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Optimal Error Estimates of the Penalty Finite Element Method for Micropolar Fluids Equations

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Abstract

An optimal error estimate of the numerical velocity, pressure and angular velocity, is proved for the fully discrete penalty finite element method of the micropolar equations, when the parameters $\epsilon, \Delta t$ and h are sufficiently small. In order to obtain above we present the time discretization of the penalty micropolar equation which is based on the backward Euler scheme; the spatial discretization of the time discretized penalty Micropolar equation is based on a finite elements space pair (H_h, L_h) which satisfies some approximate assumption.

1 Introduction

The equations that describes the motion of a viscous incompressible micropolar fluids in a bounded domain $\Omega \subset \mathbb{R}^3$, with boundary $\partial\Omega$ smooth a time interval $[0,T]$, $0 < T < +\infty$ are given by (see [4])

$$(P) = \begin{cases} \mathbf{u}_t - \nu_1 \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 2\mu_r \text{ rot } \mathbf{w} + \mathbf{f}, \\ \text{div } \mathbf{u} = 0, \\ \mathbf{w}_t + L\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + 4\mu_r \mathbf{w} = 2\mu_r \text{ rot } \mathbf{u} + \mathbf{g}, \\ \mathbf{u}(x,t) = 0, \qquad \mathbf{w}(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{a}(x), \quad \mathbf{w}(x,0) = \mathbf{b}(x) \quad \text{in} \quad \Omega, \end{cases}$$

where $L\,\mathbf{w} = -\nu_2\,\Delta\,\mathbf{w} - \nu_3\,\nabla \text{ div } \mathbf{w}$, with $\nu_1 = \mu + \mu_r$, $\nu_2 = c_a + c_d$, $\nu_3 = c_0 + c_d - c_a$.

The functions $\mathbf{u}, \mathbf{w}$ and p denote the velocity vector, the angular velocity vector of rotation of particles, and the pressure of the fluid, respectively. The functions $\mathbf{f}$ and

$\mathbf{g}$ denote external sources of linear and angular momentum, respectively. The positive constants μ, μ_r, c_0, c_a and c_d are viscosities and $c_0 + c_d > c_a$. Also, $\mathbf{a}$, $\mathbf{b}$ are given functions in Ω.

The penalty method applied to (P) is to approximate $(\mathbf{u}, p, \mathbf{w})$ by $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{w}_\epsilon)$ satisfying the following penalty micropolar equations:

$$(P)_\epsilon = \begin{cases} \partial_t \mathbf{u}_\epsilon - \nu_1 \Delta \mathbf{u}_\epsilon + B(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon) + \nabla p_\epsilon = 2\mu_r \mathrm{rot}\, \mathbf{w}_\epsilon + \mathbf{f}, \\ \mathrm{div}\ \mathbf{u}_\epsilon + \dfrac{\epsilon}{\nu_1} p_\epsilon = 0, \\ \partial_t \mathbf{w}_\epsilon + L\, \mathbf{w}_\epsilon + B(\mathbf{u}_\epsilon, \mathbf{w}_\epsilon) + 4\mu_r \mathbf{w}_\epsilon = 2\mu_r \mathrm{rot}\, \mathbf{u}_\epsilon + \mathbf{g}, \\ \mathbf{u}_\epsilon(x,t) = 0, \qquad \mathbf{w}_\epsilon(x,t) = 0 \quad \text{on} \quad S_T, \\ \mathbf{u}_\epsilon(x,0) = \mathbf{a}(x), \quad \mathbf{w}_\epsilon(x,0) = \mathbf{b}(x) \quad \text{in} \quad \Omega, \end{cases}$$

where $B(\mathbf{u}, \mathbf{w}) = (\mathbf{u} \cdot \nabla)\mathbf{w} + \frac{1}{2}(\ \mathrm{div}\ \mathbf{u})\, \mathbf{w}$ is the modified bilinear term introduced by Temam [7] and it is well known that $\lim_{\epsilon \to 0}(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{w}_\epsilon) = (\mathbf{u}, p, \mathbf{w})$ with error bound

$$\|\mathbf{u} - \mathbf{u}_\epsilon\|_{L^2(0,T;H^1)} + \|\mathbf{w} - \mathbf{w}_\epsilon\|_{L^2(0,T;H^1)} + \|p - p_\epsilon\|_{L^2(0,T;L^2)} \leq C\, \epsilon^{1/2},$$

where $C > 0$ is a general positive constant depending on the data $\nu_1, \nu_2, \nu_3, \mathbf{a}, \mathbf{b}, \mathbf{f}, \mathbf{g}, T$.

2 Preliminaries

By simplicity we denote $\mathbf{L}^k = \mathbf{L}^k(\Omega)$, $\mathbf{H}^m = \mathbf{H}^m(\Omega)$ and $L^k(\mathbf{H}^m) = L^k(0, T; \mathbf{H}^m(\Omega))$. For the mathematical setting of (P), to consider the following function spaces

$$L_0^2 = \{q \in L^2(\Omega)\ ;\ \int_\Omega q\, dx = 0\}, \qquad V = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega)\ ;\ \mathrm{div}\, \mathbf{v} = 0\}$$

We define $A\mathbf{u} = -\triangle \mathbf{u}$ and $A_\epsilon \mathbf{u} = -\triangle \mathbf{u} - \frac{1}{\epsilon}\nabla$ div $\mathbf{u}$, which are positive self-adjoint operators associated with the micropolar and penalty micropolar fluid equations, and they are defined from $D(A) = \mathbf{H}^2 \cap \mathbf{H}_0^1$ onto $\mathbf{L}^2$. Also, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1$ are well defined

$$\begin{aligned} (A_\epsilon^{1/2}\mathbf{u}, A_\epsilon^{1/2}\mathbf{v}) &= (\nabla \mathbf{u}, \nabla\, \mathbf{v}) + \frac{1}{\epsilon}(\mathrm{div}\, \mathbf{u}, \mathrm{div}\, \mathbf{v}), \\ (L^{1/2}\mathbf{w}, L^{1/2}\mathbf{v}) &= \nu_2\, (\nabla \mathbf{w}, \nabla \mathbf{v}) + \nu_3\, (\mathrm{div}\, \mathbf{w}, \mathrm{div}\, \mathbf{v}). \end{aligned}$$

Moreover, by Shen [6] there exists a constant $M_0 > 0$ such that if $\epsilon M_0 \leq 1$, then

$$\|A\mathbf{v}\| \leq M_0 \|A_\epsilon \mathbf{v}\|, \qquad \|\nabla \mathbf{v}\| \leq M_0 \|A_\epsilon^{1/2}\mathbf{v}\|, \qquad \|\nabla \mathbf{w}\| \leq \nu_2^{-1}\, \|L_\epsilon^{1/2}\mathbf{w}\|. \tag{1}$$

We define the following trilinear forms on $\mathbf{H}_0^1 \times \mathbf{H}_0^1 \times \mathbf{H}_0^1$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad = \quad \langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{H_0^1 \times H_0^1} = \frac{1}{2}((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}) - \frac{1}{2}((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v}),$$

$\forall\ \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1$, and satisfies the property $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$.

We make the following assumption on the prescribed data $\mathbf{a}, \mathbf{b}, \mathbf{f}$ and $\mathbf{g}$.

(A1) The initial velocity $\mathbf{a} \in D(A) \cap \mathbf{V}$, the initial angular velocity $\mathbf{b} \in \mathbf{H}^2 \cap \mathbf{H}_0^1$, and the external forces $\mathbf{f}, \mathbf{g} \in W^{1,\infty}(\mathbf{L}^2)$.

With the above notation, the variational formulation of the micropolar equations (P) is given by: Find $\mathbf{u}$, $\mathbf{w} \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}_0^1)$ and $p \in L^2(L_0^2)$ such that

$$(PV) = \begin{cases} (\mathbf{u}_t, \mathbf{v}) + \nu_1 (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) + (\operatorname{div} \mathbf{u}, q) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) \\ = 2\,\mu_r\,(\operatorname{rot} \mathbf{w}, \mathbf{v}) + (\mathbf{f}, \mathbf{v}), \quad \forall\ \mathbf{v} \in \mathbf{H}_0^1,\ q \in L_0^2 \\ (\mathbf{w}_t, \mathbf{z}) + (L^{1/2}\,\mathbf{w}, L^{1/2}\,\mathbf{z}) + 4\mu_r(\mathbf{w}, \mathbf{z}) + b(\mathbf{u}, \mathbf{w}, \mathbf{z}) \\ = 2\,\mu_r\,(\operatorname{rot} \mathbf{u}, \mathbf{z}) + (\mathbf{g}, \mathbf{z}), \quad \forall\ \mathbf{z} \in \mathbf{H}_0^1, \\ \mathbf{u}(0) = \mathbf{a}, \qquad \mathbf{w}(0) = \mathbf{b} \quad \text{in} \quad \Omega, \end{cases}$$

and the penalty micropolar variational formulation of $(\mathrm{P})_\epsilon$ is defined as follows: Find $\mathbf{u}_\epsilon$, $\mathbf{w}_\epsilon \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}_0^1)$ and $p_\epsilon \in L^2(L_0^2)$ such that

$$(PV)_\epsilon = \begin{cases} (\partial_t \mathbf{u}_\epsilon, \mathbf{v}) + \nu_1 (\nabla\, \mathbf{u}_\epsilon, \nabla\, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_\epsilon) + (\operatorname{div} \mathbf{u}_\epsilon, q) + \dfrac{\epsilon}{\nu_1}(p_\epsilon, q) \\ +b(\mathbf{u}_\epsilon, \mathbf{u}_\epsilon, \mathbf{v}) = 2\,\mu_r\,(\operatorname{rot} \mathbf{w}_\epsilon, \mathbf{v}) + (\mathbf{f}, \mathbf{v}), \quad \forall\ \mathbf{v} \in \mathbf{H}_0^1,\ q \in L_0^2 \\ (\partial_t \mathbf{w}_\epsilon, \mathbf{z}) + (L^{1/2}\,\mathbf{w}_\epsilon, L^{1/2}\,\mathbf{z}) + 4\mu_r(\mathbf{w}_\epsilon, \mathbf{z}) + b(\mathbf{u}_\epsilon, \mathbf{w}_\epsilon, \mathbf{z}) \\ = 2\,\mu_r\,(\operatorname{rot} \mathbf{u}_\epsilon, \mathbf{z}) + (\mathbf{g}, \mathbf{z}), \quad \forall\ \ \mathbf{z} \in \mathbf{H}_0^1, \\ \mathbf{u}_\epsilon(0) = \mathbf{a}, \qquad \mathbf{w}_\epsilon(0) = \mathbf{b} \quad \text{in} \quad \Omega. \end{cases}$$

Rather than assuming that the data are small, we make the existence of the solution on some interval $[0, T)$ an assumption

($\mathbf{A2}$) The solution $(\mathbf{u}, p, \mathbf{w})$ of problem (PV) exists on $[0, T)$, and $\mathbf{u}, \mathbf{w} \in L^\infty(\mathbf{H}^1)$.
($\mathbf{A3}$) The solution $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{w}_\epsilon)$ of problem $(PV)_\epsilon$ exists on $[0, T)$, and $\mathbf{u}_\epsilon, \mathbf{w}_\epsilon \in L^\infty(\mathbf{H}^1)$.

3 Time discretization and regularity

For the problem $(\mathrm{PV})_\epsilon$ we consider the time discretization by the backward Euler scheme

$$(PV)_\epsilon^n = \begin{cases} (d_t\, \mathbf{u}_\epsilon^n, \mathbf{v}) + \nu_1 (\nabla\, \mathbf{u}_\epsilon^n, \nabla\, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_\epsilon^n) + (\operatorname{div} \mathbf{u}_\epsilon^n, q) + \dfrac{\epsilon}{\nu_1}(p_\epsilon^n, q) \\ +b(\mathbf{u}_\epsilon^n, \mathbf{u}_\epsilon^n, \mathbf{v}) = 2\,\mu_r\,(\operatorname{rot} \mathbf{w}_\epsilon^n, \mathbf{v}) + (\mathbf{f}(t_n), \mathbf{v}),\ \forall\ \mathbf{v} \in \mathbf{H}_0^1,\ q \in L_0^2, \\ (d_t \mathbf{w}_\epsilon^n, \mathbf{z}) + (L^{1/2}\,\mathbf{w}_\epsilon^n, L^{1/2}\,\mathbf{z}) + 4\mu_r(\mathbf{w}_\epsilon^n, \mathbf{z}) + b(\mathbf{u}_\epsilon^n, \mathbf{w}_\epsilon^n, \mathbf{z}) \\ = 2\,\mu_r\,(\operatorname{rot} \mathbf{u}_\epsilon^n, \mathbf{z}) + (\mathbf{g}(t_n), \mathbf{z}), \qquad \forall\ \mathbf{z} \in \mathbf{H}_0^1, \\ \mathbf{u}_\epsilon^0 = \mathbf{a}, \qquad p_\epsilon^0 = 0, \quad \mathbf{w}_\epsilon^0 = \mathbf{b} \quad \text{in} \quad \Omega, \end{cases}$$

for all $1 \le n \le N$, where $t_n = n\Delta t$ and $0 < \Delta t < 1$ is the time step size, $t_N = T$, $(\mathbf{u}_\epsilon^n, p_\epsilon^n, \mathbf{w}_\epsilon^n)$ is an approximation of $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{w}_\epsilon)$ at time t_n, and $d_t \mathbf{u}_\epsilon^n = \frac{1}{\Delta t}(\mathbf{u}_\epsilon^n - \mathbf{u}_\epsilon^{n-1})$ for $1 \le n \le N$, with $d_t \mathbf{u}_\epsilon^0,\ d_t \mathbf{w}_\epsilon^0 \in \mathbf{L}^2$.

We can refer to Shen [5] for the proof of the following result.

Theorem 3.1 *Suppose that (*$\mathbf{A1}$*), (*$\mathbf{A2}$*), (*$\mathbf{A3}$*) and* $\epsilon M_0 \le 1$ *are valid. The following error estimate holds*

$$\tau^2(t_m)\|\nabla\, \mathbf{u}(t_m) - \nabla\, \mathbf{u}_\epsilon^m\|^2 + \tau^2(t_m)\|\nabla\, \mathbf{w}(t_m) - \nabla\, \mathbf{w}_\epsilon^m\|^2$$
$$+\Delta t \sum_{n=1}^{m} \tau^2(t_n)\|p(t_n) - p_\epsilon^n\|^2 \le C(\epsilon^2 + \Delta t^2) \tag{2}$$

for all $1 \le m \le N$*. Where* M_0 *is given in (1),* $\tau(t) = \min\{\, t, 1\,\}$*, and* $C > 0$*, is a general positive constant depending on the data* ν_1*,* ν_2*,* ν_3*,* $\mathbf{f}$*,* $\mathbf{g}$*,* Ω*,* T*.*

In order to obtain the error bound of the finite element solution related to the problem $(\mathrm{PV})^n_\epsilon$, we are going to provide with some regularity results for the solutions of $(\mathrm{PV})^n_\epsilon$.

Theorem 3.2 *Suppose that (**A1**), (**A3**) and $\epsilon M_0 \le 1$ are valid. There is a constant $M > 0$ such that if $\Delta t\, M \le 1$, then*

$$\|A_\epsilon^{1/2}\mathbf{u}_\epsilon^m\|^2 + \|L^{1/2}\,\mathbf{w}_\epsilon^m\|^2 + \Delta t\sum_{n=1}^m(\|A_\epsilon\mathbf{u}_\epsilon^n\|^2 + \|L\,\mathbf{w}_\epsilon^n\|^2 + \|p_\epsilon^n\|^2_{H^1}) \le C, \tag{3}$$

$$\|A_\epsilon\mathbf{u}_\epsilon^n\|^2 + \|L\,\mathbf{w}_\epsilon^n\|^2 + \|p_\epsilon^n\|^2_{H^1} + \Delta t\sum_{n=1}^m(\|A_\epsilon^{1/2}\,d_t\,\mathbf{u}_\epsilon^n\|^2 + \|L^{1/2}\,d_t\,\mathbf{w}_\epsilon^n\|^2) \le C, \tag{4}$$

$$\Delta t\sum_{n=1}^m \tau(t_n)(\|A_\epsilon d_t\mathbf{u}_\epsilon^n\|^2 + \|L d_t\mathbf{w}_\epsilon^n\|^2) \le C, \tag{5}$$

for all $1 \le m \le N$.

The proof of the Theorem 3.2 can be done without no difficulty.

4 Finite element penalty method

To avoid technical difficulties, the bounded domain Ω is assumed to be a polyhedron. Let $\pi_h = \{K\}$ be a discretization of mesh size h, $0 < h < 1$ of the polyhedral domain $\overline{\Omega}$ into closed subsets K, and the family π_h satisfies the usual regularity assumptions.

For each h, let $\mathbf{H}_h$ and L_h be the finite dimensional spaces to be used for approximating the "velocity space" $\mathbf{H}_0^1$ and the "pressure space" L_0^2, respectively. The spaces $\mathbf{H}_h$ and L_h satisfy several approximations properties and one compatibility condition (see Girault and Raviart [2], Ciarlet [1], Heywood and Rannacher [3]), thus the continuous and discrete spaces are relate by the following hypotheses:

(**S1**) There exists a continuous mapping $r_h\,:\,\mathbf{H}^2\cap\mathbf{H}_0^1 \longrightarrow \mathbf{H}_h$ such that

$$\begin{aligned}&(i)\quad (q_h, \operatorname{div}(\mathbf{v} - r_h\mathbf{v})) = 0, \quad \forall\, q_h \in L_h,\ \forall\,\mathbf{v}\in\mathbf{H}^2\cap\mathbf{H}_0^1,\\ &(ii)\ \|\mathbf{v} - r_h\mathbf{v}\| + h\,\|\mathbf{v} - r_h\mathbf{v}\|_{H^1} \le C\,h^2\,\|\mathbf{v}\|_{H^2}, \quad \forall\,\mathbf{v}\in\mathbf{H}^2\cap\mathbf{H}_0^1.\end{aligned}$$

(**S2**) The orthogonal projection operator $j_h\,:\,L_0^2(\Omega)\longrightarrow L_h$ satisfies

$$\|q - j_h q\| \le C\,h\,\|q\|_{H^1}, \quad \forall\, q\in\mathbf{H}^1\cap L_0^2.$$

(**S3**) (Inf-sup condition) There exists a constant $\beta > 0$, independent of h, such that

$$\sup_{\mathbf{v}_h\in\mathbf{H}_h}\frac{(q_h, \operatorname{div}\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|} \ge \beta\,\|q_h\|, \quad \forall\, q_h\in L_h\,.$$

Also, it is true the inverse inequality $\quad \|\nabla\,\mathbf{v}_h\| \le C\,h^{-1}\,\|\mathbf{v}_h\|, \quad \forall\mathbf{v}_h\in\mathbf{H}_h.$

Now, we define $(\mathbf{u}_{\epsilon h}^n, p_{\epsilon h}^n, \mathbf{w}_{\epsilon h}^n)$ as the finite element approximations of $(\mathbf{u}_\epsilon^n, p_\epsilon^n, \mathbf{w}_\epsilon^n)$, which satisfied the following penalty finite element system

$$(PV)^n_{\epsilon h} = \begin{cases} (d_t\,\mathbf{u}^n_{\epsilon h}, \mathbf{v}_h) + \nu_1\,(\nabla\,\mathbf{u}^n_{\epsilon h}, \nabla\,\mathbf{v}_h) - (\mathrm{div}\,\mathbf{v}_h, p^n_{\epsilon h}) \\ +(\mathrm{div}\,\mathbf{u}^n_{\epsilon h}, q_h) + \dfrac{\epsilon}{\nu_1}(p^n_{\epsilon h}, q_h) + b(\mathbf{u}^n_{\epsilon h}, \mathbf{u}^n_{\epsilon h}, \mathbf{v}_h) \\ = 2\,\mu_r\,(\mathrm{rot}\,\mathbf{w}^n_{\epsilon h}, \mathbf{v}_h) + (\mathbf{f}(t_n), \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{H}_h,\ q_h \in L_h, \\ (d_t\mathbf{w}^n_{\epsilon h}, \mathbf{z}_h) + (L^{1/2}\,\mathbf{w}^n_{\epsilon h}, L^{1/2}\,\mathbf{z}_h) + 4\mu_r(\mathbf{w}^n_{\epsilon h}, \mathbf{z}_h) + b(\mathbf{u}^n_{\epsilon h}, \mathbf{w}^n_{\epsilon h}, \mathbf{z}_h) \\ = 2\,\mu_r\,(\mathrm{rot}\,\mathbf{u}^n_{\epsilon h}, \mathbf{z}_h) + (\mathbf{g}(t_n), \mathbf{z}_h) \quad \forall\ \ \mathbf{z}_h \in \mathbf{H}_h, \\ \mathbf{u}^0_{\epsilon h} = \mathbf{a}, \qquad p^0_{\epsilon h} = 0, \quad \mathbf{w}^0_{\epsilon h} = \mathbf{b} \quad \text{in} \quad \Omega, \end{cases}$$

We write the following theorem, in which we obtain similar results that in [8].
Theorem 4.1 *Assume that* (**S1**)*-*(**S3**) *and that the hypotheses of the Theorem 3.2 are valid. Then it holds that*

$$\|\mathbf{u}^m_{\epsilon h}\|^2 + \|\mathbf{w}^m_{\epsilon h}\|^2 + \Delta t\sum_{n=1}^{m}(\|\nabla\,\mathbf{u}^n_{\epsilon h}\|^2 + \|\nabla\,\mathbf{w}^n_{\epsilon h}\|^2) \le C, \tag{6}$$

$$\begin{aligned} &\|\nabla\,\mathbf{u}^m_{\epsilon h}\|^2 + \|L^{1/2}\,\mathbf{w}^m_{\epsilon h}\|^2 + \Delta t\sum_{n=1}^{m}(\|d_t\mathbf{u}^n_{\epsilon h}\|^2 + \|d_t\mathbf{w}^n_{\epsilon h}\|^2) \\ &\le C + C\,h^{-3}\Delta t\,\sum_{n=1}^{m}[\|\mathbf{u}^n_{\epsilon h}\|^2(\|\nabla\,\tilde{\mathrm{e}}^n\|^2 + \|\nabla\,\tilde{\theta}^n\|^2) + \|\nabla\,\mathbf{u}^n_{\epsilon h}\|^2(\|\tilde{\mathrm{e}}^n\|^2 + \|\tilde{\theta}^n\|^2)] \end{aligned} \tag{7}$$

for all $1 \le m \le N$ *and* $\tilde{\mathrm{e}}^n = \mathbf{u}^n_\epsilon - \mathbf{u}^n_{\epsilon h}, \quad \tilde{\theta}^n = \mathbf{w}^n_\epsilon - \mathbf{w}^n_{\epsilon h}$.

5 Optimal error analysis

We establish the optimal error for $\mathbf{u}(t_n) - \mathbf{u}^n_{\epsilon h}$, $\mathbf{w}(t_n) - \mathbf{w}^n_{\epsilon h}$ and $p(t_n) - p^n_{\epsilon h}$. With this purpose, and since $\mathbf{u}(t_n) - \mathbf{u}^n_{\epsilon h} = (\mathbf{u}(t_n) - \mathbf{u}^n_\epsilon) + (\mathbf{u}^n_\epsilon - \mathbf{u}^n_{\epsilon h})$, firstly we are going to do estimates for the error $\tilde{\mathrm{e}}^n$, $\tilde{\xi}^n = p^n_\epsilon - p^n_{\epsilon h}$ and $\tilde{\theta}^n$.
Lemma 5.1 *Under the hypothesis of Theorem 4.1, it is holds the error estimate*

$$\|\tilde{\mathrm{e}}^m\|^2 + \|\tilde{\theta}^m\|^2 + \Delta\,t\sum_{n=1}^{m}(\|\nabla\,\tilde{\mathrm{e}}^m\|^2 + \|L^{1/2}\,\tilde{\theta}^m\|^2) \le C\,h^2, \tag{8}$$

for all $1\le m \le N$.

Proof. Denoting $\mathrm{e}^n_h = (I - r_h)\,\mathbf{u}^n_\epsilon$, $\theta^n_h = (I - r_h)\,\mathbf{w}^n_\epsilon$, $\xi^n_h = (I - \rho_h)p^n_\epsilon$, $\mathrm{e}^n = r_h\mathbf{u}^n_\epsilon - \mathbf{u}^n_{\epsilon h}$, $\xi^n = \rho_h p^n_\epsilon - p^n_{\epsilon h}$ and $\theta^n = r_h\mathbf{w}^n_\epsilon - \mathbf{w}^n_{\epsilon h}$, we have $\tilde{\mathrm{e}}^n = \mathrm{e}^n_h + \mathrm{e}^n$, $\tilde{\theta}^n = \theta^n_h + \theta^n$, $\tilde{\xi}^n = \xi^n_h + \xi^n$.
From (**S1**)-*ii* and considering (1) and (3), we obtain

$$\|\mathrm{e}^m_h\|^2 + \|\theta^m_h\|^2 + C\,\Delta t\sum_{n=1}^{m}(\|\nabla\,\mathrm{e}^n_h\|^2 + \|L^{1/2}\,\theta^n_h\|^2) \le C\,h^2. \tag{9}$$

Then, taking into account (9), to prove (8) it is enough to show

$$\|\mathrm{e}^m\|^2 + \|\theta^m\|^2 + \Delta t\,\sum_{n=1}^{m}(\|\nabla\,\mathrm{e}^n\|^2 + \|L^{1/2}\theta^n\|^2) \le C\,h^2. \tag{10}$$

Subtracting $(\mathrm{PV})_{\epsilon h}^n$ from $(\mathrm{PV})_\epsilon^n$ with $\mathbf{v}=\mathbf{v}_h = 2\,\mathrm{e}^n\Delta t, q=q_h=2\,\xi^n\Delta t$ and $\mathbf{z}=\mathbf{z}_h = 2\,\theta^n\Delta t$, using (**S1**), (1), (4), definition of $d_t\mathbf{u}^n$ and the Hölder and Young's inequalities, we obtain

$$\begin{aligned}
&\|\mathrm{e}^n\|^2 - \|\mathrm{e}^{n-1}\|^2 + \|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \Delta t\,(\mu\,\|\nabla\,\mathrm{e}^n\|^2 + \|L^{1/2}\theta^n\|^2)\\
&\quad\le \Delta t\,C\,h^2\,(\|\nabla\,d_t\mathbf{u}_\epsilon^n\|^2 + \|L^{1/2}d_t\mathbf{w}_\epsilon^n\|^2) + \Delta t\,C\,h^2\,(\|A\mathbf{u}_\epsilon^n\|^2 + \|p_\epsilon^n\|_{H^1}^2 + \|L\mathbf{w}_\epsilon^n\|^2)\\
&\quad+\Delta t\,C\,h^2\,(\|\nabla\,\mathbf{u}_{\epsilon h}^n\|^2 + \|\nabla\,\mathbf{u}_\epsilon^n\|^2 + \|L^{1/2}\mathbf{w}_\epsilon^n\|^2) + \Delta t\,C\,d_n\,(\|\mathrm{e}^n\|^2 + \|\theta^n\|^2), \qquad (11)
\end{aligned}$$

where $d_n = C(\nu_1,\nu_2)(\|\nabla\mathbf{u}_\epsilon^n\|^4 + \|\nabla\,\mathbf{w}_\epsilon^n\|^4)$ and $\Delta\,t$ is chosen such that $2\Delta t\sum_{n=1}^m d_n \le C$.

Then, summing (11) from 1 to m, by using Theorem 3.2 and Theorem 4.1, and the discrete Gronwall's Lemma, is followed (10).

Remark 1. From Theorem 3.2, Theorem 4.1 and Lemma 5.1, we can conclude

$$\|\nabla\,\mathbf{u}_{\epsilon h}^m\|^2 + \|\nabla\,\tilde{\mathrm{e}}^m\|^2 \le C\,h^{-1}, \qquad \|\tilde{\mathrm{e}}^m\|_{L^3}^2 \le C\,\|\tilde{\mathrm{e}}^m\|\|\nabla\,\tilde{\mathrm{e}}^m\| \le C\,h^{1/2}. \qquad (12)$$

Lemma 5.2 *Under the hypotheses of Theorem 4.1 is true*

$$\|\nabla(\boldsymbol{u}_\epsilon^m - \boldsymbol{u}_{\epsilon h}^m)\|^2 + \|L^{1/2}(\boldsymbol{w}_\epsilon^m - \boldsymbol{w}_{\epsilon h}^m)\|^2 + \Delta\,t\sum_{n=1}^m \|p_\epsilon^n - p_{\epsilon h}^n\|^2 \le C\,h^{1/2}, \qquad (13)$$

for all $1\le m \le N$.

Proof. We will consider to the notations done in the proof of the Lemma 5.1.
Subtracting $(\mathrm{PV})_{\epsilon h}^n$ from $(\mathrm{PV})_\epsilon^n$ with $\mathbf{v}=\mathbf{v}_h, \mathbf{z}=\mathbf{z}_h$ and $q=q_h$, we obtain

$$\begin{aligned}
&(d_t\tilde{\mathrm{e}}^n,\mathbf{v}_h) + \nu_1\,(\nabla\,\tilde{\mathrm{e}}^n,\nabla\,\mathbf{v}_h) - (\operatorname{div}\mathbf{v}_h,\tilde{\xi}^n) + (\operatorname{div} d_t\tilde{\mathrm{e}}^n, q_h) + \frac{\epsilon}{\nu_1}(d_t\tilde{\xi}^n, q_h)\\
&\quad+b(\tilde{\mathrm{e}}^n,\mathbf{u}_\epsilon^n,\mathbf{v}_h) + b(\mathbf{u}_\epsilon^n,\tilde{\mathrm{e}}^n,\mathbf{v}_h) - b(\tilde{\mathrm{e}}^n,\tilde{\mathrm{e}}^n,\mathbf{v}_h) = 2\,\mu_r(\operatorname{rot}\tilde{\theta}^n,\mathbf{v}_h), \qquad (14)
\end{aligned}$$

$$\begin{aligned}
&(d_t\tilde{\theta}^n,\mathbf{z}_h) + (L^{1/2}\,\tilde{\theta}^n, L^{1/2}\,\mathbf{z}_h) + 4\mu_r(\tilde{\theta}^n,\mathbf{z}_h) + b(\tilde{\mathrm{e}}^n,\mathbf{w}_\epsilon^n,\mathbf{z}_h)\\
&+b(\mathbf{u}_\epsilon^n,\tilde{\theta}^n,\mathbf{z}_h) - b(\tilde{\mathrm{e}}^n,\tilde{\theta}^n,\mathbf{z}_h) = 2\,\mu_r\,(\operatorname{rot}\tilde{\mathrm{e}}^n,\mathbf{z}_h). \qquad (15)
\end{aligned}$$

Now, we observe that $\|d_t\tilde{\phi}^n\|^2 = \|d_t\phi_h^n\|^2 + \|d_t\phi^n\|^2 + 2(d_t\phi_h^n, d_t\phi^n)$, and then

$$2\,\Delta\,t\,\|d_t\tilde{\phi}^n\|^2 \ge \Delta\,t\,\|d_t\tilde{\phi}^n\|^2 + \Delta\,t\,\|d_t\phi^n\|^2 + 2\Delta\,t\,(d_t\phi_h^n, d_t\phi^n). \qquad (16)$$

Considering inequality (16) and setting $\mathbf{v}_h = 2\Delta t\,d_t\tilde{\mathrm{e}}^n,\ q_h = 2\Delta t\,\tilde{\xi}^n,\ \mathbf{z}_h = 2\Delta t\,d_t\tilde{\theta}^n$ in (14)-(15), we have

$$\begin{aligned}
&\Delta t\,(\|d_t\tilde{\mathrm{e}}^n\|^2 + \|d_t\mathrm{e}^n\|^2) + \Delta t\,(\|d_t\tilde{\theta}^n\|^2 + \|d_t\theta^n\|^2) + \nu_1\,(\|\nabla\,\tilde{\mathrm{e}}^n\|^2 - \|\nabla\,\tilde{\mathrm{e}}^{n-1}\|^2)\\
&+\|L^{1/2}\tilde{\theta}^n\|^2 - \|L^{1/2}\tilde{\theta}^{n-1}\|^2 + \frac{\epsilon}{\nu_1}(\|\tilde{\xi}^n\|^2 - \|\tilde{\xi}^{n-1}\|^2) + 4\,\mu_r\,(\|\tilde{\theta}^n\|^2 - \|\tilde{\theta}^{n-1}\|^2)\\
&\le -2\Delta\,t\,(d_t\mathrm{e}_h^n, d_t\mathrm{e}^n) - 2\Delta\,t\,(d_t\theta_h^n, d_t\theta^n) - 2\Delta t\,b(\tilde{\mathrm{e}}^n,\mathbf{u}_\epsilon^n, d_t\tilde{\mathrm{e}}^n) - 2\Delta t\,b(\mathbf{u}_\epsilon^n,\tilde{\mathrm{e}}^n, d_t\tilde{\mathrm{e}}^n)\\
&\quad+2\Delta t\,b(\tilde{\mathrm{e}}^n,\tilde{\mathrm{e}}^n, d_t\tilde{\mathrm{e}}^n) - 2\Delta t\,b(\tilde{\mathrm{e}}^n,\mathbf{w}_\epsilon^n, d_t\tilde{\theta}^n) - 2\Delta t\,b(\mathbf{u}_\epsilon^n,\tilde{\theta}^n, d_t\tilde{\theta}^n)\\
&\quad+2\Delta t\,b(\tilde{\mathrm{e}}^n,\tilde{\theta}^n, d_t\tilde{\theta}^n) + 4\,\Delta t\,\mu_r(\operatorname{rot}\tilde{\theta}^n, d_t\tilde{\mathrm{e}}^n) + 4\Delta t\,\mu_r(\operatorname{rot}\tilde{\mathrm{e}}^n, d_t\tilde{\theta}^n). \qquad (17)
\end{aligned}$$

Now, by using the Hölder and Young's inequalities, Theorem 3.2 and (12), we obtain the following inequalities

$$2\,(d_t\mathrm{e}_h^n, d_t\mathrm{e}^n) + 2(d_t\theta_h^n, d_t\theta^n) \le C\,\|d_t\mathrm{e}_h^n\|^2 + \frac{1}{2}\,\|d_t\mathrm{e}^n\|^2 + \frac{1}{2}\,\|d_t\theta^n\|^2, \tag{18}$$

$$\begin{aligned} &2\,b(\tilde{\mathrm{e}}^n, \mathbf{u}_\epsilon^n, d_t\tilde{\mathrm{e}}^n) + 2\,b(\mathbf{u}_\epsilon^n, \tilde{\mathrm{e}}^n, d_t\tilde{\mathrm{e}}^n) + 4\,\mu_r(\mathrm{rot}\,\tilde{\theta}^n, d_t\tilde{\mathrm{e}}^n) \\ &\qquad \le C\,(\|\nabla\,\tilde{\mathrm{e}}^n\|^2 + \|\nabla\,\tilde{\theta}^n\|^2) + \frac{1}{2}\,\|d_t\tilde{\mathrm{e}}^n\|^2, \end{aligned} \tag{19}$$

$$\begin{aligned} 2\,b(\tilde{\mathrm{e}}^n, \tilde{\mathrm{e}}^n, d_t\tilde{\mathrm{e}}^n) &\le C\,\|\tilde{\mathrm{e}}^n\|_{L^3}\|\nabla\,\tilde{\mathrm{e}}^n\|(\|d_t\mathrm{e}_h^n\|_{L^6} + \|d_t\mathrm{e}^n\|_{L^6}) \\ &\le C\,h^{1/4}\|\nabla\,\tilde{\mathrm{e}}^n\|(\|\nabla\,d_t\mathrm{e}_h^n\| + h^{-1}\|d_t\mathrm{e}^n\|) \\ &\le C\,h^{-3/2}\|\nabla\,\tilde{\mathrm{e}}^n\|^2 + C\,h\|\nabla\,d_t\mathrm{e}_h^n\|^2 + \frac{1}{2}\,\|d_t\mathrm{e}^n\|^2, \end{aligned} \tag{20}$$

$$\begin{aligned} &2\,b(\tilde{\mathrm{e}}^n, \mathbf{w}_\epsilon^n, d_t\tilde{\theta}^n) + 2\,b(\mathbf{u}_\epsilon^n, \tilde{\theta}^n, d_t\tilde{\theta}^n) + 4\,\mu_r(\mathrm{rot}\,\tilde{\mathrm{e}}^n, d_t\tilde{\theta}^n) \\ &\qquad \le C\,(\|\nabla\,\tilde{\mathrm{e}}^n\|^2 + \|\nabla\,\tilde{\theta}^n\|^2) + \|d_t\tilde{\theta}^n\|^2, \end{aligned} \tag{21}$$

$$\begin{aligned} 2\,b(\tilde{\mathrm{e}}^n, \tilde{\theta}^n, d_t\tilde{\theta}^n) &\le C\,(\|\tilde{\mathrm{e}}^n\|_{L^3}\|\nabla\,\tilde{\theta}^n\| + \|\nabla\,\tilde{\mathrm{e}}^n\|\|\tilde{\theta}^n\|_{L^3})(\|d_t\theta_h^n\|_{L^6} + \|d_t\theta^n\|_{L^6}) \\ &\le C\,h^{1/4}(\|\nabla\,\tilde{\theta}^n\| + \|\nabla\,\tilde{\mathrm{e}}^n\|)(\|\nabla\,d_t\theta_h^n\| + h^{-1}\|d_t\theta^n\|) \\ &\le C\,h^{-3/2}(\|\nabla\,\tilde{\theta}^n\|^2 + \|\nabla\,\tilde{\mathrm{e}}^n\|^2) + C\,h\|\nabla\,d_t\theta_h^n\|^2 + \frac{1}{2}\,\|d_t\theta^n\|^2. \end{aligned} \tag{22}$$

The fact that $h^2 \le C\,\Delta t \le C\,\tau(t_n)$ together (**S1**)-(ii), implies

$$\|\nabla\,d_t\mathrm{e}_h^n\|^2 + \|\nabla\,d_t\theta_h^n\|^2 \le C\,\tau(t_n)\,(\|A\,d_t\mathbf{u}_\epsilon^n\|^2 + \|L\,d_t\mathbf{w}_\epsilon^n\|^2). \tag{23}$$

Then, carrying (18)-(22) in (17), and taking into account (**S1**)-(ii) and (23), we get

$$\begin{aligned} &\frac{\Delta t}{2}\|d_t\tilde{\mathrm{e}}^n\|^2 + \nu_1\,(\|\nabla\,\tilde{\mathrm{e}}^n\|^2 - \|\nabla\,\tilde{\mathrm{e}}^{n-1}\|^2) + \|L^{1/2}\tilde{\theta}^n\|^2 - \|L^{1/2}\tilde{\theta}^{n-1}\|^2 \\ &+\frac{\epsilon}{\nu_1}(\|\tilde{\xi}^n\|^2 - \|\tilde{\xi}^{n-1}\|^2) + 4\,\mu_r\,(\|\tilde{\theta}^n\|^2 - \|\tilde{\theta}^{n-1}\|^2) \\ &\le C\,h^2\,\Delta t\,(\|\nabla\,d_t\mathbf{u}_\epsilon^n\|^2 + \|L^{1/2}\,d_t\mathbf{w}_\epsilon^n\|^2) + C\,\Delta t\,h^{-3/2}\,(\|\nabla\,\tilde{\theta}^n\|^2 + \|\nabla\,\tilde{\mathrm{e}}^n\|^2) \\ &\quad +C\,\Delta t\,h\,\tau(t_n)\,(\|A\,d_t\mathbf{u}_\epsilon^n\|^2 + \|L\,d_t\mathbf{w}_\epsilon^n\|^2). \end{aligned} \tag{24}$$

Thus, summing (24) from 1 to m and by using Theorem 3.2 and Lemma 5.1, we obtain

$$\|\nabla\,\tilde{\mathrm{e}}^m\|^2 + \|L^{1/2}\tilde{\theta}^m\|^2 + \Delta t\sum_{n=1}^{m}\|d_t\tilde{\mathrm{e}}^n\|^2 \le C\,h^{1/2}. \tag{25}$$

To pressure, again subtracting $(\mathrm{PV})_{\epsilon h}^n$ from $(\mathrm{PV})_\epsilon^n$ with $\mathbf{v} = \mathbf{v}_h$ and $q = q_h$, and by inf-sup condition, we deduce

$$\begin{aligned} \|\xi^n\| \le \frac{(\mathrm{div}\,\mathbf{v}_h, \xi^n)}{\|\nabla\,\mathbf{v}_h\|} &\le C\,\|d_t\tilde{\mathrm{e}}^n\| + C\,\|\nabla\,\tilde{\mathrm{e}}^n\| + C\,\|\xi_h^n\| + C\,\|L^{1/2}\tilde{\theta}^n\| \\ &\quad +C\,\|\nabla\,\tilde{\mathrm{e}}^n\|(\|\nabla\,\mathbf{u}_\epsilon^n\| + \|\nabla\,\mathbf{u}_{\epsilon h}^n\|), \end{aligned} \tag{26}$$

and by *Remark 1*, from (26) we have

$$\|\xi^n\|^2 \le C\,(\|d_t\tilde{\mathrm{e}}^n\|^2 + \|\nabla\,\tilde{\mathrm{e}}^n\|^2 + \|\xi_h^n\|^2 + \|L^{1/2}\tilde{\theta}^n\|^2 + h^{-1}\|\nabla\,\tilde{\mathrm{e}}^n\|^2). \qquad (27)$$

Then, since

$$\Delta t\,\|p_\epsilon^n - p_{\epsilon h}^n\|^2 \le C\,\Delta t\,(\|\xi_h^n\|^2 + \|\xi^n\|^2),$$

by (**S2**) together Lemma 5.1, (27) and (25), we conclude

$$\Delta t \sum_{n=1}^{m} \|p_\epsilon^n - p_{\epsilon h}^n\|^2 \le C\,h^{1/2}$$

and Lemma 5.2 is complete.

Finally, from Theorem 3.1, Lemma 5.2 and the triangles inequality, we establish the following result

Theorem 5.3 *Under the hypotheses of Theorem 4.1 is hold the following optimal error estimate*

$$\tau^2(t_m)\|\nabla\,(\mathbf{u}(t_m) - \mathbf{u}_{\epsilon h}^m)\|^2 + \tau^2(t_m)\|L^{1/2}(\mathbf{w}(t_m) - \mathbf{w}_{\epsilon h}^m)\|^2$$
$$+\Delta\,t \sum_{n=1}^{m} \tau^2(t_n)\|p(t_n) - p_{\epsilon h}^n\|^2 \le C\,(\epsilon^2 + \Delta t^2 + h^{1/2}),$$

for all $1 \le m \le N$.

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