KATZ-RADON TRANSFORM OF $\ell$-ADIC REPRESENTATIONS

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Abstract. We prove a simple explicit formula for the local Katz-Radon transform of an $\ell$-adic representation of the Galois group of the fraction field of a strictly henselian discrete valuation ring with positive residual characteristic, which can be defined as the local additive convolution with a fixed tame character. The formula is similar to one proved by D. Arinkin in the $\mathcal{D}$-module setting, and answers a question posed by N. Katz.

1. Introduction

In [10, 3.4.1], N. Katz defines some functors on the category of continuous $\ell$-adic representations of the inertia groups $I_0$ and $I_\infty$ of the projective line over $\bar{k}$ at 0 and infinity, where $\bar{k}$ is the algebraic closure of a finite field of characteristic $p$ and $\ell$ is a prime different from $p$. These functors arise during his study of middle convolution of sheaves on the affine line and, roughly speaking, correspond to locally convolving a representation with a fixed tame character $L_\chi$ of $I_0$ or $I_\infty$. They are defined using G. Laumon’s local Fourier transform functors, and in fact correspond to taking the tensor product with the conjugate tame character $L_{\bar{\chi}}$ on the other side of the equivalence of categories given by these functors. Katz asks [10, 3.4.1] whether there is a simple expression for the functors defined in this way.

Recently, D. Arinkin [1] has studied the analog of Katz’s functor in $\mathcal{D}$-module theory: if $K$ is a field of characteristic 0, $K((x))$ is the field of Laurent series over $K$ and $\mathcal{D}_x$ the ring of differential operators with coefficients in $K((x))$, the local Katz-Radon transform for a given $\chi \in K - \mathbb{Z}$ is an equivalence of categories $\rho_\chi : \mathcal{D}_x\text{-mod} \to \mathcal{D}_x\text{-mod}$, originally defined in [3]. Arinkin proves the simple formula [1 Theorem C]

$$\rho_\chi(F) \cong F \otimes K^{\chi_1 + 1}$$

for any $F \in \mathcal{D}_x\text{-mod}$ with a single slope $\mu$, where $K^\mu$ is the Kummer $\mathcal{D}_x$-module of rank 1 generated by $e$, on which the derivative acts by

$$\frac{d}{dx} e = \frac{\mu}{x} e.$$

In this article we will prove a similar formula in the $\ell$-adic case. More precisely, for a fixed tame $\ell$-adic character $L_\chi$ and an $\ell$-adic representation $F$ of $I_0$, let

$$\rho_\chi(F) := FT^{\psi}_{(0,\infty)}(L_\chi \otimes FT^{\psi}_{(0,\infty)} F)$$

where $FT^{\psi}_{(0,\infty)}$ denotes Laumon’s local Fourier transform functor. If $F$ has a single slope $a = c/d$ (with $c, d$ relatively prime positive integers), we will prove that there is an isomorphism of $I_0$-representations

$$\rho_\chi(F) \cong F \otimes L_\chi^{\otimes (a+1)}$$

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where $L^\otimes(a+1)$ is any $d$-th root of the character $L^\otimes(c+d)$.

For a large class of representations $\mathcal{F}$ of $I_0$ (in particular for many of those who appear in applications), the isomorphism can be proven via the explicit formulas for the local Fourier transforms given by L. Fu [3] and A. Abbes and T. Saito [2]. In this article we take a different approach that works for any $\mathcal{F}$, and is independent of any explicit expression for the local Fourier transforms.

2. The Katz-Radon transform

Fix a finite field $k$ of characteristic $p > 0$ and an algebraic closure $\bar{k}$. Let $\mathbb{P}^1_k$ be the projective line over $\bar{k}$ and, for every $t \in \mathbb{P}^1(\bar{k}) = \bar{k} \cup \{\infty\}$, denote by $I_t$ its inertia group at $t$: for $t \neq \infty$, if $x - t$ denotes a local coordinate at $t$, it is the Galois group of the fraction field of the henselization of the local ring $\bar{k}[x]_{(x-t)}$. We have an exact sequence [8, 1.0]

$$0 \to P_t \to I_t \to \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 0$$

for every $t \in \mathbb{P}^1(\bar{k})$, where $P_t$ is the only $p$-Sylow subgroup of $I_t$. Moreover, there is a canonical filtration of $I_t$ by the higher ramification groups

$$I_t^{(r)} \supseteq I_t^{(s)} \text{ for } 0 \leq r < s \in \mathbb{R}$$

which are normal in $I_t$.

Fix a prime $\ell \neq p$, and denote by $\mathcal{R}_t$ the abelian category of continuous $\ell$-adic representations of $I_t$ (i.e. continuous representations $\mathcal{F} : I_t \to \text{GL}_n(\mathbb{Q}_\ell)$, whose image is in $\text{GL}_n(E_\lambda)$ for some finite extension $E_\lambda$ of $\mathbb{Q}_\ell$). For every irreducible $\mathcal{F} \in \mathcal{R}_t$, the slope of $\mathcal{F}$ is $\inf\{r \geq 0|\mathcal{F}_{I_t^{(r)}} \text{ is trivial}\}$. It is a non-negative rational number. In general, the slopes of $\mathcal{F}$ are the slopes of the irreducible components of $\mathcal{F}$. For every $\mathcal{F}$ there is a canonical direct sum decomposition [8, Lemma 1.8]

$$\mathcal{F} \cong \bigoplus_{r \geq 0} \mathcal{F}^r$$

with $\mathcal{F}^r$ having a single slope $r$. The slope 0 (tame) part will be denoted by $\mathcal{F}^t$. $\mathcal{F}$ is said to be tame (respectively totally wild) if $\mathcal{F} = \mathcal{F}^t$ (resp. $\mathcal{F}^t = 0$).

For every $r \geq 0$ let $\mathcal{R}^r_t$ denote the full subcategory of $\mathcal{R}_t$ consisting of representations with a single slope $r$. We have a decomposition

$$\mathcal{R}_t = \bigoplus_{r \geq 0} \mathcal{R}^r_t$$

in the sense that every $\mathcal{F} \in \mathcal{R}_t$ has a decomposition [1] and $\text{Hom}_{\mathcal{R}_t}(\mathcal{F}, \mathcal{G}) = 0$ if $\mathcal{F} \in \mathcal{R}^r_t$, $\mathcal{G} \in \mathcal{R}^s_t$ and $r \neq s$ [8, Proposition 1.1].

Let $k' \subseteq \bar{k}$ be a finite extension of $k$, and $\chi : k'^\times \to \hat{\mathbb{Q}}^\times_\ell$ a multiplicative character. By [4, 1.4-1.8] there is an associated smooth Kummer sheaf $L_\chi$ on $\mathbb{G}_{m,k}$, which is a tame character of $I_0$ (and of $I_\infty$) of the same order as $\chi$. If $k'' \subseteq k'$ is another extension, the sheaves defined by $\chi$ and $\chi \circ \text{Nm}_{k''/k'} : k''^\times \to \hat{\mathbb{Q}}^\times_\ell$ are isomorphic.

Moreover, every tame character of $I_0$ (and of $I_\infty$) can be obtained in this way. Whenever we speak about a tame character of $I_0$, we will implicitly assume that we have made a choice of such a finite extension of $k$ and of a character.
Fix a non-trivial additive character \( \psi : k \to \bar{Q}_p^\times \). The local Fourier transform functors, defined by G. Laumon in [11], give equivalences of categories
\[
\text{FT}^\psi_{(0,\infty)} : \mathcal{R}_0 \to \mathcal{R}_{<1}^{<1},
\]
\[
\text{FT}^\psi_{(\infty,0)} : \mathcal{R}_{>1}^{>1} \to \mathcal{R}_\infty^{<1}
\]
and
\[
\text{FT}^\psi_{(\infty,0)} : \mathcal{R}_{<1}^{<1} \to \mathcal{R}_0
\]
(where \( \mathcal{R}_{<1}^{<1} = \bigoplus_{r<1} \mathcal{R}_r^{<1} \) and \( \mathcal{R}_{>1}^{>1} = \bigoplus_{r>1} \mathcal{R}_r^{>1} \)) that describe the relationship between the local monodromies of an \( \ell \)-adic sheaf on \( \mathbb{A}^1_k \) and its Fourier transform with respect to \( \psi \). The Katz-Radon transform is defined in terms of them.

**Definition 2.1.** Fix a tame character \( \mathcal{L}_\chi \) of \( I_0 \). The (local) Katz-Radon transform (with respect to \( \mathcal{L}_\chi \)) is the functor \( \rho_\chi : \mathcal{R}_0 \to \mathcal{R}_0 \) given by
\[
\rho_\chi(\mathcal{F}) = \text{FT}^{-1}_{(0,\infty)}(\text{FT}^\psi_{(0,\infty)} \mathcal{L}_\chi \otimes \text{FT}^\psi_{(0,\infty)} \mathcal{F}) = \text{FT}^{-1}_{(0,\infty)}(\mathcal{L}_\chi \otimes \text{FT}^\psi_{(0,\infty)} \mathcal{F}).
\]

The Katz-Radon transform is an auto-equivalence of the category \( \mathcal{R}_0 \) (since it is a composition of three equivalences of categories). It preserves dimensions and slopes, and for tame \( \mathcal{F} \) it is given by \( \rho_\chi(\mathcal{F}) = \mathcal{F} \otimes \mathcal{L}_\chi \) [10, 3.4.1]. For totally wild \( \mathcal{F} \), it can be interpreted as the “local additive convolution” of \( \mathcal{F} \) and \( \mathcal{L}_\chi \) [10, 3.4.3]: if we extend \( \mathcal{F} \) to a smooth sheaf on \( \mathbb{G}_{m,k} \), tamely ramified at infinity, then \( \rho_\chi(\mathcal{F}) \) is the wild part of the local monodromy at 0 of \( \mathcal{F} \otimes \mathcal{L}_\chi \), where
\[
\mathcal{F} \otimes \mathcal{L}_\chi = \mathcal{R}^1 \sigma_!(\mathcal{F} \boxtimes \mathcal{L}_\chi)
\]
and \( \sigma : \mathbb{A}^2_k \to \mathbb{A}^1_k \) denotes the addition map (in [10], the “middle convolution” is used instead, but that one differs from the one used here only by Artin-Shreier components, which are smooth at 0 and therefore do not affect the local monodromy). Notice that, in particular, \( \rho_\chi \) is independent of the choice of the additive character \( \psi \).

More intrinsically, it can be described in terms of vanishing cycles functors [11, 2.7.2]: If \( X = \mathbb{A}^2_{m,(1,1)} \) (respectively \( S = \mathbb{A}^1_{m,(1)} \)) denotes the henselization of \( \mathbb{A}^2_k \) at \( (0,0) \) (resp. the henselization of \( \mathbb{A}^1_k \) at 0) then \( \rho_\chi(\mathcal{F}) \cong \mathcal{R}^1 \Phi(\sigma, \mathcal{F} \boxtimes \mathcal{L}_\chi)_{(0,0)} \), where \( \mathcal{R} \Phi(\sigma, \mathcal{F} \boxtimes \mathcal{L}_\chi) \) is the vanishing cycles complex for the addition map \( \sigma : X \to S \) with respect to the sheaf \( \mathcal{F} \boxtimes \mathcal{L}_\chi \) on \( X \).

Similarly, it also has an interpretation as a “local multiplicative convolution” [12, Corollary 5.6]: If \( X = \mathbb{G}^2_{m,(1,1)} \) (respectively \( S = \mathbb{G}_{m,(1)} \)) denotes the henselization of \( \mathbb{G}_{m,k} \) at \( (1,1) \) (resp. the henselization of \( \mathbb{G}_{m,k} \) at 1) then \( \rho_\chi(\mathcal{F}) \cong \mathcal{R}^1 \Phi(\mu, \mathcal{F} \boxtimes \mathcal{L}_\chi)(1,1) \), where \( \mathcal{R} \Phi(\mu, \mathcal{F} \boxtimes \mathcal{L}_\chi) \) is the vanishing cycles complex for the multiplication map \( \mu : X \to S \) with respect to the sheaf \( \mathcal{F} \boxtimes \mathcal{L}_\chi \) on \( X \), and \( \mathcal{F} \) and \( \mathcal{L}_\chi \) are viewed as representations of \( I_1 \) via the isomorphism \( I_0 \cong I_1 \) that maps the uniformizer \( x \) at 0 to the uniformizer \( x-1 \) at 1.

The main result of this article is the following simple expression for \( \rho_\chi \):

**Theorem 2.2.** Let \( \mathcal{F} \in \mathcal{R}_0 \) be totally wild with a single slope \( a > 0 \). Write \( a = c/d \), where \( c \) and \( d \) are relatively prime positive integers. Let \( \mathcal{L}_\eta \) be any tame character of \( I_0 \) such that \( \mathcal{L}_\eta^{\otimes d} = \mathcal{L}_\chi^{\otimes (c+d)} \). Then
\[
\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta.
\]
Lemma 3.2. Let \( \sigma \) where \( n \) and \( \tau \) where \( r \). We have
\[
\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi^{(\sigma+1)}
\]
where \( \mathcal{L}_\chi^{(\sigma+1)} \) stands for “any character that can reasonably be called \( \mathcal{L}_\chi^{(\sigma+1)} \).”

By the decomposition \( \mathcal{R}_0 = \bigoplus_{r \geq 0} \mathcal{R}_0^r \), this determines \( \rho_\chi(\mathcal{F}) \) for any \( \mathcal{F} \in \mathcal{R}_0 \), thus answering the question posed by N. Katz in [10, 3.4.1].

A question that remains open is the following: in the article we prove that \( \rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_n \), independently for any \( \mathcal{F} \) with slope \( a \). So the functors \( \mathcal{R}_0^0 \to \mathcal{R}_0^0 \) given by \( \rho_\chi \) and \( (-) \otimes \mathcal{L}_n \) map any \( \mathcal{F} \) to isomorphic objects. Is there an actual isomorphism of functors between them? In the affirmative case, is there a simple way to construct it?

3. Proof of the main theorem

In this section we will prove theorem 2.2. We will start with the case where \( \mathcal{F} \in \mathcal{R}_0 \) is irreducible.

**Lemma 3.1.** Let \( \mathcal{F} \in \mathcal{R}_0 \). Then \( \mathcal{F}^I \neq 0 \) if and only if there exists \( \epsilon > 0 \) such that for every \( \mathcal{G} \in \mathcal{R}_0 \) with a single slope \( b \in (0, \epsilon) \) we have
\[
\text{Swan}(\mathcal{F} \otimes \mathcal{G}) > \text{Swan}(\mathcal{F}) \dim(\mathcal{G}).
\]

**Proof.** Suppose that \( \mathcal{F}^I \neq 0 \), and let \( a_0 = 0 < a_1 < \cdots < a_r \) be the slopes of \( \mathcal{F} \), with multiplicities \( n_0, n_1, \ldots, n_r \). Then \( \text{Swan}(\mathcal{F}) = \sum n_i a_i \). Let \( \epsilon = a_1 \). Then for every \( \mathcal{G} \in \mathcal{R}_0 \) with a single slope \( b \in (0, \epsilon) \) the tensor product \( \mathcal{F} \otimes \mathcal{G} \) has slopes \( b < a_1 < \cdots < a_r \) with multiplicities \( n_0 m, n_1 m, \ldots, n_r m \) where \( m = \dim(\mathcal{G}) \) by [S] Lemma 1.3. Therefore
\[
\text{Swan}(\mathcal{F} \otimes \mathcal{G}) = n_0 mb + \sum_{i=1}^r n_i ma_i > \sum_{i=1}^r n_i ma_i = \text{Swan}(\mathcal{F}) \dim(\mathcal{G}).
\]
Conversely, suppose that \( \mathcal{F}^I = 0 \), and let \( a_1 < \cdots < a_r \) be the slopes of \( \mathcal{F} \). Then for every \( \mathcal{G} \in \mathcal{R}_0 \) with a single slope \( b \in (0, a_1) \) the tensor product \( \mathcal{F} \otimes \mathcal{G} \) has the same slopes as \( \mathcal{F} \) by [S] Lemma 1.3, and in particular \( \text{Swan}(\mathcal{F} \otimes \mathcal{G}) = \text{Swan}(\mathcal{F}) \dim(\mathcal{G}) \). This proves the lemma, since for every \( \epsilon > 0 \) there exist representations in \( \mathcal{R}_0 \) with slope \( b \in (0, \epsilon) \) (for instance, one may take \([n_1, \mathcal{H}] \), where \( \mathcal{H} \in \mathcal{R}_0 \) has slope \( a > 0 \) and \( n \) is a prime to \( p \) integer greater than \( a/\epsilon \) [S 1.13.2]).

For any two objects \( K, L \in \mathcal{D}_c^0(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell) \), we will denote by \( K \ast L \in \mathcal{D}_c^0(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell) \) their additive convolution:
\[
K \ast L = \text{R}\sigma_1(\text{K} \boxtimes \text{L})
\]
where \( \sigma : \mathbb{A}_k^2 \to \mathbb{A}_k^1 \) is the addition map.

**Lemma 3.2.** Let \( K, L, M \in \mathcal{D}_c^0(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell) \). Then
\[
\text{R}\Gamma_c(\mathbb{A}_k, (K \ast L) \otimes M) \cong \text{R}\Gamma_c(\mathbb{A}_k, K \otimes ((\tau_{-1} \ast L) \ast M))
\]
where \( \tau_{-1} : \mathbb{A}_k^1 \to \mathbb{A}_k^1 \) is the additive inversion.

**Proof.** We have
\[
\text{R}\Gamma_c(\mathbb{A}_k, (K \ast L) \otimes M) = \text{R}\Gamma_c(\mathbb{A}_k, \text{R}\sigma_1(\text{K} \boxtimes \text{L}) \otimes M) = \text{R}\Gamma_c(\mathbb{A}_k, \text{R}\sigma((\text{K} \boxtimes \text{L}) \otimes \sigma^* M)) = \text{R}\Gamma_c(\mathbb{A}_k, (\text{K} \boxtimes \text{L}) \otimes \sigma^* M)
\]
by the projection formula. If \( \pi_1, \pi_2 : \mathcal{A}_k^2 \to \mathcal{A}_k^1 \) are the projections then
\[
\text{RG}_c(\mathcal{A}_k^2, (K \boxtimes L) \otimes \sigma^* M) = \text{RG}_c(\mathcal{A}_k^2, \pi_1^* K \otimes \pi_2^* L \otimes \sigma^* M).
\]
Consider the automorphism \( \phi : \mathcal{A}_k^2 \to \mathcal{A}_k^2 \) given by \((x, y) \mapsto (x + y, -y)\). Then \( \sigma = \pi_1 \circ \phi, \pi_1 = \sigma \circ \phi \) and \( \tau_{-1} \circ \pi_2 = \pi_2 \circ \phi \). It follows that
\[
\text{RG}_c(\mathcal{A}_k^2, \pi_1^* K \otimes \pi_2^* L \otimes \sigma^* M) \cong \text{RG}_c(\mathcal{A}_k^2, \phi^* \pi_1^* K \otimes \phi^* \pi_2^* L \otimes \phi^* \sigma^* M) = \\
\cong \text{RG}_c(\mathcal{A}_k^2, \sigma^* K \otimes \pi_2^* \tau_{-1} L \otimes \pi_1^* M) = \text{RG}_c(\mathcal{A}_k^2, \text{RG}(\sigma^* K \otimes \pi_2^* \tau_{-1} L \otimes \pi_1^* M)) \cong \\
\cong \text{RG}_c(\mathcal{A}_k^2, K \otimes \text{RG}(\tau_{-1} L \otimes M)) = \text{RG}_c(\mathcal{A}_k^2, K \otimes ((\tau_{-1} L) * M)).
\]

\[\square\]

If \( \mathcal{F} \) is a smooth \( \mathbb{Q}_p \)-sheaf on \( G_{m,k} \) which is totally wild at 0, then for every \( t \in \bar{k} \) the sheaf \( \mathcal{F} \otimes \mathcal{L}_\chi(t-x) \) (extended by zero to \( \mathcal{A}_k^1 \)) is totally wild at 0 and has no punctual sections (where \( \mathcal{L}_\chi(t-x) \) is the pull-back of the Kummer sheaf \( \mathcal{L}_\chi \) under the map \( x \mapsto t-x \)), so its only non-zero cohomology group with compact support is \( H_1^1 \). We conclude that the only non-zero cohomology sheaf of \( \mathcal{F}[0] * \mathcal{L}_\chi[0] \in D^b_c(\mathcal{A}_k^1, \mathbb{Q}_p) \) is \( H_1^1 = R^1\sigma(\mathcal{F} \otimes \mathcal{L}_\chi) \). We will denote this sheaf by \( \mathcal{F} * \mathcal{L}_\chi \).

**Lemma 3.3.** Let \( \mathcal{F}, \mathcal{G} \in \mathcal{R}_0 \). Then
\[
\text{Swan}(\rho(x) (\mathcal{F} \otimes \mathcal{G})) = \text{Swan}(\mathcal{F} \otimes \mathcal{G}).
\]

**Proof.** By additivity of the Swan conductor, we may assume that \( \mathcal{F} \) is irreducible, and in particular that it has a single slope \( a \geq 0 \). If \( a = 0 \) then \( \rho(x) (\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi \), so the equality is clear. Suppose that \( a > 0 \). By [7, Theorem 1.5.6], \( \mathcal{F} \) and \( \mathcal{G} \) can be extended to smooth sheaves on \( G_{m,k} \), namely ramified at infinity, which we will also denote by \( \mathcal{F} \) and \( \mathcal{G} \). Let \( \mathcal{F} \) and \( \mathcal{G} \) be also their extensions by zero to \( \mathcal{A}_k^1 \).

Using the compatibility between Fourier transform with respect to \( \psi \) and convolution [11, Proposition 1.2.2.7], we have
\[
\mathcal{F} * \mathcal{L}_\chi = \text{FT}^\psi(\text{FT}^\psi \mathcal{F} \otimes \text{FT}^\psi \mathcal{L}_\chi) = \text{FT}^\psi(\text{FT}^\psi \mathcal{F} \otimes \mathcal{L}_\chi),
\]
where \( \text{FT}^\psi \mathcal{F} \) denotes the “naive” Fourier transform in the sense of [8, 8.2], that is, the \((-1)\)-th cohomology sheaf of the Fourier transform of \( \mathcal{F}[1] \in D^b_c(\mathcal{A}_k^1, \mathbb{Q}_p) \) (which is its only non-zero cohomology sheaf, since \( \mathcal{F} \) is totally wild at zero and therefore it is Fourier [8, Lemma 8.3.1]).

Let \( n \) be the rank of \( \mathcal{F} \), and denote by \( \mathcal{F}(\infty) \in \mathcal{R}_\infty \) its local monodromy at infinity, which is a tame representation of \( I_\infty \). By Ogg-Shafarevic [3, Exposé X, Corollaire 7.12], \( \text{FT}^\psi \mathcal{F} \) is smooth on \( G_{m,k} \) of rank \( na + n = n(a+1) \). By Laumon’s local Fourier transform theory [9, Theorem 13], \( \text{FT}^\psi \mathcal{F} \) has a single slope \( \frac{a}{a+1} \) at infinity, with multiplicity \( n(a+1) \), and its monodromy at 0 has a trivial part of dimension \( na \) and its quotient is the dual \( \mathcal{F}_\infty(\infty) \) of \( \mathcal{F}_\infty \). Then \( \text{FT}^\psi \mathcal{F} \otimes \mathcal{L}_\chi \) also has a single slope \( \frac{a}{a+1} \) at infinity with multiplicity \( n(a+1) \), and its monodromy \( \mathcal{M} \) at 0 sits in an exact sequence
\[
0 \to \mathcal{L}_\chi^{na} \to \mathcal{M} \to \mathcal{F}_\infty(\infty) \otimes \mathcal{L}_\chi \to 0.
\]
Its inverse Fourier transform, by Ogg-Shafarevic, is smooth of rank \( n(a+1) \) on \( G_{m,k} \), and by local Fourier transform its wild part at 0 has slope \( a \) with multiplicity \( n \).
In fact, this wild part is simply $\rho_\chi(\mathcal{F})$ by the additive convolution interpretation of $\rho_\chi$. Its monodromy at infinity sits in an exact sequence
\begin{equation}
0 \rightarrow \mathcal{L}_\chi^{\otimes n} \rightarrow (\mathcal{F} \ast \mathcal{L}_\chi)(\infty) \rightarrow \mathcal{F}(\infty) \otimes \mathcal{L}_\chi \rightarrow 0
\end{equation}

obtained from (3) by local Fourier transform.

So $\mathcal{F} \ast \mathcal{L}_\chi$ has rank $n(a + 1)$ on $\mathbb{G}_{m,k}$, and its monodromy at $0$ is the direct sum of $\rho_\chi(\mathcal{F})$ and a constant part of dimension $na = \text{Swan}(\mathcal{F})$. So

$$\text{Swan}_0((\mathcal{F} \ast \mathcal{L}_\chi) \otimes \mathcal{G}) = \text{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) + \text{Swan}(\mathcal{F})\text{Swan}(\mathcal{G}).$$

In particular, by Ogg-Shafarevic, the Euler characteristic of the sheaf $(\mathcal{F} \ast \mathcal{L}_\chi) \otimes \mathcal{G}$ (extended by zero to $A^1_k$) is $-\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) - \text{Swan}(\mathcal{F})\text{Swan}(\mathcal{G})$. Using lemma \ref{lem:3.2} proper base change, and the fact that $\chi(G_m \otimes K \otimes \mathcal{L}_\chi) = \chi(G_m \otimes K)$ for any object $K \in D^b(G_m,k,\mathbb{Q}_\ell)$, we get

\begin{align*}
\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) + \text{Swan}(\mathcal{F})\text{Swan}(\mathcal{G}) & = \chi(A^1_k, (\mathcal{F}[1] \ast \mathcal{L}_\chi[1]) \otimes \mathcal{G}) = \\
& = \chi(A^1_k, \mathcal{L}_\chi \otimes (\tau_1^* \mathcal{F}[1] \ast \mathcal{G}[1])) = \chi(G_m,k, \tau_1^* \mathcal{F}[1] \ast \mathcal{G}[1]) = \\
& = \chi(A^1_k, \tau_1^* \mathcal{F}[1] \ast \mathcal{G}[1]) - \text{rank}_0(\tau_1^* \mathcal{F}[1] \ast \mathcal{G}[1]) = \\
& = \chi(A^1_k, \mathcal{F}[1]\chi(A^1_k, \mathcal{G}[1]) - \chi(A^1_k, \mathcal{F}[1] \otimes \mathcal{G}[1]) = \\
& = \text{Swan}(\mathcal{F})\text{Swan}(\mathcal{G}) + \text{Swan}(\mathcal{F} \otimes \mathcal{G})
\end{align*}

where rank$_0$ of a derived category object denotes the alternating sum of the ranks at 0 of its cohomology sheaves, so Swan$(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) = \text{Swan}(\mathcal{F} \otimes \mathcal{G})$.

\begin{proposition}
Let $\mathcal{F} \in \mathcal{R}_0$ be totally wild and irreducible. Then there exists a tame character $\mathcal{L}_\eta$ of $I_0$ such that $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$.
\end{proposition}

\begin{proof}
Let $\widehat{\mathcal{F}}$ be the dual representation. We claim that the tame part of $\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}$ is non-zero. By lemma \ref{lem:3.1} it suffices to show that there is an $\epsilon > 0$ such that, for any $\mathcal{G} \in \mathcal{R}_0$ with slope $b \in (0, \epsilon)$, $\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}} \otimes \mathcal{G}) > \text{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}) \dim(\mathcal{G})$.

But by lemma \ref{lem:3.3} we have

$$\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}} \otimes \mathcal{G}) = \text{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}} \otimes \mathcal{G})$$

and

$$\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}) = \text{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}})$$

and, since $\widehat{\mathcal{F}}$ is the dual of $\mathcal{F}$, the tensor product $\mathcal{F} \otimes \widehat{\mathcal{F}}$ has a trivial quotient and, in particular, has non-trivial tame part. By lemma \ref{lem:3.1} there exists $\epsilon > 0$ such that, for any $\mathcal{G} \in \mathcal{R}_0$ with slope $b \in (0, \epsilon)$, $\text{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}} \otimes \mathcal{G}) > \text{Swan}(\mathcal{F} \otimes \widehat{\mathcal{F}}) \dim(\mathcal{G})$.

Since the tame part of $\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}}$ is non-zero and it is a direct summand, it contains a tame character $\mathcal{L}_\eta$ of $I_0$ as a subrepresentation. Then

$$\rho_\chi(\mathcal{F}) \otimes \widehat{\mathcal{F}} \otimes \mathcal{L}_\eta = \rho_\chi(\mathcal{F}) \otimes \mathcal{F} \otimes \mathcal{L}_\eta = \text{Hom}(\mathcal{F} \otimes \mathcal{L}_\eta, \rho_\chi(\mathcal{F}))$$

contains a trivial subrepresentation, so $\text{Hom}_{I_0}(\mathcal{F} \otimes \mathcal{L}_\eta, \rho_\chi(\mathcal{F})) \neq 0$. Since both $\rho_\chi(\mathcal{F})$ and $\mathcal{F} \otimes \mathcal{L}_\eta$ are irreducible, any non-zero $I_0$-equivariant map $\mathcal{F} \otimes \mathcal{L}_\eta \rightarrow \rho_\chi(\mathcal{F})$ must be an isomorphism. \qedhere

\begin{proposition}
Let $\mathcal{F} \in \mathcal{R}_0$ be totally wild and irreducible of dimension $n$ and slope $a$, and let $\mathcal{L}_\eta$ be a tame character of $I_0$ such that $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$. Then $\mathcal{L}_\eta^{\otimes n} \cong \mathcal{L}_\chi^{\otimes n(a+1)}$.
\end{proposition}
Proof. Extend \( F \) to a smooth \( \ell \)-adic sheaf on \( \mathbb{G}_{m, \bar{k}} \), tamely ramified at infinity, also denoted by \( F \). Let \( F \) also denote its extension by zero to \( \bar{k}_1 \). By the proof of lemma 3.3, the sheaf \( F * L_\chi \) is smooth on \( \mathbb{G}_{m, \bar{k}} \), its monodromy at 0 is the direct sum of \( \rho_\chi(F) \cong F \otimes L_\eta \) and a trivial part of dimension \( na \), and its monodromy at infinity sits in the exact sequence (4). Its determinant is then a smooth sheaf of tame fundamental group of rank 1 on \( \mathbb{G}_{m, \bar{k}} \), whose monodromy at 0 is \( \det(F) \otimes L_\eta^{\otimes n} \), and whose monodromy at \( \infty \) is \( \det(F(\infty)) \otimes L_\chi^{\otimes(n+1)} \).

Then \( \det(F) \otimes L_\eta^{\otimes n} \otimes \det(F * L_\chi) \) is a rank 1 smooth sheaf on \( \mathbb{G}_{m, \bar{k}} \), with trivial monodromy at 0 and tamely ramified at infinity. Since the tame fundamental group of \( \bar{k}_1 \) is trivial, we conclude that

\[ \det(F * L_\chi) \cong \det(F) \otimes L_\eta^{\otimes n} \]

as sheaves on \( \mathbb{G}_{m, \bar{k}} \). Comparing their monodromies at infinity gives the desired isomorphism.

It remains to show that any such \( L_\eta \) works.

Lemma 3.6. Let \( F \in R_0 \) be irreducible of dimension \( n \), and let \( L_\eta \) be a tame character of \( I_0 \) such that \( L_\eta^{\otimes n} \) is trivial. Then \( F \otimes L_\eta \cong F \).

Proof. Write \( n = n_0 p^a \), where \( \alpha \geq 0 \) and \( n_0 \) is prime to \( p \). Since the \( p \)-th power operation permutes the tame characters of \( I_0 \) preserving their order, \( L_\eta^{\otimes n_0} \) must be the trivial character. Now by [8, 1.14.2], \( F \) is induced from a \( p^a \)-dimensional representation \( G \) of \( I_0(n_0) \), the unique open subgroup of \( I_0 \) of index \( n_0 \). Then

\[ F \otimes L_\eta = \left( \text{Ind}_{I_0(n_0)}^{I_0} G \right) \otimes L_\eta \cong \text{Ind}_{I_0(n_0)}^{I_0} (G \otimes \text{Res}^{I_0}_{I_0(n_0)} L_\eta) = \text{Ind}_{I_0(n_0)}^{I_0} (G) = F \]

since the restriction of \( L_\eta \) to \( I_0(n_0) \) is trivial.

We can now finish the proof of theorem 2.2 for irreducible representations.

Proposition 3.7. Let \( F \in R_0 \) be irreducible of slope \( a > 0 \). Write \( a = c/d \), where \( c \) and \( d \) are relatively prime positive integers. Let \( L_\eta \) be any tame character of \( I_0 \) such that \( L_\eta^{\otimes d} = L_\chi^{\otimes(c+d)} \). Then

\[ \rho_\chi(F) \cong F \otimes L_\eta. \]

Proof. Let \( n \) be the dimension of \( F \). By propositions 3.4 and 3.5, there exists a tame character \( L_\eta' \) of \( I_0 \) such that \( \rho_\chi(F) \cong F \otimes L_\eta' \), and \( L_\eta^{n_0} \cong L_\chi^{n_0(a+1)} \). Since the Swan conductor \( na = nc/d \) of \( F \) is an integer, \( n \) must be divisible by \( d \). Then

\[ (L_\eta' \otimes L_\eta)^{\otimes n} = L_\eta'^{\otimes n} \otimes L_\eta^{\otimes d(n/d)} = L_\chi^{\otimes(n(a+1))} \otimes L_\chi^{\otimes(c+d)(n/d)} = L_\chi^{\otimes(n(a+1))} \otimes L_\chi^{\otimes(n(a+1))} = 1 \]

so, by lemma 3.6

\[ \rho_\chi(F) \cong F \otimes L_\eta' \cong (F \otimes L_\eta') \otimes (L_\eta' \otimes L_\eta) = F \otimes L_\eta. \]

Proof of theorem 2.2. The functors \( R_0^0 \to R_0^0 \) given by \( F \mapsto \rho_\chi(F) \) and \( F \mapsto F \otimes L_\eta \) are equivalences of categories, so they preserve direct sums. It is enough then to prove the isomorphism for indecomposable representations.
So let $\mathcal{F} \in \mathcal{R}_m$ be indecomposable of length $m$. Then by [10] Lemma 3.1.6, Lemma 3.1.7(3)] there exist an irreducible $\mathcal{F}_0 \in \mathcal{R}_m$ and a (necessarily tame) indecomposable unipotent $\mathcal{U}_m \in \mathcal{R}_m$ of dimension $m$ such that $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{U}_m$. Since $\mathcal{F}$ is a successive extension of $m$ copies of $\mathcal{F}_0$, by exactness $\rho_\chi(\mathcal{F})$ is a successive extension of $m$ copies of $\rho_\chi(\mathcal{F}_0) \cong \mathcal{F}_0 \otimes \mathcal{L}_\eta$, which is irreducible. By [10] Lemma 3.1.7(2)], there is a unipotent $\mathcal{U} \in \mathcal{R}_m$ of dimension $m$ such that $\rho_\chi(\mathcal{F}) \cong \mathcal{F}_0 \otimes \mathcal{L}_\eta \otimes \mathcal{U}$.

Since $\rho_\chi$ is an equivalence of categories, $\rho_\chi(\mathcal{F})$ must be indecomposable, so $\mathcal{U}$ itself must be indecomposable. Therefore $\mathcal{U} \cong \mathcal{U}_m$ and

$$\rho_\chi(\mathcal{F}) \cong \mathcal{F}_0 \otimes \mathcal{L}_\eta \otimes \mathcal{U}_m \cong \mathcal{F} \otimes \mathcal{L}_\eta.$$  

□

4. Some variants

We will consider now representations of the inertia group $I_\infty$ at infinity. For any $\mathcal{F} \in \mathcal{R}_\infty$ of slope $> 1$, we can take its local Fourier transform $\text{FT}^\psi_{(\infty, \infty)} \mathcal{F}$, which is again in the same category. In [10] 3.4.4, N. Katz asks about a simple formula for

$$\rho_\chi'(\mathcal{F}) := \text{FT}^\psi_{(\infty, \infty)}(\mathcal{L}_\chi \otimes \text{FT}^\psi_{(\infty, \infty)} \mathcal{F}),$$

which is an auto-equivalence of the category of continuous $\ell$-adic representations of $\mathcal{R}_\infty$ with slopes $> 1$. It can be interpreted as the wild part of the monodromy at infinity of the (additive) convolution $\mathcal{F} \ast \mathcal{L}_\chi$ [10] 3.4.6], where $\mathcal{F}$ is any extension of the representation $\mathcal{F}$ to a smooth sheaf on $\mathbb{G}_{m, k}$ tamely ramified at 0. In this section we will prove

**Theorem 4.1.** Let $\mathcal{F} \in \mathcal{R}_\infty$ be totally wild with a single slope $a > 1$. Write $a = c/d$, where $c$ and $d$ are relatively prime positive integers. Let $\mathcal{L}_\eta$ be any tame character of $I_\infty$ such that $\mathcal{L}_\eta^{\otimes d} = \mathcal{L}_\chi^{\otimes (c-d)}$. Then

$$\rho_\chi'(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta.$$

In other words, we have the formula

$$\rho_\chi'(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi^{\otimes (a-1)}$$

where $\mathcal{L}_\chi^{\otimes (a-1)}$ stands for “any character that can reasonably be called $\mathcal{L}_\chi^{\otimes (a-1)}$.”

The proof is very similar to the one for $\rho_\chi$. Since every representation in $\mathcal{R}_\infty$ is a direct sum of representations with single slopes, we can assume that $\mathcal{F}$ has a single slope $a$.

**Lemma 4.2.** Let $\mathcal{F}, \mathcal{G} \in \mathcal{R}_\infty$ be totally wild, with $\mathcal{F}$ having all slopes $> 1$. Then

$$\text{Swan}(\rho_\chi'(\mathcal{F}) \otimes \mathcal{G}) = \text{Swan}(\mathcal{F} \otimes \mathcal{G}).$$

**Proof.** We can assume that $\mathcal{F}$ has a single slope $a > 1$. Extend $\mathcal{F}$ and $\mathcal{G}$ to smooth sheaves on $\mathbb{G}_{m, \bar{k}}$, tamely ramified at 0, which we will also denote by $\mathcal{F}$ and $\mathcal{G}$ (as well as their extensions by zero to $\mathbb{A}_k^1$).

Let $n$ be the rank of $\mathcal{F}$, and denote by $\mathcal{F}(0)$ its local monodromy at 0, which is a tame representation of $I_0$. Since all slopes of $\mathcal{F}$ at infinity are $> 1$, it is a Fourier sheaf [3] Lemma 8.3.1, so its Fourier transform is a single sheaf that we will denote by $\text{FT}^\psi \mathcal{F}$. By Ogg-Shafarevic, $\text{FT}^\psi \mathcal{F}$ is smooth on $\mathbb{G}_{m, \bar{k}}$ of rank $na$. By Laumon’s local Fourier transform theory [9] Remark 9], it has a single positive slope $\frac{a}{a-1}$ at infinity with multiplicity $n(a-1)$ and tame part isomorphic to $\mathcal{F}(0)$,
and it is unramified at 0. Then FT$^\psi$ $\mathcal{F}$ $\otimes$ $\mathcal{L}_\chi$ also has a single slope $\frac{a}{a-1}$ at infinity with multiplicity $n(a-1)$, tame part isomorphic to $\mathcal{L}_\chi$ $\otimes$ $\mathcal{F}(0)$, and its monodromy at 0 is a direct sum of $na$ copies of $\mathcal{L}_\chi$.

Its inverse Fourier transform, by Ogg-Shafarevic, is smooth of rank $n(a-1)\frac{a}{a-1} + n = n(a+1)$ on $\mathbb{G}_{m,k}$, and by local Fourier transform its monodromy at infinity is the direct sum of $\rho'_\chi(\mathcal{F})$ and $na = \text{Swan}(\mathcal{F})$ copies of $\mathcal{L}_\chi$. At 0 is has trivial part of rank $na$, while quotient isomorphic to $\mathcal{L}_\chi$ $\otimes$ $\mathcal{F}(0)$. So

$$\text{Swan}_\infty((\mathcal{F} \ast \mathcal{L}_\chi) \otimes \mathcal{G}) = \text{Swan}(\rho'_\chi(\mathcal{F}) \otimes \mathcal{G}) + \text{Swan}(\mathcal{F})\text{Swan}(\mathcal{G}).$$

We conclude exactly as in lemma 3.3.

Using lemma 3.1 as in proposition 3.4 we deduce

**Proposition 4.3.** Let $\mathcal{F} \in \mathcal{R}_\infty$ be irreducible with slope $> 1$. Then there exists a tame character $\mathcal{L}_\eta$ of $I_\infty$ such that $\rho'_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$.

**Proposition 4.4.** Let $\mathcal{F} \in \mathcal{R}_\infty$ be irreducible of dimension $n$ and slope $a > 1$, and let $\mathcal{L}_\eta$ be a tame character of $I_\infty$ such that $\rho'_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$. Then $\mathcal{L}_\eta^{\otimes n} \cong \mathcal{L}_\chi^{\otimes n(a-1)}$.

**Proof.** Extend $\mathcal{F}$ to a smooth $\ell$-adic sheaf on $\mathbb{G}_{m,k}$, tamely ramified at 0, also denoted by $\mathcal{F}$, and let $\mathcal{F}$ also denote its extension by zero to $\mathbb{A}^1_k$. By the proof of lemma 4.2, the sheaf $\mathcal{F} \ast \mathcal{L}_\chi$ is smooth on $\mathbb{G}_{m,k}$, its monodromy at infinity is the direct sum of $\rho'_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta$ and $na$ copies of $\mathcal{L}_\chi$, and its monodromy at 0 has trivial part of dimension $na$ with quotient isomorphic to $\mathcal{L}_\chi \otimes \mathcal{F}(0)$. Its determinant is then a smooth sheaf of rank 1 on $\mathbb{G}_{m,k}$, whose monodromy at infinity is $\text{det}(\mathcal{F}) \otimes \mathcal{L}_\eta^{\otimes n} \otimes \mathcal{L}_\chi^{\otimes na}$, and whose monodromy at 0 is $\text{det}(\mathcal{F}(0)) \otimes \mathcal{L}_\chi^{\otimes n}$.

We conclude, as in proposition 3.5 that

$$\text{det}(\mathcal{F} \ast \mathcal{L}_\chi) \cong \text{det}(\mathcal{F}) \otimes \mathcal{L}_\eta^{\otimes n} \otimes \mathcal{L}_\chi^{\otimes na}$$

as sheaves on $\mathbb{G}_{m,k}$. Comparing their monodromies at 0 gives the desired isomorphism.

The remainder of the proof of theorem 4.1 is identical to the one for $\rho_\chi$.

We have a third variant, for representations $\mathcal{F} \in \mathcal{R}_\infty$ with slopes $< 1$:

$$\rho''_\chi(\mathcal{F}) := \text{FT}_{(\infty,0)}^{\psi,-1}(\mathcal{L}_\chi \otimes \text{FT}_{(\infty,0)}^{\psi}(\mathcal{F})),$$

which is again an auto-equivalence of the category of continuous $\ell$-adic representations of $\mathcal{R}_\infty$ with slopes $< 1$. As in the $\rho_\chi$ case we have $\rho''_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi$ for $\mathcal{F}$ tame. The corresponding formula for wild $\mathcal{F}$ is

**Theorem 4.5.** Let $\mathcal{F} \in \mathcal{R}_\infty$ be totally wild with a single slope $a < 1$. Write $a = \frac{c}{d}$, where $c$ and $d$ are relatively prime positive integers. Let $\mathcal{L}_\eta$ be any tame character of $I_\infty$ such that $\mathcal{L}_\eta^{\otimes d} = \mathcal{L}_\chi^{\otimes (d-c)}$. Then

$$\rho''_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\eta.$$

**Proof.** Let $\mathcal{G} := \text{FT}_{(\infty,0)}^{\psi}(\mathcal{F}) \in \mathcal{R}_0$, which has slope $\frac{a}{1-a} = \frac{c}{d-c}$ [9, Theorem 13]. The statement is then equivalent to

$$\text{FT}_{(\infty,0)}^{\psi,-1}(\mathcal{L}_\chi \otimes \mathcal{G}) \cong \mathcal{L}_\eta \otimes \text{FT}_{(\infty,0)}^{\psi,-1}(\mathcal{G})$$

or

$$\text{FT}_{(\infty,0)}^{\psi}(\mathcal{L}_\eta \otimes \text{FT}_{(\infty,0)}^{\psi,-1}(\mathcal{G})) \cong \mathcal{G} \otimes \mathcal{L}_\chi.$$
But the left hand side is just $\rho_{\overline{\eta}}(\mathcal{G})$, since the inverse of $\text{FT}^{\psi}_{(\infty,0)}$ is $\text{FT}^{\overline{\psi}}_{(0,\infty)}$ with respect to the conjugate additive character, and $\rho_{\chi}$ does not depend on the choice of the non-trivial additive character $\psi$. So the isomorphism follows from theorem 2.2.

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