Lie symmetries and multiple solutions in $\lambda - \omega$ reaction-diffusion systems.

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Abstract

Lie theory of transformation groups is applied to the study of $\lambda - \omega$ reaction-diffusion systems in two-dimensional media. Our study proves that they are invariant with respect to a five-parameter symmetry group. Multiple types of invariant solutions with physical interest are possible, some of them can be found in the literature applied to particular models.

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1 Introduction.

Non-linear reaction-diffusion equations have been widely studied throughout the passed years. These equations arise naturally as description models of many evolution problems in the real world, as in chemistry [1], biology [2], ecology [3], etc.

As is well known, complex behaviour is a peculiarity of systems modelled by reaction-diffusion equations, and the Belousov-Zhabotinskii reaction [4-6] provides a classic example.

Reaction-diffusion equations have been investigated for certain boundary and initial conditions and in most cases explicit solutions cannot been found.

This paper deals with the application of Lie group theory to non-linear reaction-diffusion equations. Although group analysis of differential equations has been applied a great deal in many fields of mathematical physics [7-11], much less has been applied in connection with problems related to reaction-diffusion models. We think that application of these techniques to systems of reaction-diffusion equations may help to elucidate many types of solutions, specially for models which possess the appropriate symmetries.

We have selected for investigation the denominated \( \lambda - \omega \) models, introduced some years ago by Koppell and Howard [12], which have been widely used in prototype studies of reaction-diffusion processes. Their importance lie in the fact that \( \lambda - \omega \) systems arise naturally as the dominant part in the asymptotic analysis of many general reaction-diffusion systems [13]. Spiral wave solutions of particular \( \lambda - \omega \) systems have been investigated, for example, by Greenberg [14], Hagan [15] and Kuramoto and Koga [16]. Many other solutions are also known and the list of references is extensive.

We show that the \( \lambda - \omega \) systems in two-dimensional media are invariant with respect to a five-parameter symmetry group. The invariance properties give rise to multiple types of solutions and to the reduced equations, which are essential in the study of bifurcating solutions applied to particular models.

2 Lie symmetries and \( \lambda - \omega \) reaction-diffusion models

The \( \lambda - \omega \) reaction-diffusion systems with two reactants are described by systems of partial differential equations (SPDE) of the form:

\[
\begin{align*}
    u_t &= D \nabla^2 u + \lambda(z) u - \omega(z) v, \\
    v_t &= D \nabla^2 v + \omega(z) u + \lambda(z) v, \\
    z &= (u^2 + v^2)^{1/2},
\end{align*}
\] (1)

\( D \) is the diffusion coefficient, and \( \lambda(z) \) and \( \omega(z) \) are functions which depend on the position in the domain.
where $\lambda(z)$ is a positive function of $z$ for $0 \leq z < z_0$ and negative for $z > z_0$, 
$\omega(z)$ is a positive function of $z$; $u = u(x, y, t)$ and $v = v(x, y, t)$ represent, for 
example, concentrations of two chemical reactants which at the same time diffuse 
through the plane $(x, y)$. $D$ represents the diffusion coefficient, $\lambda(z)u - \omega(z)v$ 
and $\omega(z)u + \lambda(z)v$ are nonlinear functions that describe the kinetics of the 
reaction. The spatially homogeneous system, has a limit cycle solution with 
amplitude $z$ and frequency $w(z_0)$, thus, $\lambda - \omega$ systems have been proposed as 
models for chemical or biological systems which exhibit oscillating behaviour in 
homogeneous situations.

We have found, using Lie group theory of transformations [8], that this 
system is invariant with respect to the five-parameter group which has associated 
the following characteristics:

$$
Q^u = a_1 u_x + a_2 u_y + a_3 u_z + a_4 (x u_y - y u_x) + a_5 v,
Q^v = a_1 v_x + a_2 v_y + a_3 v_z + a_4 (x v_y - y v_x) - a_5 u,
$$

where the set $\{a_i\}^5_{i=1}$ represents arbitrary constants. Every set $\{a_i\}^5_{i=1}$ is 
associated to a one-parameter group of transformations.

Five simple one-parameter groups can be obtained by making $a_i = 1$, 
$i = 1, \ldots, 5$, and $a_j = 0$ with $j \neq i$. We denote each of these groups by 
$G_i$, and the associated characteristics by $Q^u_i$ and $Q^v_i$:

$$
G_1 : \quad Q^u_1 = u_x, \quad Q^v_1 = v_x,
G_2 : \quad Q^u_2 = u_y, \quad Q^v_2 = v_y,
G_3 : \quad Q^u_3 = u_z, \quad Q^v_3 = v_z,
G_4 : \quad Q^u_4 = x u_y - y u_x, \quad Q^v_4 = x v_y - y v_x,
G_5 : \quad Q^u_5 = v, \quad Q^v_5 = -u.
$$

The characteristics associated to $G_1, G_2$ and $G_3$ correspond to translations 
in the coordinates $x, y$, and $t$, respectively. The associated to $G_4$ and $G_5$ 
correspond to rotations in the planes $(x, y)$ and $(u, v)$, respectively.

Also, we denote by $G_{ij}$ the one-parameter groups obtained by making $a_i \neq 0$, 
$a_j \neq 0$ and $a_k = 0$ with $k \neq i, j$.

It is convenient to change the variables $(x, y)$ to polar variables $(r, \theta)$, 
and $(u, v)$ to polar variables $(z, \phi)$. The characteristics of $G_4$ and $G_5$ are:

$$
G_4 : \quad Q^u_4 = a_4 u_\theta, \quad Q^v_4 = a_4 v_\theta,
G_5 : \quad Q^u_5 = 0, \quad Q^v_5 = 1.
$$

In terms of the variables $(z, \phi)$, system (1) reads:

$$
\nabla^2 z + z (\lambda(z) - |\nabla \phi|^2) - z_t = 0 \quad , \quad \nabla^2 \phi + 2 \nabla \phi \frac{\nabla z}{z} + \omega(z) - \phi_t = 0.
$$

Let us now consider the general reaction-diffusion systems of the form:

$$
F \equiv \nabla^2 u + f(u, v) - u_t = 0 \quad , \quad G \equiv \nabla^2 v + g(u, v) - v_t = 0.
$$
If these systems are invariant under the groups associated to the characteristics (2), we can demonstrate that they are of the type \( \lambda - \omega \).

These systems are invariant with respect to \( G_1, G_2, G_3 \) and \( G_4 \), because \( F \) and \( G \) do not depend explicitly on \((x, y, t)\). The condition of invariance with respect to \( G_5 \) is:

\[
V_5(F) = 0, \quad V_5(G) = 0,
\]

(7)

when \( u \) is a solution of the system (6). We represent by \( V_5 \) the prologation of the Lie operator for \( G_5 \):

\[
V_5 = -v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v} + \sum_I -v_I \frac{\partial}{\partial u_I} + u_I \frac{\partial}{\partial v_I},
\]

(8)

\( I \) is a multi-index referring to the multiple derivatives of \( u \) and \( v \), with \(|I| > 0\).

Then:

\[
-\nabla^2 v - v \frac{\partial f}{\partial u} + u \frac{\partial f}{\partial v} + v_t = 0,
\]

(9)

\[
\nabla^2 u - v \frac{\partial g}{\partial u} + u \frac{\partial g}{\partial v} - u_t = 0.
\]

That is, substituting \( u_t \) and \( v_t \) from (6):

\[
-v \frac{\partial f}{\partial u} + u \frac{\partial f}{\partial v} = -g,
\]

(10)

\[
-v \frac{\partial g}{\partial u} + u \frac{\partial g}{\partial v} = f.
\]

These equations may be written in the variables \((z, \phi)\) as:

\[
\frac{\partial f}{\partial \phi} = -g, \quad \frac{\partial g}{\partial \phi} = f,
\]

(11)

thus,

\[
f + \frac{\partial^2 f}{\partial \phi^2} = 0, \quad g = -\frac{\partial f}{\partial \phi}.
\]

Consequently the functions \( f \) and \( g \) take the form of the kinetics of the \( \lambda - \omega \) systems:

\[
f = \lambda(z)z \cos(\phi) - \omega(z)z \sin(\phi) = \lambda(z)u - \omega(z)v,
\]

\[
g = \omega(z)z \cos(\phi) + \lambda(z)z \sin(\phi) = \omega(z)u + \lambda(z)v.
\]

(12)

Thus, we have proved that \( \lambda - \omega \) systems are characterized, among reaction diffusion systems, by their symmetry properties. In the next section we show that the study of solutions invariant with respect to some subgroups of the full symmetry group may be useful to describe pattern formation.
3 Invariant solutions

Invariant solutions, \( u \) and \( v \), for a subgroup of the full symmetry group, i.e. partially invariant solutions [17], must satisfy system (1) and the characteristic equations:

\[
Q^u(u, v) = 0, \quad Q^v(u, v) = 0. \tag{13}
\]

for some set of the constants \( \{a_i\} \). This requirement imposes special forms to the solutions. Substitution in system (1) gives rise to the reduced equations, these are PDE’s with a number of independent variables reduced in one.

There exist two types of invariant solutions according to the value of the constant \( a_5 \):

I) If \( a_5 = 0 \), it is possible to change the variables \((x, y, t)\) to new variables \((\xi_1, \xi_2, \eta)\), such that the characteristic equations are:

\[
u_\eta = 0, \quad v_\eta = 0. \tag{14}
\]

Hence, invariant solutions depend only on \((\xi_1, \xi_2)\). Substitution in (1) leads to a new system of PDE’s with two independent variables.

II) If \( a_5 \neq 0 \), it is possible to change variables from \((x, y, t)\) to \((\xi_1, \xi_2, \eta)\) such that the characteristic equations are:

\[
\alpha u_\eta + v = 0, \quad \alpha v_\eta - u = 0, \tag{15}
\]

then:

\[
u = z(\xi_1, \xi_2) \cos\left(\frac{\eta}{\alpha} + \beta(\xi_1, \xi_2)\right), \quad v = z(\xi_1, \xi_2) \sin\left(\frac{\eta}{\alpha} + \beta(\xi_1, \xi_2)\right). \tag{16}
\]

That is, invariant solutions are periodic functions with respect to \( \eta \). Substitution of \( z = z(\xi_1, \xi_2) \) and \( \phi = \frac{\eta}{\alpha} + \beta(\xi_1, \xi_2) \) in (5) leads to the reduced equations for \( z \) and \( \beta \).

If a solution is invariant with respect to a two-parameter group, the reduced equations are ordinary differential equations.

In the following a solution invariant with respect to a group \( G_I \), will be called a \( G_I \)-solution. If it is invariant with respect to two groups \( G_I \) and \( G_J \), it will be called a \( G_I + G_J \)-solution.

4 Multiple solutions

In this section we consider solutions invariant with respect to different subgroups.
4.1 Homogeneous solutions

These are $G_1 + G_2$-solutions.

The reduced equations are:

$$z \lambda(z) - z_t = 0 \quad \phi_t - \omega(z) = 0. \quad (17)$$

As $\lambda(z)$ has a zero with negative derivative in $z_0$, then, there exists a stable limit cycle defined by the equations:

$$u = z_0 \cos(\omega(z_0) + \phi_0) \quad v = z_0 \sin(\omega(z_0) + \phi_0). \quad (18)$$

4.2 Travelling waves

These are $G_{15}$-solutions, and the characteristic equations take the form:

$$z_x = 0 \quad a_1 \phi_x - 1 = 0. \quad (19)$$

then,

$$u = z(y, t) \cos \left( \frac{x}{a_1} + \beta(y, t) \right) \quad v = z(y, t) \sin \left( \frac{x}{a_1} + \beta(y, t) \right). \quad (20)$$

If in addition they are $G_{15}$-solutions:

$$u = z(y) \cos \left( \frac{x}{a_1} + \frac{t}{a_3} + \alpha(y) \right) \quad v = z(y) \sin \left( \frac{x}{a_1} + \frac{t}{a_3} + \alpha(y) \right), \quad (21)$$

which are travelling waves trains solutions.

The reduced equations are:

$$z_{yy} + z \left( \lambda(z) - \frac{1}{a_1} \alpha_y - \alpha_y^2 \right) = 0 \quad \alpha_{yy} + 2 \alpha_y \frac{\partial \phi}{\partial z} + (\omega(z) - \frac{1}{a_3}) = 0. \quad (22)$$

4.3 Stationary bands

The characteristic equations for the two-parameter group $G_{15} + G_3$ are:

$$z_x = 0 \quad z_t = 0, \quad \phi_x = \frac{1}{a_1}, \quad \phi_t = 0. \quad (23)$$

The solutions take the form:

$$u = z(y) \cos \left( \frac{x}{a_1} + \beta(y) \right) \quad v = z(y) \sin \left( \frac{x}{a_1} + \beta(y) \right). \quad (24)$$

The reduced equations are:

$$z_{yy} + z \left( \lambda(z) - \frac{1}{a_1} \beta_y^{2} \right) = 0 \quad \beta_{yy} + 2 \beta_y \frac{\partial \phi}{\partial z} + \omega(z) = 0. \quad (25)$$
4.4 Wave Packets

The characteristic equations for $G_{135}$ solutions are:

\[
\begin{align*}
   a_1 z_x + a_3 z_t & = 0 \quad ; \quad a_3 \phi_x + a_1 \phi_t - 1 = 0, \\
\end{align*}
\]

where $c_g = \frac{a_1}{a_3}$. A change of variables to $x' = x - c_g t$ and $t' = t$, leads to new characteristic equations:

\[
\begin{align*}
   a_1 z_t & = 0 \quad ; \quad a_3 \phi_t' - 1 = 0, \\
\end{align*}
\]

The amplitude and phase in the new variables take the form:

\[
\begin{align*}
   z & = z(x', y) \quad ; \quad \phi = \Omega t' + \beta(x', y), \\
\end{align*}
\]

where $\Omega = \frac{1}{a_3}$. The invariant solutions are:

\[
\begin{align*}
   u & = z(x', y) \cos(\Omega t' + \beta(x', y)) \quad ; \quad v = z(x', y) \sin(\Omega t' + \beta(x', y)). \\
\end{align*}
\]

We define the complex function:

\[
\begin{align*}
   \bar{u} = u + iv & = z(x', y)e^{i(\beta(x', y) + \Omega t')} \\
\end{align*}
\]

It is easy to compare this expression with the wave packet travelling in the $x$-direction:

\[
\bar{u}' = \int G(k, y)e^{i(kx - \omega(k)t')}dk \\
\]

The group speed $c_g = \frac{dw}{dk}$ is supposed approximately constant in the interval where $G$ is significantly different from zero. Then $w(k) = w_0 + c_g k'$, with $k' = k - k_0$ for some arbitrary wave number $k_0$ in that interval, so:

\[
\bar{u}' = \int G'(k', y)e^{i(k_0 x + k' x - w_0 t - c_g k' t')}dk' = \\
A(x', y)e^{i\alpha(x', y)}e^{i(k_0 x - w_0 t)} \\
\]

with $G'(k', y) = G(k_0 + k', y)$, and:

\[
A(x', y)e^{i\alpha(x', y)} = (\int G'(k', y)e^{i(k' x - c_g t')}dk') \\
\]

This expression may be identified with the $G_{135}$-solutions if:

\[
\omega_0 = k_0 c_g - \Omega \quad ; \quad \alpha(x', y) = \beta(x', y) - k_0 x' \\
\]

The reduced equations are:
\[ z_{xx'} + z_{yy} + z (\lambda(z) - \beta_{x'}^2 - \beta_y^2) + c_y z_{x'} = 0 \]

\[ \beta_{xx'} + \beta_{yy} + 2 \beta_{x'} \frac{z_{xx'}}{z} + 2 \beta_y \frac{z_y}{z} + \omega(z) - \Omega + c_y \beta_{x'} = 0 \]

If the solutions are \( G_2 + G_{135} \) invariant, the functions \( z \) and \( \beta \) do not depend on \( y \), that is, the wave fronts are straight lines.

### 4.5 Solutions with rotational symmetry

These are \( G_4 \)-solutions, with characteristic equations:

\[ u_\theta = 0, \quad v_\theta = 0. \quad (36) \]

#### 4.5.1 Stationary target patterns

These are \( G_4 + G_3 \)-solutions. The additional characteristic equations are:

\[ u_t = 0, \quad v_t = 0. \quad (37) \]

The solutions are of the form \( u = u(r), v = v(r) \). The reduced equations are:

\[ u_{rr} + \frac{u_r}{r} + u\lambda(z) - \omega(z)v = 0, \quad v_{rr} + \frac{v_r}{r} + u\omega(z) + \lambda(z)v = 0. \quad (38) \]

#### 4.5.2 Travelling circular waves

These are \( G_4 + G_{35} \)-solutions, with characteristic equations:

\[ z_t = 0, \quad a_3 \phi_t - 1 = 0. \quad (39) \]

The solutions are of the form:

\[ u = z(r) \cos \left( \frac{t}{a_3} + \beta(r) \right), \quad v = z(r) \sin \left( \frac{t}{a_3} + \beta(r) \right). \quad (40) \]

The reduced equations are:

\[ z_{rr} + \frac{z_r}{r} + z (\lambda(z) - \beta_{r}^2(z)) = 0, \quad \beta_{rr} + \frac{\beta_r}{r} + 2 \beta_r \frac{z_r}{z} + \omega(z) - \frac{1}{a_3} = 0. \quad (41) \]

If \( \beta(r) \) is not constant these solutions are travelling circular waves with speed

\[ c = -\frac{1}{a_3 \beta_r}. \]
4.5.3 Stationary circular waves

If \( \beta(r) \) is a constant \( \beta \), then \( \omega(z) \) must also be constant with value \( \frac{1}{a_3} \), the solutions are of the form:

\[
u = z(r) \cos(t \frac{a_4}{a_3} + \beta), \quad \nu = z(r) \sin(t \frac{a_4}{a_3} + \beta),
\]

which are stationary circular waves.

The reduced equation is:

\[
z_{rr} + \frac{z_r}{r} + z \lambda(z) = 0.
\]

4.6 Rotating waves

These are \( G_{34} \)-solutions, with characteristic equations:

\[
a_4 u_\theta + a_3 u_t = 0, \quad a_4 v_\theta + a_3 v_t = 0,
\]

which may be written in the variables \( \theta' = \theta - \Omega t \) and \( t' = t \), where \( \Omega = \frac{a_4}{a_3} \), as:

\[
a_3 u'_{\theta} = 0, \quad a_3 v'_{\theta} = 0.
\]

Then, \( u = u(r, \theta') \) and \( v = v(r, \theta') \). The reduced equations are:

\[
\nabla' u + u\omega(z) - \omega(z) v + u_{\theta'} \Omega = 0, \quad \nabla' v + u\omega(z) + \lambda(z) v + v_{\theta'} \Omega = 0,
\]

where \( \nabla' \) is the nabla operator in the new variables.

4.7 Solutions with \( S_n \) symmetry

These are \( G_{45} \)-solutions. The characteristic equations are:

\[
a_4 z_{\theta} = 0, \quad a_4 \phi_{\theta} - 1 = 0.
\]

Then, \( z = z(r, t) \) and \( \phi = \theta \frac{a_4}{a_4} + \beta(r, t) \). The solutions are of the form:

\[
u = z(r, t) \cos(\theta \frac{a_4}{a_4} + \beta(r, t)), \quad \nu = z(r, t) \sin(\theta \frac{a_4}{a_4} + \beta(r, t)).
\]

These solutions must be continuous in the plane \( (x, y) \), that is, \( u(r, \theta, t) = u(r, \theta + 2\pi, t) \) and \( v(r, \theta, t) = v(r, \theta + 2\pi, t) \), then \( a_4 = 1/n \), where \( n \) is an integer. The solutions are of the form:

\[
u = z(r, t) \cos(n \theta + \beta(r, t)), \quad \nu = z(r, t) \sin(n \theta + \beta(r, t)).
\]
There are \( n \) equations for the curves of constant phase \( 2\pi \):
\[
\theta = -\frac{1}{n}\beta(r,t) + 2\pi \frac{m}{n} , \quad m = 0,1,2,...,n-1.
\] (50)

The reduced equations are:
\[
z_{rr} + \frac{z_r}{r} + z \left( \lambda(z) - \beta_r^2 - \frac{n^2}{r^2} \right) - z_t = 0 , \quad \beta_{rr} + \frac{\beta_r}{r} + 2\beta_r \frac{z_r}{z} + \omega(z) - \beta_t = 0.
\] (51)

### 4.7.1 Stationary solutions with \( S_n \) symmetry

The solutions and reduced equations with \( G_{45} + G_3 \) symmetry have the same form as above, with the condition that \( \beta \) and \( z \) are \( t \) independent.

### 4.7.2 Multiarmed rotating spiral waves

These are \( G_{45} + G_{35} \)-solutions. The characteristic equations for \( G_{35} \) are:
\[
z_t = 0 , \quad a_3 \phi_t - 1 = 0.
\]

Then, the solutions are of the form:
\[
u = z(r) \cos(n\theta + \Omega t + \beta(r)) , \quad v = z(r) \sin(n\theta + \Omega t + \beta(r)),
\] (52)

with \( \Omega = \frac{1}{a_3} \). The reduced equations are:
\[
z_{rr} + \frac{z_r}{r} + z \left( \lambda(z) - \beta_r^2 - \frac{n^2}{r^2} \right) = 0 , \quad \beta_{rr} + \frac{\beta_r}{r} + 2\beta_r \frac{z_r}{z} + \omega(z) - \Omega = 0.
\] (53)

The phase curves rotate rigidly with angular speed \( \Omega \).

## 5 Conclusions

The \( \lambda - \omega \) systems are characterized among reaction-diffusion systems by its symmetry group. Solutions invariant with respect to different subgroups of the full symmetry group exhibit many different patterns with physical interest. The study of the reduced equations with appropriate boundary conditions applied to specifics models is necessary to delimitate the ranges of the parameters values, inherent to each model, associated to differences types of solutions. We are now concluding a study relative to a model for the Belousov-Zhabotinskii reaction [18], which is a \( \lambda - \omega \) system.
Appendix: Determination of Lie Symmetries

In this appendix we briefly sketch, without technical details, the method used for obtaining the characteristics of $\lambda - \omega$ systems. A complete reference can be found in [8].

**Group of transformations**

Let $G$ be a local Lie Group, $x = (x_1, x_2, \ldots, x_n)$ the set of independent variables, and $u = (u_1, u_2, \ldots, u_m)$ the set of dependent variables, in a space of functions $u = u(x)$. A local Lie group of transformations in the space $(x,u)$ is given by the set of equations:

$$x^\epsilon = X(x, u, \epsilon) \quad ; \quad u^\epsilon = U(x, u, \epsilon),$$

(A1)

where $\epsilon$ is a continuous parameter of a local group, being $\epsilon = 0$ the value of the parameter for the identity element. The expression local means that the group properties are valid at least in some neighbourhood of $\epsilon = 0$. If the functions $X$ and $U$ depend not only on $x$ and $u$ but also on some derivatives, the transformations (A1) have no geometrical interpretation, and must be seeing as transformations in the space of functions $u(x)$. In this case they are called generalized transformations.

**Infinitesimals**

For every transformation (A1) there is an infinitesimal transformation given by

$$\delta x = \xi(x, u)\epsilon \quad ; \quad \delta u = \eta(x, u)\epsilon,$$

(A2)

with $\epsilon$ small enough; $\xi = (\xi^1, \xi^2, \ldots, \xi^n)$ and $\eta = (\eta^1, \eta^2, \ldots, \eta^m)$ are called the infinitesimals of the transformation and are given by

$$\xi = \left(\frac{\partial X}{\partial \epsilon}\right)_{\epsilon=0} \quad ; \quad \eta = \left(\frac{\partial U}{\partial \epsilon}\right)_{\epsilon=0}$$

(A3)

**Characteristics**
The characteristic of the transformation group is defined as \( Q = \eta - \xi^i u_i = . \)

An equivalent transformation [8] to (A1) that leaves invariant the \( x \) variables is given infinitesimally by

\[
\delta u = Q(x, u, \{ u_i \}) \epsilon \quad \text{where} \quad Q = \left( \frac{\partial U}{\partial \epsilon} \right)_{\epsilon=0} \quad (A4)
\]

This is a generalized transformation which has an equivalent geometrical transformation. The expression \( \{ u_i \} \) represents the set of derivatives \( \frac{\partial u_\alpha}{\partial x^i} \) with \( \alpha = 1, 2, ..., m \) and \( i = 1, 2, ..., n \).

We represent by \( \{ u_I \} \), where \( I = (i_1, i_2, ..., i_n) \) is a multiindex, the set of derivatives, given explicitly by the expressions

\[
\{ u_I \} \rightarrow \frac{\partial^{\mid I \mid} u_\alpha}{\partial x_{i_1}^{i_1} \partial x_{i_2}^{i_2} ... \partial x_{i_n}^{i_n}} \quad \alpha = 1, 2, ..., m; |I| = \sum_{j=1}^{n} i_j > 0
\]

The infinitesimal transformation for \( u_I \) is given by

\[
\delta u_I = (D_I Q) \epsilon,
\]

where \( D_I \) is the total derivative operator

\[
D_I = \frac{\partial}{\partial x^I} + u_I \frac{\partial}{\partial u} + \sum_J u_{J,I} \frac{\partial}{\partial u_J}, \quad |J| > 0
\]

where \( \frac{\partial}{\partial x^I} = \frac{\partial^{\mid I \mid}}{\partial x_{i_1}^{i_1} \partial x_{i_2}^{i_2} ... \partial x_{i_n}^{i_n}} \).

Invariant functions

A function \( u(x) \) is said to be invariant if it is left unchanged the action of the transformation group, that is \( \frac{\partial u}{\partial \epsilon} = 0 \), or equivalently

\[
Q(x, u, \{ u_i \}) = 0 \quad (A5)
\]

Symmetry Group

A system of partial differential equations,

\[
F(x, u, \{ u_I \}) = 0 \quad (A6)
\]
is said to be invariant under a transformation group if every solution \( u \) is transformed by the group into other solution \( u^\varepsilon \), that is, \( F(x, u^\varepsilon, \{u^\varepsilon_F\}) = 0 \). The corresponding infinitesimal condition is

\[
Q \frac{\partial F}{\partial u} + D_I(Q) \frac{\partial F}{\partial u_I} = 0, \quad |I| > 0,
\]

(A7)

whenever \( u \) is a solution of the SPDE.

**Invariant Solutions**

Invariant solutions are solutions of the SPDE that are invariant with respect to a symmetry group. Then they must be solutions of equations (A5) and (A6). When the SPDE models a physical system, invariant solutions are very often functions that exhibits interesting patterns with physical interest.

**Procedure**

In order to find a symmetry group of a SPDE we first substitute the partial differential equations into (A7). The resulting equations are treated as forms in the derivatives of \( u \), whose coefficients depend on \((u, x, t)\) and the infinitesimals \((\eta, \xi)\). After the substitution we collect together the coefficients of like derivative terms in \( u \) and set all of them equal to zero. The resulting equations are called the determining equations of the group. In practice these equations are solvable and thus the infinitesimals and characteristics of the group are determined. The subsequent study is clearly shown in this paper.

**Mathematical Packages**

These calculations, though not difficult in itself, are clearly complicated as the order of the SPDE and the number of equations increase, so a software for symbolic mathematics becomes really useful. To our knowledge, the best package for these kind of calculations is Macsyma. Programs written by the authors in Macsyma 4.0, running in a Convex, have been used to get the results shown in this paper.
References


