EXTREMAL GRAPHS WITHOUT TOPOLOGICAL COMPLETE SUBGRAPHS

M. CERA, A. DIÁNEZ, AND A. MÁRQUEZ

Abstract. The exact values of the function $ex(n; TK_p)$ are known for $\left\lceil \frac{2n+5}{3} \right\rceil \leq p < n$ (see [Cera, Diánez, and Márquez, SIAM J. Discrete Math., 13 (2000), pp. 295–301]), where $ex(n; TK_p)$ is the maximum number of edges of a graph of order $n$ not containing a subgraph homeomorphic to the complete graph of order $p$. In this paper, for $\left\lceil \frac{2n+6}{3} \right\rceil \leq p < n - 3$, we characterize the family of extremal graphs $EX(n; TK_p)$, i.e., the family of graphs with $n$ vertices and $ex(n; TK_p)$ edges not containing a subgraph homeomorphic to the complete graph of order $p$.

Key words. extremal graph theory, topological complete subgraphs

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1. Introduction. The study of the function $ex(n; TK_p)$—i.e., the maximum number of edges of a graph of order $n$ not containing a subgraph homeomorphic to $K_p$, where $K_p$ is the complete graph with $p$ vertices—is one of the most general extremal problems, as pointed out by Bollobás in [1]. Exact values for this function are known only in some cases, as can be seen in Table 1.1.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$ex(n; TK_p)$</th>
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<tbody>
<tr>
<td>3</td>
<td>$n - 1$</td>
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<tr>
<td>4</td>
<td>$2n - 3$</td>
<td>[3]</td>
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<td>5</td>
<td>$3n - 6$</td>
<td>[4], [8], [9]</td>
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<tr>
<td>$\left\lceil \frac{2n+5}{3} \right\rceil$</td>
<td>$\left( \frac{n}{2} \right) - (5n - 6p + 3)$</td>
<td>[2]</td>
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<tr>
<td>$\left\lceil \frac{2n+6}{3} \right\rceil$</td>
<td>$\left( \frac{n}{2} \right) - (2n - 2p + 1)$</td>
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The aim of this work is to characterize a family of extremal graphs $EX(n; TK_p)$ for appropriate values of $n$ and $p$, i.e., the set of graphs of order $n$, with $ex(n; TK_p)$
edges and not containing any subgraph homeomorphic to $K_p$. Actually, we characterize the family $EX(n;TK_p)$ for $\left[\frac{2n+6}{3}\right] \leq p < n-3$:

$$EX(n;TK_p) = \begin{cases} 
(3n-4p+2)\overline{K_3} + (6p-4n-3)\overline{K_2} & \text{for } \left[\frac{2n+6}{3}\right] \leq p < \left[\frac{3n+2}{4}\right], \\
K_{4p-3n-2} + (2n-2p+1)\overline{K_2} & \text{for } \left[\frac{3n+2}{4}\right] \leq p < n-3.
\end{cases}$$

2. Definitions and notation. Given a graph $H$ and a set $\{v_1,\ldots,v_q\}$ of vertices of $H$, we denote by $H_0 = H$ and by $H_k$ for $k = 1,\ldots,q$ the induced subgraph in $H$ by the set of vertices $V(H)-\{v_1,\ldots,v_k\}$. We denote by $\Delta(H)$ the maximum degree of the graph $H$ and by $\delta_H(v)$ the degree of the vertex $v$ in the graph $H$. The complement graph of $H$ will be denoted by $\overline{H}$.

Let $q$ and $s$ be a pair of nonnegative integers; $C_q^s$ denotes the set of graphs $H$ such that there exists a set $\{v_1,\ldots,v_q\}$ of vertices of $H$ verifying the following:

1. $\delta_{H_{q-1}}(v_j) \geq \delta_{H_k}(v_{j+1})$ for $j = 1,\ldots,q-1$.
2. For each positive integer $h$, if there exists $k \in \{1,\ldots,q\}$ and $v \in H_k$ such that $\delta_{H_k}(v) \geq h$, then $\delta_{H_q}(v_{j+1}) \geq h$ for all $j = 1,\ldots,k$.
3. $H_q$ has at most $s$ edges (i.e., $|E(H_q)| \leq s$).

The next results show different conditions to guarantee that a graph belongs to the family described above (see [2]).

**Lemma 2.1** (see [2]). Let $H$ be a graph with $n$ vertices. Then, for any $q \leq n$, there exists $s$ such that $H$ is in $C_q^s$.

When $s = q$, we know sufficient conditions for the edges of a graph to belong to the class $C_q^q$.

**Lemma 2.2** (see [2]). Let $n$ and $q$ be two positive integers, with $q < n$. If $H$ is a graph with $n$ vertices and $2q$ edges, then

1. $H \in C_q^q$,
2. $\delta_{H_q}(v) \leq 1$ for $v \in V(H_q)$.

**Lemma 2.3** (see [2]). Let $q$ and $k$ be two positive integers with $k \leq q-2$. Let $H$ be a graph with $4q-k+1$ vertices and $2q+k+1$ edges. Then $H \in C_q^q$.

Notation and terminology not given here can be found in [1] and [2].

3. The family of extremal graphs. In this section, we will characterize the family $EX(n;TK_n)$ for $\left[\frac{2n+6}{3}\right] \leq p < n-3$. This problem is equivalent to characterizing $EX(n;TK_{n-q})$ for $n \geq 4q+2$ with $q \geq 4$ (case $\left[\frac{2n+6}{4}\right] \leq p < n-3$) and $n = 4q-k+1$ with $q \geq 5$, $0 \leq k \leq q-5$ (the case $\left[\frac{2n+6}{3}\right] \leq p < \left[\frac{3n+2}{4}\right]$).

In order to avoid excessive repetition, we define the graphs $H(n;TK_{n-q})$:

$$H(n;TK_{n-q}) = \begin{cases} 
K_{n-(4q+2)} + (2q+1)\overline{K_2} & \text{for } n \geq 4q+2, \\
(k+1)\overline{K_3} + (2(q-k)-1)\overline{K_2} & \text{for } n = 4q-k+1, 0 \leq k \leq q-5.
\end{cases}$$

For $n \geq 4q+2$, a graph $G$ belongs to the family $\{H(n;TK_{n-q})\}$ if $G$ has $n$ vertices and $\overline{G}$ is formed by $2q+1$ nonadjacent edges (see Figure 3.1).

For $n = 4q-k+1$ with $q \geq 5$ and $0 \leq k \leq q-5$, a graph $G$ belongs to the family $\{H(n;TK_{n-q})\}$ if it has $4q-k+1$ vertices and $\overline{G}$ is formed by $k+1$ nonadjacent triangles and $2(q-k)-1$ nonadjacent edges, as Figure 3.2 shows.

In the next two sections, we will prove the following theorem.

**Theorem 3.1.** $EX(n;TK_p) = \{H(n;TK_p)\}$ for $\left[\frac{2n+6}{3}\right] \leq p < n-3$. 

4. Case $\left\lceil \frac{3n+2}{4} \right\rceil \leq p < n - 3$. The aim of this section is to prove Theorem 3.1 when $n$ and $p$ are related by the expression $\left\lceil \frac{3n+2}{4} \right\rceil \leq p < n - 3$.

**Proposition 4.1.** Let $n$ and $p$ be two positive integers such that $\left\lceil \frac{3n+2}{4} \right\rceil \leq p < n - 3$. It is verified that

$$EX(n;TK_p) = \{H(n;TK_p)\}.$$  

In order to provide this proposition, we need some previous results. First, we recall the following results about the function $ex(n;TK_{n-q})$ (see [2]).

**Theorem 4.2** (see [2]). Let $n$ and $q$ be two positive integers. If $n \geq 4q + 2$, then

$$ex(n;TK_{n-q}) = \left(\frac{n}{2}\right) - (2q + 1).$$

Also, we recall that, given a graph $H$ and $v \in H$, the set of vertices adjacent to $v$ in $H$ is denoted by $\Gamma(v)$ (see [1]). Given a bipartite graph $B$ whose classes are $X$ and $Y$ with $|X| \leq |Y|$, we say that $B$ has a complete matching if there exists a set of nonadjacent edges in $B$ with cardinality $|X|$. If we need to show the existence of a complete matching in a bipartite graph, then we can use Hall’s condition.

**Theorem 4.3** (see [5]). Given a bipartite graph with classes $X$ and $Y$, if $|\Gamma(A)| \geq |A|$ for all $A \subseteq X$, where $\Gamma(A) = \bigcup_{v \in A} \Gamma(v)$, then there exists a complete matching.

The next result asserts that for any graph $G \in EX(n;TK_{n-q})$ its complement graph $\overline{G}$ is extremal for $C_q^{q+1}$ in the sense that $\overline{G} \in C_q^{q+1}$ and $\overline{G} \not\in C_q^{q}$.

**Lemma 4.4.** Let $n$ and $q$ be two nonnegative integers with $q \geq 4$ and $n \geq 4q + 2$. For every graph $G$ from the family of graphs $EX(n;TK_{n-q})$, we have

$$\overline{G} \in C_q^{q+1} - C_q^q.$$  

**Proof.** Let $G$ be a graph such that $G \in EX(n;TK_{n-q})$. The graph $G$ does not contain a subgraph homeomorphic to $K_{n-q}$, so by Theorem 4.2, we know that

$$|E(G)| = \left(\frac{n}{2}\right) - (2q + 1).$$
Hence, \(|E(H)| = 2q + 1\), where \(H = \overline{C}\).

By Lemma 2.1, there exists an integer \(s\) such that \(H \in C^*_q\). This means that there exists a subset \(\{v_1, \ldots, v_q\}\) of vertices of \(G\) verifying \(|E(H_q)| \leq s\), where \(H_q = H - \{v_1, \ldots, v_q\}\). If \(s \leq q + 1\), then \(H \in C^*_q\). Otherwise \((s > q + 1)\), let \(H^*\) be the graph obtained from \(H\) by removing one of the edges of the subgraph \(H_q\). The graph \(H^*\) has \(n \geq 4q + 2\) vertices and \(2q\) edges, and applying Lemma 2.2 results in \(H^* \in C^*_q\). Furthermore, by the construction of the graph \(H^*\), the set of vertices chosen to prove that \(H^*\) belongs to the class of graphs \(C^*_q\) is the same as the one we chose previously in \(H\); thus \(|E(H_q)| \leq q + 1\) and \(H \in C^*_q\).

Now we will prove that the number of edges of \(H_q\) may not be equal to or less than \(q\), i.e., \(H \notin C^*_q\). Suppose that \(H \notin C^*_q\). This means there exists a set of vertices \(\{v_1, \ldots, v_q\}\) guaranteeing this assertion. Let \(e_1 = (a_1, b_1), \ldots, e_s = (a_s, b_s)\) be the edges of \(H_q\) with \(1 \leq s \leq q\).

We consider the bipartite graph \(B\) whose classes are \(X = \{e_1, \ldots, e_s\}\) and \(Y = \{v_1, \ldots, v_q\}\) such that \(e_i\) is adjacent to \(v_j\) in \(B\) if the path \(a_i v_j b_i\) exists in \(G\). We note that if there exists a complete matching in \(B\), then we have that \(G\) contains a subgraph homeomorphic to \(K_{n-q}\). Now Hall’s condition implies the existence of a complete matching. Thus, we will prove that \(|\Gamma(A)| \geq |A|\) for each \(A \subseteq X\).

Let \(A = \{e_i\}\) be a subset of \(X\) with \(|A| = 1\) for \(i \in \{1, \ldots, s\}\). If \(|\Gamma(A)| = 0\), then \(e_i\) is nonadjacent to any vertex of the set \(\{v_{q-2}, v_{q-1}, v_q\}\) in \(B\). Hence, no vertex \(v \in \{v_{q-2}, v_{q-1}, v_q\}\) is adjacent to both \(a_i\) and \(b_i\) in \(G\). Consequently, \(\delta_{H_{q-1}}(a_i) \geq 2\) or \(\delta_{H_{q-1}}(b_i) \geq 2\) and, furthermore, \(\delta_{H_{q-3}}(a_i) \geq 3\) or \(\delta_{H_{q-3}}(b_i) \geq 3\). Thus, using property (2) of the definition of \(C^*_q\), we obtain that \(\delta_{H_{j-1}}(v_j) \geq 3\) for \(j = 1, \ldots, q-2\) and \(\delta_{H_{j-1}}(v_j) \geq 2\) for \(j = q-1, q\). Therefore, since \(s \geq 1\) we have that

\[|E(H)| \geq 3(q-2) + 2 \cdot 2 + s \geq 2q + 2\]

for \(q \geq 3\). But this is not possible since \(|E(H)| = 2q + 1\).

We consider \(A = \{e_i, e_j\} \subseteq X\) for \(i, j \in \{1, \ldots, s\}\) with \(i \neq j\), and we suppose \(|\Gamma(A)| \leq 1\). This means that at least three vertices of the set \(\{v_{q-3}, v_{q-2}, v_{q-1}, v_q\}\) are nonadjacent to \(e_i\) and to \(e_j\) in \(B\). Taking into account property (2) of the definition of \(C^*_q\), we have that \(\delta_{H_{j-1}}(v_j) \geq 3\) for \(j = 1, \ldots, q-3\), \(\delta_{H_{j-1}}(v_j) \geq 2\) for \(j = q-2, q-1\), and \(\delta_{H_{q-1}}(v_q) \geq 1\) (see Figure 4.1). Hence,

\[|E(H)| \geq 3(q-3) + 2 \cdot 2 + 1 + s \geq 2q + 2\]

for \(q \geq 4\), and this is a contradiction, as in the previous case.

Let \(m\) be an integer with \(3 \leq m \leq s\). Let \(A\) be the set of vertices \(\{e_{i_1}, \ldots, e_{i_m}\} \subseteq \{e_1, \ldots, e_s\}\) with \(i_1 < i_2 < \cdots < i_m\). If \(|\Gamma(A)| \leq m - 1\), then there

![Fig. 4.1](#)
exists \( i \in \{ q - (m - 1), \ldots, q \} \) in such a way that \( v_i \) is not adjacent to any vertex of the set \( A \) in the graph \( B \). By applying condition (2) of the definition of \( \mathcal{C}_q \), we obtain that \( \delta_{H_{q-m}}(v_{q-(m-1)}) \geq m \) and, therefore, \( \delta_{H_{q-1}}(v_j) \geq m \) for \( 1 \leq j \leq q - (m - 1) \) (see Figure 4.2). Furthermore, \( \delta_{H_{q-1}}(v_j) \geq 1 \) for \( q - (m - 2) \leq j \leq q \) and \( |E(H_q)| = s \geq m \). Consequently,

\[
|E(H)| \geq m(q - (m - 1)) + m - 1 + s \\
\geq mq - m^2 + 3m - 1.
\]

Since \( E(H) = 2q + 1 \), we have that \( 2q + 1 \geq mq - m^2 + 3m - 1 \) and, therefore, \( q \leq \frac{m^2 - 3m + 2}{m - 1} < m - 1 < m \leq s \), but this is not possible. Therefore, \( |\Gamma(A)| \geq |A| \) for each \( A \subseteq X \). Thus, by Hall’s condition, there exists a complete matching in \( B \) and, thereby, the graph \( G \) contains a subgraph homeomorphic to \( K_{n-q} \). This is not possible, and the result follows. \( \square \)

Now we can prove Proposition 4.1.

**Proof of Proposition 4.1.** It is equivalent to prove that

\[
EX(n; TK_{n-q}) = \{ H(n; TK_{n-q}) \}
\]

for \( q \geq 4 \) and \( n \geq 4q + 2 \).

Let \( G \) be a graph belonging to \( \{ H(n; TK_{n-q}) \} \) with \( n \geq 4q + 2 \). It is easy to check that \( G \) does not contain a subgraph homeomorphic to \( K_{n-q} \). Furthermore, by denoting \( |E(G)| \) as the number of edges of \( G \), we have that

\[
|E(G)| = ex(n; TK_{n-q}) = \binom{n}{2} - (2q + 1).
\]

Thus, by Theorem 4.2, \( G \) is maximal on edges and

\[
\{ H(n; TK_{n-q}) \} \subseteq EX(n; TK_{n-q}).
\]

In order to prove that \( EX(n; TK_{n-q}) \subseteq \{ H(n; TK_{n-q}) \} \), let \( G \) be a graph belonging to \( EX(n; TK_{n-q}) \), and we set \( H = \overline{G} \). By Theorem 4.2 we have that \( |E(H)| = 2q + 1 \). By Lemma 2.1, we know there exists \( s \) such that \( H \in \mathcal{C}_q^s \). Let \( \{ v_1, \ldots, v_q \} \) be a set of \( q \) vertices guaranteeing this property. We know that there exists a vertex \( v \in H_q \) such that \( \delta_{H_q}(v) \geq 1 \), because otherwise \( H_q \) is empty and \( H \in \mathcal{C}_q^s \). But this is not possible because, by Lemma 4.4, we know that \( H \notin \mathcal{C}_q^s \). If \( \delta(v_1) \geq 2 \), then \( |E(H_q)| \leq 2q + 1 - (2 + q - 1) = q \) and therefore \( H \in \mathcal{C}_q^q \), a contradiction. Therefore, \( \delta(v_1) \leq 1 \).

Thus, as \( v_1 \) is the vertex of maximum degree in \( H \), we have that \( \delta(v) \leq 1 \) for all \( v \in H \), and then the graph \( H \) is formed by \( 2q + 1 \) nonadjacent edges. Therefore, the result follows. \( \square \)
5. Case $\left[ \frac{2n+6}{3} \right] \leq p < \left[ \frac{3n+2}{4} \right]$. In this section, we will characterize the family of extremal graphs $EX(n;TK_{n-q})$ for $n = 4q - k + 1$ with $0 \leq k \leq q - 5$ in such a way that we will show that $EX(n;TK_{n-q}) = \{H(n;TK_{n-q})\}$, applying techniques based on the same ideas as in the previous section.

**Theorem 5.1.** Let $n$ and $p$ be two positive integers with $\left[ \frac{2n+6}{3} \right] \leq p < \left[ \frac{3n+2}{4} \right]$. Then

$$EX(n;TK_p) = \{H(n;TK_p)\}.$$ 

In order to prove this result, we also need to recall some results about the function $ex(n;TK_{n-q})$ (see [2]).

**Lemma 5.2** (see [2]). Let $k$ be a nonnegative integer and $H$ be a graph with maximum degree 2 and at least $3k + 1$ vertices of maximum degree. Then there exist at least $k + 1$ nonadjacent vertices with degree 2.

**Theorem 5.3** (see [2]). Let $n$, $k$, and $q$ be three nonnegative integers with $0 \leq k \leq q - 4$ and $n = 4q - k + 1$. It is verified that

$$ex(n;TK_{n-q}) = \left( \frac{n}{2} \right) - (2q + k + 2).$$

Now we will show, as in Lemma 4.4, that if $G \in EX(n;TK_{n-q})$ with $n = 4q - k + 1$, then $G \in C_{q+1}^p$ but $G \notin C_q^p$.

**Lemma 5.4.** Let $k$, $n$, and $q$ be three nonnegative integers such that $q \geq 5$, $0 \leq k \leq q - 5$, and $n = 4q - k + 1$. If $G \in EX(n;TK_{n-q})$, then

$$G \in C_{q+1}^p - C_q^p.$$ 

**Proof.** Let $G$ be a graph belonging to $EX(n;TK_{n-q})$. This graph does not contain a graph homeomorphic to $K_{n-q}$, and by Theorem 5.3 we know that

$$|E(G)| = \left( \frac{n}{2} \right) - (2q + k + 2).$$

Thus, $H = G$ has $2q + k + 2$ edges.

Let $H^*$ be the graph obtained from $H$ by removing one edge, similar to what we have done in Lemma 4.4. Since $H^*$ is a graph formed by $4q - k + 1$ vertices and $2q + k + 1$ edges, then applying Lemma 2.3 yields $H^* \in C_q^s$, and then

$$H \in C_{q+1}^s.$$ 

Now we will show that $H \notin C_q^s$. To the contrary, suppose $H \in C_q^s$ and let $\{v_1, \ldots, v_q\}$ be a set of vertices of $H$ guaranteeing that $H \in C_q^s$. Let $e_1 = (a_1, b_1), \ldots, e_s = (a_s, b_s)$ be the edges of $H_q$ with $s \leq q$. We consider the bipartite graph $B$ constructed as in Lemma 4.4, i.e., the graph whose classes are $X = \{e_1, \ldots, e_s\}$ and $Y = \{v_1, \ldots, v_q\}$ in such a way that $e_i$ is adjacent to $v_j$ if the path $a_i v_j b_s$ exists in the graph $G$. In this case, if we show the existence of a complete matching in $B$, then we would have that $G$ contains a subgraph homeomorphic to $K_{n-q}$. Therefore, we will show that $|\Gamma(A)| \geq |A|$ for each $A \subseteq X$.

If $|A| = m = 1$, by reasoning as in the proof of Lemma 4.4, we have that

$$|E(H)| \geq 3(q - 2) + 4 + s = 3q + s - 2 \geq 3q - 1.$$
Since \( k \leq q - 4 \), it is verified that \( 3q - 1 \geq 2q + k + 4 - 1 > 2q + k + 2 \), but this is not possible.

For \( m = 2 \), by considering as done previously, we have that
\[
|E(H)| \geq 3(q - 3) + 4 + 1 + s = 3q - 4 + s \geq 3q - 2.
\]

Taking into account that \( k \leq q - 5 \), it is verified that \( |E(H)| > 2q + k + 2 \), and this is a contradiction.

We consider \( m = 3 \). Let \( A = \{e_1, e_2, e_3\} \) be a subset of vertices of \( X \) with \( 1 \leq i_1 < i_2 < i_3 \leq s \). If \( |\{i\}| \leq m - 1 \), then there exists \( i \in \{q - (m - 1), \ldots, q\} \) in such a way that \( v_i \) is not adjacent to any vertex of the set \( A \) in the graph \( B \). Hence, by applying property (2) of the definition of \( C_q^2 \), we have that \( \delta_{H_q, m}(v_q - (m - 1)) \geq 3 \). Thus,
\[
|E(H)| \geq 3(q - 2) + 2 + s \geq 3q - 1 > 2q + k + 2
\]
since \( k \leq q - 4 \).

In general, if \( 4 \leq m \leq s \), then we consider \( A \) as the set of vertices \( \{e_1, \ldots, e_m\} \subseteq \{e_1, \ldots, e_s\} \) with \( i_1 < i_2 < \cdots < i_m \), if \( |\{i\}| \leq m - 1 \), then there exists \( i \in \{q - (m - 1), \ldots, q\} \) in such a way that \( v_i \) is not adjacent to any vertex of the set \( A \) in the graph \( B \). Hence, as in the proof of Lemma 4.4, we have that \( \delta_{H_{q, m}}(v_{q - (m - 1)}) \geq m \) and, therefore,
\[
|E(H)| \geq m(q - (m - 1)) + m - 1 + s \geq mq - m^2 + 3m - 1.
\]

But \( |E(H)| = 2q + k + 2 \leq 3q - 3 \) for \( k \leq q - 5 \). Thus, \( 3q - 3 \geq mq - m^2 + 3m - 1 \) and, thereby, \( q \leq m - \frac{2}{m-3} \leq m \), but this is not possible.

Thus, using Hall’s condition, there exists a complete matching in \( B \), and consequently, \( G \) contains a subgraph homeomorphic to \( K_{n-q} \), but this is not possible. Hence, \( H \notin C_q^2 \) and the result follows.

The next result is devoted to proving the existence of nonadjacent triangles in graphs with maximum degree 2 and the prescribed number of vertices of maximum degree.

**Lemma 5.5.** Let \( r \) be a nonnegative integer, and let \( H \) be a graph with maximum degree 2. If \( H \) has \( 3r + 3 \) vertices of degree 2 and \( r + 1 \) of them form an independent set, then \( H \) contains \( r + 1 \) nonadjacent triangles.

**Proof.** We apply induction on \( r \). For \( r = 0 \) the result is obvious, because the triangle is the unique graph formed by 3 vertices of degree 2 and all of them are adjacent among themselves.

Now suppose that \( r + 1 \geq 2 \) and the result holds for \( r \). Let \( H \) be a graph with \( 3(r + 1) + 3 = 3(r + 2) \) vertices of degree 2, and let \( w_1, \ldots, w_{r+2} \) be \( r + 2 \) nonadjacent vertices of \( H \).

If there exist \( i, j \in \{1, \ldots, r + 2\} \) with \( i \neq j \) such that \( \Gamma(w_i) \cap \Gamma(w_j) \neq \emptyset \), then \( |\bigcup_{k=1}^{r+2}(\Gamma(w_k) \cup \{w_k\})| < 3(r + 2) \). Thus, there exists \( w \in H \) with degree 2 nonadjacent to \( w_i \) for all \( i \). Hence, \( \{w, w_1, \ldots, w_{r+2}\} \) is a set of \( r + 3 \) nonadjacent vertices of degree 2, but this is a contradiction. Therefore, \( \Gamma(w_i) \cap \Gamma(w_j) = \emptyset \) for all \( i \neq j \). Furthermore, if \( w \in H \) is adjacent to any \( w_i \) for \( i \in \{1, \ldots, r + 2\} \), then \( w \) has degree 2; otherwise, since the number of vertices of degree 2 is \( 3(r + 2) \), there exists \( v \in H \) with degree 2 nonadjacent to \( w_i \) for all \( i \), and we have seen above that this is not possible.

Now, let \( a \) and \( b \) be the vertices adjacent to \( w_{r+2} \). If the edge \( (a, b) \) does not belong to \( H \), we have that \( \{w_1, \ldots, w_{r+1}, a, b\} \) is a set of \( r + 3 \) nonadjacent vertices of degree 2. Thus, the vertices \( w_1, a \), and \( b \) form a triangle.
Denote by $H^*$ the graph obtained from $H$, removing the previous triangle. Therefore, $H^*$ is a graph with $3r + 3$ vertices of degree 2, and $r + 1$ of them are nonadjacent; by induction hypothesis, $H^*$ contains $r + 1$ nonadjacent triangles. Thus, $H$ contains $r + 2$ nonadjacent triangles.

To finish this section, we give the proof of Theorem 5.1, using the previous results.

**Proof of Theorem 5.1.** It is equivalent to show that

$$EX(n;TK_{n-q}) = \{H(n;TK_{n-q})\}$$

for $n = 4q - k + 1$ with $q \geq 5$, $0 \leq k \leq q - 5$.

Let $G$ be a graph belonging to the set $\{H(n;TK_{n-q})\}$. By checking the structure of this graph $G$, it is easy to prove that $G$ does not contain a subgraph homeomorphic to $K_{n-q}$. Since $|E(G)| = ex(n;TK_{n-q}) = \binom{n}{2} - (2q + k + 2)$, we have that $G \in EX(n;TK_{n-q})$.

In order to show that $EX(n;TK_{n-q}) \subseteq \{H(n;TK_{n-q})\}$, let $G$ be a graph belonging to $EX(n;TK_{n-q})$. We denote by $H = G$. By Theorem 5.3, $|E(H)| = 2q + k + 2$.

First, we will prove that $\Delta(H) \leq 2$. Suppose the contrary, that $\Delta(H) \geq 3$.

By applying Lemma 5.4, we have $H \in C_q^{q+1} - C_q^q$. Hence, there exists a subset of vertices $\{v_1, \ldots, v_q\}$ of $H$ guaranteeing this property. Furthermore, $|E(H_q)| = q + 1$.

We claim there exists $j \in \{1, \ldots, q\}$ such that $\Delta(H_{j-1}) \geq 3$ and $\Delta(H_{j}) \leq 2$, because otherwise we have $\delta_{H_{j-1}}(v_i) \geq 3$ for each $1 \leq i \leq q$, and

$$|E(H)| \geq 3q + (q + 1) > 2q + k + 2,$$

but this is not possible. Now we distinguish the cases $j \geq k + 1$ and $j \leq k$.

For $j \geq k + 1$, we consider the fact that $\Delta(H_{j-1}) \geq 3$ and $\Delta(H_{j}) \leq 2$. Taking into account property (2) of the definition of $C_q^{q+1}$ and $|E(H_q)| > 0$, we have $\delta_{H_{j-1}}(v_i) \geq 3$ for $1 \leq i \leq j$ and $\delta_{H_{j-1}}(v_i) \geq 1$ for $j + 1 \leq i \leq q$. Hence,

$$|E(H_q)| \leq 2q + k + 2 - (3j + (q - j)) \leq q - j + 1 \leq q.$$

But this is not possible since $|E(H_q)| = q + 1$.

For $j \leq k$, we have that $\delta_{H_{j-1}}(v_i) \geq 3$ for $1 \leq i \leq j$. If $\Delta(H_{k}) \leq 1$, then $2|E(H_k)| \leq |V(H_k)|$ and

$$4q - 2k + 1 = |V(H_k)| \geq 2|E(H_k)| \geq 2(q - k + q + 1) = 4q - 2k + 2,$$

and this is a contradiction. Thus, $\Delta(H_{k}) = 2$ and $\delta_{H_{j-1}}(v_i) \geq 2$ for $j + 1 \leq i \leq k$. Hence,

$$|E(H_q)| \leq 2q + k + 2 - (3j + 2(k - j + 1) + (q - k + 1)) = q - j + 1 \leq q,$$

and this not possible. Thus, $\Delta(H) \leq 2$.

Since $2|E(H)| > |V(H)|$, we have $\Delta(H) \geq 2$ and, consequently, $\Delta(H) = 2$.

Next we are going to study the structure of $H$. On the one hand, if $H$ has at least $3(k + 1) + 1$ vertices of degree 2, then by Lemma 5.2 we have that $k + 2$ of those vertices $\{w_1, \ldots, w_{k+2}\}$ are nonadjacent. Let $w_{k+3}, \ldots, w_q$ be $q - (k + 2)$ vertices of $H$ such that the set $\{w_1, \ldots, w_{k+2}, w_{k+3}, \ldots, w_q\}$ verifies properties (1) and (2) of the definition of $C_q^q$. For this set of vertices, we have that

$$|E(H_q)| \leq 2q + k + 2 - (2(k + 2) + q - (k + 2)) = q.$$
and therefore, $H \in C_q^2$, a contradiction. Thus, $H$ has at most $3k + 3$ vertices of degree 2. On the other hand, if we denote by $n_i$ the number of vertices of degree $i$ in $H$, we have that

$$\begin{align*}
2n_2 + n_1 &= 2(2q + k + 2) \\
n_2 + n_1 + n_0 &= 4q - k + 1
\end{align*}$$

Thus, $n_2 = 3k + 3 + n_0 \geq 3k + 3$ and the number of vertices of degree 2 in $H$ is $n_2 = 3k + 3$.

Furthermore, as we have shown previously, $H$ may not have $k + 2$ nonadjacent vertices of degree 2. Since $H$ has $3k + 3 \geq 3k + 1$ vertices of degree 2, by Lemma 5.2 we have that $H$ has at least $k + 1$ nonadjacent vertices. Hence, $H$ has maximum degree 2 and $3k + 3$ vertices of degree 2, and $k + 1$ of them are nonadjacent. Therefore, by applying Lemma 5.5, $H$ contains $k + 1$ nonadjacent triangles. Additionally, $n_0 = 0$, $n_1 = 4q - 4k - 2$, and the result follows.

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REFERENCES