Boundary controllability of parabolic coupled equations

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Abstract

This paper is concerned with the boundary controllability of non-scalar linear parabolic systems. More precisely, two coupled one-dimensional parabolic equations are considered. We show that, in this framework, boundary controllability is not equivalent and is more complex than distributed controllability. In our main result, we provide necessary and sufficient conditions for the null controllability.

1 Introduction

This paper deals with the controllability properties of some systems of two coupled one-dimensional parabolic equations where the control is exerted at one boundary point for all times.

Thus, let us fix $T > 0$ and let us consider the linear system

$$\begin{cases} y_t - y_{xx} = Ay & \text{in } \ Q = (0,1) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(1,\cdot) = 0 & \text{in } (0,T), \\ y(\cdot\,,0) = y_0 & \text{in } \ (0,1), \end{cases} \tag{1}$$

where $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are given and $y_0 \in H^{-1}(0,1)^2$. Here, $v \in L^2(0,T)$ is a control function (to be determined) and $y = (y_1, y_2)^*$ is the state variable. Observe that, for every $v \in L^2(0,T)$ and $y_0 \in H^{-1}(0,1)^2$, (1) admits a unique weak solution (defined by transposition) that satisfies

$$y \in L^2(Q)^2 \cap C^0([0,T]; H^{-1}(0,1)^2);$$

see Section 2.

It will be said that (1) is *approximately controllable* in $H^{-1}(0,1)^2$ at time T if, for any $y_0, y_d \in H^{-1}(0,1)^2$ and any $\varepsilon > 0$, there exists a control function $v \in L^2(0,T)$ such that the associated solution satisfies

$$\|y(\cdot\,,T) - y_d\|_{H^{-1}(0,1)} \le \varepsilon.$$

On the other hand, it will be said that (1) is *null controllable* at time T if, for each $y_0 \in H^{-1}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that the associated solution satisfies

$$y(\cdot\,,T) = 0 \quad \text{in } H^{-1}(0,1)^2. \tag{2}$$

Since (1) is linear, this second property is equivalent to *the exact controllability to the trajectories* at time T, that is to say, to the following property: for any trajectory $\widehat{y}$ (i.e. any solution to (1) corresponding to $v \equiv 0$ and $\widehat{y}_0 \in H^{-1}(0,1)^2$) and any $y_0 \in H^{-1}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that the associated solution to (1) satisfies

$$y(\cdot\,,T) = \widehat{y}(\cdot\,,T) \quad \text{in} \quad H^{-1}(0,1)^2.$$

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The controllability properties of similar scalar problems are nowadays well known; see for instance [8], [21], [7], [18], [12] and [11].

To be precise, let $\Omega \subset \mathbb{R}^N$ be a nonempty regular bounded open set with $N \geq 1$, let $\omega \subset \Omega$ be a nonempty open subset, and let $\gamma \subset \partial\Omega$ be a nonempty relative open set. Let us consider the following scalar problems:

$$\begin{cases} y_t - \Delta y = v 1_\omega & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot\,,0) = y_0 & \text{in } \Omega \end{cases} \tag{3}$$

and

$$\begin{cases} y_t - \Delta y = 0 & \text{in } \Omega \times (0,T), \\ y = v 1_\gamma & \text{on } \partial\Omega \times (0,T), \\ y(\cdot\,,0) = y_0 & \text{in } \Omega. \end{cases} \tag{4}$$

Here, 1_ω and 1_γ are, respectively, the characteristic functions of ω and γ, $y_0 \in L^2(\Omega)$ is given and v is the control.

Under the previous assumptions, for every Ω, ω, γ and T, both systems (3) and (4) are approximately controllable in $L^2(\Omega)$ and also null controllable at any time T (see for instance [18] and [12]). In fact, the boundary controllability results for system (4) can be easily obtained from the corresponding distributed controllability results for system (3) and viceversa. We will see that the situation is quite different for similar non-scalar systems.

There are not many works devoted to the controllability of parabolic systems of PDEs. To our knowledge, all them deal with distributed controls, exerted on a small open set ω; see for instance [23], [6], [2], [5], [13], [14], [15], [3] and [4]. In these papers, almost all the results have been established for 2×2 systems where the control is exerted on the first equation. The most general results in this context seem to be those in [14], [3] and [4]. In [14], the authors study a *cascade* parabolic system of n equations ($n \geq 2$) controlled with one single distributed control. In [3] and [4], the authors provide necessary and sufficient conditions for the controllability of $n \times n$ parabolic linear systems with constant or time-dependent coefficients.

It is worth mentioning that, in [17], an approximate boundary controllability result is obtained for a particular system of two parabolic coupled equations as a consequence of a unique continuation principle. The result is valid in several dimensions but only for a very particular kind of coupling. It is also interesting to recall the boundary controllability results for a system of two wave equations obtained by Alabau-Boussouira in [1].

For completeness, let us recall the main result proved in [3] and [4] for the problem

$$\begin{cases} y_t - \Delta y = Ay + Bv 1_\omega & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{5}$$

where $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$ (with $n, m \geq 1$) and $y_0 \in L^2(\Omega)^n$. It is the following:

> *Let* $[A\,|\,B]$ *be the following matrix in* $\mathcal{L}(\mathbb{R}^{n\times m};\mathbb{R}^n)$*:*
>
> $$[A\,|\,B] = [B\,|\,AB\,|\,A^2B\,|\,\cdots\,|\,A^{n-1}B].$$
>
> *Then,* (5) *is null controllable if and only the so called Kalman's rank condition*
>
> $$\operatorname{rank}[A\,|\,B] = n$$
>
> *is satisfied. In that case, null controllability holds at any time* $T > 0$.

In this paper, our main aim is to characterize the boundary controllability properties of (1) (a system of 2 equations) when we apply just one control on a part of the boundary. Our main result is the following:

Theorem 1.1. *Let $A \in \mathcal{L}(\mathbb{R}^2;\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A. Then* (1) *is exactly controllable to the trajectories at any time $T > 0$ if and only if*

$$\mathrm{rank}\,[B\,|\,AB] = 2 \tag{6}$$

and

$$\pi^{-2}(\mu_1 - \mu_2) \neq j^2 - k^2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j. \tag{7}$$

In view of Theorem 1.1, we find two different situations: when the matrix A in (1) has one double real eigenvalue or a couple of conjugate complex eigenvalues, (6) is a necessary and sufficient condition for the null controllability at any time (as in the distributed case); otherwise, if A has two different real eigenvalues, an additional condition is needed for null controllability, independently of the vector B we are considering.

As a consequence of this result, we observe that the Kalman's rank condition is necessary, but not sufficient, for the boundary controllability of (1). This is a crucial discrepancy between boundary and distributed controllability for coupled parabolic systems and shows that, for a given system, these two properties can be independent.

The proof of Theorem 1.1 is based on the proof of Fattorini and Russell [8] of the boundary controllability of the one-dimensional heat equation. They reduce the task to construct a biorthogonal family in $L^2(0,\infty)$ to a given family of exponential functions and, then, to deduce appropriate estimates of the corresponding norms. Recall that two families $\{p_n : n \geq 1\}$ and $\{q_n : n \geq 1\}$ in $L^2(0,\infty)$ are said to be biorthogonal in this space if

$$(p_n, q_k)_{L^2(0,\infty)} = \delta_{nk} \quad \forall n, k \geq 1.$$

In our case, we have to construct and estimate appropriately in $L^2(0,\infty)$ a family that must be biorthogonal to a larger set of functions. We use techniques similar to those in [8], but adapted to this new situation. The constructed family is then used, together with (6) and (7), to prove an observability inequality for the solutions to the adjoint system. As a consequence, we get the null controllability of (1).

On the other hand, we prove that (6) and (7) are necessary by analyzing some particular systems that serve as counter-examples to unique continuation.

The rest of the paper is organized as follows. In the next Section, we give some basic and preliminary results concerning the existence of a solution and the controllability properties of (1); the proofs of some of them are postponed to Appendix A and Appendix B. In Section 3, we present some results related to the Fattorini-Russell method. In particular, we give details on the construction and estimates of certain biorthogonal families and we show how they can be used to prove some inequalities. In Section 4 we prove Theorem 1.1. Finally, Section 5 deals with some further results and open problems.

The main results in this paper have been announced in [10].

2 Preliminary results

This Section is devoted to establish some results for (1) that will be needed in the proof of Theorem 1.1. In the sequel, C denotes a generic positive constant; sometimes, we will make emphasis on the dependence of C on T, by writing $C(T)$. We will also use the following notation: $\|\cdot\|_X$ stands for the norm of the normed space X or X^m, with $m \geq 2$; also, $\|\cdot\|_{L^p(X)}$ stands for the norm in $L^p(0,T;X)$ ($p \geq 1$).

We begin by clarifying what is a *solution by transposition* to (1). To this end, let us consider the linear backwards in time problem

$$\begin{cases} -\varphi_t - \varphi_{xx} = A^*\varphi + g & \text{in } Q, \\ \varphi(0,\cdot) = 0, \quad \varphi(1,\cdot) = 0 & \text{in } (0,T), \\ \varphi(\cdot\,,T) = 0 & \text{in } (0,1), \end{cases} \tag{8}$$

where $g \in L^2(Q)^2$. It is well known that, for every $g \in L^2(Q)^2$, (8) possesses exactly one (strong) solution

$$\varphi \in L^2(0,T;H^2(0,1)^2) \cap C^0([0,T];H_0^1(0,1)^2).$$

Hence, the following definition makes sense:

Definition 2.1. *Let $y_0 \in H^{-1}(0,1)^2$ and $v \in L^2(0,T)$ be given. It will be said that $y \in L^2(Q)^2$ is a solution by transposition to* (1) *if, for each $g \in L^2(Q)^2$, one has*

$$\iint_Q y \cdot g \, dx\, dt = \langle y_0, \varphi(\cdot,0)\rangle + \int_0^T B \cdot \varphi_x(0,t)\, v(t)\, dt, \tag{9}$$

where φ is the solution to (8) *associated to g and $\langle \cdot\,,\cdot\rangle$ stands for the usual duality pairing between $H^{-1}(0,1)^2$ and $H_0^1(0,1)^2$.*

Thus, one has:

Proposition 2.2. *Assume that $y_0 \in H^{-1}(0,1)^2$ and $v \in L^2(0,T)$ are given. Then* (1) *admits a unique solution by transposition y that satisfies:*

$$\begin{cases} y \in L^2(Q)^2 \cap C^0([0,T];H^{-1}(0,1)^2), \quad y_t \in L^2(0,T;(D(-\Delta)')^2), \\ y_t - y_{xx} = Ay \quad in \quad L^2(0,T;(D(-\Delta)')^2), \\ y(\cdot\,,0) = y_0 \quad in \quad H^{-1}(0,1)^2 \quad and \\ \|y\|_{L^2(Q)} + \|y_t\|_{L^2(D(-\Delta)')} \le C\left(\|y_0\|_{H^{-1}(0,1)} + \|v\|_{L^2(0,T)}\right). \end{cases}$$

In the sequel, it will be said that y is the state associated to y_0 and v.

Results of this kind are well known. For completeness, we recall the proof of Proposition 2.2 in Appendix A.

Now, let us consider the adjoint of system (1):

$$\begin{cases} -\varphi_t - \varphi_{xx} = A^*\varphi & \text{in } \ Q, \\ \varphi(0,\cdot) = 0, \quad \varphi(1,\cdot) = 0 & \text{in } (0,T), \\ \varphi(\cdot\,,T) = \varphi_0 & \text{in } \ (0,1), \end{cases} \tag{10}$$

where $\varphi_0 \in H_0^1(0,1)^2$. In the sequel, the solution to (10) will be called the *adjoint state* associated to φ_0. The controllability of (1) can be characterized in terms of appropriate properties of the solutions to (10). More precisely, we have:

Proposition 2.3. *The following properties are equivalent:*

1. *There exists a positive constant C such that, for any $y_0 \in H^{-1}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that*
$$\|v\|^2_{L^2(0,T)} \le C\|y_0\|^2_{H^{-1}(0,1)} \tag{11}$$
and the associated state satisfies (2).

2. *There exists a positive constant C such that, for any trajectory $\widehat{y} \in C^0([0,T];H^{-1}(0,1)^2)$ of* (1) *and any $y_0 \in H^{-1}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that*
$$\|v\|^2_{L^2(0,T)} \le C\|y_0 - \widehat{y}(\cdot\,,0)\|^2_{H^{-1}(0,1)} \tag{12}$$
and the associated state satisfies
$$y(\cdot\,,T) = \widehat{y}(\cdot\,,T) \quad in \quad H^{-1}(0,1)^2.$$

3. There exists a positive constant C such that the observability inequality

$$\|\varphi(\cdot,0)\|^2_{H^1_0(0,1)} \le C\int_0^T |B^*\varphi_x(0,t)|^2\,dt \tag{13}$$

holds for every $\varphi_0 \in H^1_0(0,1)^2$. In (13), φ is the adjoint state associated to φ_0.

Again, this result is well known. For completeness, the proof is presented in Appendix B, at the end of the paper.

Remark 2.1. It is also well known that the approximate controllability of (1) can be characterized in terms of a property of the solutions to (10). More precisely, (1) is approximately controllable if and only if the following unique continuation property holds:

> *"Let $\varphi_0 \in H^1_0(0,1)^2$ be given and let φ be the associated adjoint state. Then, if $B^*\varphi_x(0,t) = 0$ on $(0,T)$, one has $\varphi \equiv 0$ on Q."*

■

3 Biorthogonal families: construction, estimates and applications

In this Section, some technical results are given. They will be used below to prove Theorem 1.1.

Let us first present a fundamental lemma whose first part was essentially proved by Luxemburg and Korevaar in [20]. For the sake of completeness, we have included the proof below. As far as we know, the second part of this lemma is new.

Lemma 3.1. *Suppose that $\{\Lambda_n\}_{n\ge1}$ is a sequence of complex numbers such that, for some $\delta,\rho>0$, one has:*

$$\begin{cases} \Re(\Lambda_n) \ge \delta|\Lambda_n|, \quad |\Lambda_n-\Lambda_k| \ge |n-k|\rho \quad \forall n,k\ge 1,\\ \displaystyle\sum_{n=1}^\infty \frac{1}{|\Lambda_n|} < \infty. \end{cases} \tag{14}$$

Then,

a) There exists a sequence $\{h_n\}$ biorthogonal to $\{e^{-\Lambda_n t}\}$ such that, for every $\varepsilon>0$, one has

$$\|h_n\|_{L^2(0,\infty)} \le K(\varepsilon)e^{\varepsilon\,\Re(\Lambda_n)} \quad \forall n\ge1. \tag{15}$$

b) There exists a sequence $\{q_n,\tilde q_n\}$ biorthogonal to $\{e^{-\Lambda_n t}, te^{-\Lambda_n t}\}$ such that, for every $\varepsilon>0$, one has

$$\|(q_n,\tilde q_n)\|_{L^2(0,\infty)} \le K(\varepsilon)e^{\varepsilon\,\Re(\Lambda_n)} \quad \forall n\ge1. \tag{16}$$

As a consequence, we also have:

Lemma 3.2. *Let us assume that (14) holds. Then:*

a) For every $T>0$, there exists $C(T)>0$ such that, for all $m\ge1$ and $A_j\in\mathbb{C}$, one has:

$$\int_0^T |\sum_{j=1}^m A_j e^{-\Lambda_j t}|^2\,dt \ge C(T)\int_0^\infty |\sum_{j=1}^m A_j e^{-\Lambda_j t}|^2\,dt.$$

b) For every $T>0$, there exists $C(T)>0$ such that, for all $m\ge1$ and $A_j,B_j\in\mathbb{C}$, one has:

$$\int_0^T |\sum_{j=1}^m (A_j+tB_j) e^{-\Lambda_j t}|^2\,dt \ge C(T)\int_0^\infty |\sum_{j=1}^m (A_j+tB_j) e^{-\Lambda_j t}|^2\,dt.$$

Let us introduce the (closed) spaces

$$E_T = [e^{-\Lambda_j t} : j \geq 1]_{L^2(0,T)}, \quad E_\infty = [e^{-\Lambda_j t} : j \geq 1]_{L^2(0,\infty)},$$

$$F_T = [e^{-\Lambda_j t}, te^{-\Lambda_j t} : j \geq 1]_{L^2(0,T)}, \quad F_\infty = [e^{-\Lambda_j t}, te^{-\Lambda_j t} : j \geq 1]_{L^2(0,\infty)},$$

spanned by the functions $e^{-\Lambda_j t}$ (and $te^{-\Lambda_j t}$) in $L^2(0,T)$ and $L^2(0,\infty)$, respectively.

Let us also introduce the canonical mappings $\Gamma : E_\infty \mapsto E_T$ and $\widetilde{\Gamma} : F_\infty \mapsto F_T$, with

$$\Gamma v = v|_{(0,T)} \quad \forall v \in E_\infty \quad \text{and} \quad \widetilde{\Gamma} w = w|_{(0,T)} \quad \forall w \in F_\infty.$$

A trivial consequence of Lemma 3.2 is the following:

Lemma 3.3. *Let us assume that* (14) *holds. Then:*

a) For every $T > 0$, $\Gamma : E_\infty \mapsto E_T$ is an isomorphism. In particular, there exists $C(T) > 0$ such that

$$\|v\|_{L^2(0,\infty)} \leq C(T)\|\Gamma v\|_{L^2(0,T)} \quad \forall v \in E_\infty.$$

b) For every $T > 0$, $\widetilde{\Gamma} : F_\infty \mapsto F_T$ is an isomorphism. In particular, there exists $C(T) > 0$ such that

$$\|w\|_{L^2(0,\infty)} \leq C(T)\|\widetilde{\Gamma} w\|_{L^2(0,T)} \quad \forall w \in F_\infty.$$

These lemmas are crucial for the proof of the main result in this Section, that is the following:

Proposition 3.4. *Let us assume that* (14) *holds. Then:*

a) For every $T > 0$, there exists $C(T) > 0$ such that

$$\int_0^T |\sum_{j\geq 1} A_j e^{-\Lambda_j t}|^2 \, dt \geq C(T) \sum_{j\geq 1} \frac{|A_j|^2}{|\Lambda_j|} \, e^{-\Re(\Lambda_j)T}, \tag{17}$$

whenever the sum in the left hand side makes sense.

b) For every $T > 0$, there exists $C(T) > 0$ such that

$$\int_0^T |\sum_j (A_j + tB_j) e^{-\Lambda_j t}|^2 dt \geq C(T) \sum_{j\geq 1} \frac{|A_j|^2 + |B_j|^2}{|\Lambda_j|} \, e^{-\Re(\Lambda_j)T}, \tag{18}$$

whenever the sum in the left hand side makes sense.

We will first give the proof of Proposition 3.4 assuming that Lemmas 3.1, 3.2 and 3.3 hold true. Then, we will present the proofs of these lemmas.

Proof of Proposition 3.4: Let us prove part b). Part a) is simpler and can be established in a similar way.

Let us take q_k and $\widetilde{q}_k$ as in Lemma 3.1 b) and let us assume that the left hand side of (18) is meaningful, i.e.

$$\sum_j (A_j + tB_j) \, e^{-\Lambda_j t} \in L^2(0,T).$$

In view of Lemma 3.3 b), we also have $\sum_j (A_j + tB_j) \, e^{-\Lambda_j t} \in L^2(0,\infty)$ and

$$\int_0^\infty |\sum_j (A_j + tB_j) e^{-\Lambda_j t}|^2 dt \leq C(T) \int_0^T |\sum_j (A_j + tB_j) e^{-\Lambda_j t}|^2 dt. \tag{19}$$

For all $k \geq 1$, we have

$$\begin{cases} \displaystyle\int_0^\infty |\sum_j (A_j + tB_j)e^{-\Lambda_j t}|^2\,dt \geq \frac{1}{\|q_k\|^2}\left|\int_0^\infty \sum_j (A_j+tB_j)e^{-\Lambda_j t}\overline{q_k(t)}\,dt\right|^2 \\ \displaystyle\qquad\qquad = \frac{\left|(A_k e^{-\Lambda_k t}, q_k)\right|^2}{\|q_k\|^2} = \frac{|A_k|^2}{\|q_k\|^2}\,. \end{cases}$$

Consequently,

$$\left(\int_0^\infty |\sum_j (A_j + tB_j)e^{-\Lambda_j t}|^2\,dt\right)\left(\sum_k \frac{1}{|\Lambda_k|}\right) \geq \sum_k \frac{1}{|\Lambda_k|}\,\frac{|A_k|^2}{\|q_k\|^2}\,.$$

Let us fix $\varepsilon > 0$. Then this inequality together with (16) imply

$$\int_0^\infty |\sum_j (A_j + tB_j)e^{-\Lambda_j t}|^2\,dt \geq \frac{C}{K(\varepsilon)^2}\sum_k \frac{1}{|\Lambda_k|}|A_k|^2 e^{-2\varepsilon\Re(\Lambda_k)}.$$

Taking $\varepsilon = T/2$, we see that

$$\int_0^\infty |\sum_j (A_j + tB_j)e^{-\Lambda_j t}|^2\,dt \geq C(T)\sum_k \frac{1}{|\Lambda_k|}|A_k|^2 e^{-\Re(\Lambda_k)T} \tag{20}$$

for some $C(T) > 0$. Proceeding as before, but using $\widetilde{q}_k$ instead of q_k, we also get

$$\int_0^\infty |\sum_j (A_j + tB_j)e^{-\Lambda_j t}|^2\,dt \geq C(T)\sum_k \frac{1}{|\Lambda_k|}|B_k|^2 e^{-\Re(\Lambda_k)T}. \tag{21}$$

Now, combining (20) and (21), we find that

$$\int_0^\infty |\sum_j (A_j + tB_j)e^{-\Lambda_j t}|^2\,dt \geq C(T)\sum_{j\geq 1} \frac{|A_j|^2 + |B_j|^2}{|\Lambda_j|}\,e^{-\Re(\Lambda_j)T}.$$

Finally, from (19), we get (18). ■

Let us now present the proofs of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1:

In this proof, $\|\cdot\|$ will stand for the norm in $L^2(0,+\infty)$. Part a) can be deduced from the proof given in [9] (see also [8]); it can also be deduced from part b). However, for clarity and completeness, we will include here the proof.

Thus, let us set $p_n(t) = e^{-\Lambda_n t}$ and let us introduce the space

$$E_n = [p_k : k \neq n]_{L^2(0,\infty)}\,,$$

that is to say, the closed span in $L^2(0,\infty)$ of the functions p_k with $k \neq n$. Thanks to Müntz's Theorem (see [22], p. 24), $p_n \notin E_n$ and there exists a unique $r_n \in E_n$ such that

$$\|p_n - r_n\| = \mathrm{dist}\,(p_n, E_n).$$

Of course, r_n is characterized by

$$r_n \in E_n \quad \text{and} \quad (p_n - r_n)\perp E_n\,.$$

Let us choose

$$h_n = \frac{p_n - r_n}{\|p_n - r_n\|^2}\,.$$

It is then clear that $(h_n, p_k) = \delta_{kn}$ for all k and n, i.e. the sequence $\{h_n\}$ is biorthogonal to$\{e^{-\Lambda_n t}\}$.

Let us now prove the inequalities (15) or, equivalently, let us estimate $\|p_n - r_n\|$ from below.

For each $m \in \mathbb{N}$ let us denote by r_n^m the projection of p_n over

$$E_n^m = [p_k : k \neq n,\ 1 \leq k \leq m]_{L^2(0,\infty)}\,.$$

Then

$$r_n^m \to r_n \text{ in } L^2(0,\infty) \quad \text{as} \quad m \to \infty$$

and $\|p_n - r_n\| = \lim_{m\to\infty} \|p_n - r_n^m\|$. We also have

$$\|p_n - r_n^m\|^2 = (e^{-\Lambda_n t}, e^{-\Lambda_n t} - r_n^m) = \int_0^\infty e^{-\Lambda_n t}\left(e^{-\overline{\Lambda}_n t} - \overline{r}_n^m(t)\right)\,dt = \Phi(\Lambda_n),$$

where Φ is given by

$$\Phi(\Lambda) = \int_0^\infty e^{-\Lambda t}\left(e^{-\overline{\Lambda}_n t} - \overline{r}_n^m(t)\right)\,dt \quad \forall \Lambda \in \mathbb{C} \quad \text{with} \quad \Re(\Lambda) \geq 0.$$

Observe that Φ depends on n and m; however, in order to simplify the notation, from now on we will not indicate explicitly this dependence.

Since $r_n^m \in E_n^m$, we can write

$$\Phi(\Lambda) = \frac{1}{\Lambda + \overline{\Lambda}_n} - \sum_{j=1, j\neq n}^{m} \frac{\overline{a}_j^m}{\Lambda + \overline{\Lambda}_j} = \frac{g(\Lambda)}{(\Lambda + \overline{\Lambda}_n)\prod_{j=1,j\neq n}^m (\Lambda + \overline{\Lambda}_j)}$$

for some $a_j^m \in \mathbb{C}$. Here, g is a polynomial of degree $\leq m-1$. The orthogonality properties of r_n^m imply that $\Phi(\Lambda_j) = 0$ for all j with $1 \leq j \leq m$ and $j \neq n$. As a consequence, this is also satisfied by g and we have

$$g(\Lambda) = K \prod_{j=1, j\neq n}^{m} (\Lambda - \Lambda_j) \tag{22}$$

for some $K \in \mathbb{C}$. On the other hand, we also have

$$g(\Lambda) = \prod_{j=1,j\neq n}^{m} (\Lambda + \overline{\Lambda}_j) - (\Lambda + \overline{\Lambda}_n) \sum_{j=1,\, j\neq n}^{m} \left(\overline{a}_j^m \prod_{i=1,\, i\neq n,j}^{m} (\Lambda + \overline{\Lambda}_i) \right),$$

whence

$$g(-\overline{\Lambda}_n) = \prod_{j=1,j\neq n}^{m} (\overline{\Lambda}_j - \overline{\Lambda}_n).$$

This and (22) together imply that

$$K = \prod_{j=1,j\neq n}^{m} \frac{\overline{\Lambda}_n - \overline{\Lambda}_j}{\overline{\Lambda}_n + \Lambda_j} \quad \text{and} \quad \Phi(\Lambda) = \frac{1}{\Lambda + \overline{\Lambda}_n} \prod_{j=1,j\neq n}^{m} \frac{(\overline{\Lambda}_n - \overline{\Lambda}_j)(\Lambda - \Lambda_j)}{(\overline{\Lambda}_n + \Lambda_j)(\Lambda + \overline{\Lambda}_j)}\,.$$

In particular, we see that

$$\Phi(\Lambda_n) = \frac{1}{2\Re(\Lambda_n)} \prod_{j=1,j\neq n}^{m} \frac{\left|1 - \dfrac{\Lambda_n}{\Lambda_j}\right|^2}{\left|1 + \dfrac{\Lambda_n}{\overline{\Lambda}_j}\right|^2}\,. \tag{23}$$

Taking limits as $m \to \infty$ in (23), we get $\|p_n - r_n\| = P_n$, where

$$P_n = \left(\frac{1}{2\Re(\Lambda_n)}\right)^{1/2} \prod_{j=1, j\neq n}^{\infty} \frac{\left|1 - \dfrac{\Lambda_n}{\Lambda_j}\right|}{\left|1 + \dfrac{\Lambda_n}{\overline{\Lambda}_j}\right|}. \tag{24}$$

Following the ideas in [9] and [20], it can be proved that, for every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$P_n \geq C(\varepsilon) e^{-\varepsilon\Re(\Lambda_n)}. \tag{25}$$

For completeness, we give a proof below. From these inequalities, taking into account the definition of h_n , we directly obtain (15).

This ends the proof of part a).

Let us now prove (25). Let us fix $\epsilon > 0$. From (14), there exists $N_0(\epsilon) \in \mathbb{N}$ such that

$$\sum_{j \geq N_0(\epsilon)} \frac{1}{|\Lambda_j|} \leq \epsilon.$$

Thus, using the inequality $1 + x \leq e^x$, $x \in \mathbb{R}$, we can estimate the denominator of (24) as follows:

$$\begin{aligned}
\prod_{j=1, j\neq n}^{\infty} \left|1 + \frac{\Lambda_n}{\overline{\Lambda}_j}\right| &\leq \prod_{j=1, j\neq n}^{\infty} \left(1 + \frac{|\Lambda_n|}{|\Lambda_j|}\right) = \prod_{j=1,}^{N_0(\epsilon)-1} \left(1 + \frac{|\Lambda_n|}{|\Lambda_j|}\right) \prod_{j=N_0(\epsilon)}^{\infty} \left(1 + \frac{|\Lambda_n|}{|\Lambda_j|}\right) \\
&\leq \prod_{j=1,}^{N_0(\epsilon)-1} \left(1 + \frac{|\Lambda_n|}{c}\right) \prod_{j=N_0(\epsilon)}^{\infty} e^{\frac{|\Lambda_n|}{|\Lambda_j|}} \leq \left(1 + \frac{|\Lambda_n|}{c}\right)^{N_0(\epsilon)-1} e^{\epsilon|\Lambda_n|} \\
&\leq C_1(\epsilon) e^{2\epsilon|\Lambda_n|} \quad \forall n \in \mathbb{N},
\end{aligned} \tag{26}$$

for a positive constant $C_1(\epsilon)$. In the previous inequality we have used that, for some constant $c > 0$, one has $|\Lambda_j| \geq c > 0$ for every $j \in \mathbb{N}$.

Let us now work on the numerator of (24). We introduce

$$S_1(n) = \{j : |\Lambda_j| \leq \frac{1}{2}|\Lambda_n|\}, \quad S_2(n) = \{j \neq n : \frac{1}{2}|\Lambda_n| < |\Lambda_j| \leq 2|\Lambda_n|\} \quad \text{and}$$
$$S_3(n) = \{j : |\Lambda_j| > 2|\Lambda_n|\}.$$

Then

$$\prod_{j \in S_1(n)} \left|1 - \frac{\Lambda_n}{\Lambda_j}\right| \geq \prod_{j \in S_1(n)} \left(\frac{|\Lambda_n|}{|\Lambda_j|} - 1\right) \geq 1 \quad \forall n \in \mathbb{N}. \tag{27}$$

On the other hand,

$$\prod_{j \in S_3(n)} \left|1 - \frac{\Lambda_n}{\Lambda_j}\right| \geq \prod_{j \in S_3(n)} \left(1 - \frac{|\Lambda_n|}{|\Lambda_j|}\right) \geq \prod_{j \in S_3(n)} e^{-2\frac{|\Lambda_n|}{|\Lambda_j|}} = e^{-2|\Lambda_n| \sum_{j \in S_3(n)} \frac{1}{|\Lambda_j|}}. \tag{28}$$

In this inequality we have used that $e^{-2x} \leq 1 - x$ if $x \in [0, 1/2]$. Using (14), we deduce that there exists $N_1(\epsilon) \in \mathbb{N}$ such that, if $n \geq N_1(\epsilon)$, one has

$$\sum_{j \in S_3(n)} \frac{1}{|\Lambda_j|} \leq \epsilon.$$

From (28) and the previous inequality we deduce that, if $n \geq N_1(\epsilon)$, then

$$\prod_{j \in S_3(n)} \left|1 - \frac{\Lambda_n}{\Lambda_j}\right| \geq e^{-2\epsilon|\Lambda_n|}.$$

Evidently, if $n \le N_1(\epsilon)$, we get

$$\prod_{j\in S_3(n)} \left|1-\frac{\Lambda_n}{\Lambda_j}\right| \ge C_2(\epsilon)$$

for a new positive constant $C_2(\epsilon)$. Therefore,

$$\prod_{j\in S_3(n)} \left|1-\frac{\Lambda_n}{\Lambda_j}\right| \ge C_2(\epsilon)e^{-2\epsilon|\Lambda_n|} \quad \forall n\in\mathbb{N}. \tag{29}$$

Finally, using again (14), we see that

$$\prod_{j\in S_2(n)} \left|1-\frac{\Lambda_n}{\Lambda_j}\right| = \prod_{j\in S_2(n)} \left|\frac{\Lambda_j-\Lambda_n}{\Lambda_j}\right| \ge \prod_{j\in S_2(n)} \frac{|j-n|\rho}{2|\Lambda_n|} \ge r_n!s_n!\left(\frac{\rho}{2|\Lambda_n|}\right)^{r_n+s_n}, \tag{30}$$

where r_n (resp. s_n) is the number of elements $j\in S_2(n)$ such that $j<n$ (resp. $j>n$).

Following [9], we find that

$$\frac{r_n+s_n}{|\Lambda_n|} = \sum_{j\in S_2(n)} \frac{1}{|\Lambda_n|} \le \sum_{j\in S_2(n)} \frac{2}{|\Lambda_j|} = \sum_{\{j:|\Lambda_j|>|\Lambda_n|/2\}} \frac{2}{|\Lambda_j|} \to 0 \quad \text{as} \quad n\to\infty,$$

i.e. we can write $r_n = \eta_n|\Lambda_n|$ and $s_n = \nu_n|\Lambda_n|$, with $\lim \eta_n = \lim \nu_n = 0$.

Let us now estimate the right hand side of (30). If the sequence $\{r_n\}_{n\ge 1}$ is bounded, then

$$r_n!\left(\frac{\rho}{2|\Lambda_n|}\right)^{r_n} \ge \left(\frac{\rho}{2|\Lambda_n|}\right)^{M} \ge C(\epsilon)\, e^{-\epsilon|\Lambda_n|}, \quad \forall n\in\mathbb{N}.$$

If $r_n\to\infty$ we can use the Stirling formula $r_n! = \beta_n(r_n/e)^{r_n}\sqrt{2\pi r_n}$, with $\beta_n\to 1$ as $n\to\infty$ and deduce that

$$\begin{aligned} r_n!\left(\frac{\rho}{2|\Lambda_n|}\right)^{r_n} &= \beta_n\left(\frac{r_n\rho}{2e|\Lambda_n|}\right)^{r_n}\sqrt{2\pi r_n} \equiv \beta_n\left(\frac{\eta_n\rho}{2e}\right)^{\eta_n|\Lambda_n|}\sqrt{2\pi\eta_n}|\Lambda_n|^{1/2} \\ &= \beta_n\alpha_n^{|\Lambda_n|}\sqrt{2\pi\eta_n}|\Lambda_n|^{1/2}, \end{aligned}$$

with $\lim\alpha_n = \lim\beta_n = 1$ and $\lim\eta_n = 0$. Thus, if we fix $\epsilon>0$, there exists $\widetilde{C}(\epsilon)>0$ such that

$$r_n!\left(\frac{\rho}{2|\Lambda_n|}\right)^{r_n} \ge \widetilde{C}(\epsilon)e^{-\epsilon|\Lambda_n|}, \quad \forall n\in\mathbb{N}.$$

A similar inequality can be obtained for

$$s_n!\left(\frac{\rho}{2|\Lambda_n|}\right)^{s_n}.$$

Therefore, we have proved the existence of a positive constant $C_3(\epsilon)$ such that

$$\prod_{j\in S_2(n)} \left|1-\frac{\Lambda_n}{\Lambda_j}\right| \ge C_3(\epsilon)e^{-2\epsilon|\Lambda_n|} \quad \forall n\in\mathbb{N}. \tag{31}$$

Coming back to (24), using (14) and putting together the inequalities (26), (27), (29) and (31), we find that

$$P_n \ge C_4(\epsilon)\left(\frac{1}{2\Re(\Lambda_n)}\right)^{1/2} e^{-6\epsilon|\Lambda_n|} \ge C_4(\epsilon)\left(\frac{1}{2\Re(\Lambda_n)}\right)^{1/2} e^{-6\epsilon\Re(\Lambda_n)/\delta} \ge C_5(\epsilon)e^{-7\epsilon\Re(\Lambda_n)/\delta},$$

for some new positive constants $C_4(\epsilon)$ and $C_5(\epsilon)$. This last inequality shows (25).

Let us now prove part b). Let us set $p_n(t) = e^{-\Lambda_n t}$, $\widetilde{p}_n(t) = te^{-\Lambda_n t}$,

$$F_n = [p_k : k \neq n;\ \widetilde{p}_k, k \geq 1]_{L^2(0,\infty)} \quad \text{and} \quad \widetilde{F}_n = [p_k : k \geq 1;\ \widetilde{p}_k : k \neq n]_{L^2(0,\infty)}.$$

Then there exists $s_n \in F_n$ satisfying $\|p_n - s_n\| = \text{dist}(p_n, F_n)$. The function s_n is characterized by

$$s_n \in F_n \quad \text{and} \quad (p_n - s_n) \perp F_n.$$

In a similar way, there exists $\widetilde{s}_n \in \widetilde{F}_n$ such that $\|\widetilde{p}_n - \widetilde{s}_n\| = \text{dist}(\widetilde{p}_n, \widetilde{F}_n)$, characterized by

$$\widetilde{s}_n \in \widetilde{F}_n \quad \text{and} \quad (\widetilde{p}_n - \widetilde{s}_n) \perp \widetilde{F}_n.$$

We will see later that $s_n \neq p_n$ and $\widetilde{s}_n \neq \widetilde{p}_n$. Thus, we can introduce

$$q_n = \frac{p_n - s_n}{\|p_n - s_n\|^2} \quad \text{and} \quad \widetilde{q}_n = \frac{\widetilde{p}_n - \widetilde{s}_n}{\|\widetilde{p}_n - \widetilde{s}_n\|^2}$$

and then $(q_n, p_k) = \delta_{nk}$, $(\widetilde{q}_n, p_k) = 0$, $(q_n, \widetilde{p}_k) = 0$ and $(\widetilde{q}_n, \widetilde{p}_k) = \delta_{nk}$ for all n and k.

In this way, we have obtained a family $\{q_n, \widetilde{q}_n\}$ that is biorthogonal to $\{p_n, \widetilde{p}_n\}$. To conclude the proof we have to estimate the norms $\|q_n\|$ and $\|\widetilde{q}_n\|$. These are the goals of the next two paragraphs.

Estimate of $\|q_n\|$: For any $m \geq 1$, let us introduce the space

$$F_n^m = [p_k : 1 \leq k \leq m,\ k \neq n;\ \widetilde{p}_k : 1 \leq k \leq m]_{L^2(0,\infty)}.$$

Let s_n^m be the unique function in F_n^m satisfying $\|p_n - s_n^m\| = \min_{r \in F_n^m} \|p_n - r\|$. Then $s_n^m \to s_n$ in $L^2(0,\infty)$ as $m \to \infty$ and, consequently,

$$\|p_n - s_n\| = \lim_{m \to \infty} \|p_n - s_n^m\|.$$

So, let us look for an estimate of

$$\|p_n - s_n^m\|^2 = (e^{-\Lambda_n t}, e^{-\Lambda_n t} - s_n^m) = F(\Lambda_n),$$

where, for $\Re(\Lambda) \geq 0$, we have set

$$\left\{\begin{aligned} F(\Lambda) &= (e^{-\Lambda t}, e^{-\Lambda_n t} - s_n^m) \\ &= \int_0^\infty e^{-\Lambda t} \left[e^{-\overline{\Lambda}_n t} - \left(\sum_{j=1, j\neq n}^m \overline{a}_j^n\, e^{-\overline{\Lambda}_j t} + \sum_{j=1}^m \overline{b}_j^n\, te^{-\overline{\Lambda}_j t} \right) \right] dt \\ &= \frac{1}{\Lambda + \overline{\Lambda}_n} - \sum_{j=1, j\neq n}^m \frac{\overline{a}_j^n}{\Lambda + \overline{\Lambda}_j} - \sum_{j=1}^m \frac{\overline{b}_j^n}{(\Lambda + \overline{\Lambda}_j)^2} = \frac{G(\Lambda)}{\prod_{j=1}^m (\Lambda + \overline{\Lambda}_j)^2} = \frac{G(\Lambda)}{R(\Lambda)}. \end{aligned}\right. \tag{32}$$

Here, G is a polynomial of degree $\leq 2m - 1$.

We have $F'(\Lambda) = (-te^{-\Lambda t}, e^{-\Lambda_n t} - s_n^m)$. Furthermore, the orthogonality relations satisfied by $e^{-\Lambda_n t} - s_n^m$ give

$$F(\Lambda_k) = 0 \quad \forall 1 \leq k \leq m,\ k \neq n, \quad \text{and} \quad F'(\Lambda_k) = 0 \quad \forall 1 \leq k \leq m.$$

In terms of G, this can be rewritten as follows:

$$G(\Lambda_k) = 0 \quad \forall 1 \leq k \leq m,\ k \neq n \quad \text{and} \quad \begin{cases} G'(\Lambda_k) = 0 \quad \forall 1 \leq k \leq m,\ k \neq n, \\ G'(\Lambda_n)R(\Lambda_n) = G(\Lambda_n)R'(\Lambda_n). \end{cases} \tag{33}$$

Consequently, we can write

$$G(\Lambda) = (A\Lambda + B) \prod_{j=1, j\neq n}^{m} (\Lambda - \Lambda_j)^2 \tag{34}$$

for some complex coefficients A and B.

In view of (32), we also have

$$G(\Lambda) = (\Lambda + \overline{\Lambda}_n) \prod_{j=1, j\neq n}^{m} (\Lambda + \overline{\Lambda}_j)^2 + (\Lambda + \overline{\Lambda}_n)^2 G_1(\Lambda) - \overline{b}_n^n \prod_{j=1, j\neq n}^{m} (\Lambda + \overline{\Lambda}_j)^2 \tag{35}$$

for some polynomial G_1. From this expression and (34) (written for $\Lambda = -\overline{\Lambda}_n$), we deduce that

$$\overline{b}_n^n \prod_{j=1, j\neq n}^{m} (\overline{\Lambda}_j - \overline{\Lambda}_n)^2 = \prod_{j=1, j\neq n}^{m} (\Lambda_j + \overline{\Lambda}_n)^2 (A\overline{\Lambda}_n - B),$$

that is to say,

$$\overline{b}_n^n = \frac{1}{\overline{S}} \left(A\overline{\Lambda}_n - B\right), \quad \text{with} \quad S = \prod_{j=1, j\neq n}^{m} \left(\frac{\Lambda_n - \Lambda_j}{\Lambda_n + \overline{\Lambda}_j} \right)^2. \tag{36}$$

On the other hand, from (34) we get the following:

$$G'(\Lambda) = A \prod_{j=1, j\neq n}^{m} (\Lambda - \Lambda_j)^2 + 2(A\Lambda + B) \sum_{j=1, j\neq n}^{m} \left[(\Lambda - \Lambda_j) \prod_{k=1, k\neq n, j}^{m} (\Lambda - \Lambda_k)^2 \right] \tag{37}$$

and

$$G'(-\overline{\Lambda}_n) = A \prod_{j=1, j\neq n}^{m} (\overline{\Lambda}_n + \Lambda_j)^2 + 2(A\overline{\Lambda}_n - B) \sum_{j=1, j\neq n}^{m} \left[(\overline{\Lambda}_n + \Lambda_j) \prod_{k=1, k\neq n, j}^{m} (\overline{\Lambda}_n + \Lambda_k)^2 \right].$$

From (35), we also have that

$$G'(-\overline{\Lambda}_n) = \prod_{j=1, j\neq n}^{m} (\overline{\Lambda}_j - \overline{\Lambda}_n)^2 - 2\overline{b}_n^n \sum_{j=1, j\neq n}^{m} \left[(\overline{\Lambda}_j - \overline{\Lambda}_n) \prod_{k=1, k\neq n, j}^{m} (\overline{\Lambda}_k - \overline{\Lambda}_n)^2 \right].$$

In view of these equalities and (36), we get:

$$A + 2(A\overline{\Lambda}_n - B) \sum_{j=1, j\neq n}^{m} \left(\frac{1}{\overline{\Lambda}_n + \Lambda_j} + \frac{1}{\overline{\Lambda}_j - \overline{\Lambda}_n} \right) = \overline{S}. \tag{38}$$

Notice that

$$\frac{R'(\Lambda)}{R(\Lambda)} = \frac{d}{d\Lambda} \log R(\Lambda) = \frac{2}{\Lambda + \overline{\Lambda}_n} + 2 \sum_{j=1, j\neq n}^{m} \frac{1}{\Lambda + \overline{\Lambda}_j}.$$

Consequently, using (33) and (34), we also get:

$$\begin{aligned} G'(\Lambda_n) &= G(\Lambda_n) \frac{R'(\Lambda_n)}{R(\Lambda_n)} \\ &= 2(A\Lambda_n + B) \left(\prod_{j=1, j\neq n}^{m} (\Lambda_n - \Lambda_j)^2 \right) \left(\frac{1}{2\Re(\Lambda_n)} + \sum_{j=1, j\neq n}^{m} \frac{1}{\Lambda_n + \overline{\Lambda}_j} \right). \end{aligned}$$

But, from (37), we also have

$$G'(\Lambda_n) = A \prod_{j=1,j\neq n}^{m} (\Lambda_n - \Lambda_j)^2 + 2(A\Lambda_n + B) \sum_{j=1,j\neq n}^{m} \left[(\Lambda_n - \Lambda_j) \prod_{k=1,k\neq n,j}^{m} (\Lambda_n - \Lambda_k)^2 \right].$$

Comparing these last two inequalities, we see that

$$-iA\frac{\Im(\Lambda_n)}{\Re(\Lambda_n)} - \frac{B}{\Re(\Lambda_n)} - 2(A\Lambda_n + B) \sum_{j=1,j\neq n}^{m} \left(\frac{1}{\Lambda_j - \Lambda_n} + \frac{1}{\Lambda_n + \overline{\Lambda}_j} \right) = 0. \tag{39}$$

Let us consider the equalities (38) and (39). Introducing

$$D_n^m = 2 \sum_{j=1,j\neq n}^{m} \left(\frac{1}{\Lambda_j - \Lambda_n} + \frac{1}{\Lambda_n + \overline{\Lambda}_j} \right), \quad \alpha = A\overline{\Lambda}_n - B \quad \text{and} \quad \beta = A\Lambda_n + B, \tag{40}$$

it is not difficult to rewrite these identities in the form

$$\begin{cases} \dfrac{1}{2\Re(\Lambda_n)}(\alpha + \beta) + \overline{D}_n^m \alpha = \overline{S}, \\ \dfrac{1}{2\Re(\Lambda_n)}(\beta - \alpha) + D_n^m \beta = 0. \end{cases}$$

In particular, we get

$$\beta = \frac{2\Re(\Lambda_n)\overline{S}}{1 + |1 + 2\Re(\Lambda_n) D_n^m|^2}$$

and, recalling (32), we see that

$$\|p_n - s_n^m\|^2 = F(\Lambda_n) = \frac{G(\Lambda_n)}{R(\Lambda_n)} = \frac{1}{4\Re(\Lambda_n)^2}\beta S = \frac{|S|^2}{2\Re(\Lambda_n)\left(1 + |1 + 2\Re(\Lambda_n) D_n^m|^2\right)}, \tag{41}$$

where S is given in (36).

Let us recall that $\|p_n - s_n\| = \lim_{m\to\infty} \|p_n - s_n^m\|$. In view of (36), (40) and (41), this means that

$$\|q_n\|^{-1} \equiv \|p_n - s_n\| = \frac{\left(2\Re(\Lambda_n)\right)^{1/2} P_n^2}{\left[1 + |1 + 2\Re(\Lambda_n) D_n|^2\right]^{1/2}},$$

where P_n is given by (24) and

$$D_n \equiv 2 \sum_{j=1,j\neq n}^{\infty} \left(\frac{1}{\Lambda_n + \overline{\Lambda}_j} + \frac{1}{\Lambda_j - \Lambda_n} \right).$$

We are looking for a lower bound of $\|q_n\|^{-1}$. Thanks to (25), we know how to bound P_n from below. Consequently, it will suffice to find an upper bound for $|D_n|$.

We can write $|D_n| \leq D_{n,1} + D_{n,2}$ with

$$D_{n,1} = 2 \sum_{j\geq 1} \frac{1}{|\overline{\Lambda}_j + \Lambda_n|} \leq 2 \sum_{j\geq 1} \frac{1}{\Re(\Lambda_j) + \Re(\Lambda_n)} \leq 2 \sum_{j\geq 1} \frac{1}{\Re(\Lambda_j)} \leq \frac{2}{\delta} \sum_{j\geq 1} \frac{1}{|\Lambda_j|}$$

and

$$D_{n,2} = 2 \sum_{j=1,j\neq n} \frac{1}{|\Lambda_j - \Lambda_n|}.$$

As in the proof of (25), let us introduce

$$A_1(n) = \{j : j \neq n,\ |\Lambda_j| \leq 2|\Lambda_n|\} \quad \text{and} \quad A_2(n) = \{j : |\Lambda_j| > 2|\Lambda_n|\}.$$

It is then clear that

$$D_{n,2} = 2 \sum_{j \in A_1(n)} \frac{1}{|\Lambda_j - \Lambda_n|} + 2 \sum_{j \in A_2(n)} \frac{1}{|\Lambda_j - \Lambda_n|} = S_{n,1} + S_{n,2}.$$

Let us estimate each term in the right hand side. Thanks to (14),

$$S_{n,1} \leq \frac{2}{\rho} \sum_{j \in A_1(n)} \frac{1}{|j-n|} \leq \frac{2}{\rho} c_n,$$

where $c_n = \operatorname{card} A_1(n)$. Observe that

$$\frac{c_n}{2|\Lambda_n|} \leq \sum_{j \in A_1(n)} \frac{1}{|\Lambda_j|} \leq \sum_{j \geq 1} \frac{1}{|\Lambda_j|}\,,$$

whence

$$S_{n,1} \leq C|\Lambda_n| \leq C\Re(\Lambda_n).$$

On the other hand,

$$S_{n,2} \leq 2 \sum_{j \in A_2(n)} \frac{1}{|\Lambda_j| - |\Lambda_n|} \leq 4 \sum_{j \geq 1} \frac{1}{|\Lambda_j|}\,.$$

Therefore, $|D_n| \leq D_{n,1} + D_{n,2} \leq C + C\Re(\Lambda_n)$ and

$$\|q_n\|^{-1} = \frac{\left(2\Re(\Lambda_n)\right)^{1/2} P_n^2}{\left[1 + |1 + 2\Re(\Lambda_n) D_n|^2\right]^{1/2}} \geq C(\varepsilon) e^{-\varepsilon \Re(\Lambda_n)}$$

for all $\varepsilon > 0$.

This proves the estimate (16) for $\|q_n\|$.

Estimate of $\|\widetilde{q}_n\|$: This will be easier. Let us introduce the space

$$\widetilde{F}_n^m = [p_k : 1 \leq k \leq m;\ \widetilde{p}_k : 1 \leq k \leq m,\ k \neq n]_{L^2(0,\infty)}$$

and let $\widetilde{s}_n^m$ be the unique function in $\widetilde{F}_n^m$ such that $\|\widetilde{p}_n - \widetilde{s}_n^m\| = \min_{\widetilde{r} \in \widetilde{F}_n^m} \|\widetilde{p}_n - \widetilde{r}\|$.

Again, we readily have $\lim_{m\to\infty} \|\widetilde{p}_n - \widetilde{s}_n^m\| = \|\widetilde{p}_n - \widetilde{s}_n\|$. It suffices to estimate from below

$$\|\widetilde{p}_n - \widetilde{s}_n^m\|^2 = (te^{-\Lambda_n t}, te^{-\Lambda_n t} - s_n^m) = -\widetilde{F}'(\Lambda_n),$$

where we have set

$$\widetilde{F}(\Lambda) = (e^{-\Lambda t}, te^{-\Lambda_n t} - \widetilde{s}_n^m).$$

Taking into account that $\widetilde{s}_n^m \in \widetilde{F}_n^m$, one has

$$\widetilde{F}(\Lambda) = \frac{1}{(\Lambda + \overline{\Lambda}_n)^2} - \sum_{j=1}^{m} \frac{\widetilde{a}_j^n}{\Lambda + \overline{\Lambda}_j} - \sum_{j=1, j\neq n}^{m} \frac{\widetilde{b}_j^n}{(\Lambda + \overline{\Lambda}_j)^2} = \frac{\widetilde{G}(\Lambda)}{R(\Lambda)}\,. \tag{42}$$

for appropriate coefficients $\widetilde{a}_j^n$ and $\widetilde{b}_j^n$. Again, in (42) $\widetilde{G}$ is a polynomial of degree $2m-1$ and $R(\Lambda) = \prod_{j=1}^m (\Lambda + \overline{\Lambda}_j)^2$.

The orthogonality properties of $\widetilde{r}_n^m$ imply

$$\widetilde{F}(\Lambda_j) = 0 \quad \forall 1 \leq j \leq m \quad \text{and} \quad \widetilde{F}'(\Lambda_j) = 0 \quad \forall 1 \leq j \leq m,\ j \neq n.$$

It is not difficult to see that this can be rewritten in terms of $\widetilde{G}$ as follows:

$$\widetilde{G}(\Lambda_j) = 0 \quad \forall 1 \leq j \leq m \quad \text{and} \quad \widetilde{G}'(\Lambda_j) = 0 \quad \forall 1 \leq j \leq m,\ j \neq n.$$

Consequently,

$$\widetilde{G}(\Lambda) = K(\Lambda - \Lambda_n) \prod_{j=1, j\neq n}^{m} (\Lambda - \Lambda_j)^2$$

for some $K \in \mathbb{C}$. Notice that this gives

$$\widetilde{G}(-\overline{\Lambda}_n) = -2K\Re(\Lambda_n) \prod_{j=1, j\neq n}^{m} (\overline{\Lambda}_n + \Lambda_j)^2.$$

But we also have from (42) that

$$\widetilde{G}(-\overline{\Lambda}_n) = \prod_{j=1, j\neq n}^{m} (\overline{\Lambda}_n - \overline{\Lambda}_j)^2.$$

Therefore,

$$K = -\frac{1}{2\Re(\Lambda_n)} \prod_{j=1, j\neq n}^{m} \left(\frac{\overline{\Lambda}_n - \overline{\Lambda}_j}{\overline{\Lambda}_n + \Lambda_j}\right)^2 \quad \text{and} \quad \widetilde{F}'(\Lambda_n) = \frac{-1}{(2\Re(\Lambda_n))^3} \prod_{j=1, j\neq n}^{m} \left|\frac{\Lambda_n - \Lambda_j}{\overline{\Lambda}_n + \Lambda_j}\right|^4.$$

Since $\|\widetilde{q}_n\|^2 = \lim_{m\to\infty} \|\widetilde{p}_n - \widetilde{s}_n^m\|^{-2}$ and $\|\widetilde{p}_n - \widetilde{s}_n^m\|^2 = -F'(\Lambda_n)$, we see in particular that

$$\|\widetilde{q}_n\|^{-1} = \frac{1}{[2\Re(\Lambda_n)]^{1/2}}\, P_n^2,$$

where P_n is again given by (24). The estimate (16) for $\|\widetilde{q}_n\|$ can be deduced from (25).

This ends the proof of Lemma 3.1. ∎

Proof of Lemma 3.2: As in the case of Proposition 3.4, we will only prove part b). Part a) is a direct consequence of part b).

Let us consider the linear spaces

$$\begin{cases} \mathcal{D}_\infty = \{\varphi : \varphi(t) = \displaystyle\sum_{j=1}^{N} (a_j + tb_j) e^{-\Lambda_j t} \ \forall t \in (0,\infty), \ \text{with } N \in \mathbb{N} \text{ and } a_j, b_j \in \mathbb{C}\}; \\ \mathcal{D}_T = \{\varphi : \varphi(t) = \displaystyle\sum_{j=1}^{N} (a_j + tb_j) e^{-\Lambda_j t} \ \forall t \in (0,T), \ \text{with } N \in \mathbb{N} \text{ and } a_j, b_j \in \mathbb{C}\}. \end{cases}$$

Evidently, $\Gamma : \mathcal{D}_\infty \mapsto \mathcal{D}_T$ is bijective. What we have to prove is that, for some $C(T)$, one has

$$\|\varphi\|_{L^2(0,\infty)} \leq C(T)\|\Gamma\varphi\|_{L^2(0,T)} \quad \forall \varphi \in \mathcal{D}_\infty.$$

We reason by contradiction. Thus, suppose that for every $m \geq 1$ there exist $N(m)$ and $k_m = \sum_{j=1}^{N(m)} (a_{m,j} + tb_{m,j})\, e^{-\Lambda_j t}$ such that

$$\|k_m\|_{L^2(0,\infty)} > m\|\Gamma k_m\|_{L^2(0,T)}.$$

Let us set

$$\widetilde{k}_m = \frac{1}{\|k_m\|_{L^2(0,\infty)}}\, k_m.$$

Then we can write $\widetilde{k}_m = \sum_{j=1}^{N(m)} (\widetilde{a}_{m,j} + t\widetilde{b}_{m,j})\, e^{-\Lambda_j t}$ for some complex numbers $\widetilde{a}_{m,j}$ and $\widetilde{b}_{m,j}$. Observe that, in view of (16), for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|\widetilde{a}_{m,j}| = \left|(\widetilde{k}_m, q_j)_{L^2(0,\infty)}\right| \leq \|\widetilde{k}_m\|_{L^2(0,\infty)} \|q_j\|_{L^2(0,\infty)} \leq C_\varepsilon e^{\varepsilon\Re(\Lambda_j)}$$

and

$$|\widetilde{b}_{m,j}| = \left|(\widetilde{k}_m, \widetilde{q}_j)_{L^2(0,\infty)}\right| \le \|\widetilde{k}_m\|_{L^2(0,\infty)} \|\widetilde{q}_j\|_{L^2(0,\infty)} \le C_\varepsilon e^{\varepsilon \Re(\Lambda_j)}$$

for all $m \ge 1$ and all $1 \le j \le N(m)$.

We also have

$$\|\Gamma \widetilde{k}_m\|_{L^2(0,T)} \to 0 \tag{43}$$

and

$$\|\widetilde{k}_m\|_{L^2(0,\infty)} = 1 \quad \forall m \ge 1. \tag{44}$$

Let $0 < \varepsilon < T/3$ be given and let us introduce the set

$$\mathfrak{U}_\varepsilon = \{z \in \mathbb{C} : \Re(z) > 3\varepsilon, \quad |\Im(z)| < \left(\delta^{-2} - 1\right)^{-1/2} \varepsilon\},$$

where $\delta > 0$ is given in (14). Using assumption (14), we have $|\Im(\Lambda_j)| \le \left(\delta^{-2} - 1\right)^{1/2} \Re(\Lambda_j)$ for all $j \ge 1$. Therefore, if $z \in \mathfrak{U}_\varepsilon$, one has:

$$|e^{-\Lambda_j z}| = e^{\Im(\Lambda_j)\Im(z) - \Re(\Lambda_j)\Re(z)} \le e^{-(\Re(z)-\varepsilon)\Re(\Lambda_j)} \quad \forall j \ge 1.$$

Observe that, thanks to (14), we have $\lim_{j\to\infty} \Re(\Lambda_j) = \infty$. Consequently,

$$\begin{aligned}
|\widetilde{k}_m(z)| &\le \sum_{j=1}^{N(m)} \left(|\widetilde{a}_{m,j}| + |z||\widetilde{b}_{m,j}|\right) |e^{-\Lambda_j z}| \le C_\varepsilon \sum_{j=1}^{N(m)} e^{-\Re(\Lambda_j)(\Re(z)-2\varepsilon)} (1+|z|) \\
&= C_\varepsilon e^{-\Re(\Lambda_1)(\Re(z)-2\varepsilon)} (1+|z|) \sum_{j=1}^{N(m)} e^{-[\Re(\Lambda_j)-\Re(\Lambda_1)](\Re(z)-2\varepsilon)} \\
&\le C_\varepsilon e^{-\Re(\Lambda_1)(\Re(z)-2\varepsilon)} (1+|z|) \sum_{j=1}^{\infty} e^{-\varepsilon[\Re(\Lambda_j)-\Re(\Lambda_1)]} \equiv \widetilde{C}_\varepsilon e^{-\Re(\Lambda_1)(\Re(z)-2\varepsilon)} (1+|z|)
\end{aligned}$$

for all $z \in \mathfrak{U}_\varepsilon$.

We deduce that the holomorphic function $\widetilde{k}_m(z)$ is uniformly bounded in $\mathfrak{U}_\varepsilon$. Therefore, there exist a subsequence (still denoted by $\widetilde{k}_m$) and a holomorphic function $\widetilde{k}$ in $\mathfrak{U}_\varepsilon$ such that $\widetilde{k}_m \to \widetilde{k}$ uniformly on the compacts of $\mathfrak{U}_\varepsilon$. In particular, $\widetilde{k}_m(t) \to \widetilde{k}(t)$ for all $t \in (3\varepsilon, \infty)$ and

$$|\widetilde{k}_m(t)| \le \widetilde{C}_\varepsilon e^{-\Re(\Lambda_1)(t-2\varepsilon)} (1+t) \quad \forall t \in (3\varepsilon, \infty).$$

Using Lebesgue's Theorem, we also deduce that $\widetilde{k}_m \to \widetilde{k}$ in $L^2(3\varepsilon, \infty)$ (strongly). In view of (43), $\widetilde{k}(t) = 0$ for all $t \in (3\varepsilon, T)$. Since $\widetilde{k}$ is holomorphic, we must have $\widetilde{k} \equiv 0$ in $\mathfrak{U}_\varepsilon$, whence

$$\int_T^\infty |\widetilde{k}_m(t)|^2 \, dt \to 0.$$

But this and (43) imply $\|\widetilde{k}_m\|_{L^2(0,\infty)} \to 0$, which contradicts (44).

This ends the proof of Lemma 3.2. ∎

4 Proof of Theorem 1.1

We will devote this Section to the proof of Theorem 1.1.

First of all, observe that the Kalman's rank condition (6) is a necessary condition for the controllability of system (1).

Indeed, if $B = 0$, it is clear that (1) is not null controllable at time T. Therefore, let us assume that $B \not\equiv 0$ and

$$\operatorname{rank}[B \,|\, AB] = 1,$$

i.e. $AB = \alpha B$ for some $\alpha \in \mathbb{R}$.

Let us choose $\widetilde{B} \in \mathbb{R}^2$ such that $\det \widetilde{P} = \det [B \,|\, \widetilde{B}] \neq 0$. Then it is not difficult to see that

$$\widetilde{P}^{-1}B = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \widetilde{P}^{-1}A\widetilde{P} = \widetilde{C} := \begin{bmatrix} \alpha & \beta_1 \\ 0 & \beta_2 \end{bmatrix}$$

for some $\beta_1, \beta_2 \in \mathbb{R}$. The change of variables $z = \widetilde{P}^{-1}y$ leads to the following reformulation of (1):

$$\begin{cases} z_t - z_{xx} = \widetilde{C}z & \text{in } Q\,, \\ z(0,\cdot) = e_1 v, \quad z(1,\cdot) = 0 & \text{in } (0,T)\,, \\ z(\cdot\,,0) = \widetilde{P}^{-1}y_0 & \text{in } (0,1)\,. \end{cases}$$

But it is clear that this system is neither approximately nor null controllable, since the second component of z is independent of v. Consequently, this is also the case for (1) and (6) is certainly a necessary condition.

Henceforth, it will be assumed that (6) is satisfied.

Let us take $P = [B \,|\, AB]$ and let us introduce $\widetilde{y} = P^{-1}y$. Arguing as above, we obtain

$$P^{-1}B = e_1 \quad \text{and} \quad P^{-1}AP = \widetilde{A} := \begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix},$$

where a_1 and a_2 are the coefficients of the characteristic polynomial of A:

$$p_A(\mu) = \mu^2 - a_2\mu - a_1.$$

The eigenvalues of A and $\widetilde{A}$ are the same. The system satisfied by $\widetilde{y}$ is

$$\begin{cases} \widetilde{y}_t - \widetilde{y}_{xx} = \widetilde{A}\widetilde{y} & \text{in } Q\,, \\ \widetilde{y}(0,\cdot) = e_1 v, \quad \widetilde{y}(1,\cdot) = 0 & \text{in } (0,T)\,, \\ \widetilde{y}(\cdot\,,0) = P^{-1}y_0 & \text{in } (0,1)\,, \end{cases}$$

Therefore, the previous change of variables reduces the situation to the case where

$$A = \begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{45}$$

For simplicity, it will be assumed in the rest of the proof that A and B are given by (45).

In view of Proposition 2.3, we just have to see whether or not the solutions to the adjoint system (10) satisfy the observability inequality (13). In order to deal with this inequality, we are going to reformulate the original control problem in a more simple way.

Thus, let F be a *fundamental matrix* of the linear ordinary differential system $\xi_t = A\xi$. Let us introduce w and ψ, with

$$y = F(t)w \quad \text{and} \quad \varphi = F(t)^*\psi,$$

where y is the solution to (1) (i.e. the state associated to y_0 and v) and φ is the solution to (10) (i.e. the adjoint state associated to φ_0). Then the functions w and ψ respectively satisfy

$$\begin{cases} w_t - w_{xx} = 0 & \text{in } Q\,, \\ w(0,\cdot) = \widetilde{B}(t)v(t), \ w(1,\cdot) = 0 & \text{in } (0,T)\,, \\ w(\cdot\,,0) = y_0 & \text{in } (0,1) \end{cases}$$

(where $\widetilde{B}(t) = F(t)^{-1}e_1$ for all t) and

$$\begin{cases} \psi_t + \psi_{xx} = 0 & \text{in } Q\,, \\ \psi(0,\cdot) = 0, \ \psi(1,\cdot) = 0 & \text{in } (0,T)\,, \\ \psi(\cdot\,,T) = \psi_0 & \text{in } (0,1)\,. \end{cases} \tag{46}$$

Obviously, (1) is null controllable at time T if and only if this is the case for the system satisfied by w. Clearly, this property is also equivalent to the fact that the solutions to (46), where the final data ψ_0 belong to $H_0^1(0,1)^2$, satisfy the observability inequality

$$\|\psi(\cdot\,,0)\|^2_{H_0^1(0,1)} \leq C \int_0^T |\widetilde{B}^*(t)\psi_x(0,t)|^2 dt. \tag{47}$$

We can now distinguish three different cases, depending on the spectrum of A:

Case 1: A has two different real eigenvalues.

Let μ_1 and μ_2 denote the eigenvalues of A, with $\mu_1 < \mu_2$, i.e. $a_2^2 + 4a_1 > 0$ and

$$\mu_1 = \frac{a_2}{2} - \frac{\sqrt{a_2^2+4a_1}}{2}, \qquad \mu_2 = \frac{a_2}{2} + \frac{\sqrt{a_2^2+4a_1}}{2}.$$

We can then choose M such that F is given by

$$F(t) = \begin{bmatrix} -\mu_2 e^{\mu_1 t} & -\mu_1 e^{\mu_2 t} \\ e^{\mu_1 t} & e^{\mu_2 t} \end{bmatrix}.$$

Consequently, it can be assumed that

$$\widetilde{B}(t) = F(t)^{-1} e_1 = \frac{1}{\mu_2-\mu_1} \begin{pmatrix} -e^{-\mu_1 t} \\ e^{-\mu_2 t} \end{pmatrix}$$

and (47) reads

$$\|\psi(\cdot\,,0)\|^2_{H_0^1(0,1)} \leq \frac{C}{|\mu_2-\mu_1|^2} \int_0^T |e^{-\mu_1 t}\partial_x\psi_1(0,t) - e^{-\mu_2 t}\partial_x\psi_2(0,t)|^2\, dt. \tag{48}$$

The eigenvalues and eigenfunctions of the (one-dimensional) Dirichlet Laplacian in $(0,1)$ are

$$\lambda_j = \pi^2 j^2, \quad \theta_j(x) = \sin(\pi j x), \quad j = 1, 2, \dots.$$

Hence, if $\psi^0 \in L^2(0,1)^2$ is given, the associated solution ψ is

$$\psi(x,t) = \sum_{j\geq 1} \begin{pmatrix} a_j \\ b_j \end{pmatrix} e^{-\lambda_j (T-t)} \sin(\pi j x),$$

where the a_j, b_j are the Fourier coefficients of the components of ψ_0. Replacing this expression in (48) and performing the change of variables $t \to T-t$, we readily see that (48) is equivalent to

$$\sum_{j\geq 1} \lambda_j \left(a_j^2 + b_j^2\right) e^{-2\lambda_j T} \leq C \int_0^T |\sum_{j\geq 1} j \left(a_j e^{-\mu_1(T-t)} - b_j e^{-\mu_2(T-t)}\right) e^{-\lambda_j t}|^2\, dt. \tag{49}$$

for every a_j, b_j such that $\sum_{j\geq 1} j^2 \left(a_j^2+b_j^2\right) < \infty$.

Case 1.1: there exists $j_0, k_0 \in \mathbb{N}$ with $j_0 \neq k_0$ such that $\mu_2 - \mu_1 = \lambda_{j_0} - \lambda_{k_0}$.

Setting

$$\begin{cases} a_{k_0} = j_0 e^{-\mu_2 T} \quad \text{and} \quad a_j = 0, \quad \forall j \neq k_0, \\ b_{j_0} = k_0 e^{-\mu_1 T} \quad \text{and} \quad b_j = 0, \quad \forall j \neq j_0, \end{cases}$$

we see that the observability inequality (49) fails. Therefore, system (1) is not null controllable at time T in this case.

Remark 4.1. Actually, in view of this argument, when (7) is not satisfied, (1) is not approximately controllable, since the solutions to the adjoint system (10) do not necessarily satisfy the related unique continuation property; see Remark 2.1. ■

CASE 1.2: $\mu_2 - \mu_1 \neq \lambda_j - \lambda_k$ FOR ALL $j, k \geq 1$.

Let us show that in this case (49) holds and, consequently, (1) is null controllable.

Let us introduce the sequence $\{\Lambda_j\}$, where

$$\{\Lambda_j : j \geq 1\} = \{\lambda_k : k \geq 1\} \cup \{\lambda_k + \mu_2 - \mu_1 : k \geq 1\}$$

and the indexes are fixed in such a way that $\Lambda_j \leq \Lambda_{j+1}$ for all j. Observe that $\Lambda_1 = \pi^2 > 0$, since $\mu_2 - \mu_1 > 0$. Denoting by $[x]$ the integer part of x, we see that, whenever

$$i \geq i_0 := \left[\frac{\mu_2 - \mu_1}{2\pi^2} + \frac{1}{2}\right] + 1,$$

one has $\lambda_{i-1} + \mu_2 - \mu_1 < \lambda_i < \lambda_i + \mu_2 - \mu_1 < \lambda_{i+1}$. Therefore,

$$\Lambda_{2k-1} = \lambda_k \quad \text{and} \quad \Lambda_{2k} = \lambda_k + \mu_2 - \mu_1 \quad \forall k \geq i_0 .$$

Let us check that the sequence $\{\Lambda_j\}$ satisfies the assumptions of Lemma 3.1. Clearly, if $k \leq 2i_0 - 2$, $\Lambda_{k+1} - \Lambda_k \geq \rho_1 > 0$, where ρ_1 only depends on $\mu_2 - \mu_1$. On the other hand, if $k \geq 2i_0 - 1$, it is not difficult to check that

$$\Lambda_{k+1} - \Lambda_k \geq \min\{\mu_2 - \mu_1, (2i_0 + 1)\pi^2 - \mu_2 + \mu_1\} = \rho_2 > 0$$

and, again, ρ_2 only depends on $\mu_2 - \mu_1$. Hence, condition (14) is fulfilled by taking $\rho = \min\{\rho_1, \rho_2\}$. Finally,

$$\sum_{n \geq 1} \frac{1}{\Lambda_n} < \infty.$$

Thus, we can apply Proposition 3.4 to the sequence $\{\Lambda_j\}$ with

$$C_k = \begin{cases} j a_j e^{-\mu_1 T} & \text{if} \quad \Lambda_k = \lambda_j + \mu_2 - \mu_1, \\ -j b_j e^{-\mu_2 T} & \text{if} \quad \Lambda_k = \lambda_j. \end{cases}$$

Observe that, if $k \geq i_0$,

$$\frac{|C_{2k}|^2}{\Lambda_{2k}} = \frac{k^2 a_k^2 e^{-2\mu_1 T}}{\pi^2 k^2 + \mu_2 - \mu_1} \geq C|a_k|^2 e^{-2\mu_1 T}, \quad \frac{|C_{2k-1}|^2}{\Lambda_{2k-1}} = \frac{1}{\pi^2}|b_k|^2 e^{-2\mu_2 T},$$

where $C = C(\mu_1, \mu_2)$ is a positive constant. Consequently, for $1 \leq k \leq 2(i_0 - 1)$ we have

$$\frac{|C_k|^2}{\Lambda_k} \geq \begin{cases} \dfrac{1}{\pi^2(i_0-1)^2 + \mu_2 - \mu_1}|a_j|^2 e^{-2\mu_1 T} & \text{if} \quad \Lambda_k = \lambda_j + \mu_2 - \mu_1 , \\ \dfrac{1}{\pi^2(i_0-1)^2}|b_j|^2 e^{-2\mu_2 T} & \text{if} \quad \Lambda_k = \lambda_j . \end{cases}$$

Let us now prove the observability inequality (49). The following holds:

$$\begin{aligned} I &= \int_0^T |\sum_{j\geq 1} j \left(a_j e^{-\mu_1(T-t)} - b_j e^{-\mu_2(T-t)}\right) e^{-\lambda_j t}|^2 \, dt \\ &= \int_0^T e^{2\mu_2 t} |\sum_{j\geq 1} j a_j e^{-\mu_1 T} e^{-(\lambda_j + \mu_2 - \mu_1)t} - \sum_{j\geq 1} j b_j e^{-\mu_2 T} e^{-\lambda_j t}|^2 \, dt \\ &\geq \min\{1, e^{2\mu_2 T}\} \int_0^T |\sum_{j\geq 1} j a_j e^{-\mu_1 T} e^{-(\lambda_j + \mu_2 - \mu_1)t} - \sum_{j\geq 1} j b_j e^{-\mu_2 T} e^{-\lambda_j t}|^2 \, dt \\ &= \min\{1, e^{2\mu_2 T}\} \int_0^T |\sum_{k\geq 1} C_k e^{-\Lambda_k t}|^2 \, dt. \end{aligned}$$

In view of (17), taking into account the properties of the sequence $\{C_k\}_{k\geq 1}$, we get (C is a positive constant which depends on μ_1 and μ_2):

$$
\begin{aligned}
I &\geq C(T)\min\{1, e^{2\mu_2 T}\} \sum_{k\geq 1} \frac{|C_k|^2}{\Lambda_k} e^{-\Lambda_k T} \\
&\geq CC(T)\min\{1, e^{2\mu_2 T}\} \sum_{j\geq 1} \left(a_j^2 e^{-(\mu_1+\mu_2)T} + b_j^2 e^{-2\mu_2 T}\right) e^{-\lambda_j T} \\
&\geq CC(T)\min\{1, e^{-2\mu_2 T}\} \sum_{j\geq 1} \left(a_j^2 + b_j^2\right) e^{-\lambda_j T} \geq \widetilde{C}'_T \min\{1, e^{-2\mu_2 T}\} \sum_{j\geq 1} \lambda_j \left(a_j^2 + b_j^2\right) e^{-2\lambda_j T},
\end{aligned}
$$

where $\widetilde{C}'_T > 0$ is such that

$$0 < \widetilde{C}'_T \leq \frac{1}{\lambda_j} C(T) e^{\lambda_j T} \quad \forall j \geq 1. \tag{50}$$

This proves (49) and concludes the proof in this first case.

Case 2: A has two complex eigenvalues.

In this case, $a_2^2 + 4a_1 < 0$,

$$\mu_1 = \alpha + i\beta \text{ and } \mu_2 = \alpha - i\beta, \text{ where } \alpha = a_2/2 \text{ and } \beta = \sqrt{-(a_2^2 + 4a_1)}/2.$$

We can choose M such that

$$F(t) = e^{\alpha t} \begin{bmatrix} -(\alpha\cos\beta t + \beta\sin\beta t) & -(\alpha\sin\beta t - \beta\cos\beta t) \\ \cos\beta t & \sin\beta t \end{bmatrix}.$$

Consequently,

$$\widetilde{B}(t) = F(t)^{-1} e_1 = \frac{e^{-\alpha t}}{\beta} \begin{pmatrix} -\sin\beta t \\ \cos\beta t \end{pmatrix}.$$

Again, (47) and (49) are equivalent. Now, we consider the complex sequence $\{\Lambda_k\}$, with

$$\Lambda_{2k-1} = \lambda_k = \pi^2 k^2, \quad \Lambda_{2k} = \lambda_k - 2i\beta = \pi^2 k^2 + 2i\beta \quad \forall k \geq 1.$$

The assumptions in Lemma 3.1 are again fulfilled. Indeed, one has $\Re(\Lambda_{2k-1}) = \lambda_k = |\Lambda_{2k-1}|$ and

$$\Re(\Lambda_{2k}) = \pi^2 k^2 \geq \delta(\pi^4 k^4 + 4\beta^2)^{1/2} = \delta|\Lambda_{2k}|$$

for some $\delta \in (0,1)$ (which depends on β). On the other hand,

$$\left\{ \begin{aligned} |\Lambda_{2k} - \Lambda_{2n}| &= |\Lambda_{2k-1} - \Lambda_{2n-1}| = \pi^2|k^2 - n^2| \geq 3\pi^2|k-n| \\ &= \frac{3\pi^2}{2}|2k - 2n| = \frac{3\pi^2}{2}|(2k-1) - (2n-1)| \end{aligned} \right.$$

and

$$\left\{ \begin{aligned} |\Lambda_{2k-1} - \Lambda_{2n}|^2 &= \pi^4|k^2 - n^2|^2 + 4\beta^2 \geq 9\pi^4|k-n|^2 + 4\beta^2 \\ &\geq \min\{9\pi^4/8, 2\beta^2\}|2k - 1 - 2n|^2 \end{aligned} \right.$$

for every $k, n \geq 1$. Finally,

$$\sum_{n\geq 1} \frac{1}{|\Lambda_n|} < \infty.$$

As a consequence, we can apply Proposition 3.4 a), with

$$C_{2k-1} = k a_k e^{-\mu_1 T} \quad \text{and} \quad C_{2k} = -k b_k e^{-\mu_2 T} \quad \forall k \geq 1,$$

which satisfies

$$\frac{|C_{2k-1}|^2}{|\Lambda_{2k-1}|} \geq Ce^{-2\alpha T}|a_k|^2 \quad \text{and} \quad \frac{|C_{2k}|^2}{|\Lambda_{2k}|} \geq Ce^{-2\alpha T}|b_k|^2$$

for a positive constant $C = C(\beta)$. This gives:

$$\begin{aligned} I &= \int_0^T |\sum_{j\geq 1} j\left(a_j e^{-\mu_1(T-t)} - b_j e^{-\mu_2(T-t)}\right) e^{-\lambda_j t}|^2 \, dt \\ &\geq \min\{1, e^{2\alpha T}\} \int_0^T |\sum_{k\geq 1} C_k e^{-\Lambda_k t}|^2 \, dt. \end{aligned}$$

If we now apply (17) to these Λ_k and C_k, we deduce that

$$\begin{cases} I \geq C(T) \min\{1, e^{2\alpha T}\} \sum_{k\geq 1} \frac{|C_k|^2}{|\Lambda_k|} e^{-\Re(\Lambda_k)T} \\ \quad \geq CC(T)\min\{1, e^{-2\alpha T}\} \sum_{j\geq 1} \left(a_j^2 + b_j^2\right) e^{-\lambda_j T} \\ \quad \geq C\widetilde{C}(T)\min\{1, e^{-2\alpha T}\} \sum_{j\geq 1} \lambda_j \left(a_j^2 + b_j^2\right) e^{-2\lambda_j T}, \end{cases}$$

where $\widetilde{C}(T)$ is a positive constant satisfying (50). This proves (49) in this case.

Case 3: A has a double real eigenvalue.

We denote by μ the eigenvalue of A. One has $\mu = a_2/2 \in \mathbb{R}$ and we can assume that

$$F(t) = e^{\mu t} \begin{bmatrix} -\mu & 1-\mu(t-T) \\ 1 & t-T \end{bmatrix} \quad \text{and} \quad \widetilde{B}(t) = e^{-\mu t} \begin{pmatrix} -(t-T) \\ 1 \end{pmatrix}, \quad \forall t \in [0,T].$$

The observability inequality (47) is now equivalent to prove that

$$\sum_{j\geq 1} \lambda_j \left(a_j^2 + b_j^2\right) e^{-2\lambda_j T} \leq C \int_0^T |\sum_{j\geq 1} j e^{\mu t}\left(t a_j + b_j\right) e^{-\lambda_j t}|^2 \, dt$$

for all a_j and b_j such that $\sum_{j\geq 1} j^2 \left(a_j^2 + b_j^2\right) < \infty$. But this inequality can be readily obtained from (18) by applying Proposition 3.4 to the sequences $\{\lambda_j\}$, $\{a_j\}$ and $\{b_j\}$ and taking into account (50).

This ends the proof of Theorem 1.1. ■

Combining Proposition 2.3 and Theorem 1.1, we deduce the following:

In the conditions of Theorem 1.1, there exists a positive constant C, only depending on T, such that the observability inequality (13) *is satisfied by the solutions to* (10) *if and only if conditions* (6) *and* (7) *hold.*

5 Further results and open problems

5.1 Some changes in Theorem 1.1

Obviously, the statement of Theorem 1.1 is valid if, instead of (1), we consider the controlled problem

$$\begin{cases} y_t - y_{xx} = Ay & \text{in } Q, \\ y(0,\cdot) = 0, \quad y(1,\cdot) = Bv & \text{in } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,1). \end{cases}$$

Again, $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are given, $y_0 \in H^{-1}(0,1)^2$ and $v \in L^2(0,T)$ is a control function to be determined.

On the other hand, system (1) can be posed in a general spatial interval $(0,\ell)$ with $\ell > 0$. In this case, the additional condition (7) must be changed by

$$\frac{\ell^2}{\pi^2}(\mu_1 - \mu_2) \neq j^2 - k^2 \quad \forall k, j \in \mathbb{N} \text{ with } k \neq j. \tag{51}$$

Finally, it is possible to identify the natural numbers $n \geq 1$ that can be expressed in the form $n = j^2 - k^2$ with $k, j \in \mathbb{N}$ and $j > k \geq 1$. It is not difficult to see that, given $n \in \mathbb{N}$, there exist $j, k \in \mathbb{N}$ with $j > k \geq 1$ such that $n = j^2 - k^2$ if and only if $n = 4(m+1)$ or $n = 2m+1$ for some $m \geq 1$. Thus, the controllability result for the coupled parabolic system (1) in the spatial interval $(0,\ell)$ (with $\ell > 0$) reads as follow:

Theorem 5.1. *Let $\ell > 0$, $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A. Then, system* (1) *is exactly controllable to the trajectories at time T if and only if*

$$\operatorname{rank}[B \mid AB] = 2,$$

and $(\ell/\pi)^2(\mu_1 - \mu_2)$ is not an integer of the form $4(m+1)$ or $2m+1$ for some $m \geq 1$.

5.2 Approximate controllability

As a consequence of the result stated at the end of Section 4 and the arguments in the proof of Theorem 1.1, the conditions (6) and (7) are also equivalent to the approximate controllability at time T of system (1). To be precise, one has the following result, that we state without proof:

Theorem 5.2. *Let $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ be given and let us denote by μ_1 and μ_2 the eigenvalues of A. Then, system* (1) *is approximately controllable in $H^{-1}(0,1)^2$ at time T if and only if* (6) *and* (7) *hold.*

5.3 The case of m control forces

Theorem 1.1 can be generalized to the case in which m control forces, with $m \geq 2$, appear in system (1), i.e. to the case $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^2)$ and $v \in L^2(0,T)^m$. There are two possible situations:

• $\operatorname{rank} B \leq 1$: it is then easy to check that the controllability properties of system (1) are determined by (6) and (7), as in Theorems 1.1 and 5.2.

• $\operatorname{rank} B = 2$: then (6) is automatically satisfied. Let us see that system (1) is exactly controllable to the trajectories independently of (7).

In fact, we will deduce this property as a consequence of a similar (and well known) result for scalar parabolic problems. For convenience, this will be proved in a more general framework.

Thus, let us assume that $N \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded connected open set with boundary $\partial\Omega$ of class C^2 and γ is a nonempty relative open subset of $\partial\Omega$. For $n, m \geq 2$, we consider the controlled system

$$\begin{cases} y_t - \Delta y = Ay & \text{in } Q = \Omega \times (0,T), \\ y = Bv1_\gamma & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{52}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ and $y_0 \in H^{-1}(\Omega)^n$ are given and 1_γ is the characteristic function on γ. In (52), $y = (y_1, \dots, y_n)^*$ is the state and $v \in L^2(\Sigma)^m$ is the control function.

As in Section 2, it can be shown that, for every $y_0 \in H^{-1}(\Omega)^n$ and $v \in L^2(\Sigma)^m$, the linear system (52) possesses exactly one solution (defined by transposition)

$$y \in L^2(Q)^n \cap C^0([0,T]; H^{-1}(\Omega)^n).$$

Our main assumption reads as follows:

$$\operatorname{rank} B = n. \tag{53}$$

Then, one has:

Theorem 5.3. *In the previous conditions, if* (53) *holds, then* (52) *is exactly controllable to the trajectories and approximately controllable in* $H^{-1}(\Omega)^n$ *at time* $T > 0$.

For the proof, we will use the global Carleman inequality given in the following result, by Fursikov and Imanuvilov [12]:

Theorem 5.4. *There exist a positive function* $\alpha_0 \in C^2(\overline{\Omega})$ *and two positive constants* σ_0 *and* C_0 *(only depending on* Ω *and* γ*) such that, for every* $s \geq s_0 = \sigma_0 \left(T + T^2\right)$ *and every* $z \in L^2(0,T;H_0^1(\Omega))$ *with* $z_t \pm \Delta z \in L^2(Q)$*, the following (global Carleman) estimate holds:*

$$I(z) \leq C_0 \left(\iint_Q e^{-2s\alpha} \left|z_t \pm \Delta z\right|^2 \, dx\, dt + s \iint_{\gamma\times(0,T)} e^{-2s\alpha} \rho \left|\frac{\partial z}{\partial n}\right|^2 \, d\Gamma\, dt \right).$$

Here, $I(z)$ *and the functions* α *and* ρ *are given as follows:*

$$I(z) = \iint_Q (s\rho)^{-1} e^{-2s\alpha} \left(|z_t|^2 + |\Delta z|^2 + (s\rho)^2 |\nabla z|^2 + (s\rho)^4 |z|^2 \right) dx\, dt,$$

$$\alpha(x,t) = \frac{\alpha_0(x)}{t(T-t)} \quad \forall (x,t) \in Q, \quad \rho(t) = (t(T-t))^{-1} \quad \forall t \in (0,T).$$

Let us now present the main ideas of the proof of Theorem 5.3.

Let us consider the adjoint problem

$$\begin{cases} -\varphi_t - \Delta\varphi = A^*\varphi & \text{in } Q, \\ \varphi = 0 \text{ on } \Sigma, \quad \varphi(\cdot,T) = \varphi_0 & \text{in } \Omega, \end{cases} \tag{54}$$

where $\varphi_0 \in H_0^1(\Omega)^n$. If φ is the (strong) solution to (54) associated to $\varphi_0 \in H_0^1(\Omega)^n$, it is possible to apply Theorem 5.4 to each component of φ and deduce that

$$I(\varphi) := \sum_{i=1}^n I(\varphi_i) \leq C_1 \left(\iint_Q e^{-2s\alpha} |\varphi|^2 \, dx\, dt + s \iint_{\gamma\times(0,T)} e^{-2s\alpha} \rho \left|\frac{\partial \varphi}{\partial n}\right|^2 d\Gamma\, dt \right)$$

for all $s \geq s_0 = \sigma_0 \left(T + T^2\right)$, where C_1 is a new constant which depends on n, Ω, γ and A. If we now take $s^3 \geq 2^{-5} T^6 C_1$, then $C_1 \leq (s\rho)^3/2$ and we can write

$$I(\varphi) \leq C_2 s \iint_{\gamma\times(0,T)} e^{-2s\alpha} \rho \left|\frac{\partial \varphi}{\partial n}\right|^2 d\Gamma\, dt$$

for all $s \geq s_1 = \sigma_1 \left(T + T^2\right)$, where $C_2 = 2C_1$ and $\sigma_1 = \max\{\sigma_0, 2^{-5/3} C_1^{1/3}\}$.

Taking into account (53), we get the following global Carleman estimates for the solutions to (54):

$$I(\varphi) \leq C_1 s \iint_{\gamma\times(0,T)} e^{-2s\alpha} \rho \left| B^* \frac{\partial \varphi}{\partial n}\right|^2 d\Gamma\, dt \quad \forall s \geq \sigma_1 \left(T + T^2\right), \tag{55}$$

where C_3 is a positive constant depending on n, Ω, γ, A and B.

The Carleman inequality (55) leads to a unique continuation property for the solutions to (54):

"If $\varphi \in C^0([0,T]; H_0^1(\Omega)^n)$ is a solution to (54) and $B^* \frac{\partial\varphi}{\partial n} = 0$ on $\gamma \times (0,T)$, then $\varphi \equiv 0$."

This is equivalent to the approximate controllability of (52) in $H^{-1}(\Omega)^n$ at time T.

We turn now to the exact controllability to trajectories. As above, this property is equivalent to the observability of the adjoint problem (54), i.,e. to the following property: there exists a positive constant $C > 0$ such that

$$\|\varphi\|^2_{H_0^1(\Omega)} \leq C \iint_{\gamma\times(0,T)} \left| B^* \frac{\partial \varphi}{\partial n}\right|^2 d\Gamma\, dt \quad \forall \varphi_0 \in H_0^1(\Omega)^n.$$

But, combining the global Carleman inequality (55) and the energy inequality satisfied by the solutions to (54), it is easy to show that this is true; see for instance [11] for a detailed presentation of the argument in the case of a similar scalar problem.

This ends the proof.

5.4 Possible generalizations to $n \times n$ coupled systems

It would be interesting to generalize the results presented in this work to the case of a $n \times n$ coupled system (with $n \geq 3$), controlled by m boundary control forces ($m \geq 1$).

To be precise, let us consider the system (1) with $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ and $y_0 \in H^{-1}(0,1)^n$. Then, it can be proved that the Kalman's rank condition

$$\operatorname{rank}[A \,|\, B] = n$$

is a necessary condition for the approximate controllability and also for the exact controllability to the trajectories; for a proof, see for instance [3] and [4].

On the other hand, as in the case $n = 2$, there are some necessary conditions which arise in the study of the controllability properties. Thus, let us assume that $n \geq 3$ and $m = 1$, i.e., $B \in \mathbb{R}^n$. Let us also suppose that there exist $j_0, k_0 \geq 1$ with $j_0 \neq k_0$ and two eigenvalues μ and $\widetilde{\mu}$ of A such that

$$\pi^{-2}(\mu - \widetilde{\mu}) = j_0^2 - k_0^2. \tag{56}$$

Then, (1) is neither null nor approximately controllable in $H^{-1}(\Omega)^n$ at time T. Indeed, there must exist $P \in \mathcal{L}(\mathbb{C}^n)$ (with $\det P \neq 0$) and $J \in \mathcal{L}(\mathbb{C}^{n-2})$, two matrices, such that

$$A = P \begin{pmatrix} \mu & 0 & 0 \\ 0 & \widetilde{\mu} & 0 \\ 0 & 0 & J \end{pmatrix} P^{-1}.$$

If $\varphi_0 \in H_0^1(\Omega)^n$, then the solution φ to the adjoint problem (10) satisfies

$$B^*\varphi_x(0,t) = \pi \sum_{j\geq 1} B^* j \left(P^*\right)^{-1} \begin{pmatrix} e^{(-\pi j^2+\mu)(T-t)} & 0 & 0 \\ 0 & e^{(-\pi j^2+\widetilde{\mu})(T-t)} & 0 \\ 0 & 0 & e^{(-\pi j^2 \mathbf{Id}.+J^*)(T-t)} \end{pmatrix} P^* a_j,$$

where **Id**. is the identity matrix in $\mathcal{L}(\mathbb{C}^{n-2})$ and the $a_j \in \mathbb{R}^n$ are the Fourier coefficients

$$a_j = \int_0^1 \varphi_0(x) \sin(\pi j x)\, dx.$$

Let us set $B^* \left(P^*\right)^{-1} = (\beta_1, \beta_2, \widetilde{\beta}^*)$ and $P^* a_j = (\alpha_j^1, \alpha_j^2, \widetilde{\alpha}_j^*)^*$, with $\widetilde{\beta}, \widetilde{\alpha}_j \in \mathbb{C}^{n-2}$. Then

$$B^*\varphi_x(0,t) = \pi \sum_{j\geq 1} j \left(\beta_1 e^{(-\pi j^2+\mu)(T-t)} \alpha_j^1 + \beta_2 e^{(-\pi j^2+\widetilde{\mu})(T-t)} \alpha_j^2 + \widetilde{\beta}^* e^{(-\pi j^2 \mathbf{Id}.+J^*)(T-t)} \widetilde{\alpha}_j \right).$$

Choosing $\alpha_j = 0$, for every $j \geq 1$, $\alpha_j^1 = 0$, for every $j \neq j_0$, $\alpha_j^2 = 0$, for every $j \neq k_0$, and $\alpha_{j_0}^1$ and $\alpha_{k_0}^2$ such that

$$j_0 \beta_1 \alpha_{j_0}^1 = -k_0 \beta_2 \alpha_{k_0}^2 \quad \text{and} \quad \left(\alpha_{j_0}^1\right)^2 + \left(\alpha_{k_0}^2\right)^2 \neq 0$$

and taking into account the equality (56) we deduce that $\varphi \not\equiv 0$ in Q and nevertheless $B^*\varphi_x(0,\cdot) \equiv 0$ on $(0,T)$. Therefore, the function φ does not satisfy the unique continuation property, nor the observability inequality (13). Summarizing, we have proved that the opposite to (56) is a necessary condition for the controllability of (1) when $n \geq 3$ and $m = 1$.

5.5 An example

In this paper, up to now, we have assumed that all the diffusion coefficients in the considered systems are the same. In this Section we give a simple example which shows that, when the diffusion matrix is not **Id.**, the situation can be much more complex and, again, results valid for distributed controls are no longer valid for boundary controls; see [14] and [4].

This gives an idea of the, in some sense, unnatural difficulties that arise when we try to control a non-scalar system from the boundary.

We will be concerned with the following cascade system, where $\nu > 0$:

$$\left\{ \begin{array}{ll} y_t - D y_{xx} = Ay & \text{in } Q, \\ y(0,\cdot) = Bv, \; y(1,\cdot) = 0 & \text{in } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,1), \end{array} \right. \tag{57}$$

where

$$D = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We address the following approximate controllability question: Let $\varepsilon > 0$, $y_0 \in H^{-1}(0,1)^2$ and $y_1 \in H^{-1}(0,1)^2$ be given; then, does there exist $v \in L^2(0,T)$ such that the corresponding solution to (57) satisfies

$$\|y(\cdot,T) - y_1\|_{H^{-1}} \le \varepsilon?$$

In the present situation, the adjoint system is

$$\left\{ \begin{array}{ll} -\varphi_t - D\Delta\varphi = A^*\varphi & \text{in } Q, \\ \varphi(0,\cdot) = \varphi(1,\cdot) = 0 & \text{in } (0,T), \\ \varphi(.,T) = \varphi_0 & \text{in } (0,1) \end{array} \right. \tag{58}$$

and the previous controllability property is equivalent to the following:

$$B^*\varphi_x|_{x=0} = 0 \text{ in } L^2(0,T) \text{ implies } \varphi \equiv 0 \text{ in } Q. \tag{59}$$

We then have:

Theorem 5.5. *Suppose that $\nu \neq 1$. Then (57) is approximately controllable at time $T > 0$ if and only if $\sqrt{\nu} \notin \mathbb{Q}$.*

Proof: The proof is given of two parts. In the first part we prove the unique continuation property when $\sqrt{\nu} \notin \mathbb{Q}$. In the second one, we give a counter-example to (59) when $\nu \neq 1$ and $\sqrt{\nu} \in \mathbb{Q}$.

In what follows, λ_j denotes the j-th eigenvalue of the Dirichlet Laplacian in $(0,1)$ and w_j is the associated eigenfunction of norm 1 in $L^2(0,1)$. That is, $\lambda_j = \pi^2 j^2$ and $w_j(x) \equiv \sin(\pi j x)$ for all $j \ge 1$.

First Part: Let $\varphi_0 \in H^{-1}(0,1)^2$ be given. Since $\nu \neq 1$, we have the following expression for the solution to (58):

$$\varphi(x,t) = \sum_{j\ge1} \begin{pmatrix} \left(a_j - \dfrac{b_j}{(\nu-1)\lambda_j}\right) e^{-\nu\lambda_j(T-t)} + \dfrac{b_j}{(\nu-1)\lambda_j} e^{-\lambda_j(T-t)} \\ \dfrac{b_j}{(\nu-1)\lambda_j} e^{-\lambda_j(T-t)} \end{pmatrix} w_j(x),$$

with

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \int_0^1 \varphi_0(x)\sin(\pi j x)\,dx \in \mathbb{R}^2.$$

Then,

$$B^*\varphi_x(0,t) = \sum_{j\ge1} (j\pi) \left(\left(a_j - \frac{b_j}{(\nu-1)\lambda_j}\right) e^{-\nu\lambda_j(T-t)} + \frac{b_j}{(\nu-1)\lambda_j} e^{-\lambda_j(T-t)} \right).$$

If ν is such that $\sqrt{\nu} \notin \mathbb{Q}$, the $\nu\lambda_j$ and λ_j can be reordered as an increasing sequence

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$$

and

$$B^*\varphi_x(0,t) = \sum_{j=1}^{\infty} \alpha_j e^{-\mu_j(T-t)} \quad \forall t \in (0,T).$$

It is well known that $f(t) = B^*\varphi_x(0,t)$ is an analytic function in $(0,T)$ which can be extended to the interval $(-\infty, T)$. Since $\{\mu_j\}$ is strictly increasing, it is not difficult to show that

$$B^*\varphi_x(0,\cdot) = 0 \ \text{ in } \ (-\infty,T) \Rightarrow \alpha_j = 0 \quad \forall j \geq 1.$$

Consequently, $b_j = 0$ for all j and, therefore, $a_j = 0$ for all j. In particular, $\varphi_0 = 0$ and the unique continuation property is proved.

Second Part: Suppose now that $\sqrt{\nu} \in \mathbb{Q}$. That means that $\nu = i_0^2/j_0^2$ for some $i_0, j_0 \geq 1$ and

$$\nu j_0^2 = i_0^2, \qquad \nu\lambda_{j_0} = \lambda_{i_0}.$$

Let us now take

$$b_{j_0} = 0, \quad a_{j_0} = 1,$$

$$b_{i_0} = -j_0(\nu-1)\pi^2 i_0, \quad a_{i_0} = \frac{b_{i_0}}{(\nu-1)\lambda_{i_0}}$$

and all the other coefficients a_j and b_j equal to zero. Then, $B^*\varphi_x(0,\cdot) = 0$ in $(0,T)$, but $\varphi \not\equiv 0$. ■

Remark 5.1. Observe that the arguments used in the case $\sqrt{\nu} \notin \mathbb{Q}$ are valid in a more general context. Thus, suppose that A_1 and A_2 are two self-adjoint elliptic operators involving Dirichlet conditions such that:

1. They have the same eigenfunctions but no common spectral value.
2. For both operators A_1 and A_2, the associated evolution equation satisfies the unique continuation property.

Then the coupled system associated to the equations $y_t + A_1 y = 0$ and $q_t + A_2 q = y$ is approximately controllable. ■

Appendix A: Proof of Proposition 2.2

First, observe that if $g \in L^2(Q)^2$ the solution to (8) satisfies

$$\begin{cases} \varphi \in L^2(0,T;D(-\Delta)^2) \cap C^0([0,T];H_0^1(0,1)^2) \quad \text{and} \\ \|\varphi\|_{L^2(D(-\Delta))} + \|\varphi\|_{C^0(H_0^1)} \leq C\|g\|_{L^2(Q)}. \end{cases}$$

From this regularity property it is immediate that, for any given $y_0 \in H^{-1}(0,1)^2$ and $v \in L^2(0,T)$, there exists a unique solution by transposition to (1). It is also clear that this solution satisfies the equality $y_t - y_{xx} = Ay$ in $\mathcal{D}'(Q)^2$ and the estimate

$$\|y\|_{L^2(Q)} \leq C\left(\|y_0\|_{H^{-1}(0,1)} + \|v\|_{L^2(0,T)}\right).$$

Next, we are going to show that we also have $y_{xx} \in L^2(0,T;(D(-\Delta)')^2)$ and

$$\|y_{xx}\|_{L^2(D(-\Delta)')} \leq C\left(\|y_0\|_{H^{-1}(0,1)} + \|v\|_{L^2(0,T)}\right). \tag{60}$$

To this end, let us consider two sequences $\{y_0^m\}_{m\geq 1} \subset L^2(0,1)^2$ and $\{v_m\}_{m\geq 1} \subset H^1(0,T)$ such that

$$y_0^m \to y_0 \text{ in } H^{-1}(0,1)^2 \quad \text{and} \quad v_m \to v \text{ in } L^2(0,T).$$

It is not difficult to see that problem (1) for y_0^m and v_m has a unique weak solution $y_m \in L^2(0,T;H^1(0,1))^2$ which satisfies

$$\iint_Q y_m \cdot g\,dx\,dt = \langle y_0^m, \varphi(\cdot,0)\rangle + \int_0^T B\cdot\varphi_x(0,t)\,v_m(t)\,dt,$$

for every $g \in L^2(Q)^2$ and where φ is the solution to (8). Using this last identity and (9) we get

$$\begin{cases} \|y_m\|_{L^2(Q)} \leq C\left(\|y_0\|_{H^{-1}(0,1)} + \|v\|_{L^2(0,T)}\right), \\ y_m \to y \text{ in } L^2(Q)^2 \quad \text{and} \quad y_{m,xx} \to y_{xx} \text{ in } \mathcal{D}'(Q)^2, \end{cases} \tag{61}$$

where C is a positive constant.

On the other hand, one has

$$\int_0^T \langle y_{m,xx}, \varphi\rangle = \iint_Q y_m\varphi_{xx} - \int_0^T Bv_m(t)\varphi_x(0,t)\,dt,$$

for every $\varphi \in L^2(0,T;D(-\Delta))^2$. From this equality we get $\{y_{m,xx}\}_{m\geq 1}$ is bounded in $L^2(0,T;D(-\Delta)')^2$. This property together with (61) gives $y_{xx} \in L^2(0,T;(D(-\Delta)')^2)$ and (60).

Combining the identity $y_t = y_{xx} + Ay$ and the previous property, we also see that $y_t \in L^2(0,T;(D(-\Delta)')^2)$ and

$$\|y_t\|_{L^2(D(-\Delta)')} \leq C\left(\|y_0\|_{H^{-1}(0,1)} + \|v\|_{L^2(0,T)}\right).$$

Therefore, $y \in C^0([0,T];X^2)$, where X is the interpolation space $X = [L^2(0,1), D(-\Delta)']_{1/2}$ (see [19], Theorem 3.1, p. 19). Notice that $X = [D(-\Delta), L^2(0,1)]'_{1/2} \equiv H^{-1}(0,1)$ (see also [19], Theorem 6.1, p. 29). In conclusion, we get

$$\|y\|_{C^0(H^{-1}(0,1))} \leq C\left(\|y_0\|_{H^{-1}(0,1)} + \|v\|_{L^2(0,T)}\right).$$

Finally, it is not difficult to check that $y(\cdot\,,0) = y_0$ in $H^{-1}(0,1)^2$. This ends the proof.

Appendix B: Proof of Proposition 2.3

As mentioned above, since (1) is linear, the first and second assertions are equivalent. The details are left to the reader.

Let $y_0 \in H^{-1}(0,1)^2$, $\varphi_0 \in H_0^1(0,1)^2$ and $v \in L^2(0,T)$ be given. Let y be the state associated to y_0 and v and let φ be the adjoint state associated to φ_0. Then:

$$\langle y(\cdot,t), \varphi(\cdot,t)\rangle - \langle y_0, \varphi(\cdot,0)\rangle = \int_0^t B^*\varphi_x(0,s)\,v(s)\,ds \quad \forall t \in [0,T]. \tag{62}$$

This is a straightforward consequence of the properties of y stated in Proposition 2.2.

Let us prove that the exact controllability to the trajectories together with (12) imply (13).

Indeed, let us take $\widehat{y} \equiv 0$ and let us choose y_0 arbitrarily in $H^{-1}(0,1)^2$ and φ_0 arbitrarily in $H_0^1(0,1)^2$. There exists $v \in L^2(0,T)$ such that

$$\|v\|^2_{L^2(0,T)} \leq C\|y_0\|^2_{H^{-1}(0,1)}$$

and the associated state satisfies $y(\cdot\,,T) = 0$ in $H^{-1}(0,1)^2$. Then, from (62) (with $t = T$), we get:

$$\begin{cases} \langle y_0, \varphi(\cdot,0)\rangle = -\displaystyle\int_0^T B^*\varphi_x(0,t)\,v(t)\,dt \leq \|B^*\varphi_x(0,\cdot)\|_{L^2(0,T)}\|v\|_{L^2(0,T)} \\ \qquad\qquad \leq \sqrt{C}\|B^*\varphi_x(0,\cdot)\|_{L^2(0,T)}\|y_0\|_{H^{-1}(0,1)}. \end{cases}$$

Since y_0 and φ_0 are arbitrary, we deduce (13).

Now, let us assume that (13) holds. Recall that it is enough to prove that (1) is null controllable, with controls v satisfying (12) (with $\widehat{y} \equiv 0$).

Let us fix $y_0 \in H^{-1}(0,1)^2$ and $\varepsilon > 0$ and let us consider the optimal control problem

$$\min_{v \in L^2(0,T)} \left(\frac{1}{2} \int_0^T |v(t)|^2 \, dt + \frac{1}{2\varepsilon} \|y(\cdot, T)\|^2_{H^{-1}(0,1)} \right),$$

where $y \in L^2(Q)^2$ is the state associated to y_0 and v. We can deduce easily that this control problem possesses exactly one solution $v_\varepsilon \in L^2(0,T)$, characterized by the following optimality system:

$$\begin{cases} y_{\varepsilon,t} - y_{\varepsilon,xx} = Ay_\varepsilon & \text{in } \ Q, \\ y_\varepsilon(0,\cdot) = Bv_\varepsilon, \quad y_\varepsilon(1,\cdot) = 0 & \text{in } (0,T), \\ y_\varepsilon(\cdot\,,0) = y_0 & \text{in } \ (0,1), \end{cases}$$
$$\begin{cases} -\varphi_{\varepsilon,t} - \varphi_{\varepsilon,xx} = A^*\varphi_\varepsilon & \text{in } \ Q, \\ \varphi_\varepsilon(0,\cdot) = 0, \quad \varphi_\varepsilon(1,\cdot) = 0 & \text{in } (0,T), \\ \varphi(\cdot\,,T) = \dfrac{1}{\varepsilon}(-\Delta)^{-1} y_\varepsilon(\cdot\,,T) & \text{in } \ (0,1), \end{cases}$$
$$v_\varepsilon = -\,B^*\varphi_{\varepsilon,x}(0,\cdot).$$

From (62) written for y_ε and φ_ε at $t = T$, we obtain

$$\frac{1}{\varepsilon}\langle y_\varepsilon(\cdot\,,T), (-\Delta)^{-1} y_\varepsilon(\cdot\,,T)\rangle - \langle y_0, \varphi_\varepsilon(\cdot\,,0)\rangle = -\int_0^T |B^*\varphi_{\varepsilon,x}(0,t)|^2 \, dt.$$

Observe that $\langle y_\varepsilon(\cdot\,,T), (-\Delta)^{-1} y_\varepsilon(\cdot\,,T)\rangle = \|y_\varepsilon(\cdot\,,T)\|^2_{H^{-1}(0,1)}$. Therefore,

$$\begin{cases} \displaystyle\int_0^T |B^*\varphi_{\varepsilon,x}(0,t)|^2 \, dt + \frac{1}{\varepsilon}\|y_\varepsilon(\cdot,T)\|^2_{H^{-1}(0,1)} = \langle y_0, \varphi_\varepsilon(\cdot,0)\rangle \\ \qquad \displaystyle\leq \frac{C}{2}\|y_0\|^2_{H^{-1}(0,1)} + \frac{1}{2C}\|\varphi_\varepsilon(\cdot,0)\|^2_{H^1_0(0,1)} \\ \qquad \displaystyle\leq \frac{C}{2}\|y_0\|^2_{H^{-1}(0,1)} + \frac{1}{2}\int_0^T |B^*\varphi_{\varepsilon,x}(0,t)|^2 \, dt, \end{cases}$$

where C is the constant in (13). Taking into account that $v_\varepsilon = -B^*\varphi_{\varepsilon,x}(0,\cdot)$, we deduce that

$$\|v_\varepsilon\|^2_{L^2(0,T)} + \frac{2}{\varepsilon}\|y_\varepsilon(\cdot,T)\|^2_{H^{-1}(0,1)} \leq C\|y_0\|^2_{H^{-1}(0,1)}. \tag{63}$$

This estimate allows us to extract a subsequence (still indexed with ε) such that

$$v_\varepsilon \rightharpoonup v \quad \text{weakly in} \quad L^2(0,T).$$

Let y be the state associated to y_0 and v. Thanks to Proposition 2.2, $y_\varepsilon \rightharpoonup y$ weakly in in $L^2(Q)^2$ and $y_\varepsilon(\cdot,T) \rightharpoonup y(\cdot,T)$ weakly in $H^{-1}(0,1)^2$.

Consequently, using (63), we see that we have found a control v satisfying (12) for $\widehat{y} \equiv 0$ such that the associated state satisfies (2). This ends the proof.

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