bar{k}}$ (resp. its generic rank, its Swan conductor at $t \in \mathbb{P}^1_{\bar{k}}$) will be the alternating sum of the dimensions of the fibers of the cohomology sheaves at t (resp. of their generic ranks, of their

Swan conductors at $t \in \mathbb{P}^1_{\bar{k}}$). With these conventions, the Grothendieck-Néron-Ogg-Shafarevic formula for the Euler characteristic of a sheaf on $\mathbb{A}^1_{\bar{k}}$ (cf. [11], Exposé X, Corollaire 7.12) is also true for derived category objects, by additivity:

$$\chi_c(\mathbb{A}^1_{\bar{k}}, K) = \dim(K_{\bar{\eta}}) - \sum_{t \in \bar{k}} \mathrm{drop}_t(K) - \sum_{t \in \mathbb{P}^1(\bar{k})} \mathrm{Swan}_t(K)$$

where $\dim(K_{\bar{\eta}})$ is the generic rank of K and $\mathrm{drop}_t(K) = \dim(K_{\bar{\eta}}) - \dim(K_t)$.

The main properties of the object $K' := \mathrm{R}f_{d!}\mathcal{L}_{\psi(f_{d'})}$ are summarized in the following result:

Proposition 14 *The object $K' \in \mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ is smooth on $\mathbb{G}_{m,\bar{k}}$. At 0 it is tamely ramified. At ∞ it is totally wild, and as a representation of the wild inertia group P_∞ it has a unique break equal to d'/d. Its Euler characteristic is $(1-d')^n$.*

Proof. The Euler characteristic is easy to compute:

$$\chi_c(\mathbb{A}^1_{\bar{k}}, \mathrm{R}f_{d!}\mathcal{L}_{\psi(f_{d'})}) = \chi_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f_{d'})}) = (1-d')^n$$

by [4], Lemme 8.5, since $f_{d'} = 0$ defines a non-singular hypersurface in $\mathbb{P}^{n-1}$. The rest of the properties concern only the restriction of K' to $\mathbb{G}_{m,\bar{k}}$. Let $U = f_d^{-1}(\mathbb{G}_{m,\bar{k}}) \subset \mathbb{A}^n_{\bar{k}}$, then $K'_{|\mathbb{G}_{m,\bar{k}}} = \mathrm{R}f_{d!}(\mathcal{L}_{\psi(f_{d'})|U})$. It will be convenient to modify K' slightly as follows.

Let X be the hypersurface defined in $\mathbb{A}^n_{\bar{k}}$ by $f_d(x) = 1$. On $X \times \mathbb{A}^1_{\bar{k}}$ consider the sheaf $\mathcal{L}_{\psi(tf_{d'}(x))}$ (where t is the coordinate in $\mathbb{A}^1$). Let $\pi : X \times \mathbb{A}^1_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ be the projection, and $K \in \mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ the object $\mathrm{R}\pi_!(\mathcal{L}_{\psi(tf_{d'}(x))})$. For every positive integer b, let $[b] : \mathbb{G}_{m,\bar{k}} \to \mathbb{G}_{m,\bar{k}}$ denote the b-th power map.

Lemma 15 *The objects $[d]^\star K'$ and $[d']^\star K$ are isomorphic on $\mathbb{G}_{m,\bar{k}}$.*

Proof. By proper base change, the object $[d']^\star K$ is just $\mathrm{R}\pi_!(\mathcal{L}_{\psi(t^{d'}f_{d'}(x))})$. Since $f_{d'}$ is homogeneous of degree d', $t^{d'}f_{d'}(x) = f_{d'}(tx)$, so this can also be written as $\mathrm{R}\pi_!(\mathcal{L}_{\psi(f_{d'}(tx))})$.

Let ϕ be the $\mathbb{G}_{m,\bar{k}}$-automorphism of $\mathbb{A}^n_{\bar{k}} \times \mathbb{G}_{m,\bar{k}}$ given by $\phi(x,t) = (tx, t)$. The image of $X \times \mathbb{G}_{m,\bar{k}}$ under this automorphism is the set Y of $(x,t) \in \mathbb{A}^n_{\bar{k}} \times \mathbb{G}_{m,\bar{k}}$ such that $f_d(x/t) = 1$. Since f_d is homogeneous of degree d, this is equivalent to $f_d(x) = t^d$. And $\mathcal{L}_{\psi(f_{d'}(tx))}$ is just the pull-back $\phi^\star\mathcal{L}_{\psi(f_{d'}(x))}$ of the sheaf $\mathcal{L}_{\psi(f_{d'}(x))}$ on Y, so $[d']^\star K = \mathrm{R}\pi_!(\mathcal{L}_{\psi(f_{d'}(tx))}) = \mathrm{R}(\varpi \circ \phi)_!(\phi^\star\mathcal{L}_{\psi(f_{d'}(x))}) = \mathrm{R}\varpi_!(\mathcal{L}_{\psi(f_{d'}(x))})$ where $\varpi : Y \to \mathbb{G}_{m,\bar{k}}$ is the projection.

We have the following cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{1\times[d]} & Z \\ \downarrow{\scriptstyle\varpi} & & \downarrow{\scriptstyle\pi'} \\ \mathbb{G}_{m,\bar{k}} & \xrightarrow{[d]} & \mathbb{G}_{m,\bar{k}} \end{array}$$

where $Z \subset \mathbb{A}^n_{\bar{k}} \times \mathbb{G}_{m,\bar{k}}$ is defined by $f_d(x) = t$, $\pi' : Z \to \mathbb{G}_{m,\bar{k}}$ is the projection and $1 \times [d]$ takes (x,t) to (x,t^d). Using again proper base change, we deduce that $[d']^\star K = \mathrm{R}\varpi_!(\mathcal{L}_{\psi(f_{d'}(x))}) = [d]^\star \mathrm{R}\pi'_!(\mathcal{L}_{\psi(f_{d'}(x))})$ (where, in the last expression, $\mathcal{L}_{\psi(f_{d'}(x))}$ is regarded as a sheaf on Z). But Z is just the graph of $f_d : U \to \mathbb{G}_{m,\bar{k}}$ and therefore isomorphic to U:

$$\begin{array}{ccc} U & \xrightarrow{(1,f_d)} & Z \\ \downarrow{\scriptstyle f_d} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{G}_{m,\bar{k}} & \xrightarrow{1} & \mathbb{G}_{m,\bar{k}} \end{array}$$

so $[d]^\star \mathrm{R}\pi'_!(\mathcal{L}_{\psi(f_{d'}(x))}) = [d]^\star \mathrm{R}f_{d!}(\mathcal{L}_{\psi(f_{d'}(x))|U}) = [d]^\star K'_{|\mathbb{G}_{m,\bar{k}}}$. □

The isomorphism is easier to see at the level of traces. The trace of the geometric Frobenius element acting on the stalk of K' at $t \in \bar{k}^\star$ is

$$\sum_{f_d(x)=t} \psi(f_{d'}(x)).$$

After taking the pull-back with respect to $[d]$ and making the change of variables $x \to tx$, the trace becomes

$$\sum_{f_d(x)=t^d} \psi(f_{d'}(x)) = \sum_{f_d(x)=1} \psi(f_{d'}(tx)) = \sum_{f_d(x)=1} \psi(t^{d'} f_{d'}(x))$$

and this is the pull-back with respect to $[d']$ of

$$\sum_{f_d(x)=1} \psi(t f_{d'}(x)),$$

which is the trace of Frobenius acting on the stalk of K at t.

Since d and d' are prime to p, pulling back with respect to the d-th or d'-th power maps does not affect tame ramification. As for the breaks, pulling back with respect to the d-th power map multiplies all breaks by d (cf. [12], 1.13.1). Therefore, proving the remainder of Proposition 14 is equivalent to proving

Proposition 16 *The object $K \in \mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ is smooth on $\mathbb{G}_{m,\bar{k}}$. At 0 it is tamely ramified. At ∞ it is totally wild, and as a representation of P_∞ it has a unique break equal to* 1.

For the proof we will use Laumon's local Fourier transform theory (cf. [19], [13] 7.4). It will be convenient to work in the perverse category. The truth of the proposition is invariant under semisimplification, so we will assume that K is already semisimple (i.e. it is the direct sum of its (shifted) perverse cohomology sheaves, which are all semisimple). First of all let us give an explicit description of the Fourier transform of K. Let $X \subset \mathbb{A}^n_{\bar{k}}$ be the hypersurface defined by $f_d(x) = 1$, and $L := \mathrm{R}f_{d'!}(\bar{\mathbb{Q}}_{\ell,X})$ the direct image with compact supports of the constant sheaf on X.

Lemma 17 *The object K is the Fourier transform of $L[-1]$ with respect to the additive character ψ.*

Proof. Straightforward from the definitions. If $\pi_i : \mathbb{A}^1_{\bar{k}} \times \mathbb{A}^1_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ $(i = 1, 2)$ are the projections and s, t the coordinates in $\mathbb{A}^1_{\bar{k}} \times \mathbb{A}^1_{\bar{k}}$, the Fourier transform of $L[-1]$ is

$$\mathrm{R}\pi_{2!}(\pi_1^\star(\mathrm{R}f_{d'!}(\bar{\mathbb{Q}}_{\ell,X})) \otimes \mathcal{L}_{\psi(st)}) =$$
$$= \mathrm{R}\pi_{2!}\mathrm{R}(f_{d'}, 1)_!((f_{d'}, 1)^\star(\mathcal{L}_{\psi(st)}) = \mathrm{R}\pi_!(\mathcal{L}_{\psi(tf_{d'}(x))}) = K$$

by the projection formula, where $\pi : X \times \mathbb{A}^1_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ is the projection and $(f_{d'}, 1) : X \times \mathbb{A}^1_{\bar{k}} \to \mathbb{A}^1_{\bar{k}} \times \mathbb{A}^1_{\bar{k}}$ is the product map of $f_{d'}$ and the identity. □

Recall that on a smooth curve over $\bar{k}$, an irreducible perverse object is either punctual or it is (a shift of) the extension by direct image of a smooth sheaf on a dense open subset (cf. [3], 4.3). Grouping similar perverse irreducible components together, we can decompose K as a direct sum

$$K = K_0 \oplus K_{pct} \oplus K_{cons} \oplus K_{AS} \oplus K_F$$

Where K_0 is punctual supported at 0, K_{pct} is punctual supported outside 0, K_{cons} is constant, K_{AS} is a direct sum of shifted Artin-Schreier sheaves $\mathcal{L}_{\psi(ax)}$ with $a \neq 0$ and K_F is a direct sum of shifted irreducible Fourier sheaves in the sense of [13], 7.3.5 (i.e. extensions by direct image to $\mathbb{A}^1_{\bar{k}}$ of irreducible smooth sheaves on open subsets of $\mathbb{A}^1_{\bar{k}}$, which are not constant or isomorphic to an Artin-Schreier sheaf). Similarly, write

$$L = L_0 \oplus L_{pct} \oplus L_{cons} \oplus L_{AS} \oplus L_F.$$

Since the Fourier transform interchanges objects supported at 0 with constant objects, punctual objects supported outside 0 with direct sums of Artin-Schreier sheaves and takes Fourier sheaves to Fourier sheaves (cf. [13], 7.3.8), we have

$$\begin{aligned}
K_0 &= \mathrm{FT}_\psi(L_{cons})[-1] \\
K_{pct} &= \mathrm{FT}_\psi(L_{AS})[-1] \\
K_{cons} &= \mathrm{FT}_\psi(L_0)[-1] \\
K_{AS} &= \mathrm{FT}_\psi(L_{pct})[-1] \\
K_F &= \mathrm{FT}_\psi(L_F)[-1]
\end{aligned}$$

What does Proposition 16 say in terms of L? The fact that K is smooth on $\mathbb{G}_{m,\bar{k}}$ means that $K_{pct} = 0$ and K_F is smooth on $\mathbb{G}_{m,\bar{k}}$. Taking Fourier transform, this is equivalent to $L_{AS} = 0$ and the fact that 1 is not a break of L_F at ∞ (cf. [12], 8.5.8). Equivalently, since all breaks of L_{AS} at ∞ are 1, we want 1 not to be a break of L at ∞.

The only piece of K that can possibly not be tame at 0 is K_F. And K_F being tame at 0 is equivalent to L_F not having any ∞-break strictly between 0 and 1 (cf. [13], 7.4.5). Since this is the only piece of L that can have an ∞-break different than 0 or 1, this is equivalent to L not having any ∞-break in the interval $(0, 1)$.

Finally, we want 1 to be the only break of K at infinity. This implies that K_{cons} must vanish, or equivalently, that $L_0 = 0$. The only other piece that can have an ∞-break different than 1 is K_F. So we must also require that the only ∞-break of K_F is 1.

The Fourier transform interchanges the ∞-breaks greater then 1 of K_F and L_F (cf. [13], 7.5.4). Therefore, K_F not having any ∞-break greater than 1 is equivalent to L_F (and therefore L) not having any ∞-break greater than 1. On the other hand, by [12], 8.5.8, K_F not having any break smaller than 1 is equivalent to L_F (and therefore L, since $L_0 = 0$) being smooth at 0.

We have shown that Proposition 16 is equivalent to

Proposition 18 *The object $L \in \mathcal{D}^b_c(\mathbb{A}^1_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$ is smooth at* 0 *and tamely ramified at* ∞.

Proof: Take the standard compactification of $f_{d'} : \mathbb{A}^n_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$: Z is defined in $\mathbb{P}^n_{\bar{k}} \times \mathbb{A}^1_{\bar{k}}$ by the equation $f_{d'}(x) - \lambda x_0^{d'} = 0$, and $\tilde{f}_{d'} : Z \to \mathbb{A}^1_{\bar{k}}$ is the projection. Its restriction to the closure $\bar{X}$ of X in Z is a compactification $\tilde{f}_{d'} : \bar{X} \to \mathbb{A}^1_{\bar{k}}$ of $f_{d'} : X \to \mathbb{A}^1_{\bar{k}}$. Let $j : X \to \bar{X}$ be the inclusion, then $f_{d'} = \tilde{f}_{d'} \circ j$.

To show that L is smooth at 0 we will show, as in the proof of Lemma 4, that the map $\tilde{f}_{d'}$ is locally acyclic in a neighborhood of $\tilde{f}_{d'}^{-1}(0)$ in $\bar{X}$ for the sheaf $j_!(\bar{\mathbb{Q}}_\ell)$ by showing that, locally for the étale topology, it is isomorphic to a projection $Y \times \mathbb{A}^1_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$, with the sheaf $j_!(\bar{\mathbb{Q}}_\ell)$ correspoding under this isomorphism to the inverse image of a sheaf on Y.

So let $(x, 0) \in \tilde{f}_{d'}^{-1}(0)$. If $x \in X$, we claim that $\tilde{f}_{d'}$ (or, equivalently, $f_{d'}$) is smooth at $(x, 0)$ (resp. at x). Otherwise, the Jacobian matrix

$$\begin{pmatrix} \partial f_d/\partial x_1 & \cdots & \partial f_d/\partial x_n \\ \partial f_{d'}/\partial x_1 & \cdots & \partial f_{d'}/\partial x_n \end{pmatrix}$$

would have rank at most 1 at x. Since the partial derivatives of f_d can not vanish simultaneously at x (by the Euler relation, since $f_d(x) = 1$) this implies that there is $a \in \bar{k}$ with $\partial f_{d'}/\partial x_i = a \cdot \partial f_d/\partial x_i$ for all i. Again by the Euler relation, we deduce that $0 = d' f_{d'}(x) = a d f_d(x) = ad$. So a must be 0, and therefore all partial derivatives of $f_{d'}$ vanish at x. Since $f_{d'}$ is non-singular, this can only happen if $x = 0$, which is not in X. This shows that $\tilde{f}_{d'}$ is smooth at $(x, 0)$, and therefore locally acyclic with respect to $j_!(\bar{\mathbb{Q}}_\ell)$, which is constant in a neighborhood of $(x, 0)$.

Now let $(x, 0) \in \tilde{f}_{d'}^{-1}(0) \cap (\bar{X} - X)$. Then $x_0 = 0$ and $f_d(x) = f_{d'}(x) = 0$ (where $f_d(0, x_1, \ldots, x_n) := f_d(x_1, \ldots, x_n)$ and similarly for $f_{d'}$). Without loss of generality, suppose that there is some $k = 1, \ldots, r$ such that $g_i(x) = 0$ for $i \leq k$ and $g_i(x) \neq 0$ for $i > k$. Pick $j = 1, \ldots, n$ such that $x_j \neq 0$ and let $z = x_0/x_j$, $a = f_{d'}/x_j^{d'}$ and $b_i = g_i/x_j^{e_i}$, then $\lambda, z, a, b_1, \ldots, b_k$ form a regular sequence in the local ring of $\mathbb{P}^n_{\bar{k}} \times \mathbb{A}^1_{\bar{k}}$ at $(x, 0)$. Take the étale neighborhood (Y, y) of $(x, 0)$ defined by $v^{\alpha_k} = b_{k+1}^{\alpha_{k+1}} \cdots b_r^{\alpha_r}$ with y mapping to $(x, 0)$. The maps $\lambda, z, a - \lambda z^{d'}, b_1, \ldots, b_{k-1}, b_k v$ form a regular sequence in the local ring of Y at y, which can be extended to a complete system of parameters $t_0, t_1, \ldots, t_n$

at y. With respect to this system of parameters the inverse image of $\bar{X}$ in Y is defined by the equations $t_2 = 0$ and $t_3^{\alpha_1} \cdots t_{k+2}^{\alpha_k} = t_1^d$, the restriction of the map $\tilde{f}_{d'}$ is given by t_0 and the inverse image of the sheaf $j_!(\bar{\mathbb{Q}}_\ell)$ is the extension by zero of the constant sheaf on the open set defined by $t_1 \neq 0$. If we map Y to $\mathbb{A}_{\bar{k}}^{n+1}$ using the étale map ϕ given by this system of parameters, the triplet $(\bar{X}, \tilde{f}_{d'}, j_!(\bar{\mathbb{Q}}_\ell))$ on Y is just the pull-back with respect to ϕ of the triplet $(\{t_2 = 0, t_3^{\alpha_1} \cdots t_{k+2}^{\alpha_k} = t_1^d\}, t_0, \bar{\mathbb{Q}}_{\ell|\{t_1 \neq 0\}})$, which is just the projection to $\mathbb{A}_{\bar{k}}^1$ of the product of $\mathbb{A}_{\bar{k}}^1$ and a scheme endowed with a sheaf. So $\tilde{f}_{d'}$ is locally isomorphic (for the étale topology) to a projection in a neighborhood of $(x, 0)$ and therefore locally acyclic at $(x, 0)$.

This shows that L is smooth at 0. The reason why it is tamely ramified at infinity is, roughly speaking, that the map $\tilde{f}_{d'}$ can be lifted to characteristic zero. More precisely, let R be a characteristic 0 discrete valuation ring with residue field $\bar{k}$. Lift $g_1, \ldots, g_r$ and $f_{d'}$ to non-singular homogeneous forms $\hat{g}_1, \ldots, \hat{g}_r$ and $\hat{f}_{d'}$ in $R[x_1, \ldots, x_n]$ such that $\hat{g}_1 \cdots \hat{g}_r \hat{f}_{d'} = 0$ defines a divisor with normal crossings on $\mathbb{P}_R^{n-1}$ (which is possible since smoothness is generic), and let $\hat{f}_d = \hat{g}_1^{\alpha_1} \cdots \hat{g}_r^{\alpha_r}$. Let $\hat{X} \subset \mathbb{A}_R^n$ be the subscheme defined by $\hat{f}_d = 1$. Take the compactification $\tilde{\hat{f}}_{d'} : \bar{\hat{X}} \to \mathbb{A}_R^1$ of $\hat{f}_{d'} : \hat{X} \to \mathbb{A}_R^1$ defined as above. Then, exactly as we did over $\bar{k}$, one shows that $\tilde{\hat{f}}_{d'}$ is locally isomorphic to a projection with respect to the étale topology in a neighborhood of $\tilde{\hat{f}}_{d'}^{-1}(0)$ in $\bar{\hat{X}}$, and the sheaf $\hat{j}_!(\bar{\mathbb{Q}}_\ell)$ (where $\hat{j} : \hat{X} \hookrightarrow \bar{\hat{X}}$ is the inclusion) corresponds under this isomorphism to the pull-back of a sheaf on the other factor, so $\tilde{\hat{f}}_{d'}$ is locally acyclic for the sheaf $\hat{j}_!(\bar{\mathbb{Q}}_\ell)$. Therefore $\mathfrak{L} := \mathrm{R}\hat{f}_{d'!}(\bar{\mathbb{Q}}_{\ell,\hat{X}})$ is smooth in a neighborhood U of 0 in $\mathbb{A}_R^1$, and in particular outside a closed subscheme $S \subset \mathbb{A}_R^1$ proper over R. By [9], Proposition 4.1, we conclude that $L = \mathfrak{L}_{\mathbb{A}_{\bar{k}}^1}$ is tamely ramified at infinity. □

This completes the proof of Proposition 18, and therefore also of Propositions 16 and 14.

Since the only ∞-break of K' is $d'/d < 1$ and the only ∞-break of $\mathcal{L}_\psi$ is 1, by [12], Lemma 1.3, we deduce that all ∞-breaks of $\mathcal{L}_\psi \otimes K'$ are equal to 1, and therefore $\mathrm{Swan}_\infty(\mathcal{L}_\psi \otimes K') = \dim K'_{\bar{\eta}}$. Applying the Grothendieck-Néron-Ogg-Shafarevic formula to both K' and $\mathcal{L}_\psi \otimes K'$ we get

$$(1 - d')^n = \chi_c(\mathbb{A}_{\bar{k}}^1, K') =$$

$$= \dim(K'_{\bar{\eta}}) - \mathrm{drop}_0(K') - \mathrm{Swan}_\infty(K') = \dim(K'_0) - \frac{d'}{d} \dim(K'_{\bar{\eta}})$$

and

$$\chi_c(\mathbb{A}_{\bar{k}}^n, \mathcal{L}_{\psi(f)}) = \chi_c(\mathbb{A}_{\bar{k}}^1, \mathcal{L}_\psi \otimes K') =$$

$$= \dim(K'_{\bar{\eta}}) - \mathrm{drop}_0(K') - \mathrm{Swan}_\infty(\mathcal{L}_\psi \otimes K') = \dim(K'_0) - \dim(K'_{\bar{\eta}}),$$

so

$$d' \cdot \chi_c(\mathbb{A}_{\bar{k}}^n, \mathcal{L}_{\psi(f)}) = d(1 - d')^n - (d - d') \dim(K'_0). \qquad (2)$$

It only remains to compute $\dim(K_0') = \chi_c(Y, \mathcal{L}_{\psi(f_{d'})})$, where $Y \subset \mathbb{A}^n_{\bar{k}}$ is the cone defined by the equation $f_d(x) = 0$. We have

$$\chi_c(Y, \mathcal{L}_{\psi(f_{d'})}) = 1 + \chi_c(Y - \{0\}, \mathcal{L}_{\psi(f_{d'})}).$$

Let $\pi : \mathbb{A}^n_{\bar{k}} - \{0\} \to \mathbb{P}^{n-1}_{\bar{k}}$ be the canonical projection, $Z := \pi(Y - \{0\})$ the hypersurface defined by $f_d(x) = 0$ and $P := \mathrm{R}\pi_!(\mathcal{L}_{\psi(f_{d'})}) \in \mathcal{D}^b_c(\mathbb{P}^{n-1}_{\bar{k}}, \bar{\mathbb{Q}}_\ell)$. Let $D \subset Z$ be the divisor with normal crossings defined by $f_{d'} = 0$.

Proposition 19 *The object P is smooth of rank $-d'$ on $Z - D$ and tamely ramified along D. Its restriction to D is constant of rank 0.*

Proof. The assertion is local on Z, so we can check it on each affine chart. Let U for instance be the subset defined by $x_n \neq 0$ (the proof is identical for any other affine chart). Then $(x_1, \ldots, x_{n-1}, x_n) \to (x_1/x_n, \ldots, x_{n-1}/x_n, 1)$ defines a section of the map $\pi : \pi^{-1}(U) \to U$ and gives an isomorphism $\pi^{-1}(U) \cong U \times \mathbb{G}_{m,\bar{k}}$ under which π corresponds to the first projection and $\mathcal{L}_{\psi(f_{d'})}$ corresponds to $\mathcal{L}_{\psi(\lambda^{d'} \cdot f_{d'}(x)/x_n^{d'})}$. We claim that the Galois cover V of $U - D$ defined on $\mathbb{A}^1_{\bar{k}} \times (U - D)$ by the equation $t^{d'} = f_{d'}(x)/x_n^{d'}$ trivializes P. This implies that P is smooth on $U - D$ and tamely ramified along D, since the cover has degree d' prime to p.

By proper base change, the restriction of the object P to V is given by $\mathrm{R}\pi'_!(\mathcal{L}_{\psi(\lambda^{d'} \cdot f_{d'}(x)/x_n^{d'})})$, where $\pi' : V \times \mathbb{G}_{m,\bar{k}} \to V$ is the projection. But on V

$$\mathcal{L}_{\psi(\lambda^{d'} \cdot f_{d'}(x)/x_n^{d'})} = \mathcal{L}_{\psi(\lambda^{d'} t^{d'})} = \mathcal{L}_{\psi((\lambda t)^{d'})}.$$

Now consider the V-automorphism ϕ of $V \times \mathbb{G}_{m,\bar{k}}$ given by $\phi((t,x),\lambda) = ((t,x),\lambda/t)$. We have

$$\mathrm{R}\pi'_!(\mathcal{L}_{\psi((\lambda t)^{d'})}) = \mathrm{R}(\pi' \circ \phi)_!(\phi^\star \mathcal{L}_{\psi((\lambda t)^{d'})}) = \mathrm{R}\pi'_! \mathcal{L}_{\psi(\lambda^{d'})}$$

which is clearly constant of rank $\chi_c(\mathbb{G}_{m,\bar{k}}, \mathcal{L}_{\psi(\lambda^{d'})}) = \chi_c(\mathbb{A}^1_{\bar{k}}, \mathcal{L}_{\psi(\lambda^{d'})}) - 1 = (1 - d') - 1 = -d'$.

On D, the object P is $\mathrm{R}\pi_!(\mathcal{L}_{\psi(f_{d'})|\{f_{d'}=0\}}) = \mathrm{R}\pi_!(\bar{\mathbb{Q}}_{\ell|\pi^{-1}(D)})$. Again by restricting to the affine charts, one checks that π is locally isomorphic to a projection $U \times \mathbb{G}_{m,\bar{k}} \to U$, and therefore P restricted to D is constant of rank $\chi_c(\mathbb{G}_{m,\bar{k}}, \bar{\mathbb{Q}}_\ell) = 0$. □

Since P is tamely ramified along D, we get

$$\chi_c(Y - \{0\}, \mathcal{L}_{\psi(f_{d'})}) = \chi_c(Z, P) = \chi_c(Z - D, P) + \chi_c(D, P) =$$

$$= \mathrm{rank}(P_{|Z-D}) \cdot \chi_c(Z - D) + \mathrm{rank}(P_{|D}) \cdot \chi_c(D) = -d' \cdot \chi_c(Z - D)$$

Substituting this value in (2), we conclude that

$$d' \cdot \chi_c(\mathbb{A}^n_{\bar{k}}, \mathcal{L}_{\psi(f)}) = d(1 - d')^n - (d - d')(1 - d' \cdot \chi_c(Z - D)),$$

so

$$\chi_c(\mathbb{A}^n_k, \mathcal{L}_{\psi(f)}) = 1 + (-1)^n d \frac{(d'-1)^n - (-1)^n}{d'} + (d-d')\chi_c(Z-D).$$

Theorem 12 follows by excision and the inclusion-exclusion principle, since

$$\chi_c(Z-D) = \chi_c(Z) - \chi_c(D) =$$

$$= \chi_c(\{g_1 \cdots g_r = 0\}) - \chi_c(\{g_1 \cdots g_r = 0, f_{d'} = 0\}) =$$

$$= \chi_c(\bigcup_{i=1,\ldots,r} \{g_i = 0\}) - \chi_c(\bigcup_{i=1,\ldots,r} \{g_i = 0, f_{d'} = 0\}).$$

5 Examples

In this section we will simplify the rank formula in several important particular cases.

1) Suppose that $r = 1$, so $f_d = g^\alpha$ for some non-singular homogeneous form $g \in k[x_1, \ldots, x_n]$ of degree e. We will use the following well known formulas, which can be deduced for instance from [8], Exposé XVII, 5.7.5:

$$\chi(n; d) = n + 1 + \frac{(1-d)^{n+1} - 1}{d}$$

$$\chi(n; d_1, d_2) = n + 1 + \frac{1}{d_1 - d_2} \left(d_1 \frac{(1-d_2)^{n+1} - 1}{d_2} - d_2 \frac{(1-d_1)^{n+1} - 1}{d_1} \right)$$

if $d_1 \neq d_2$, and

$$\chi(n; d, d) = n + 1 + (n+1)(1-d)^n + 2\frac{(1-d)^{n+1} - 1}{d}.$$

Then formula (1) for the rank becomes in this case

$$M(n, d, d', 1, (e), (\alpha)) =$$

$$= (-1)^n + d\frac{(d'-1)^n - (-1)^n}{d'} + (-1)^n (d-d')(\chi(n-1; e) - \chi(n-1; d', e)) =$$

$$= (-1)^n + d\frac{(d'-1)^n - (-1)^n}{d'} +$$

$$+(d-d') \left[\frac{(e-1)^n - (-1)^n}{e} - \frac{1}{d'-e} \left(d' \frac{(e-1)^n - (-1)^n}{e} - e\frac{(d'-1)^n - (-1)^n}{d'} \right) \right] =$$

$$= (-1)^n + d' \frac{d-e}{d'-e} \frac{(d'-1)^n - (-1)^n}{d'} - e\frac{d-d'}{d'-e} \frac{(e-1)^n - (-1)^n}{e} =$$

$$= \frac{1}{d'-e}[(d-e)(d'-1)^n - (d-d')(e-1)^n]$$

if $d' \neq e$. If $d' = e$, a similar computation gives

$$M(n, d, e, 1, (e), (\alpha)) = n(d-e)(e-1)^{n-1} + (e-1)^n.$$

In particular, when $d = e$ we are in the classical non-singular case, and the rank becomes

$$\frac{-(d-d')(d-1)^n}{d'-d} = (d-1)^n$$

as expected. When $e = 1$ we are in the other extremal case studied in the previous section and the rank is

$$\frac{(d-1)(d'-1)^n}{d'-1} = (d-1)(d'-1)^{n-1}$$

as we found there.

2) Now let $e_i = 1$ for all $i = 1, \ldots, r$. Then $\chi(n-1; e_I) = n - |I|$ and

$$\chi(n-1; d', e_I) = n - |I| + \frac{(1-d')^{n-|I|} - 1}{d'}$$

for all $I \subseteq \{1, \ldots, r\}$. Plugging these values in formula (1) we get

$$\chi = \sum_{\substack{I \subseteq \{1,\ldots,r\} \\ 1 \leq |I| \leq n-1}} (-1)^{|I|-1}(n-|I|) - \sum_{\substack{I \subseteq \{1,\ldots,r\} \\ 1 \leq |I| \leq n-2}} (-1)^{|I|-1} \left(n - |I| + \frac{(1-d')^{n-|I|} - 1}{d'} \right) =$$

$$= \sum_{k=1}^{n-1} \binom{r}{k} (-1)^{k-1}(n-k) - \sum_{k=1}^{n-2} \binom{r}{k} (-1)^{k-1} \left(n - k + \frac{(1-d')^{n-k} - 1}{d'} \right) =$$

$$= (-1)^n \binom{r}{n-1} - \sum_{k=1}^{n-2} \binom{r}{k} (-1)^{k-1} \frac{(1-d')^{n-k} - 1}{d'} =$$

$$= -\sum_{k=1}^{n} \binom{r}{k} (-1)^{k-1} \frac{(1-d')^{n-k} - 1}{d'}$$

so the rank is

$$(-1)^n + d\frac{(d'-1)^n - (-1)^n}{d'} + (d-d') \sum_{k=1}^{n} \binom{r}{k} \frac{(d'-1)^{n-k} - (-1)^{n-k}}{d'} =$$

$$= (d'-1)^n + (d-d') \sum_{k=0}^{n} \binom{r}{k} \frac{(d'-1)^{n-k} - (-1)^{n-k}}{d'}.$$

In order to further simplify it, we need to assume $r \leq n$. Then the sum becomes a complete binomial expansion:

$$\sum_{k=0}^{n} \binom{r}{k} \frac{(d'-1)^{n-k} - (-1)^{n-k}}{d'} =$$

$$= \frac{(d'-1)^{n-r}}{d'} \sum_{k=0}^{n} \binom{r}{k} (d'-1)^{r-k} - \frac{(-1)^{n-r}}{d'} \sum_{k=0}^{n} \binom{r}{k} (-1)^{r-k} =$$

$$= \frac{(d'-1)^{n-r}}{d'} (d'-1+1)^r - \frac{(-1)^{n-r}}{d'} (1-1)^r = (d'-1)^{n-r} d'^{r-1}$$

and the rank becomes

$$(d'-1)^n + (d-d')(d'-1)^{n-r} d'^{r-1} =$$
$$= (d'-1)^{n-r}(dd'^{r-1} - d'^r + (d'-1)^r).$$

If $r = n$, all $\alpha_i = 1$ and $d' = 1$, one particular example of admissible sum is

$$\sum_{x \in k^n} \psi(x_1 \cdots x_n + x_1 + \cdots + x_{n-1} - a x_n)$$

for any $a \neq 0$ in k. But

$$\sum_{x \in k^n} \psi(x_1 \cdots x_n + x_1 + \cdots + x_{n-1} - a x_n) =$$

$$= \sum_{x_1, \ldots, x_{n-1}} \psi(x_1 + \cdots + x_{n-1}) \sum_{x_n} \psi(x_n(x_1 \cdots x_{n-1} - a)) =$$

$$= q \sum_{x_1 \cdots x_{n-1} = a} \psi(x_1 + \cdots + x_{n-1})$$

which is just a Tate-twisted Kloosterman sum. So Theorem 1 gives a new proof of the purity of Kloosterman sums, at least in the case where the number of variables is not congruent to -1 mod p. The formula for the rank gives in this case $M = d - 1 = n - 1$, which agrees with the previously known result (cf. [6], Théorème 7.4).

Similarly, if $r = n$, $\alpha_n = 1$ and $d' = 1$, we have the sum

$$\sum_{x \in k^n} \psi(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n + x_1 + \cdots + x_{n-1} - a x_n) =$$

$$= \sum_{x_1, \ldots, x_{n-1}} \psi(x_1 + \cdots + x_{n-1}) \sum_{x_n} \psi(x_n(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} - a)) =$$

$$= q \sum_{x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} = a} \psi(x_1 + \cdots + x_{n-1})$$

which is a Tate twist of the more general kind of Kloosterman sum considered in [12]. The rank in this case is $d - 1$ which again agrees with the previously known value (cf. [12], Theorem 4.1.1). Therefore we can view the case $r = n$ as a natural generalization of a Kloosterman sum:

Theorem 20 *For any $a_1, \ldots, a_n \in k^\star$ and any prime to p positive integers $\alpha_1, \ldots, \alpha_n$ and $d' < \alpha_1 + \cdots + \alpha_n$ such that $\alpha_1 + \ldots + \alpha_n$ is prime to p, the sum*

$$\sum_{x \in k^n} \psi(x_1^{\alpha_1} \cdots x_n^{\alpha_n} + a_1 x_1^{d'} + \cdots + a_n x_n^{d'})$$

is cohomologically pure of weight n and rank $d'^{n-1}(\sum \alpha_i - d') + (d'-1)^n$.

3) The previous example includes all admissible polynomials in two variables, since every homogeneous form in two variables splits as a product of linear factors in the algebraic closure of the base field k. The condition for a polynomial to be admissible becomes much simpler in this case, it means that we can write $f = f_d + f_{d'} + f'$ with f_d homogeneous of degree d prime to p, $f_{d'}$ homogeneous and square-free of degree $d' < d$ prime to p and f' of degree $< d'$, and $\gcd(f_d, f_{d'}) = 1$. In that case, the rank of the sum $\sum_{x \in k^n} \psi(f(x))$ is

$$(-1)^2 + d(d'-2) + (d-d') \sum_{i=1}^{r} \chi(1;1) = 1 + d(d'-2) + r(d-d').$$

The highest homogeneous form f_d will be non-singular precisely when it has d distinct linear factors, i.e. when $r = d$, and then the rank becomes $1 + d(d' - 2) + d(d-d') = d^2 - 2d + 1 = (d-1)^2$, as expected.

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