Existence of a positive solution for a singular system

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abstract. We show the existence and nonexistence of positive solutions to a system of singular elliptic equations with Dirichlet boundary condition. This system arises in studies of pattern formation in biology and in the activator-inhibitor model proposed by Gierer-Meinhardt.

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1 Introduction

In this paper we study the system

\[
\begin{aligned}
-\Delta u &= \lambda u^{q_1} - \frac{u^{p_1}}{v^{\beta_1}} \quad \text{in } \Omega, \\
-\Delta v &= \mu v^{q_2} - \frac{u^{p_2}}{v^{\beta_2}} \quad \text{in } \Omega, \\
&\quad u = v = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^N, N \geq 1 \), is a bounded domain with smooth boundary \( \partial \Omega \),

\[ \lambda, \mu \in \mathbb{R}, \quad 0 < q_1, q_2, \beta_1, \beta_2 < 1 \quad \text{and} \quad p_1, p_2 > 0. \]

(1.2)
Our main goal in this paper is to show results about existence and nonexistence of positive solutions of (1.1) in terms of the parameters $\lambda$ and $\mu$. It is clear that, thanks to the maximum principle, if $\lambda \leq 0$ or $\mu \leq 0$ then (1.1) does not possess positive solutions. With respect to the existence, our main result is

**Theorem 1.1.** (A) Assume that $q_1 < p_1$. There is a constant $\lambda^*(\Omega) > 0$ depending on $\Omega$ such that for

$$\mu \geq \lambda^*(\Omega)\lambda^\sigma \quad \text{and} \quad \lambda > 0$$

where

$$\sigma = \frac{p_2(1 - q_2)}{(1 + \beta_2)(1 - q_1)},$$

there exists a positive $C^{1,\gamma}(\overline{\Omega})$, $0 < \gamma < 1$ solution of (1.1).

(B) Assume that $q_1 \geq p_1$. There is a constant $\lambda_0(\Omega) > 0$ depending on $\Omega$ such that for

$$\lambda < \lambda_0(\Omega)\mu^{-r} \quad \text{and} \quad \mu > 0,$$

where

$$r = \frac{\beta_1(1 - q_1)}{(1 - p_1)(1 - q_2)},$$

then (1.1) does not possess a positive solution.

Systems of singular equations like (1.1) are the stationary counterpart of general evolutionary problems of the form

$$\begin{cases}
    u_t = \eta \Delta u + \lambda u^{q_1} - \gamma \frac{u^{p_1}}{v^{\beta_1}} & \text{in } \Omega, \\
    v_t = \delta \Delta v + \mu v^{q_2} - \theta \frac{u^{p_2}}{v^{\beta_2}} & \text{in } \Omega, \\
    u = v = 0 & \text{on } \partial \Omega.
\end{cases} \quad (1.3)$$

In the original model proposed by Gierer-Meinhardt [10],

$$\eta, \delta > 0, \quad \lambda, \mu, \gamma, \theta < 0, \quad q_1 = q_2 = 1, \quad p_1, p_2, \beta_1, \beta_2 > 0, \quad 0 < (p_1 - 1)/\beta_1 < p_2/(\beta_2 + 1)$$

and the boundary conditions are of Neumann type. This system was motivated by biological experiments on hydra in morphogenesis, where $u$ represents the density of an activator chemical substance and $v$ is an inhibitor. The slow diffusion of $u$ and the fast diffusion of $v$ is translated into the fact that $\eta$ is small and $\delta$ is large, see also [11, 16, 18] for an account on biological applications of such systems. There are a few papers dealing with scalar equations [1, 4, 5, 8, 19] and references therein.

According to an observation made in [3], it is natural to study (1.3) with Dirichlet boundary conditions, since numerical experiments from [10] exhibit solutions approaching zero near the boundary of $\Omega$. Moreover, Neumann condition is not explicitly mentioned in the original paper [10]. Although, the majority of early papers deal with a system on a bounded domain with Neumann boundary conditions.

The stationary system with

$$\eta = \delta = 1, \quad \lambda = \mu = \gamma = \theta = -1 \quad \text{and} \quad p_1 = p_2 = q_1 = q_2 = \beta_1 = \beta_2 = 1.$$
was studied in [2]. Thus for the system
\[
\begin{align*}
-\Delta u &= -u + \frac{u^{p_1}}{v^{\beta_1}} \quad \text{in } \Omega, \\
-\Delta v &= -v + \frac{u^{p_2}}{v^{\beta_2}} \quad \text{in } \Omega, \\
u = v = 0 &\quad \text{on } \partial \Omega,
\end{align*}
\] (1.4)
they have shown existence and nonexistence of solutions and uniqueness of solution in one dimension. Another uniqueness result for (1.4) was proved in [3], in the situation
\[\eta = \delta = 1, \quad \lambda = \mu = \gamma = \theta = -1 \quad \text{and} \quad p_1 = p_2 > 1, \quad \beta_2 = 0, \quad \beta_1 = q_1 = q_2 = 1.\]

A study allowing more general singular nonlinearities was performed in [9, 13, 14].

We are interested in studying stationary states of (1.3) for a different range of parameters and constants (1.2). Notice that our results depend strongly on the size of \(q_1\) and \(p_1\). Indeed, in the existence part (A) of Theorem 1.1 we require \(q_1 < p_1\), and the conclusion holds for \(\lambda > 0\) and \(\mu \geq C\lambda^\sigma\) for some positive constants \(C\) and \(\sigma\). Part (B) demands \(q_1 \geq p_1\), thus the nonexistence of solution is inferred for \(\lambda > 0\) and \(\mu < C\lambda^{-r}\) for some positive constants \(C\) and \(r\). In order to obtain our main results we use an adequate sub-supersolution method, which will be detailed later.

The paper is organized as follows. In section 2 we show that the sub-supersolution method holds for our system, which has singular nonlinearities, generalizing classical results, see for instance [17]. In section 3 we study some auxiliary problems related to sublinear equations, singular equations and porous medium logistic equation. Section 4 is devoted to the proof of Theorem 1.1.

## 2 The sub-super method for singular systems

First of all we show that the sub-supersolution method works well for singular systems. We consider the general system
\[
\begin{align*}
-\Delta u &= f(x, u, v) \quad \text{in } \Omega, \\
-\Delta v &= g(x, u, v) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (2.1)
where \(f, g : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) are Caratheodory functions. On the other hand, we denote by
\[\rho_0(x) = \text{dist}(x, \partial \Omega),\]
and given \(w \leq z\) a.e. in \(\Omega\)
\[
[w, z] := \{u : w(x) \leq u(x) \leq z(x) \quad \text{a.e. } x \in \Omega\}.
\]
The notions of solutions and sub-supersolutions of (2.1) are:

**Definition 2.1.** We say that \((u, v) \in (L^1(\Omega))^2\) is a solution of (2.1) if
1. \(f(\cdot, u, v)\rho_0, g(\cdot, u, v)\rho_0 \in L^1(\Omega)\);
2.\[ -\int_{\Omega} u\Delta \xi = \int_{\Omega} f(x,u,v)\xi, \quad -\int_{\Omega} v\Delta \xi = \int_{\Omega} g(x,u,v)\xi, \quad \forall \xi \in C^2_0(\Omega). \]

**Definition 2.2.** We say that \((u, v), (\bar{u}, \bar{v}) \in (L^1(\Omega))^2\) is a pair of sub-supersolutions of (2.1) if

1. \(u \leq \bar{u}\) and \(v \leq \bar{v}\) in \(\Omega\);

2. \(f(\cdot, u, v)\rho_0, f(\cdot, u, v)\rho_0 \in L^1(\Omega)\) for all \(u \in [u, \bar{u}]\) and \(v \in [v, \bar{v}]\),

\[ g(\cdot, u, v)\rho_0, g(\cdot, u, v)\rho_0 \in L^1(\Omega) \quad \text{for all } u \in [u, \bar{u}] \text{ and } v \in [v, \bar{v}]; \]

3. for all \(\xi \in C^2_0(\Omega), \xi \geq 0\),

\[ -\int_{\Omega} u\Delta \xi - \int_{\Omega} f(x,u,v)\xi \leq 0 \leq -\int_{\Omega} \bar{u}\Delta \xi - \int_{\Omega} f(x,\bar{u},v)\xi, \quad \forall v \in [v, \bar{v}]; \]

and

\[ -\int_{\Omega} v\Delta \xi - \int_{\Omega} g(x,u,v)\xi \leq 0 \leq -\int_{\Omega} \bar{v}\Delta \xi - \int_{\Omega} g(x,u,\bar{v})\xi, \quad \forall u \in [u, \bar{u}]. \]

Next we prove that the existence of a pair of sub-supersolutions implies the existence of a solution of the system.

**Theorem 2.3.** Assume that there exists a pair of sub-supersolution \((u, v), (\bar{u}, \bar{v})\) of (2.1). Then, there exists a solution \((u, v)\) of (2.1) such that \(u \leq \bar{u}\) and \(v \leq \bar{v}\) in \(\Omega\).

**Proof.** First, we define the truncations

\[ Tu(x) := \begin{cases} \bar{u}(x) & \text{if } u(x) \geq \bar{u}(x), \\ u(x) & \text{if } u(x) \leq u(x) \leq \bar{u}(x), \\ \bar{u}(x) & \text{if } u(x) \leq u(x), \end{cases} \quad \text{(2.3)} \]

and

\[ Sv(x) := \begin{cases} \bar{v}(x) & \text{if } v(x) \geq \bar{v}(x), \\ v(x) & \text{if } v(x) \leq v(x) \leq \bar{v}(x), \\ \bar{v}(x) & \text{if } v(x) \leq v(x). \end{cases} \quad \text{(2.4)} \]

We denote by

\[ L^1(\rho_0, \Omega) := \{ u : u\rho_0 \in L^1(\Omega) \}. \]

We define the Nemytskii operators (well defined by (2.2))

\[ F : L^1(\Omega) \times L^1(\Omega) \rightarrow L^1(\rho_0, \Omega) \]

\[ (u, v) \mapsto F(u, v) := f(x, Tu, Sv) \]
and similarly
\[ G : L^1(\Omega) \times L^1(\Omega) \rightarrow L^1(\rho_0, \Omega) \]
\[ (u, v) \rightarrow G(u, v) := g(x, Tu, Sv). \]

We define the operator \( K : L^1(\rho_0, \Omega) \rightarrow L^1(\Omega) \) by \( h \mapsto w := K(h) \), being \( w \) the unique solution of
\[
\begin{cases}
-\Delta w = h & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]

It can be proved:

1. \( F \) and \( G \) are continuous (Theorem 2.1 in [15], the notion of equi-integrability is not needed here).
2. \([F, G](L^1(\Omega))^2\) is bounded in \( L^1(\rho_0, \Omega) \), since \( T \) and \( S \) defined by (2.3) and (2.4) are bounded.
3. \( K \circ F \) and \( K \circ G \) are continuous and compact operators from \((L^1(\Omega))^2\) to \( L^1(\Omega)\) (Theorem 3.1 in [15]).

Then, by the Schauder’s fixed point theorem, we can conclude the existence of a solution \((u, v) \in (L^1(\Omega))^2\) of
\[
\begin{cases}
-\Delta u = f(x, Tu, Sv) & \text{in } \Omega, \\
-\Delta v = g(x, Tu, Sv) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We claim that \((u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]\) and so \((u, v)\) is solution of (2.1). Indeed, let
\[ w := u - \bar{u}. \]

Then, for all \( V \in [\underline{v}, \bar{v}] \) and all \( \xi \in C^2_0(\Omega), \xi \geq 0 \), we get
\[
- \int_{\Omega} w \Delta \xi \leq \int_{\Omega} (f(x, Tu, Sv) - f(x, \bar{u}, V)) \xi
\]
and then taking \( V = Sv \)
\[
- \int_{\Omega} w \Delta \xi \leq \int_{\Omega} (f(x, Tu, Sv) - f(x, \bar{u}, Sv)) \xi.
\]

Then, applying the Kato’s inequality (see Proposition 3.1 in [15]) we obtain
\[
- \int_{\Omega} w^+ \Delta \xi \leq \int_{[u \geq \bar{u}]} (f(x, Tu, Sv) - f(x, \bar{u}, Sv)) \xi = 0 \quad \forall \xi \in C^2_0(\Omega), \xi \geq 0.
\]

We deduce that \( w^+ = 0 \) a.e.; and conclude the proof.

**Remark 2.4.** Assuming more regularity to \( f, g \) and the pair of sub-supersolution, we can obtain that the solution lies in a better space, see Section 5 in [15]. See also Remark 3.6.
3 Some auxiliary problems

In order to find a pair of sub-supersolutions of (1.1) we need to study some scalar equations. First of all, given \( \lambda \in \mathbb{R} \) and \( 0 < q < 1 \), consider

\[
\begin{cases}
-\Delta u = \lambda u^q & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3.1)

It is well-known that there exists a unique positive solution of (3.1) if, and only if, \( \lambda > 0 \). We denote this solution by \( \omega_{[\lambda,q]} \); moreover

\[ \omega_{[\lambda,q]} = \lambda^{1/(1-q)} \omega_{[1,q]} \]

It is known that there exist constants \( k \) and \( K \) with \( 0 < k < K < +\infty \) such that

\[ k \rho_0(x) \leq \omega_{[\lambda,q]}(x) \leq K \rho_0(x) \quad x \in \Omega. \]  

(3.2)

We need to study the following problem

\[
\begin{cases}
-\Delta u = \lambda f(x,u) - \frac{a(x)}{u^\beta} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(3.3)

where \( \beta \in (0,1) \) and

\[ a : \Omega \to \mathbb{R} \quad \text{is a continuous positive function}, \]

(3.4)

there is \( 1 < \gamma < 2 \) such that \( \limsup_{x \to \partial \Omega} \frac{a(x)}{\rho_0(x)^{\gamma(1+\beta)-2}} < +\infty \),

(3.5)

\[ f : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a continuous function}, \]

(3.6)

\[ f(x,s) > 0 \quad \text{for } s \neq 0, \]

(3.7)

\[ \lim_{s \to +\infty} \frac{f(x,s)}{s} = 0 \quad \text{uniformly in } x. \]

(3.8)

In the following result we characterize the existence of positive solution of (3.3).

**Proposition 3.1.** There exists \( \lambda^* \in (0, +\infty) \) such that for all \( \lambda \geq \lambda^* \), problem (3.3) has a positive a.e. weak solution and no positive solution for \( \lambda < \lambda^* \).

**Proof.** We are going to apply the sub-supersolution method from [15]. Take

\[ \underline{u} := c \varphi_1^\gamma, \quad \overline{u} := Ke, \]

for \( c, K > 0 \) such that \( \underline{u} \leq \overline{u} \) in \( \Omega \), where \( e \) is the unique positive solution of

\[
\begin{cases}
-\Delta e = 1 & \text{in } \Omega, \\
e = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(3.1)
and \( \varphi_1 > 0 \) is the first eigenfunction of the Laplacian in \( H^1_0(\Omega) \) such that \( \|\varphi_1\|_\infty = 1 \).

Recall that there exist positive constants \( 0 < c < C < \infty \) such that

\[
0 < c \rho_0(x) \leq e(x), \varphi_1(x) \leq C \rho_0(x), \quad \forall x \in \Omega.
\]

First, observe that

\[
\left| \lambda f(x, u) - \frac{a(x)}{u^\beta} \right| \rho_0 \in L^1(\Omega), \quad \forall u \in [u, \overline{u}].
\]

Indeed, for \( u \in [u, \overline{u}] \) we have

\[
|a(x)u^{-\beta}|\rho_0 \leq Ca(x)\rho_0^{-\gamma+1} \leq C\rho_0^{\gamma-1} \in L^1(\Omega)
\]

if \( \gamma - 1 > -1 \).

To show that \( u \) is subsolution, we need to verify

\[
-\Delta u + \frac{a(x)}{u^\beta} = -c\gamma(\gamma - 1)|\nabla \varphi_1|^2 + c\gamma \lambda_1 \varphi_1^2 + a(x)c^{-\beta}\varphi_1^{-\beta\gamma} \leq \lambda f(x, c\varphi_1^\gamma) \quad \text{in} \quad \Omega.
\]

We distinguish two cases:

(i) Near the boundary \( \partial \Omega \):

For every \( M > 0 \) there is a \( \delta > 0 \) such that for every \( x \in \Omega \setminus \{ x \in \Omega : \rho_0(x) < \delta \} \) one has by (3.5)

\[
-\gamma(\gamma - 1)|\nabla \varphi_1|^2 + a(x)c^{-\beta}\varphi_1^{-\beta\gamma} = c^{1+\beta} \gamma(\gamma - 1)|\nabla \varphi_1|^2 + \frac{a(x)}{\varphi_1^{\gamma-2+\beta\gamma}} \leq c^{1+\beta} \gamma(\gamma - 1)|\nabla \varphi_1|^2 + M
\]

for a sufficiently large \( c > 0 \).

In this way, taking \( \delta \) smaller if necessary, we get

\[
-\Delta u + \frac{a(x)}{u^\beta} \leq c\gamma \varphi_1^{\gamma-2}[\varphi_1^2] \leq 0.
\]

Notice that if \( M = 0 \), we can take \( c > 0 \) arbitrary.

(ii) Inner points \( x \in \Omega \setminus \overline{\Omega}_\delta \).

Once \( c \) has been fixed above, take \( \lambda \) large enough in such a way that

\[
e^{1+\beta} \gamma \lambda_1 \varphi_1^2 + a(x)\varphi_1^{-\beta\gamma} \leq \lambda c^\beta f(x, c\varphi_1^\gamma).
\]

On the other hand, with respect to the supersolution we need that

\[
-\Delta \overline{u} \geq \lambda f(x, \overline{u}) - \frac{a(x)}{\overline{u}^\beta},
\]

for which it suffices that

\[
K \geq \lambda f(x, K\varphi_1).
\]
This is promptly verified for $K$ large enough thanks to (3.8).

We claim that there is no positive solution of (3.3) if $\lambda > 0$ is small. Indeed, if $u > 0$ is an existing solution, multiply the equation by $\varphi_1$ and integrate. Hence,

$$
\int_{\Omega} (\lambda_1 \varphi_1 u + \frac{a(x)}{w^\beta} \varphi_1) = \lambda \int_{\Omega} f(x, u) \varphi_1
$$

(3.9)

Let $\delta > 0$ and $\Omega^\delta := \{ x \in \Omega : \rho_0(x) > \delta \}$. Thus

$$
c \int_{\Omega^\delta} (u + \frac{1}{w^\beta}) \varphi_1 \leq \int_{\Omega^\delta} (\lambda_1 u + \frac{a(x)}{w^\beta}) \varphi_1 < \lambda \int_{\Omega} f(x, u) \varphi_1
$$

(3.10)

where $c$ is a constant depending on $\delta$, $\Omega$ and $\|a\|_{L^\infty(\Omega^\delta)}$. Since

$$
\lambda \int_{\Omega} f(x, u) \varphi_1 \to 0 \quad \text{as} \quad \lambda \to 0
$$

we get a contradiction since $u + 1/u^\beta$ is bounded from below and $\int_{\Omega} f(x, u) \varphi_1$ is bounded.

Remark 3.2. If $\gamma - 2 + \beta \gamma > 0$, then in view of (3.5), $a(x) \to 0$ as $x \to \partial \Omega$. This is true if $\beta \geq 1$ for example.

If $\gamma - 2 + \beta \gamma < 0$, then eventually $0 < \beta < 1$ and $a(x) \to 0$ as $x \to \partial \Omega$ or $a(x) \to +\infty$ as $x \to \partial \Omega$. But with (3.5) satisfied.

We now consider a particular case of (3.3),

$$
\begin{cases}
-\Delta u = \lambda u^q - a(x) \frac{1}{u^\beta} & \text{in} \ \Omega, \\
u = 0 & \text{on} \ \partial \Omega,
\end{cases}
$$

(3.11)

where $0 < q, \beta < 1$ and $a$ verifies (3.4) and (3.5).

Proposition 3.3. There exists $\lambda^*(a) > 0$ such that a positive maximal solution of (3.11) exists if, and only if,

$$
\lambda \geq \lambda^*(a).
$$

We denote this maximal solution by $\Theta_{(\lambda, q, \beta, a)}$. Moreover, the map $a \mapsto \lambda^*(a)$ is increasing.

Furthermore, if $a \in C(\bar{\Omega})$, there exist constants $c$ and $C$ such that

$$
c \rho_0(x) \leq \Theta_{(\lambda, q, \beta, a)}(x) \leq C \rho_0(x).
$$

(3.12)

Proof. The existence of a positive solution as well as $\lambda^*(a)$ follow by Proposition 3.1. The maximality of the solution is due to the fact that any positive solution of (3.3) is a subsolution of (3.1).

The fact that $a \mapsto \lambda^*(a)$ is increasing is immediate.

The existence of the constant $c$ verifying (3.12) is due to the Hopf maximum principle and $C$ is due to the $C^1(\bar{\Omega})$ regularity of the solution, see also Remark 3.6. □
We need some properties of the porous medium logistic equation with a possibly singular weight

\[
\begin{aligned}
-\Delta u &= \lambda u^q - N(x)u^p \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(0 < q < 1, p > 0\) with

\[
0 < N \leq k \rho_0(x)^\beta, \quad k > 0, \quad (3.14)
\]

\(N \in C(\Omega)\) and \(\beta \in \mathbb{R}\) (possibly negative).

**Proposition 3.4.** Assume that \(\beta + p > -1\).

1. If \(q < p\), then there exists a unique \(C^1(\overline{\Omega})\) positive solution if, and only if, \(\lambda > 0\).

2. If \(q \geq p\), then there exists \(\lambda^* (N) \geq 0\) such that there exists a positive \(C^1(\Omega)\) solution if, and only if, \(\lambda \geq \lambda^* (N)\).

Moreover, if \(N \geq N_0 > 0\) for some \(N_0 \in \mathbb{R}\) then \(\lambda^* (N) > 0\).

**Proof.** Take \(u := Ke\) and \(u := \varepsilon \varphi_1^r\), \(r \geq 1\) and \(K, \varepsilon > 0\) positive constants to be chosen later. In order to apply the sub-supersolution method we need that

\[
|N(x)u^p|\rho_0 \in L^1(\Omega), \quad \forall u \in [u, \overline{u}].
\]

Observe that \((3.14)\) implies

\[
|N(x)u^p|\rho_0 \leq K \rho_0^{\beta + p + 1}
\]

and so \(|N(x)u^p|\rho_0 \in L^1(\Omega)\) if

\[
\beta + p > -2.
\]

First observe that \(u\) is subsolution of \((3.13)\) provided that

\[
r(1 - r)\varepsilon^{1-q} r_1^{(1-q)-2} |\nabla \varphi_1|^2 + r\varepsilon^{1-q} \lambda_1 r_1^{(1-q)} + C \varepsilon^{p-q} r_1^{(p-q)+\beta} \leq \lambda. \quad (3.15)
\]

On the other hand, \(\overline{u}\) is supersolution if \(K\) is taken large. Take also \(K\) large such that \(u \leq \overline{u}\) in \(\Omega\). So, it suffices to verify \((3.15)\). For that, we consider two cases:

1. Assume that \(p > q\). Take \(r > 1\) such that \(r(p - q) + \beta > 0\). Then, recalling that \(\|\varphi_1\|_{\infty} = 1\), \((3.15)\) is satisfied if

\[
r\varepsilon^{1-q} \lambda_1 + C \varepsilon^{p-q} \leq \lambda
\]

for which it suffices to take \(\varepsilon\) sufficiently small.

With respect to the uniqueness, the result follows applying Theorem 2.1 in [6], specifically taking \(g(t) = t^q\).

2. Assume now that \(p \leq q\). Take now \(\varepsilon = 1\). Again we distinguish two cases:

(i) Near the boundary \(\partial \Omega\):

Take in this case \(r \geq 1\) and \(r(1 - q) - 2 < r(p - q) + \beta\), or equivalently, \(r(1 - p) < \beta + 2\). Then we need that \(1 < (2 + \beta)/(1 - p)\) or equivalently \(-1 < \beta + p\). In this case, \((3.15)\) is equivalent to

\[
\varphi_1^{r(1-q)-2} \left[ r(1 - r)|\nabla \varphi_1|^2 + r\lambda_1 \varphi_1^2 + C \varphi_1^{r(p-1)+\beta+2} \right] \leq \lambda.
\]
Take \( \delta > 0 \) small enough such that
\[
 r(1 - r)|\nabla \varphi_1|^2 + r \lambda_1 \varphi_1^2 + C \varphi_1^{(p-1)+\beta+2} < 0
\]
in \( \Omega_\delta = \{ x \in \Omega : \rho_0(x) < \delta \} \).

(ii) Inner points:
In the region \( \Omega \setminus \overline{\Omega}_\delta \) we have that \( \varphi_1 \geq c(\delta) \) for some \( c(\delta) > 0 \). Hence, for (3.15) it is sufficient that
\[
 r \lambda_1 + C(\delta) \leq \lambda,
\]
for some \( C(\delta) \). Fixed \( \delta \), we can take \( \lambda \) large.

Hence, we can define
\[
 \lambda_*(N) = \inf \{ \lambda > 0 \mid \text{such that (3.13) has a positive a.e. solution} \}.
\]
Then \( \lambda_*(N) < +\infty \) and for all \( \lambda \geq \lambda_*(N) \), problem (3.13) has a positive a.e. weak solution.

Finally, assume that \( N \geq N_0 > 0 \) and \( q \geq p \). Then, multiplying the equation by \( \varphi_1 \) and integrating we have
\[
 0 = \int_\Omega \varphi_1 u^p(\lambda u^{q-p} - N - \lambda_1 u^{1-p}) \leq \int_\Omega \varphi_1 u^p(\lambda u^{q-p} - N_0 - \lambda_1 u^{1-p}).
\]
Assuming \( q > p \), the maximum of the function \( f(x) := \lambda x^{q-p} - \lambda_1 x^{1-p} \) is attained at
\[
 x_M = \left( \frac{\lambda(q-p)}{\lambda_1(1-p)} \right)^{1/(1-q)}
\]
and
\[
 f(x_M) = \lambda^{(1-p)/(1-q)} \left( \frac{q-p}{\lambda_1(1-p)} \right)^{(q-p)/(1-q)} \frac{1-q}{1-p}
\]
and so if \( \lambda \) is small we have that
\[
 \int_\Omega \varphi_1 u^p(\lambda u^{q-p} - N_0 - \lambda_1 u^{1-p}) < 0,
\]
a contradiction. A similar argument can be used in the case \( q = p \). This completes the proof.

\textbf{Remark 3.5.} Equations (3.3) and (3.11) have been studied in [5] and [19], but with different behavior of \( a(x) \) or without \( a(x) \). Also, equation (3.13) has been previously studied when \( N \) is bounded, see [7] and references therein.

\textbf{Remark 3.6.} The solutions of Propositions 3.3, 3.4 and Theorem 1.1 (A) belong to \( C^{1,\Upsilon}(\overline{\Omega}) \), \( 0 < \Upsilon < 1 \). This follows from the results in [12] which says that if \(-\Delta u = h \) in \( \Omega \) with \( u = 0 \) on \( \partial \Omega \) and \( \sup_{\Omega} |h(x)| \rho_0^\Upsilon(x) < \infty \) for some \( 0 < \Upsilon < 1 \), then \( u \in C^{1,1-\Upsilon}(\overline{\Omega}) \).
4 Proof of Theorem 1.1

We are going to apply the sub-supersolution method to system (1.1). If we denote
\[ f(u, v) := \lambda u^{q_1} - \frac{v^{p_2}}{v^{\beta_2}}, \quad g(u, v) := \mu v^{q_2} - \frac{u^{p_1}}{u^{\beta_1}}, \]
the third paragraph of the definition of sub-supersolution (Definition 2.2) is equivalent to
\[-\Delta u \leq f(u, v), \quad -\Delta u \geq f(u, v), \]
and
\[-\Delta v \leq g(u, v), \quad -\Delta v \geq g(u, v). \]

We start the proof of Theorem 1.1:

Proof. (A) Take
\[ \overline{u} := \omega_{[\lambda, q_1]}, \quad \text{and} \quad \overline{v} := \omega_{[\mu, q_2]}. \quad (4.1) \]
A subsolution is
\[ \underline{v} := \Theta_{[\mu, q_2, \beta_2, \omega_{[1, q_1]}^{p_2}]}, \quad (4.2) \]
Observe that \( \omega_{[\lambda, q_1]} = \lambda^{1/(1-q_1)}\omega_{[1, q_1]} \) and so \( \underline{v} \) verifies
\[-\Delta \underline{v} = \mu v^{q_2} - \lambda^{p_2/(1-q_1)} \frac{\omega_{[1, q_1]}^{p_2}}{v^{\beta_2}} \quad \text{in} \quad \Omega. \quad (4.3) \]
Under the change of variable
\[ V = R v, \]
where
\[ R = \frac{1}{\lambda^{p_2/(1-q_1)(1+\beta_2))}, \]
(4.3) transforms into
\[ \begin{cases} -\Delta V = \Lambda V^{q_2} - \frac{\omega_{[1, q_1]}^{p_2}}{V^{\beta_2}} \quad \text{in} \quad \Omega, \\ V = 0 \quad \text{on} \quad \partial \Omega, \end{cases} \quad (4.4) \]
where
\[ \Lambda = \mu \lambda^{-\sigma}, \]
with
\[ \sigma = \frac{p_2(1-q_2)}{1-q_1(1+\beta_2)} \]
Observe that (4.4) is in the setting of (3.11) by taking \( a = \omega_{[1, q_1]}^{p_2} \). Indeed, (3.4) and (3.5) are verified for all \( \gamma \) such that
\[ \gamma \leq \frac{p_2 + 2}{1 + \beta_2}, \]
which can be chosen \( 1 < \gamma \). Hence, applying Proposition 3.3, we conclude the existence of a positive solution of (4.4) if
\[ \Lambda \geq \lambda^*(\Omega) \]
or equivalently,

\[ \mu \geq \lambda^*(\Omega)\lambda^\sigma. \]

It is clear that \( v \leq \overline{v} \) and \( v > 0 \) if \( \mu \geq \lambda^*(\Omega)\lambda^\sigma \). It remains to check that there exists \( \underline{v} > 0 \) and satisfies

\[ -\Delta u \leq \lambda u^{q_1} - v^{-\beta_1} u^{p_1} \quad \text{in } \Omega. \]

Let \( u \) be the solution of

\[
\begin{aligned}
-\Delta u &= \lambda u^{q_1} - v^{-\beta_1} u^{p_1} \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\tag{4.5}
\]

Observe that in this case \( N(x) = v^{-\beta_1} \), being \( v \) defined in (4.2). Hence, taking into account (3.12) we obtain that \( 0 < N \leq C \rho_0^{-\beta_1} \) and so it is clear that

\[ -\beta_1 + p_1 > -1. \]

Thus we can apply Proposition 3.4 to conclude that, if \( q_1 < p_1 \), there exists a positive solution of (4.5) provided \( \lambda > 0 \). Moreover, it is clear that \( v \leq \overline{v} \).

Finally, the second paragraph of Definition 2.2 is easy to verify.

In conclusion, if \( q_1 < p_1 \) there is a positive solution of (1.1) if \( \lambda > 0 \) and \( \mu \geq \lambda^*(\Omega)\lambda^\sigma \).

(B) Finally, we assume that \( q_1 \geq p_1 \). Observe that if \((u, v)\) is a solution of (1.1), then

\[ v \leq \omega[\mu, q_2] = \mu^{1/(1-q_2)} \omega[1, q_2] \]

and then,

\[ -\Delta u \leq \lambda u^{q_1} - \mu^{-\beta_1/(1-q_2)} \omega^{-\beta_1 / [1, q_2]} u^{p_1}. \]

Under the change of variable

\[
U = Ru, \quad R = \mu^{\beta_1 / ((1-q_2)(1-p_1))}
\]

the above inequality is transformed into

\[
\begin{aligned}
-\Delta U &\leq \lambda \mu^r U^{q_1} - \omega^{-\beta_1 / [1, q_2]} U^{p_1} \quad \text{in } \Omega, \\
 U &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Hence, multiplying by \( \varphi_1 \), integrating and with a similar argument to the proof of Proposition 3.4, we can conclude that if

\[ \lambda \mu^r < \lambda_*(\Omega), \]

there is no positive solution of (1.1).

\( \square \)

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References


