Convergence to equilibrium for smectic-A liquid crystals in 3D domains without constraints for the viscosity*

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Abstract

In this paper, we focus on a smectic-A liquid crystal model in 3D domains, and obtain three main results: the proof of an adequate Lojasiewicz-Simon inequality by using an abstract result; the rigorous proof (via a Galerkin approach) of the existence of global in-time weak solutions that become strong (and unique) in long-time; and its convergence to equilibrium of the whole trajectory as time goes to infinity. Given any regular initial data, the existence of a unique global in-time regular solution (bounded up to infinite time) and the convergence to an equilibrium have been previously proved under the constraint of a sufficiently high level of viscosity. Here, all results are obtained without imposing said constraint.

Keywords: Liquid crystals, Navier-Stokes equations, Ginzburg-Landau potential, energy dissipation, convergence to equilibrium, Lojasiewicz-Simon’s inequalities.

1 Introduction

We consider the following equations ([5]), which model a smectic-A liquid crystal confined in an open bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial \Omega$ within the time interval $(0, +\infty)$:

$$
\partial_t u + (u \cdot \nabla) u - \nu \Delta u - \lambda w \nabla \varphi + \nabla q = 0,
$$

$$
\nabla \cdot u = 0,
$$

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\[ \frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi + \gamma w = 0, \]
\[ \Delta^2 \varphi - \nabla \cdot f_\varepsilon(\nabla \varphi) - w = 0, \]

where

\[ f_\varepsilon(\mathbf{n}) = \nabla \mathbf{n} F_\varepsilon(\mathbf{n}) = \frac{1}{\varepsilon^2}(|\mathbf{n}|^2 - 1)\mathbf{n}, \quad \forall \mathbf{n} \in \mathbb{R}^3 \]

and \( F_\varepsilon(\mathbf{n}) = \frac{1}{4\varepsilon^2}(|\mathbf{n}|^2 - 1)^2 \) is the Ginzburg-Landau potential. Here, \( \mathbf{u} : \Omega \times [0, +\infty) \mapsto \mathbb{R}^3 \) is the flow velocity; \( p : \Omega \times [0, +\infty) \mapsto \mathbb{R} \) describes a potential function (dependent of the fluid pressure); \( \varphi : \Omega \times [0, +\infty) \mapsto \mathbb{R} \) is the layer variable, whose level sets represent the layer structure; and \( w = \Delta^2 \varphi - \nabla \cdot f_\varepsilon(\nabla \varphi) \) is a variable related to the equilibrium equation with respect to the (smectic) elastic energy

\[ E_\varepsilon(\varphi) = \int_\Omega \left( \frac{1}{2} |\Delta \varphi|^2 + F_\varepsilon(\nabla \varphi) \right). \]

The constants \( \nu > 0, \lambda > 0, \) and \( \gamma > 0 \) are some coefficients which depend on the viscosity, the elasticity and the time relaxation, respectively. The system (1)-(4) is completed with the (Dirichlet) boundary conditions

\[ \mathbf{u}|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = \varphi_1, \quad \partial_n \varphi|_{\partial\Omega} = \varphi_2, \]

where \( \varphi_1 \) and \( \varphi_2 \) are given time-independent functions, and the initial conditions

\[ \mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \]

For compatibility, we assume \( \mathbf{u}_0|_{\partial\Omega} = 0 \) with \( \nabla \cdot \mathbf{u}_0 = 0 \) and \( \varphi_0|_{\partial\Omega} = \varphi_1, \partial_n \varphi_0|_{\partial\Omega} = \varphi_2. \)

The first mathematical results of problem (1)-(7) were obtained in [10]. For three-dimensional domains and time-independent boundary conditions, both the existence of global in-time weak solutions for the smectic-A problem (1)-(7) and pioneering research into its long-time behaviour are jointly studied in [10], and convergence of \( \mathbf{u}(t) \) and \( w(t) \) to zero as \( t \to +\infty \) is attained, although the uniqueness of limit for the trajectories \( \varphi(t) \) as \( t \uparrow \infty \) is not assured. The regularity and time-periodicity of solutions of the problem (1)-(7) with time-dependent boundary conditions is studied in [3]. These results were previously studied for nematic liquid crystals in [9] and [1].

The convergence in infinite time of the whole trajectory was first solved in [14] for a nematic model with Dirichlet boundary conditions, thereby obtaining the convergence of the director vector \( \mathbf{d}(t) \) (an average of preferential orientation of molecules) as \( t \to +\infty \) towards an equilibrium of the elastic energy. In [15], a similar problem with stretching terms and periodic boundary conditions of \( \mathbf{d} \) is treated. For these convergence results, suitable Lojasiewicz-Simon inequalities are used. In both cases above, in order to obtain a global
in-time regular solution, a uniform in-time Gronwall theorem is used (see [13]), requiring either a sufficiently high viscosity coefficient or initial conditions sufficiently near to a global minimizer.

The long-time behaviour of a nematic liquid crystal model with time-dependent boundary conditions and external forces is studied in [6], while also imposing a high level of viscosity. For nematic models including stretching terms, in the recent paper [11], the authors show that any weak solution has a ω-limit set containing a single steady solution, thereby circumventing the use of the strong regularity (hence the viscosity constraint is rendered unnecessary).

Returning to the smectic-A problem (1)-(7), its long-time behaviour has already been studied in [12], where the imposition of both a high level of viscosity and periodic boundary conditions plays a main role. On the other hand, the convergence of the whole trajectory to equilibrium for a smectic-A model modified by penalization is given in [4], without imposing constraints for the viscosity.

Consequently, with respect to the above results, the main contribution that we will present in this paper is the identification of a unique critical point as the limit of the trajectory of ϕ(t) as t approaches to infinity, for each global weak solution of the smectic-A model (1)-(7) that is strong over long periods, without imposing a high level of viscosity. Moreover, we consider of remarkable interest the following facts:

1. The proof of an adequate Lojasiewicz-Simon inequality by means of an abstract result given in [8] (see Theorem 4 below).

2. The rigorous proof, via a Galerkin approach, of the existence of weak solutions of the smectic-A problem (1)-(7), which are strong solutions in the case of long periods.

1.1 Notation

- In general, the notation will be abridged: \( L^p = L^p(\Omega), \ p \geq 1, \ H^1_0 = H^1_0(\Omega), \) etc. If \( X = X(\Omega) \) is a space of functions defined in the open set \( \Omega \), then \( L^p(X) \) denotes the Banach space \( L^p(0, T; X(\Omega)) \). Moreover, boldface letters will be used for vectorial spaces, for instance \( L^2 = L^2(\Omega)^3 \).

- The \( L^p \)-norm is denoted by \( |\cdot|_p, \ 1 \leq p \leq \infty \), and the \( H^m \)-norm by \( \|\cdot\|_m \) (in particular \( |\cdot|_2 = \|\cdot\|_0 \)). The inner product of \( L^2(\Omega) \) is denoted by \( \langle \cdot, \cdot \rangle \). The boundary \( H^s(\partial\Omega) \)-norm is denoted by \( \|\cdot\|_{s;\partial\Omega} \).

- The space formed by all fields \( u \in C_0^\infty(\Omega)^3 \) satisfying \( \nabla \cdot u = 0 \) is set as \( \mathcal{V} \). The closure of \( \mathcal{V} \) in \( L^2 \) and \( H^1 \) are denoted as \( \mathcal{H} \) and \( \mathcal{V} \), which are Hilbert spaces for the norms \( |\cdot|_2 \)
and \(\|\cdot\|_1\), respectively. Furthermore,

\[ H = \{ u \in L^2; \nabla \cdot u = 0, \ u \cdot n = 0 \text{ on } \partial \Omega \}, \quad V = \{ u \in H^1; \nabla \cdot u = 0, \ u = 0 \text{ on } \partial \Omega \}. \]

Note that if \( u \in H \), since \( u \in L^2 \) and \( \nabla \cdot u \in L^2 \), therefore \( u \cdot n = 0 \) holds in \( H^{-1/2}(\partial \Omega) \).

- We will consider a sufficiently regular \( \Omega \) in order to have the following equivalent norms:

\[
\| \varphi \|_1 \approx |\nabla \varphi|_2 + \| \varphi \|_{1/2, \partial \Omega} = |\nabla \varphi|_2 + \| \varphi_1 \|_{1/2, \partial \Omega} \quad (8)
\]

\[
\| \varphi \|_2 \approx |\Delta \varphi|_2 + \| \varphi \|_{3/2, \partial \Omega} = |\Delta \varphi|_2 + \| \varphi_1 \|_{3/2, \partial \Omega} \quad (9)
\]

\[
\| \varphi \|_4 \approx |\Delta^2 \varphi|_2 + \| \varphi_1 \|_{7/2, \partial \Omega} + \| \varphi_2 \|_{5/2, \partial \Omega} \quad (10)
\]

- In the following, \( C, K > 0 \) will denote several constants, which depend only on the fixed data of the problem.

- For the sake of simplicity, henceforth we will consider \( \nu, \lambda, \gamma = 1 \).

## 2 Some preliminary results

### 2.1 Long-time behaviour

Assume the following starting point:

Let \( E, \Phi \in L^1_{loc}(0, +\infty) \) be two positive functions with \( E \in H^1(0, T) \forall T > 0 \), satisfying

\[
E'(t) + \Phi(t) \leq 0, \quad \text{a.e. } t \in (0, +\infty). \quad (11)
\]

Therefore, \( E \) is a decreasing function with \( E \in L^\infty(0, +\infty) \) and

\[
\exists \lim_{t \to +\infty} E(t) = E_\infty \geq 0. \quad (12)
\]

Moreover, by integrating (11), one has \( \Phi \in L^1(0, +\infty) \).

The following result is proved in [2].

**Lemma 1** Let \( \Phi \in L^1(0, +\infty) \) be a positive function such that \( \Phi \in H^1(0, T) \forall T > 0 \), which satisfies

\[
\Phi'(t) \leq C_2(\Phi(t)^3 + 1). \quad (13)
\]

Therefore, there exists a sufficiently large \( T^* \geq 0 \) such that \( \Phi \in L^\infty(T^*, +\infty) \) and

\[
\exists \lim_{t \to +\infty} \Phi(t) = 0.
\]

We will extend this result for function sequences in order to uniformly bound them with respect to the index of sequence. Specifically,
Theorem 2 Let $\Phi^m, E^m$, be two positive function sequences, which satisfy (11) and (13) for some constant $C_2 > 0$ independent of $m$. Let $E(t) = \lim_{m \to +\infty} E^m(t)$ a.e. $t \in (0, +\infty)$. Therefore, for each $\varepsilon \in (0, 1)$, there exists a sufficiently large time $T^* = T^*(\varepsilon) \geq 0$, independent of $m$, such that

$$\|\Phi^m\|_{L^\infty(T^*, +\infty)} \leq \varepsilon.$$ 

Proof.

By construction, $E(t)$ is a decreasing positive function which satisfies (12) for a certain $E_\infty \geq 0$.

Let $R^*$ and $t$ be two times such that $R^* < t$. By integrating (11) in $[R^*, t]$ and taking the limit as $m \to +\infty$,

$$\int_{R^*}^{t} \Phi^m(s) \, ds \leq E^m(R^*) - E^m(t) \to E(R^*) - E(t) \leq E(R^*) - E_\infty.$$ 

For each $\delta > 0$ given, we can choose a sufficiently large $m_0(\delta) \in \mathbb{N}$ such that

$$\int_{R^*}^{t} \Phi^m(s) \, ds \leq E(R^*) - E_\infty + \delta/2 \leq \delta, \quad \forall t \geq R^*, \quad \forall m \geq m_0(\delta).$$ 

Taking $t \to +\infty$, we have

$$\int_{R^*(\delta)}^{+\infty} \Phi^m(s) \, ds \leq \delta,$$ 

(14)

where $R^*(\delta)$ does not depend on $m$. Starting from (13) and (14), we are going to finish the proof of this theorem, using the lines provided in [2]. Indeed, from (14),

$$\frac{1}{\tau} \int_{t}^{t+\tau} \Phi^m(t) \, dt \leq \frac{\delta}{\tau}, \quad \forall \tau > 0, \quad \forall t \geq R^*(\delta).$$ 

(15)

Lemma 2.1 of [2] implies that, $\forall t \geq R^*(\delta)$ and $\forall \tau > 0$, there exist times $\bar{t} \in [t, t + \tau]$ such that:

$$\Phi^m(\bar{t}) \leq \frac{2\delta}{\tau}.$$ 

(16)

On the other hand, from (13), Lemma 2.2 of [2] implies that for any $\varepsilon < 1$, if $\Phi^m(t_0) \leq \varepsilon/3$, then $\Phi^m(t) \leq \varepsilon \forall t \in [t_0, t_0 + S^*(\varepsilon)]$, where $S^*(\varepsilon) = \frac{\varepsilon}{3C_2}$ (that is independent of $m$).

By using (15) and (16) for $\delta = \frac{\varepsilon^2}{36C_2}$ and $\tau = \frac{S^*(\varepsilon)}{2}$, Theorem 2.3 of [2] gives

$$\Phi^m(t) \leq \varepsilon, \quad \forall t \geq R^*(\delta) + \frac{S^*(\varepsilon)}{2} = R^*(\delta) + \frac{\varepsilon}{6C_2} := T^*(\varepsilon).$$ 

(17)

Observe that bound (17) does not depend on $m$. Therefore, for each $\varepsilon < 1$, there exists a sufficiently large $T^* = T^*(\varepsilon)$ such that $\|\Phi^m\|_{L^\infty(T^*, +\infty)} \leq \varepsilon$. \qed
2.2 Lojasiewicz-Simon inequality

It is standard procedure to use appropriate Lojasiewicz-Simon inequalities to study the convergence of trajectories in infinite time. It is not easy to find in the literature a demonstration of these types of inequalities associated to various Euler-Lagrange equations. Here, a particular Lojasiewicz-Simon inequality associated to the critical points of the elastic energy (5) is deduced, by using the abstract Theorem 4 presented below (Theorem 4.2 of [8]). Some extensions of this Lojasiewicz-Simon inequality are commented in the Remark 6 below.

We begin by recalling the following definitions:

**Definition 3** A bounded linear operator $L : X_1 \mapsto X_2$ between two Banach spaces $X_1$ and $X_2$ is called a Fredholm operator of index zero if $L$ has a closed range $R(L)$, a finite dimensional kernel $N(L)$ and $\dim N(L) = \dim (X_2 / R(L)) < \infty$. A $C^1$ map $M : U \subset X_1 \mapsto X_2$ is called a Fredholm map of index zero if its Fréchet differential at each point are Fredholm operators of index zero.

For instance, an invertible operator plus a compact operator is a Fredholm operator of index zero.

**Theorem 4** Assume the following hypotheses:

- Let $H$ be a Hilbert space and $A : D(A) \subset H \mapsto H$ a linear self-adjoint and positive definite operator. In particular, $H_A \equiv (D(A), \langle \cdot, \cdot \rangle_A)$ is a Hilbert space endowed with the scalar product $\langle u, v \rangle_A \equiv (Au, Av)_H$ for all $u, v \in D(A)$.

- Let $X$ and $\tilde{X}$ be two Banach spaces such that the embeddings $X \hookrightarrow H_A$ and $\tilde{X} \hookrightarrow H$ are continuous. Moreover, $X \hookrightarrow \tilde{X}$ is also a continuous embedding.

- Let $E : X \mapsto \mathbb{R}$ be a Fréchet-differentiable functional.

- Let $\mathcal{M} = \mathcal{E}' : X \mapsto \tilde{X}$ be an analytic gradient map with the following properties:

  - $\mathcal{M}$ is a Fredholm map of index zero; i.e., for each $u \in X$ the bounded linear operator $\mathcal{M}'(u) \in \mathcal{L}(X, \tilde{X})$ is a Fredholm operator of index zero.

  - For each fixed $u \in X$, the bounded linear symmetric operator $\mathcal{M}'(u) : X \mapsto \tilde{X}$ has an extension $\mathcal{M}_1(u) : H_A \mapsto H$, which is a symmetric Fredholm operator of index zero.

  - The map $\mathcal{R} : u \in X \mapsto \mathcal{M}_1(u)A^{-1} \in \mathcal{L}(H)$ is continuous.

Therefore, if $\bar{u} \in X$ is a critical point of $\mathcal{E}$, i.e. $\mathcal{E}'(\bar{u}) = 0$, then positive constants $C$, $\beta_1$ and $\sigma \in [1/2, 1)$ exist such that

$$|\mathcal{E}(u) - \mathcal{E}(\bar{u})|^\sigma \leq C \|\mathcal{E}'(u)\|_H \quad \forall u \in X \text{ with } \|u - \bar{u}\|_X < \beta_1.$$
This theorem is now going to be applied to the smectic-A model, by using strong norms.

**Lemma 5 (Strong Lojasiewicz-Simon inequality for smectic-A problems)** Let $S$ be the following set of equilibrium points related to the elastic energy $E_e(\varphi) = \int_\Omega \left( \frac{1}{2} |\Delta \varphi|^2 + F_e(\nabla \varphi) \right)$:

$$S = \{ \varphi \in H^4(\Omega) : \Delta^2 \varphi - \nabla \cdot f_e(\nabla \varphi) = 0 \text{ a.e in } Q, \varphi|_{\partial \Omega} = \varphi_1, \partial_n \varphi|_{\partial \Omega} = \varphi_2 \}.$$

If $\overline{\varphi} \in S$, there are three positive constants $C$, $\beta$, and $\theta \in (0, 1/2)$ which depend on $\overline{\varphi}$, such that for all $\varphi \in H^4$ with $\varphi|_{\partial \Omega} = \varphi_1$, $\partial_n \varphi|_{\partial \Omega} = \varphi_2$ and $\|\varphi - \overline{\varphi}\|_3 \leq \beta$, then

$$|E_e(\varphi) - E_e(\overline{\varphi})|^{1-\theta} \leq C |w|^2$$

where $w = w(\varphi) := \Delta^2 \varphi - \nabla \cdot f_e(\nabla \varphi)$.

**Proof.** The proof is divided into two steps.

**Step 1** (Application of Theorem 4): $\exists \beta_1, C > 0$ such that if $\|\varphi - \overline{\varphi}\|_4 \leq \beta_1$, then (18) holds.

Let $\phi \in H^4(\Omega)$ be the “lifting” function defined as the (strong) solution of the problem:

$$\Delta^2 \phi = 0 \text{ in } \Omega, \quad \phi|_{\partial \Omega} = \varphi_1, \quad \partial_n \phi|_{\partial \Omega} = \varphi_2.$$  

(19)

Theorem 4 is going to be applied for the following spaces and operators:

$$H \equiv \bar{X} = L^2(\Omega), \quad X \equiv H_A = H_0^2(\Omega) \cap H^4(\Omega),$$

$$A = \Delta^2 : \xi \in X \mapsto A \xi = \Delta^2 \xi \in H \text{ and } (\xi, \psi)_A = (\Delta^2 \xi, \Delta^2 \psi)_{L^2} \quad \forall, \xi, \psi \in D(A),$$

$$E : \xi \in X \mapsto E(\xi) = E_e(\xi + \phi) = \int_\Omega \left( \frac{1}{2} |\Delta (\xi + \phi)|^2 + F_e(\nabla (\xi + \phi)) \right) \in \mathbb{R},$$

$$M = E' : \xi \in X \mapsto H, \text{ such that } M(\xi) = \Delta^2 \xi - \nabla \cdot f_e(\nabla (\xi + \phi)),$$

and $M_1(\xi) = M'(\xi)$, where for each $\xi \in X$,

$$M'(\xi) : \psi \in X \mapsto M'(\xi)(\psi) = \Delta^2 \psi - \nabla \cdot ((f_e)'(\nabla (\xi + \phi)) \nabla \psi) \in H.$$

Indeed, $M'(\xi)$ is a Fredholm operator of index zero, because $M'(\xi)$ is the sum of the invertible operator $A$ and the compact operator $\psi \in X \rightarrow -\nabla \cdot ((f_e)'(\nabla (\xi + \phi)) \nabla \psi) \in H$.

Moreover, the map $R : \xi \in X \mapsto M'(\xi)A^{-1} \in \mathcal{L}(H)$ is well-posed because $A^{-1} \in \mathcal{L}(H; X)$ and $M'(\xi) \in \mathcal{L}(X; H)$. It remains to be proved that $R$ is (sequentially) continuous. Let $\xi_n \rightarrow \xi$ in $X$ as $n \rightarrow \infty$. Therefore,

$$\|R(\xi_n) - R(\xi)\|_{\mathcal{L}(H)} = \|M'(\xi_n)A^{-1} - M'(\xi)A^{-1}\|_{\mathcal{L}(H)} \leq \|M'(\xi_n) - M'(\xi)\|_{\mathcal{L}(X,H)} A^{-1}\|_{\mathcal{L}(H,X)}$$
and

$$\|M'(\xi_n) - M'(|\xi)\|_{L^2(\Omega, H)} = \sup_{\psi \in X \setminus \{0\}} \frac{\|M'(\xi_n)(\psi) - M'(|\xi)(\psi)\|_H}{\|\psi\|_H}$$

$$= \sup_{\psi \in X \setminus \{0\}} \frac{\|\nabla \cdot \left( (f_\xi)'(\nabla(|\xi + \phi)) - (f_\xi)'(\nabla(|\xi + \phi)) \right) \nabla \psi \|_2}{\|\psi\|_4}$$

$$\leq \sup_{\psi \in X \setminus \{0\}} \frac{\|((f_\xi)'(\nabla(|\xi + \phi)) - (f_\xi)'(\nabla(|\xi + \phi)) \nabla \psi \|_1}{\|\psi\|_4}$$

$$\leq C \|((f_\xi)'(\nabla(|\xi + \phi)) - (f_\xi)'(\nabla(|\xi + \phi)) \|_1$$

By taking into account that \(\|((f_\xi)'(\nabla(|\xi + \phi)) - (f_\xi)'(\nabla(|\xi + \phi))\|_{H^1} \to 0\) as \(n \to \infty\) if \(\xi_n \to \xi\) in \(H^4\), then the continuity of the operator \(R\) has been proved.

In order to apply Theorem 4, the boundary conditions must be lifted by using the function \(\phi\) given in (19). In fact, function \(\bar{\xi} = \bar{\varphi} - \phi\) (recall that \(\bar{\varphi} \in S\) satisfies \(\bar{\varphi}|_{\partial \Omega} = 0\) and \(\partial_n \bar{\varphi}|_{\partial \Omega} = 0\) and represents a critical point of \(E(\xi)\). Let \(\varphi \in H^4(\Omega)\) with \(\varphi|_{\partial \Omega} = \varphi_1, \partial_n \varphi|_{\partial \Omega} = \varphi_2\) and \(\|\varphi - \bar{\varphi}\|_4 \leq \beta_1\) (\(\beta_1 > 0\) given in Theorem 4). If we define \(\xi = \varphi - \phi \in X\), then \(\|\xi - \bar{\xi}\|_4 \leq \beta_1\) and, owing to Theorem 4:

$$|E_\varepsilon(\varphi) - E_\varepsilon(\bar{\varphi})|^{1 - \theta} = |E(\xi) - E(\bar{\xi})|^{1 - \theta} \leq C \|E'(\xi)\|_H$$

$$= C |\Delta^2 \xi - \nabla \cdot f_\xi(\nabla(\xi + \phi))|_2 = C |w(\varphi)|_2.$$ 

Hence (18) holds.

**Step 2:** (Relaxing the local approximation \(\|\varphi - \bar{\varphi}\|_4 \leq \beta\) by \(\|\varphi - \bar{\varphi}\|_3 \leq \beta\)) There exists \(\beta > 0\) and \(C > 0\) such that if \(\varphi \in H^4(\Omega)\) and \(\|\varphi - \bar{\varphi}\|_3 \leq \beta\), then (18) holds.

In this step, a similar argument is followed to that in Lemma 4.4 of [12]. Since \(\varphi - \bar{\varphi} = \xi - \bar{\xi}\), this is reduced to the homogeneous functions \(\xi, \bar{\xi}\). From (10), there exists \(M > 0\) such that

$$\|\xi - \bar{\xi}\|_4 \leq M |\Delta^2 (\xi - \bar{\xi})|_2$$

and by using Sobolev’s embeddings and \(\|\xi\|_3 \leq \|\bar{\xi}\|_3 + \beta \leq C\), we obtain

$$|\nabla \cdot (f_\xi(\nabla(\xi + \phi)) - f_\xi(\nabla(\bar{\xi} + \phi)))|_2 \leq C(\beta) \|\xi - \bar{\xi}\|_3,$$

$$|E(\xi) - E(\bar{\xi})|^{1 - \theta} \leq C(\beta) \|\xi - \bar{\xi}\|_2^{1 - \theta} \leq C(\beta) \|\xi - \bar{\xi}\|_3^{1 - \theta}$$

where \(C(\beta)\) depends on \(\beta\) (and \(\|\bar{\xi}\|_3\)). In particular, since \(\|\xi - \bar{\xi}\|_3 < \beta\), then

$$|\nabla \cdot (f_\xi(\nabla(\xi + \phi)) - f_\xi(\nabla(\bar{\xi} + \phi)))|_2 + |E(\xi) - E(\bar{\xi})|^{1 - \theta} < C(\beta)(\beta + \beta^{1 - \theta}).$$

Therefore, there exists a (sufficiently small) \(\beta \in (0, 1]\) independent of \(\xi\), such that

$$C(\beta)(\beta + \beta^{1 - \theta}) < \frac{\beta_1}{2M}. $$
For any $\xi \in H^4(\Omega)$ satisfying $\|\xi - \xi\|_3 < \beta$ (that is, for any $\varphi \in H^4(\Omega)$ satisfying $\|\varphi - \varphi\|_3 < \beta$), there are only two possibilities: either $\|\xi - \xi\|_4 < \beta_1$ and then (18) holds by using Step 1; or $\|\xi - \xi\|_4 > \beta_1$. In this latter case,

$$|w(\varphi)|_2 = |\Delta^2(\xi - \xi) - \nabla \cdot (f_\xi(\nabla(\xi + \phi)) - f_\xi(\nabla(\xi + \phi)))|_2$$

$$\geq \frac{1}{M} \|\xi - \xi\|_4 - |\nabla \cdot (f_\xi(\nabla(\xi + \phi)) - f_\xi(\nabla(\xi + \phi)))|_2$$

$$> \frac{\beta_1}{M} - \frac{\beta_1}{2M} = \frac{\beta_1}{2M} > |\mathcal{E}(\xi) - \mathcal{E}(\xi)|^{1-\theta} = |E_e(\xi) - E_e(\xi)|^{1-\theta},$$

and hence (18) holds.

\[ \blacksquare \]

**Remark 6** The Lojasiewicz-Simon inequality given in Lemma 5 has been formulated in a “strong sense”. However, other versions are also possible. For example, Theorem 2.1 of [7] for homogeneous Dirichlet conditions and the comments given in [14] for the non-homogeneous “strong sense”. However, other versions are also possible. For example, Theorem 2.1 of [7] for homogeneous Dirichlet conditions and the comments given in [14] for the non-homogeneous Dirichlet case show a “weak” version where, if $\|\varphi - \varphi\|_1 \leq \beta$, then $|E_e(\varphi) - E_e(\varphi)|^{1-\theta} \leq C\|w\|_2$ holds. Furthermore, an “intermediate” version has been applied in [12] for periodic boundary conditions, where $|E_e(\varphi) - E_e(\varphi)|^{1-\theta} \leq C\|w\|_1$ if $\|\varphi - \varphi\|_2 \leq \beta$.

### 3 The Smectic Model

**Definition 7** A pair $(u, \varphi)$ is said to be a global weak solution of (1)-(7) in $(0, +\infty)$ if

$$u \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2(0, +\infty; V), \quad w \in L^2(0, +\infty; L^2(\Omega)), \quad \varphi \in L^\infty(0, +\infty; H^2(\Omega)),$$

$$\nabla \cdot u = 0 \text{ in } Q, \quad u|_{\Sigma} = 0, \quad \varphi|_{\Sigma} = \varphi_1, \quad \partial_\nu \varphi|_{\Sigma} = \varphi_2,$$

$$u(0) = u_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega,$$

and it satisfies the variational formulation:

$$\langle \partial_t u, \bar{u} \rangle + \langle u \cdot \nabla u, \bar{u} \rangle + \langle \nabla u, \nabla \bar{u} \rangle - \langle w \nabla \varphi, \bar{u} \rangle = 0 \quad \forall \bar{u} \in V, \quad (21)$$

$$\langle \partial_t \varphi, \bar{w} \rangle + \langle u \cdot \nabla \varphi, \bar{w} \rangle + \langle w, \bar{w} \rangle = 0 \quad \forall \bar{w} \in L^2 \quad (22)$$

$$\langle \Delta \varphi, \Delta \bar{\varphi} \rangle - \langle \nabla \cdot f_\xi(\nabla \varphi), \bar{\varphi} \rangle - \langle w, \bar{\varphi} \rangle = 0 \quad \forall \bar{\varphi} \in H^2. \quad (23)$$

Moreover, from the weak regularity of $(\varphi, w)$ given in (20), (23) and (10), it can be deduced that $\varphi \in L^2_{\text{loc}}(0, +\infty; H^4)$ whenever $\varphi_1 \in H^{7/2}(\partial \Omega)$ and $\varphi_2 \in H^{5/2}(\partial \Omega)$, i.e. $\varphi \in L^2(0, T; H^4)$ for all $T > 0$. 

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Definition 8 A weak solution \((u, \varphi)\) is said to be a strong solution of (1)-(7) in \((0, +\infty)\) if
\[
\begin{align*}
  u &\in L^\infty(0, +\infty; H^1(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^2(\Omega)), \\
  \varphi &\in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^2(\Omega)), \\
  \partial_t u &\in L^2_{\text{loc}}(0, +\infty; L^2(\Omega)), \\
  \partial_t \varphi &\in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2_{\text{loc}}(0, +\infty; H^2(\Omega)),
\end{align*}
\] (24)
and it satisfies the fully differential system (1)-(3) point-wise in \((0, +\infty) \times \Omega\).

Moreover, for regular domains, one has
\[
\varphi \in L^\infty(0, +\infty; H^1) \cap L^2_{\text{loc}}(0, +\infty; H^6), \quad w \in L^\infty(0, +\infty; L^2) \cap L^2_{\text{loc}}(0, +\infty; H^2)
\]
whenever \(\varphi_1 \in H^{11/2}(\partial \Omega)\) and \(\varphi_2 \in H^{9/2}(\partial \Omega)\).

3.1 Energy Equality and Weak Estimates

If \((u, \varphi, w)\) is a regular enough solution of (1)-(4), (6), (7), then by taking \(\tilde{u} = u, \tilde{w} = w\) and \(\tilde{\varphi} = \partial_t \varphi\) as a test function in (21), (22) and (23) respectively, one has
\[
\begin{align*}
  \frac{1}{2} \frac{d}{dt} |\nabla u|^2_2 + |\nabla u|^2_2 - (w \nabla \varphi, u) &= 0, \\
  (\partial_t \varphi, w) + (u \cdot \nabla \varphi, w) + |w|^2_2 &= 0, \\
  \frac{d}{dt} \left( \frac{1}{2} |\Delta \varphi|^2_2 + \int_{\Omega} F_\varepsilon(\nabla \varphi) \right) - (w, \partial_t \varphi) &= 0.
\end{align*}
\]
Through adding these three equalities, the term \((w, \partial_t \varphi)\) is cancelled and the nonlinear convective term \((u \cdot \nabla \varphi, w)\) plus the elastic term \(- (w \nabla \varphi, u)\) also vanish, thereby yielding at the following energy equality:
\[
\frac{d}{dt} E(u(t), \varphi(t)) + |\nabla u|^2_2 + |w|^2_2 = 0.
\] (25)
This energy equality illustrates the dissipative character of the model with respect to the total free energy \(E(u, \varphi) = E_k(u) + E_e(\varphi)\), where \(E_k(u) = \frac{1}{2} \int_{\Omega} |u|^2\) is the kinetic energy and \(E_e(\varphi)\) is the elastic energy defined in (5). Moreover, assuming the initial estimate \(|u_0|^2 \leq C\) and \(\|\varphi_0\|^2_2 \leq C\), the following uniform bounds at the infinite time interval \((0, +\infty)\) hold:
\[
\begin{align*}
  u &\text{ in } L^\infty(0, +\infty; H) \cap L^2(0, +\infty; V), \\
  w &\text{ in } L^2(0, +\infty; L^2), \\
  \varphi &\text{ in } L^\infty(0, +\infty; H^2).
\end{align*}
\] (26)
In particular, from the bound of \(w\) in \(L^2(0, +\infty; L^2)\) and (10), one has the finite time bound
\[
\varphi \text{ in } L^2(0, T; H^4), \quad \forall T > 0.
\]
For instance, weak solutions furnished by a limit of Galerkin approximate solutions which satisfy the corresponding energy inequality (by replacing the equality \(= 0\) with the inequality \(\leq 0\) in (25)) can be obtained, which suffices to rigorously prove all previous estimates.
3.2 Strong Estimates

From (23) and (10), we have for each $t \in (0, +\infty)$:

$$\|\varphi(t)\|_4 \leq C(\|\varphi_1\|_{7/2,\partial\Omega} + \|\varphi_2\|_{5/2,\partial\Omega} + |w(t)|_2 + |\nabla \cdot f_\varepsilon(\nabla \varphi(t))|_2).$$  \hfill (27)

By using weak estimates $\|\varphi(t)\|_2 \leq C$ and

$$|\nabla \cdot f_\varepsilon(\nabla \varphi(t))|_2 \leq C|\nabla \cdot f_\varepsilon(\nabla \varphi(t))|_3 D^2 \varphi(t)|_6 \leq C\|\varphi(t)\|_3,$$  \hfill (28)

we obtain

$$\|\varphi(t)\|_3 \leq C\|\varphi(t)\|_{1/2}^{1/2}\|\varphi(t)\|_4^{1/2} \leq C(1 + |w(t)|_2^{1/2} + \|\varphi(t)\|_3^{1/2}).$$  \hfill (29)

Hence

$$\|\varphi(t)\|_3 \leq C(1 + |w(t)|_2^{1/2}).$$  \hfill (29)

On the other hand, from (3), it follows that

$$|w(t)|_2 \leq C(|\partial_t \varphi(t)|_2 + |u(t)|_3|\nabla \varphi(t)|_6) \leq C(|\partial_t \varphi(t)|_2 + \|u(t)\|_1^{1/2}).$$  \hfill (30)

Hence, from (29) and (30)

$$\|\varphi(t)\|_3 \leq C(1 + |\partial_t \varphi(t)|_2^{1/2} + \|u(t)\|_1^{1/4}).$$  \hfill (31)

By means of taking $-Au + \partial_t u$ as a test function in the $u$-system (1) ($A$ being the Stokes operator), and by applying H"{o}lder and Young’s inequalities and the interpolation inequality

$$\|\varphi\|_{W^{1,\infty}} \leq C\|\varphi\|_{2}^{1/2}\|\varphi\|_{3}^{1/2},$$

we attain:

$$\frac{d}{dt} |\nabla u|_2^2 + |Au|_2^2 + |\partial_t u|_2^2 \leq C\left(|(\nabla \varphi_w)|_2 + |(\nabla \varphi)|_2\right) |A u|_2 + |\partial_t u|_2$$

$$\leq C\left(\|u\|_6 \|\nabla u\|_3 + |\nabla \varphi|_6 \|w\|_2\right) (\|u\|_2 + |\partial_t u|_2)$$

$$\leq C\left(\|u\|_{1}^{3/2} \|u\|_2^{3/2} + \|u\|_1^{3/2} \|u\|_2^{1/2} |\partial_t u|_2 + \|\varphi\|_1^{1/2} \|\varphi\|_3^{1/2} \|w\|_2 (\|u\|_2 + |\partial_t u|_2)\right)$$

$$\leq \frac{1}{2} \|u\|_2^2 + \frac{1}{2} |\partial_t u|_2^2 + C \left(\|u\|_1^6 + |\nabla \varphi|_3 |w|_2^2\right).$$

Therefore, by using (30) and (31), we obtain

$$\frac{d}{dt} |u|_1^2 + \frac{1}{2} |u|_3^2 + \frac{1}{2} |\partial_t u|_2^2 \leq C\left(\|u\|_1^6 + (1 + |\partial_t \varphi|_2^{1/2} + \|u\|_1^{1/4}) (|\partial_t \varphi|_2^2 + \|u\|_1)\right).$$  \hfill (32)

On the other hand, by deriving the $w$-equation (3) and $\varphi$-equation (4) with respect to $t$, taking $\partial_t \varphi$ as a test function in both these derivations, adding, and taking into account that
By denoting $\mathbf{u} \cdot \nabla \partial_t \varphi, \partial_t \varphi = 0$ and also the term $(\partial_t w, \partial_t \varphi)$ is cancelled, we then have:

$$\frac{1}{2} \frac{d}{dt} |\partial_t \varphi|^2 + |\Delta \partial_t \varphi|^2 = -(\partial_t \mathbf{u} \cdot \nabla \varphi, \partial_t \varphi) + (\partial_t (\nabla \cdot \mathbf{f}_e (\nabla \varphi)), \partial_t \varphi)$$

\leq |\partial_t \mathbf{u}|_2 |\nabla \varphi|_6 |\partial_t \varphi|_3 + (|\nabla \cdot \mathbf{f}_e (\nabla \varphi)|_6 |\nabla^2 \partial_t \varphi|_2 + |\nabla^2 \mathbf{f}_e (\nabla \varphi)|_6 |\nabla^2 \varphi|_2 |\partial_t \nabla \varphi|_6)

\leq C(|\partial_t \mathbf{u}|_2 |\partial_t \varphi|_2^3/2 |\partial_t \varphi|_1^{1/2} + |\partial_t \varphi|_2 |\partial_t \varphi|_1 + |\partial_t \varphi|_2^3/2 |\partial_t \varphi|_2^{1/2})$

\leq \frac{1}{8} |\partial_t \mathbf{u}|_2^2 + \frac{1}{2} |\partial_t \varphi|_2^2 + C |\partial_t \varphi|_2^2. \quad (33)

where (28) and $|\partial_t \varphi|_2 = |\Delta \partial_t \varphi|_2$ have been applied (because $\partial_t \varphi|_{\partial \Omega} = 0$). Therefore, from (33)

$$\frac{d}{dt} |\partial_t \varphi|^2 + |\partial_t \varphi|^2 \leq \frac{1}{4} |\partial_t \mathbf{u}|_2^2 + C |\partial_t \varphi|^2. \quad (34)$$

From the addition of (32) and (34), it follows that:

$$\frac{d}{dt} (\|\mathbf{u}\|_1^2 + |\partial_t \varphi|_2^2) + \frac{1}{2} \|\mathbf{u}\|_2^2 + \frac{1}{4} |\partial_t \mathbf{u}|_2^2 + |\partial_t \varphi|^2 \leq C \left( \|\mathbf{u}\|_1^6 + (1 + |\partial_t \varphi|_2^{1/2} + \|\mathbf{u}\|_1^{1/4})(|\partial_t \varphi|_2^2 + \|\mathbf{u}\|_1) \right). \quad (35)$$

By denoting

$$\Phi(t) := \|\mathbf{u}\|_1^2 + |\partial_t \varphi|_2^2, \quad \Psi(t) := \frac{1}{2} \|\mathbf{u}\|_2^2 + \frac{1}{4} |\partial_t \mathbf{u}|_2^2 + |\partial_t \varphi|^2,$$

then (35) can be rewritten as

$$\Phi' + \Psi \leq C(\Phi^3 + \Phi + \Phi^{1/2} + \Phi^{5/4} + \Phi^{3/4} + \Phi^{9/8}) \leq C(\Phi^3 + 1). \quad (36)$$

Observe that $\Phi \in L^1(0, +\infty)$ since $|\partial_t \varphi|_2 \in L^2(0, +\infty)$. Indeed, from the $w$-equation (3):

$$|\partial_t \varphi|_2 \leq C \left( |w|_2 + \|\mathbf{u}\|_1 \|\nabla \varphi|_1 \right) \leq C \left( |w|_2 + \|\mathbf{u}\|_1 \right),$$

and $|w|_2 + \|\mathbf{u}\|_1 \in L^2(0, +\infty)$.

Therefore, the entire hypothesis of Theorem 2 holds, then there exists a sufficiently large $T_{reg}^* \geq 0$ such that the following (regular) estimates hold in $(T_{reg}^*, +\infty)$:

$$\mathbf{u} \in L^\infty(T_{reg}^*, +\infty; H^1), \quad \partial_t \varphi \in L^\infty(T_{reg}^*, +\infty; L^2).$$

By integrating (36) in $[0, t]$ for all $t > 0$, the following local (regular) estimates in $(T_{reg}^*, +\infty)$ are obtained:

$$\mathbf{u} \in L^2_{loc}(T_{reg}^*, +\infty; H^2), \quad \partial_t \mathbf{u} \in L^2_{loc}(T_{reg}^*, +\infty; L^2), \quad \partial_t \varphi \in L^2_{loc}(T_{reg}^*, +\infty; H^2).$$

By using the $w$-equation (3), one has, for each $t \in (0, +\infty)$:

$$|w(t)|_2 \leq C(|\partial_t \varphi(t)|_2 + \|\mathbf{u}(t)\|_1), \quad (37)$$
hence

\[ w \in L^\infty(T_{\text{reg}}^*, +\infty; L^2) \]

and from (29),

\[ \varphi \in L^\infty(T_{\text{reg}}^*, +\infty; H^3). \]

Furthermore, from (3), we have

\[ \| w(t) \|_2 \leq C(\| \partial_t \varphi(t) \|_2 + \| u(t) \|_2 \| \varphi(t) \|_3), \]

hence

\[ w \in L^2_{\text{loc}}(T_{\text{reg}}^*, +\infty; H^2). \]

Observe that, through combining (3) and (4), \( \varphi(t) \) is the solution of the bilaplacian problem

\[
\begin{cases}
\Delta^2 \varphi = \nabla \cdot f_\varepsilon(\nabla \varphi) - w & \text{in } \Omega, \\
\varphi |_{\partial \Omega} = \varphi_1, \quad \partial_n \varphi |_{\partial \Omega} = \varphi_2 & \text{on } \partial \Omega.
\end{cases}
\]

By means of using the \( H^4 \) and \( H^6 \) regularity of this problem and bounding the right-hand-side terms, and from the weak regularity and the strong regularity of \( \varphi \) and \( w \) previously proved, we have

\[ \varphi \in L^\infty(T_{\text{reg}}^*, +\infty; H^4) \cap L^2_{\text{loc}}(T_{\text{reg}}^*, +\infty; H^6). \]

### 3.3 Existence of global weak solutions with long-time strong regularity

The existence of solutions of (1)-(7) can be justified by the Galerkin Method [3]. Given some fixed regular basis \( \{w^j\} \) and \( \{\phi^j\} \) of the spaces \( V \) and \( H^2_0(\Omega) \), respectively, let \( V^m \) and \( W^m \) be the finite-dimensional subspaces spanned by

\[ \{w^1, \ldots, w^m\} \quad \text{and} \quad \{\phi^1, \ldots, \phi^m\} \]

respectively. Given \( u_0 \in H \) and \( \varphi_0 \in H^2 \), for each \( m \geq 1 \), we seek an approximate solution \( (u_m, \varphi_m) \), such that \( u_m : [0, T] \mapsto V^m \) and \( \varphi_m = \tilde{\varphi} + \tilde{\varphi}_m \), where \( \tilde{\varphi} \) is an adequate lifting function of the boundary data \( \varphi_1, \varphi_2 \) and \( \tilde{\varphi}_m : [0, T] \mapsto W^m \), which satisfies the following variational formulation a.e. \( t \in (0, T) \):

\[
\begin{cases}
(\partial_t u_m(t), v_m) + (u_m(t) \cdot \nabla) u_m(t), v_m) + \nu(\nabla u_m(t), \nabla v_m)
\end{cases}
\]

\[ -(w_m(t) \nabla \varphi_m(t), v_m) = 0 \quad \forall v_m \in V^m, \]

\[
\begin{cases}
(\partial_t \varphi_m(t), e_m) + (u_m(t) \cdot \nabla \varphi_m(t), e_m) + (w_m(t), e_m)
\end{cases}
\]

\[ = (\partial_t \varphi_m(t), e_m), \quad \forall e_m \in W^m, \]

\[ u_m(0) = u_{0m} = P_m(u_0), \quad \varphi_m(0) = \varphi_{0m} = Q_m(\varphi_0) \quad \text{in } \Omega. \]
Here, $P_m : H \rightarrow V^m$ denotes the projection from $H$ onto $V^m$; $Q_m : L^2 \rightarrow W^m$ the projection from $L^2$ onto $W^m$; and the Euler-Lagrange equation $\Delta^2 \varphi_m - \nabla \cdot f(\nabla \varphi_m)$ has been projected into $W^m$ by taking

$$w_m := Q_m(\Delta^2 \varphi_m - \nabla \cdot f(\nabla \varphi_m)).$$

In particular, $u_{0m} \rightarrow u_0$ in $L^2$ and $\varphi_{0m} \rightarrow \varphi_0$ in $H^2$ (as $m \rightarrow 0$). If we write

$$u_m(t) = \sum_{i=1}^m \xi_{i,m}(t) w^i \quad \text{and} \quad \varphi_m(t) = \sum_{j=1}^m \zeta_{j,m}(t) \phi^j,$$

then (38) can be rewritten as a first-order ordinary differential system (in normal form), associated to the unknowns $(\xi_{i,m}(t), \zeta_{j,m}(t))$. By proceeding in an analogous way to [10] and [3] (local existence, a priori estimates, and tending towards the limit where the nonlinear terms are controlled by compactness), the existence of weak solutions $(u, \varphi)$ of (1)-(7) in $(0, +\infty)$ can be proved, which are also strong solutions (and unique) in $(T_{reg}^*, +\infty)$ for a sufficiently long-time $T_{reg}^* \geq 0$. Observe that $T_{reg}^*$ can be obtained by applying Theorem 2 to

$$\Phi_m(t) = \|u_m^m\|^2_1 + |\partial_t \varphi_m|^2_2,$$

and by taking into account that $T^*$ given in Theorem 2 is independent of $m$.

**Remark 9** The differential inequality (36) has been obtained with $\Phi$ depending on $u$ and $\partial_t \varphi$. Another possibility could be to deduce a similar differential inequality for a $\Phi$ depending on $u$ and $w$ (instead of for $\partial_t \varphi$). To this end, the computations could be: take $\partial_t w$ as a test function in the $w$-equation (3), derive the $\varphi$-equation (4) with respect to $t$ and take $\partial_t \varphi$ as a test function. Adding both equalities to (32) the term $(\partial_t \varphi, \partial_t w)$ is cancelled, thereby arriving at the following inequality instead of (33):

$$\frac{1}{2} \frac{d}{dt} |w|^2_2 + |\partial_t \varphi|^2_2 = -(u \cdot \nabla \varphi, \partial_t w) + (\partial_t f(\nabla \varphi), \partial_t \nabla \varphi). \quad (39)$$

Nevertheless, we do not know how to estimate the convective term $(u \cdot \nabla \varphi, \partial_t w)$ in order to deduce a differential inequality such as in (36).

### 3.4 Convergence at infinite time

We recall the definition of the elastic energy:

$$E_e(\varphi(t)) = \int_{\Omega} \left( \frac{1}{2} |\Delta \varphi(t)|^2 + F(\nabla \varphi(t)) \right)$$

and the kinetic and total energy is also defined as:

$$E_k(u(t)) = \frac{1}{2} \int_{\Omega} |u(t)|^2, \quad E(u(t), \varphi(t)) = E_k(u(t)) + E_e(\varphi(t)).$$

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Theorem 10 Assume that \((u_0, \varphi_0) \in H \times H^2\). Let \((u(t), \varphi(t), w(t))\) be a weak solution of (1)-(7) in \((0, +\infty)\) which is a strong solution in \((T_{reg}^*, +\infty)\) for some \(T_{reg}^* > 0\), then there exists a number \(E_\infty \geq 0\) such that the total energy satisfies

\[
E(u(t), \varphi(t)) \downarrow E_\infty \text{ in } \mathbb{R} \quad \text{as } t \uparrow +\infty.
\]  

Moreover, the following convergences hold:

\[
u(t) \rightarrow 0 \text{ in } H_0^1 \quad \text{and} \quad w(t) \rightarrow 0 \text{ in } L^2 \quad \text{as } t \uparrow +\infty.
\]

Proof. The (decreasing) convergence of the energy given in (40) is easy to deduce from energy equality (25) (observe (12)). By applying Lemma 1 for \(\Phi(t) := \|u\|^2 + |\partial_t \varphi|^2\), we obtain \(u(t) \rightarrow 0 \text{ in } H_0^1\) and \(\partial_t \varphi(t) \rightarrow 0 \text{ in } L^2\). Finally, from (37), \(w(t) \rightarrow 0 \text{ in } L^2\) holds.

Let \(S\) be the set of equilibrium points of (1)-(4):

\[
S = \{((\varphi, \varphi_0) : \varphi \in H^1(\Omega), \Delta^2 \varphi - \nabla \cdot f_\varepsilon(\nabla \varphi) = 0, \varphi|_{\partial \Omega} = \varphi_1, \partial_n \varphi|_{\partial \Omega} = \varphi_2\}.
\]

On the other hand, the \(\omega\)-limit set of a global weak solution, \((u, \varphi)\), associated to the initial data, \((u_0, \varphi_0) \in H \times H^2\), is defined as follows:

\[
\omega(u_0, \varphi_0) = \{(u_\infty, \varphi_\infty) \in V \times H^1 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (u(t_n), \varphi(t_n)) \rightarrow (u_\infty, \varphi_\infty) \text{ in } H^1 \times H^1\}.
\]

Theorem 11 Under the assumptions of Theorem 10, \(\omega(u_0, \varphi_0)\) is non-empty and \(\omega(u_0, \varphi_0) \subset S\). Moreover, for any \((0, \varphi) \in S\) such that \((0, \varphi) \in \omega(u_0, \varphi_0)\), then \(E_\varepsilon(\varphi) = E_\infty\) holds.

Proof. The proof is divided into two steps.

Step 1: It can been seen that \(\omega(u_0, \varphi_0) \neq \emptyset\) and \(\omega(u_0, \varphi_0) \subset S\).

From weak estimates, \((u, \varphi) \in L^\infty(0, +\infty; H \times H^2)\), hence there exists \(\{t_n\} \uparrow +\infty\) and \((u_\infty, \varphi_\infty) \in H \times H^2\) such that \((u(t_n), \varphi(t_n)) \rightarrow (u_\infty, \varphi_\infty)\) weakly in \(H \times H^2\). From (41), \(u_\infty = 0\) and \(u(t_n) \rightarrow 0 \text{ in } H_0^1\). On the other hand, \(\varphi_\infty\) will be a weak solution of the equilibrium equation \(\Delta^2 \varphi_\infty - \nabla \cdot f_\varepsilon(\nabla \varphi_\infty) = 0\). Indeed, since \(\nabla \varphi(t_n) \rightarrow \nabla \varphi_\infty\) a.e. in \(\Omega\), then

\[
f_\varepsilon(\nabla \varphi(t_n)) \rightarrow f_\varepsilon(\nabla \varphi_\infty) \text{ a.e. in } \Omega
\]

and, by using the weak estimate \(\|\varphi(t_n)\|_2 \leq C\), then

\[
|\nabla \cdot f_\varepsilon(\nabla \varphi(t_n))|_{6/5} \leq C(|\nabla \varphi(t_n)|^2_6 + 1)|D^2 \varphi(t_n)|_2 \leq C(\|\varphi(t_n)\|_2^2 + 1)\|\varphi(t_n)\|_2 \leq C,
\]

hence

\[
\nabla \cdot f_\varepsilon(\nabla \varphi(t_n)) \rightarrow \nabla \cdot f_\varepsilon(\nabla \varphi_\infty) \text{ weakly in } L^{6/5}(\Omega).
\]
By taking into account that $\varphi(t_n) \to \varphi_\infty$ weakly in $H^2$ and $w(t) \to 0$ (strongly) in $L^2$ as $t \to +\infty$, it suffices to take limits in (23) as $\{t_n\} \uparrow +\infty$ to illustrate that $\varphi_\infty$ is a weak solution of the equilibrium equation

$$\Delta^2 \varphi_\infty - \nabla \cdot f_\varepsilon(\nabla \varphi_\infty) = 0. \quad (42)$$

This step finishes by proving the convergence $\varphi(t_n) \to \varphi_\infty$ in $H^4$. Indeed, from (4), (10) and (23), it is now that

$$\|\varphi(t_n)\|_4 \leq C(\|\Delta^2 \varphi(t_n)\|_2 + 1) \leq C(\|\nabla \cdot f_\varepsilon(\nabla \varphi(t_n))\|_2 + |w(t_n)|_2 + 1). \quad (43)$$

On the other hand, by using the interpolation inequalities $|\nabla \varphi|_\infty \leq \|\varphi\|_2^{1/2}\|\varphi\|_3^{1/2}$ and $\|\varphi\|_3 \leq \|\varphi\|_2^{1/2}\|\varphi\|_4^{1/2}$, and the weak estimate $\|\varphi(t_n)\|_2 \leq C$, we obtain

$$|\nabla \cdot f_\varepsilon(\nabla \varphi(t_n))|_2 \leq C(\|\varphi(t_n)\|_2\|\varphi(t_n)\|_3 + 1)\|\varphi(t_n)\|_2 \leq C(\|\varphi(t_n)\|_4^{1/2} + 1) \leq C(\|\varphi(t_n)\|_4 + C/\delta).$$

The application of the latter inequality for a sufficiently small $\delta > 0$ in (43) yields

$$\|\varphi(t_n)\|_4 \leq C. \quad (44)$$

Moreover, from the weak estimates and (44), it is easy to attain the bound

$$\|\nabla \cdot f_\varepsilon(\nabla \varphi(t_n))\|_1 \leq C.$$ 

By compactness, $\nabla \cdot f_\varepsilon(\nabla \varphi(t_n))$ converges strongly in $L^2(\Omega)$, for at least an equally labelled subsequence. Therefore, by again using (23), $\Delta^2 \varphi(t_n) \to \Delta^2 \varphi(t_n)$ converges strongly in $L^2(\Omega)$, and hence $\varphi(t_n) \to \varphi_\infty$ converges strongly in $H^4(\Omega)$.

**Step 2:** If $(0, \varphi) \in \omega(u_0, \varphi_0)$ then $E(0, \varphi) = E_\varepsilon(\varphi) = E_\infty$ ($E_\infty$ given in Theorem 10).

From the definition of $\omega(u_0, \varphi_0)$, there exists $\{t_n\} \uparrow +\infty$ such that $(u(t_n), \varphi(t_n)) \to (0, \varphi)$ in $H^1 \times H^4$ as $n \uparrow +\infty$. In particular,

$$\lim_{n \to +\infty} E(u(t_n), \varphi(t_n)) = E_\varepsilon(\varphi).$$

Finally, from (40) and the uniqueness of the limit, one has $E_\varepsilon(\varphi) = E_\infty$. $$

Although the set of critical points $\varphi$ (with the same elastic energy) might even be a continuum of functions, the uniqueness of limit of the whole trajectory of $\varphi(t)$ can be deduced.

**Theorem 12** Under the hypotheses of Theorem 11, there exists $\varphi \in H^4$ such that $\varphi(t) \to \varphi$ in $H^1$ as $t \uparrow +\infty$, i.e. $\omega(u_0, \varphi_0) = \{(0, \varphi)\}$.

**Proof.** Let $(0, \varphi) \in \omega(u_0, \varphi_0) \subset S$, i.e., there exists $t_n \uparrow +\infty$ such that $u(t_n) \to 0$ in $H^1$ and $\varphi(t_n) \to \varphi$ in $H^4$. 

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Without any loss of generality, it can be assumed that \( E(u(t), \varphi(t)) > E(0, \varphi) (= E_\infty) \) for all \( t \), because otherwise, if there some \( \tilde{t} > 0 \) exists such that \( E(u(\tilde{t}), \varphi(\tilde{t})) = E(0, \varphi) \), then, from the energy equality (25) for each \( t \geq \tilde{t} \),

\[
E(u(t), \varphi(t)) = E(0, \varphi), \quad |\nabla u(t)|_2^2 = 0 \quad \text{and} \quad |w(t)|_2^2 = 0.
\]

Therefore, \( u(t) = 0 \) and \( w(t) = 0 \). In particular, by using the \( w \)-equation, then \( \partial_t \varphi(t) = 0 \), and hence \( \varphi(t) = \varphi \) for each \( t \geq \tilde{t} \). In this situation the convergence of the \( \varphi \)-trajectory is trivial.

The proof is now divided into three steps.

**Step 1:** Assuming there exists \( t_* > T_{\text{reg}}^* \) such that

\[
\|\varphi(t) - \varphi\|_3 \leq \beta \quad \text{and} \quad |u(t)|_2 \leq 1 \quad \forall t \geq t_*
\]

where the solution is strong in \((T_{\text{reg}}^*, +\infty)\) and \( \beta > 0 \) is the constant appearing in Lemma 5 (of Lojasiewicz-Simon’s type), then the following inequalities hold:

\[
\frac{d}{dt} \left((E(u(t), \varphi(t)) - E(0, \varphi))^{\theta}\right) + C \theta \left(|\nabla u(t)|_2 + |w(t)|_2\right) \leq 0, \quad \forall t \geq t_* \tag{45}
\]

\[
\int_{t_0}^{t_1} |\partial_t \varphi|_2 \leq \frac{C}{\theta} \left(E(u(t_0), \varphi(t_0)) - E(0, \varphi)\right)^{\theta}, \quad \forall t_1 > t_0 \geq t_*, \tag{46}
\]

where \( \theta \in (0, 1/2] \) is the constant appearing in Lemma 5.

Indeed, the energy equality (25) can be written as

\[
\frac{d}{dt} \left(E(u(t), \varphi(t)) - E_\infty\right) + C \left(|\nabla u(t)|_2^2 + |w(t)|_2^2\right) = 0.
\]

Therefore, by taking the time derivative of the (strictly positive) function

\[
H(t) := (E(u(t), \varphi(t)) - E_\infty)^{\theta} > 0,
\]

we obtain

\[
\frac{dH(t)}{dt} + \theta(E(u(t), \varphi(t)) - E_\infty)^{\theta-1} C(|\nabla u(t)|_2^2 + |w(t)|_2^2) = 0. \tag{47}
\]

On the other hand, by recalling that the unique critical point of the kinetic energy is \( u = 0 \), and by taking into account that \( |E_k(u) - E_k(0)| = \frac{1}{2}|u_2|^2_2 \) and since \( 2(1-\theta) > 1 \) and \( |u(t)|_2 \leq 1 \), then

\[
|E_k(u(t)) - E_k(0)|^{1-\theta} = \frac{1}{2^{1-\theta}} |u(t)|_2^{2(1-\theta)} \leq C|u(t)|_2 \quad \forall t \geq t_*.
\]

Therefore, by using the Lojasiewicz-Simon inequality (given in Lemma 5):

\[
(E(u(t), \varphi(t)) - E_\infty)^{1-\theta} \leq |E_k(u(t)) - E_k(0)|^{1-\theta} + |E_\varphi(\varphi(t)) - E_\varphi(\varphi)|^{1-\theta} \leq C(|u(t)|_2 + |w(t)|_2),
\]

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and hence, by using the Poincare inequality:

\[(E(u(t), \varphi(t)) - E_\infty)^{\theta - 1} \geq C(|\nabla u(t)|_2 + |w(t)|_2)^{-1} \quad \forall t \geq t_*\]  \hspace{1cm} (48)

From (47) and (48), we obtain

\[\frac{dH(t)}{dt} + \theta C(|\nabla u(t)|_2 + |w(t)|_2) \leq 0, \quad \forall t \geq t_*\]

and (45) is proved. Integrating (45) into \([t_0, t_1]\) (for any \(t_1 > t_0 \geq t_*\)) yields

\[(E(u(t_1), \varphi(t_1)) - E_\infty)^{\theta} + \theta C \int_{t_0}^{t_1} (|\nabla u(t)|_2 + |w(t)|_2) dt \leq (E(u(t_0), \varphi(t_0)) - E_\infty)^{\theta}. \hspace{1cm} (49)\]

On the other hand, since \(\partial_t \varphi + \nabla \cdot (u \otimes \varphi) - w = 0\), then, by using the weak estimate \(|\varphi(t)|_2 \leq C\), it can be deduced that

\[|\partial_t \varphi|_2 \leq C(|u \otimes \varphi|_1 + |w|_2) \leq C(|\nabla u|_2 + |w|_2)\]

By applying this inequality in (49), we obtain (46).

**Step 2:** There exists a sufficiently large \(n_0\) such that \(t_{n_0} \geq T_{reg}^*\) and \(|\varphi(t) - \overline{\varphi}|_3 \leq \beta\) and \(|u(t)|_2 \leq 1\) for all \(t \geq t_{n_0}\).

The bound \(|u(t)|_2 \leq 1\) is based on \(u(t) \to 0\) in \(H_0^1\) given in (41). We now focus on the bound for \(|\varphi(t) - \overline{\varphi}|_3\). Since \(\varphi(t_n) \to \overline{\varphi}\) in \(H^4\) and \(E(u(t_n), \varphi(t_n)) \to E_\infty = E_\epsilon(\overline{\varphi})\), then for any \(\epsilon \in (0, \beta)\), there exists an integer \(N(\epsilon)\) such that, for all \(n \geq N(\epsilon)\),

\[|\varphi(t_n) - \overline{\varphi}|_3 \leq \epsilon \quad \text{and} \quad \frac{1}{\theta}(E_\epsilon(u(t_n), \varphi(t_n)) - E_\infty)^{\theta} \leq \epsilon \hspace{1cm} (50)\]

For each \(n \geq N(\epsilon)\), we define

\[\tilde{t}_n := \sup\{t : t > t_n, |\varphi(s) - \overline{\varphi}|_3 < \beta \quad \forall s \in [t_n, t]\}.\]

It suffices to prove that \(\tilde{t}_{n_0} = +\infty\) for some \(n_0\). Assume by contradiction that \(t_n < \tilde{t}_n < +\infty\) for all \(n\). Observe that \(|\varphi(\tilde{t}_n) - \overline{\varphi}|_3 = \beta\) and \(|\varphi(t) - \overline{\varphi}|_3 < \beta\) for all \(t \in [t_n, \tilde{t}_n]\). From Step 1, for all \(t \in [t_n, \tilde{t}_n]\), from (46) and (50) we obtain

\[\int_{t_n}^{\tilde{t}_n} |\partial_t \varphi|_2 \leq C\epsilon, \quad \forall n \geq N(\epsilon).\]

Therefore,

\[|\varphi(\tilde{t}_n) - \overline{\varphi}|_2 \leq |\varphi(t_n) - \overline{\varphi}|_2 + \int_{t_n}^{\tilde{t}_n} |\partial_t \varphi|_2 \leq (1 + C)\epsilon,

which implies that \(\lim_{n \to +\infty} |\varphi(\tilde{t}_n) - \overline{\varphi}|_2 = 0\). Since \(\varphi\) is bounded in \(L^\infty(t^*, +\infty; H^4)\), \((\varphi(t))_{t \geq t^*}\) is relatively compact in \(H^3\). Therefore, there exists a subsequence of \(\varphi(\tilde{t}_n)\),
which is still denoted as $\varphi(t_n)$, that converges to $\varphi$ in $H^3$. Hence, for a sufficiently large $n$, $\|\varphi(t_n) - \varphi\|_3 < \beta$, which contradicts the definition of $t_n$.

**Step 3:** There exists a unique $\varphi$ such that $\varphi(t) \to \varphi$ in $H^4$ as $t \uparrow +\infty$.

By using Steps 1 and 2, from (46) it is deduced that, for all $t_1 > t_0 \geq t_{n_0}$,

$$|\varphi(t_1) - \varphi(t_0)|_2 \leq \int_{t_0}^{t_1} |\partial_t \varphi|_2 \to 0, \text{ as } t_0, t_1 \to +\infty.$$ 

Therefore, $(\varphi(t))_{t \geq t_{n_0}}$ is a Cauchy sequence in $L^2$ as $t \uparrow +\infty$, and hence the $L^2$-convergence of the whole trajectory is deduced, i.e. there exists a unique $\varphi \in L^2$ such that $\varphi(t) \to \varphi$ in $L^2$ as $t \uparrow +\infty$. Finally, the strong $H^4$-convergence by sequences of $\varphi(t)$ proved in Step 1 of Theorem 11, yields $\varphi(t) \to \varphi$ in $H^4$.

References


