REGULAR TIME-REPRODUCTIVE SOLUTIONS FOR GENERALIZED BOUSSINESQ MODEL WITH NEUMANN BOUNDARY CONDITIONS FOR TEMPERATURE

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Abstract. The aim of this work is to prove existence of regular time reproductive solutions for a generalized Boussinesq model (with nonlinear diffusion for velocity and temperature). The main idea is to obtain higher regularity (of $H^3$ type) for temperature than for velocity (of $H^2$ type), using specifically the Neumann boundary condition for temperature.

1. Introduction

Assume a bounded, regular open set $\Omega$ in $\mathbb{R}^N$ ($N = 2$ or 3). This paper is concerned with some equations governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are explicit functions depending on temperature. The involved equations are

\begin{align*}
\begin{cases}
\partial_t u - \nabla \cdot (\nu(\theta) \nabla u) + (u \cdot \nabla) u - \alpha g \theta + \nabla p = f, \\
\nabla \cdot u = 0, \\
\partial_t \theta - \nabla \cdot (k(\theta) \nabla \theta) + (u \cdot \nabla) \theta = 0,
\end{cases}
\end{align*}

in $\Omega \times [0, +\infty)$, where

- $u(x, t) \in \mathbb{R}^N$ denotes the velocity of the fluid at point $x \in \Omega$ and time $t \in [0, +\infty)$.
- $p(x, t) \in \mathbb{R}$ is the (hydrostatic) pressure.
- $\theta(x, t) \in \mathbb{R}$ is the temperature.
- $g(x, t) \in \mathbb{R}^N$ denotes the gravitational field and $\alpha > 0$ is a constant associated to the coefficient of volume expansion.
- $f(x, t) \in \mathbb{R}^N$ denotes the resulting of external forces.
- $\nu(\cdot) : \mathbb{R} \to \mathbb{R}$ is the kinematic viscosity.
- $k(\cdot) : \mathbb{R} \to \mathbb{R}$ is the thermal conductivity.

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We will search for a triplet \( \{ u, p, \theta \} \) regular reproductive solution of (1) in \( \Omega \times [0, \infty) \), together the follows Dirichlet-Neumann boundary conditions:

\[
\begin{align*}
  u &= 0, \\
  \partial_p \theta &= 0
\end{align*}
\]

on \( [0, \infty) \times \partial \Omega \),

and the time reproductive condition:

\[
  u(0) = u(T), \quad \theta(0) = \theta(T) \quad \text{in} \quad \Omega.
\]

Existence and uniqueness of the initial value problem related to (1) and with Dirichlet’s boundary conditions for velocity and temperature, was proved by Lorca and Boldrini in [5]. The stationary problem is studied in [6] for bounded domains and in [10] for exterior domains. On the other hand, A.C. Moretti, M.A. Rojas-Medar and M.D. Rojas-Medar [8] proved existence of reproductive weak solutions in exterior domains. The classical Boussinesq model, where \( \nu \) and \( k \) are positives constants, has been analyzed in great extent, see for instance, Morimoto [9], Ōeda [11].

The arguments used in [5] in order to obtain regular solution (and uniqueness) are not valid to find reproductivity since the initial conditions play a fundamental role. Our contribution in this paper is to obtain higher order estimates for the temperature than in [5]; namely in [5] \( H^2(\Omega) \) regularity is obtained for velocity and temperature, but now we will arrive at \( H^3(\Omega) \) regularity for the temperature. Consequently, a reproductive condition for time derivative of temperature also holds, i.e. \( \partial_t \theta(0) = \partial_t \theta(T) \). In addition, the arguments used in this paper are remarkably simpler than the used ones in [5]. By the contrary, now the regularity obtained for the solution is not sufficient to prove uniqueness.

**Notation.**

- In general, the notation will be abridged. We set \( L^p = L^p(\Omega), \ p \geq 1, \ H^k = H^k_0(\Omega), \) etc. If \( X = X(\Omega) \) is a space of functions defined in the open set \( \Omega \), we denote by \( L^p(X) \) the Banach space \( L^p(0, T; X) \). Also, boldface letters will be used for vectorial spaces, for instance \( L^2 = L^2(\Omega)^N \).
- The \( L^p \) norm is denoted by \( \| \cdot \|_p, 1 \leq p \leq \infty \). The \( H^m \) norm is denoted by \( \| \cdot \|_m \).
- We set \( V \) the space formed by all fields \( v \in C_0^\infty(\Omega)^N \) satisfying \( \nabla \cdot v = 0 \). We denote \( H \) (respectively \( V \)) the closure of \( V \) in \( L^2 \) (respectively \( H^1 \)). \( H \) and \( V \) are Hilbert spaces for the norms \( \| \cdot \|_2 \) and \( \| \cdot \|_1 \), respectively. Furthermore,
  \[
  H = \{ u \in L^2; \nabla \cdot u = 0, \ u \cdot n = 0 \text{ on } \partial \Omega \}, \\
  V = \{ u \in H^1; \nabla \cdot u = 0, \ u = 0 \text{ on } \partial \Omega \}
  \]
- It is easy deduce that \( \frac{d}{dt} \int_\Omega \theta(x, t) = 0 \) from the convection-diffusion equation for \( \theta \). Then, we can fix \( \int_\Omega \theta = 0 \). Therefore, let us consider the following spaces
  \[
  H^k_N = \left\{ \theta \in H^k; \frac{\partial \theta}{\partial n} = 0 \text{ on } \partial \Omega, \int_\Omega \theta = 0 \right\}
  \]
  where \( k = 2, 3 \). Hence, \( H^k_N \) is a closed subspace of \( H^k \). Consequently \( |\Delta \theta|_2 \) is equivalent to \( \| \theta \|_2 \) in \( H^2_N \), and \( |\nabla \Delta \theta|_2 \) is equivalent to \( \| \theta \|_3 \) in \( H^3_N \) ([12]).
Some interpolation inequalities. We will use the following classical interpolation and Sobolev inequalities (for 3D domains):

\[ |v|_6 \leq C|v|_1, \quad |v|_3 \leq |v|_2^{1/2}|v|_1^{1/2} \]

and

\[ |v|_\infty \leq C|v|_1^{1/2}|v|_2^{1/2} \]

In this work, it will be useful to use the following result (see [5]):

**Lemma 1.** Let \( u \in V \cap H^2 \) and consider the Helmholtz decomposition of \( -\Delta u \), i.e.,

\[ -\Delta u = Au + \nabla q, \]

where \( q \in H^1 \) is taken such that \( \int_{\Omega} q \, dx = 0 \) and \( A \) is the Stokes operator. Then,

\[ \|q\|_1 \leq C|Au|_2. \]

Moreover, for every \( \delta > 0 \) there exists a positive constant \( C_\delta \) (independent of \( u \)) such that

\[ |q|_2 \leq C_\delta |\nabla u|_2 + \delta|Au|_2. \]

2. **The Main Result**

**Definition 1.** It will be said that \( (u, p, \theta) \) is a regular solution of (1)–(3) in \( (0, T) \), if

\[ u \in L^2(H^2) \cap L^\infty(H^1) \quad \text{and} \quad \partial_t u \in L^2(L^2), \]

\[ p \in L^2(H^1), \]

\[ \theta \in L^2(H^2_N) \cap L^\infty(H^1_N) \quad \text{and} \quad \partial_t \theta \in L^2(H^1_N), \]

satisfying (1) a.e. in \( (0, T) \times \Omega \), boundary conditions (2) and time reproductivity conditions (3) in the sense of spaces \( V \) and \( H^2_N \) respectively.

Notice that we have imposed higher regularity for \( \theta \) than for \( u \).

**Theorem 1.** Let \( T > 0 \) and \( \Omega \) a bounded domain in \( \mathbb{R}^N \) \( (N = 2 \text{ or } 3) \) with a boundary of class \( C^{2,1} \). Let the functions \( \nu \in C^1(\mathbb{R}) \) and \( k \in C^2(\mathbb{R}) \) such that

\[ 0 < \nu_{\min} \leq \nu(s) \leq \nu_{\max} \quad 0 < k_{\min} \leq k(s) \leq k_{\max} \quad \text{in } \mathbb{R}, \]

and \( \nu', k', k'' \) are bounded in \( \mathbb{R} \) (i.e., \( |\nu'(s)| \leq \nu'_{\max}, |k'(s)| \leq k'_{\max}, |k''(s)| \leq k''_{\max} \)).

Assume that \( f \in L^2(L^2) \) and \( g \in L^\infty(L^2) \) and

\[ ||f||_{L^2(L^2)} \leq \delta \]

for \( \delta \) small enough, then there exists a regular (and small) reproductive solution of (1)–(3) in \( (0, T) \). Moreover, this solution also verifies \( \partial_t \theta(0) = \partial_t \theta(T) \).

**Remark:** The uniqueness of solutions furnished by previous Theorem remains open, because higher regularity for the velocity is necessary. To obtain \( H^3 \) regularity for the velocity seem complicated because the argument made in the proof of Lemma 3 in order to get \( H^3 \) regularity is based in the Neumann condition, but we have Dirichlet condition for \( u \).

The proof of this Theorem will be given in Section 5. The method is based on the Galerkin approximation with spectral basis (defined in Section 3) and some differential inequalities in regular norms given in Section 4.
3. The Galerkin Initial-Boundary Problem

Let \( \{\phi_i\}_{i \geq 1} \) and \( \{\varphi_i\}_{i \geq 1} \) “special” basis of \( V \) and \( H_0^1(\Omega) \), respectively, formed by eigenfunctions of the Stokes and the Poisson problems following:

\[
-\Delta \phi_i = \lambda_i \phi_i \quad \text{in } \Omega \\
\phi_i = 0 \quad \text{on } \partial \Omega
\]

\[
-\Delta \varphi_i = \mu_i \varphi_i \quad \text{in } \Omega \\
\partial_n \varphi_i = 0 \quad \text{on } \partial \Omega
\]

with \( \|\phi_i\|_1 = 1 \), \( \|\varphi_i\|_1 = 1 \) for all \( i \) and \( \int_\Omega \varphi_i = 0 \). Let \( V^m \) and \( W^m \) be the finite-dimensional subspaces spanned by \( \{\phi_1, \ldots, \phi_m\} \) and \( \{\varphi_1, \ldots, \varphi_m\} \) respectively.

For each \( m \geq 1 \), given \( u_{0m} \in V^m \) and \( \theta_{0m} \in W^m \), we seek an approximate solution \( (u_m, \theta_m) \), with \( u_m : [0, T] \mapsto V^m \) and \( \theta_m : [0, T] \mapsto W^m \), verifying the following variational formulation a.e. in \( t \in (0, T) \):

\[
\begin{cases}
(\partial_t u_m(t), v_m) + ((u_m(t) \cdot \nabla) u_m(t), v_m) + (\nu(\theta_m(t))\nabla u_m(t), \nabla v_m) \\
\quad - (\alpha \theta_m(t) g, v_m) - (f, v_m) = 0 & \forall v_m \in V^m \\
(\partial_t \theta_m(t), e_m) + ((u_m(t) \cdot \nabla) \theta_m(t), e_m) \\
\quad + (k(\theta_m(t))\nabla \theta_m(t), \nabla e_m) = 0 & \forall e_m \in W^m
\end{cases}
\]

If we put

\[
u_m(t) = \sum_{j=1}^m \xi_{i,m}(t) \phi_i \quad \text{and} \quad \theta_m(t) = \sum_{j=1}^m \xi_{i,m}(t) \varphi_i,
\]

then (5) can be rewritten as a first order ordinary differential system (in normal form) associated to the unknowns \( (\xi_{i,m}(t), \xi_{i,m}(t)) \). Then, one has existence of a maximal solution (defined in some interval \( [0, \tau_m] \subset [0, T] \)) of the related Cauchy problem. Moreover, from a priori estimates (dependent on \( m \)) which will be obtained below, in particular one has that \( \tau_m = T \). Finally, using regularity of the chosen spectral basis, uniqueness of approximate solution holds ([2]).

4. Differential inequalities in regular norms

In the sequel, \( \delta \) and \( \varepsilon \) will denote some constants sufficiently small. By \( C \) we will denote different constants, independent on data and \( \delta \) and \( \varepsilon \).

**Lemma 2.** For each \( \delta, \varepsilon > 0 \) sufficiently small, there exists a constant \( K = K(\delta, \varepsilon) > 0 \) such that

\[
\frac{d}{dt} \int_\Omega (\nu(\theta_m) + 1) |\nabla u_m|^2 + \nu_{\min} \|u_m\|^2 + |\partial_t u_m|^2 \leq \delta \|\partial_t \theta_m\|^2 + \varepsilon \|u_m\|^2 \|\theta_m\|^2 + K(\|u_m\|^2 + \|u_m\|^2 + \|\theta_m\|^2 + \|f\|^2)
\]

**Proof.**

First, taking \( v = Au_m \) as test function in the \( u \)-system of (5) \( (A \) is the Helmholtz operator mentioned in Lemma 1) one has

\[
(\partial_t u_m, Au_m) - (\nabla \cdot (\nu(\theta_m)\nabla u_m), Au_m) + ((u_m \cdot \nabla) u_m, Au_m) \\
- \alpha (g, Au_m) = (f, Au_m)
\]

We can write the first term as

\[
(\partial_t u_m, Au_m) = \frac{1}{2} \frac{d}{dt} \|u_m\|^2
\]
The second term of (6) is split as follows (using the Helmholtz decomposition $\Delta u = -Au + \nabla q$),

$$-(\nabla \cdot (\nu(\theta_m)\nabla u_m), Au_m) = (\nu(\theta_m)Au_m, Au_m) + (\nu(\theta_m)\nabla q, Au_m) - (\nu(\theta_m)\nabla \theta_m \nabla u_m, Au_m).$$

Taking into account that

$$\nu(\theta_m)\nabla q, Au_m = -(q, \nabla \cdot (\nu(\theta_m)Au_m)) = -(q, \nu'(\theta_m)\nabla \theta_m Au_m) = -(q, \nu'(\theta_m)\nabla \theta_m Au_m)$$

since $\nabla \cdot Au_m = 0$, hence the second term of (6) remains

$$-(\nabla \cdot (\nu(\theta_m)\nabla u_m), Au_m) = (\nu(\theta_m)Au_m, Au_m) - (q, \nu'(\theta_m)\nabla \theta_m Au_m) - (\nu(\theta_m)\nabla \theta_m \nabla u_m, Au_m)$$

then (6) can also be written as follows (using $\nu(\theta_m) \geq \nu_{\min} > 0$)

$$\frac{1}{2} \frac{d}{dt} \|u_m\|^2 + \nu_{\min} \|u_m\|^2 \leq -((u_m \cdot \nabla)u_m, Au_m) - \alpha (g\theta_m, Au_m)$$

$$+ (q, \nu'(\theta_m)\nabla \theta_m Au_m) + (\nu(\theta_m)\nabla \theta_m Au_m) + (f, Au_m)$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5$$

The first two terms and the last term on the right hand side of (15) are bounded respectively by

$$I_1 \leq \delta \|u_m\|^2 + C\delta \|u_m\|^6, \quad I_2 \leq \delta \|u_m\|^2 + C\delta \|g\|^2 \|\theta_m\|^2$$

and

$$I_5 \leq \delta \|u_m\|^2 + C\delta \|f\|^2.$$ 

In order to estimate the third term we use the Lemma 1 (and $|\nu'(\theta_m)| \leq \nu'_{\max}$)

$$|I_3| = |(q, \nu'(\theta_m)\nabla \theta_m Au_m)| \leq \nu'_{\max} \|g\| \|\nabla \theta_m\| \|Au_m\|$$

$$\leq C\|g\|^{1/2} \|\nu\|^{1/2} \|\theta_m\| \|u_m\|^2 \leq C(C(C \|u_m\|^{1/2} + \varepsilon \|u_m\|_2^{1/2}) \|u_m\|_2^{3/2} \|\theta_m\|_2$$

$$\leq C \|u_m\|^{1/2} \|u_m\|_2^{3/2} \|\theta_m\|_2 + \varepsilon \|u_m\|_2^{3/2} \|\theta_m\|_2$$

In what concerns to the fourth term, 

$$|I_4| = |(\nu'(\theta_m)\nabla \theta_m \nabla u_m, Au_m)| \leq \nu'_{\max} \|\nabla \theta_m\| \|\nabla u_m\| \|Au_m\|$$

$$\leq C \|\theta_m\|_2 \|u_m\|_2^{1/2} \|u_m\|_2^{3/2} \|\theta_m\|_2 \leq \delta \|u_m\|^2 + C\delta \|u_m\|_2^{5/2} \|\theta_m\|_2.$$

Consequently, choosing $\delta$ small enough we obtain that

$$\frac{d}{dt} \|u_m\|^2 + \nu_{\min} \|u_m\|^2 \leq C\|u_m\|^{1/2} + \|u_m\|_2^{3/2} \|\theta_m\|_2 + C\delta \|u_m\|^{5/2} \|\theta_m\|_2$$

On the other hand, using $\partial_t u_m$ as a test function in the $u$-system of (5), one obtains

$$\partial_t u_m + (\nu(\theta_m)\nabla u_m, \partial_t u_m) + (\nu(\theta_m)\nabla u_m, \partial_t u_m) + (\nu(\theta_m)\nabla u_m, \partial_t u_m)$$

$$-\alpha (g\theta_m, \partial_t u_m) = (f, \partial_t u_m).$$
By taking into account that the second term in (9) can be written as

$$ (\nu(\theta_m) \nabla u_m, \partial_t \nabla u_m) = \frac{1}{2} \frac{d}{dt} (\nu(\theta_m) \nabla u_m, \nabla u_m) - \frac{1}{2} (\partial_t \nu(\theta_m)) \nabla u_m, \nabla u_m), $$

we deduce from (9) that

$$ \frac{1}{2} \frac{d}{dt} \int_\Omega \nu(\theta_m)|\nabla u_m|^2 + |\partial_t u_m|^2 \leq -((u_m \cdot \nabla) u_m, \partial_t u_m) $$

$$ + \alpha(\nu \partial_t u_m) + \frac{1}{2} (\partial_t (\nu(\theta_m)) \nabla u_m, \nabla u_m) + \int_\Omega f \partial_t u_m) $$

$$ := J_1 + J_2 + J_3 + J_4 $$

The first two terms and the last term on the right side of (10) are bounded respectively, by

$$ J_1 \leq \delta(|\partial_t u_m|^2 + \|u_m\|_2^2) + C_\delta \|u_m\|_1^2, \quad J_2 \leq \delta(|\partial_t u_m|^2 + C_\delta |g|^2 \|\theta_m\|_2), $$

and

$$ J_3 \leq \delta(|\partial_t u_m|^2 + C_\delta |f|^2). $$

Lastly, we go into detail the third term:

$$ |J_3| = \int_\Omega |(\nu(\theta_m) \partial_t (\theta_m) \nabla u_m, \nabla u_m)| \leq C |\partial_t \theta_m| ||u_m||_3 |\nabla u_m|_2 $$

$$ \leq C |\partial_t \theta_m| \|u_m\|_1^2 \|\nabla u_m\|_2 \leq C \delta \|\theta_m\|_1^2 + \|u_m\|_2^2 + C_\delta \|u_m\|_1^2 $$

Consequently, choosing $\delta$ small enough,

$$ \frac{d}{dt} \int_\Omega \nu(\theta_m)|\nabla u_m|^2 + |\partial_t u_m|^2 \leq \delta(|\partial_t \theta_m|^2 + \|u_m\|_2^2) $$

$$ + C_\delta (\|u_m\|_1^2 + |g|^2 \|\theta_m\|_2^2 + |f|^2) $$

Finally, (8) and (11) prove the Lemma.

**Lemma 3.** For each $\delta > 0$ small enough, there exists $C_\delta > 0$ such that

$$ \frac{d}{dt} (\|\theta_m\|_2^2 + |\partial_t \theta_m|^2) + k_{min}(\|\theta_m\|_2^2 + \|\partial_t \theta_m\|_1^2) $$

$$ \leq \delta |\partial_t u_m|^2 + C_\delta (\|\theta_m\|_1^2 + \|\theta_m\|_2^2 |\partial_t \theta_m|^2 + \|\theta_m\|_2^2 \|u_m\|_1^2) $$

\[\square\]

**Proof.**

Differenting respect to the time the $\theta_m$-equation of (5) and using $\partial_t \theta_m$ as test function, one obtains

$$ \frac{d}{dt} |\partial_t \theta_m|^2 \leq k_{min}(\|\theta_m\|_2^2 + \|\partial_t \theta_m\|_1^2) $$

since $(u_m \cdot \nabla \partial_t \theta_m, \partial_t \theta_m) = 0$.

By taking into account that the second term in (12) can be split as

$$ (\partial_t (k(\theta_m) \nabla \theta_m), \partial_t \nabla \theta_m) = (k'(\theta_m) \partial_t \theta_m \nabla \theta_m, \partial_t \nabla \theta_m) + (k(\theta_m) \partial_t \nabla \theta_m, \partial_t \nabla \theta_m), $$

we deduce from (12) that

$$ \frac{1}{2} \frac{d}{dt} |\partial_t \theta_m|^2 + k_{min} |\partial_t \nabla \theta_m|^2 \leq -k'(\theta_m) \partial_t \theta_m \nabla \theta_m, \partial_t \nabla \theta_m) $$

$$ - (\partial_t u_m \cdot \nabla \theta_m, \partial_t \theta_m). $$
Bounding both terms on the right hand side of (13) \((k'_{\text{max}} = \max |k'|)\):

\[
|k'(\theta_m)\partial_t \theta_m \cdot \nabla \theta_m| \leq k'_{\text{max}}|\nabla \theta_m| |\partial_t \theta_m| |\nabla \theta_m| \\
\leq C \|\theta_m\|_2 |\partial_t \theta_m|^{1/2} |\theta_m|^{3/2} \leq \delta \|\theta_m\|_2^2 + C_\delta \|\theta_m\|_2^2 |\partial_t \theta_m|^{1/2}
\]

and

\[
|\partial_t u_m \cdot \nabla \theta_m| \leq |\partial_t u_m| |\nabla \theta_m| |\partial_t \theta_m| \\
\leq C |\partial_t u_m|_2 \|\theta_m\|_2 |\partial_t \theta_m|^{1/2} |\theta_m|^{1/2} \\
\leq \delta (||\partial_t \theta_m||_2^2 + |\partial_t u_m|_2^3) + C_\delta \|\theta_m\|_2^3 \|\partial_t \theta_m\|_2^2
\]

we obtain, for \(\delta\) small enough,

\[
\frac{d}{dt} |\partial_t \theta_m|^2 + k_{\min} |\partial_t \theta_m|^2 \leq \delta |\partial_t u_m|^2 + C_\delta \|\theta_m\|^2 |\partial_t \theta_m|^2 \tag{14}
\]

Now, using \(\Delta^2 \theta_m\) as test function (\(\Delta^2 \theta_m \in W^m\) thanks to the election of spectral basis) and integrating by parts in all terms (boundary terms vanish since \((\nabla \Delta \theta_m \cdot n)|_{\partial \Omega} = 0\)), one obtains:

\[
(\nabla [\nabla \cdot (k(\theta_m)\nabla \theta_m)], \nabla \Delta \theta_m) - (\nabla (u \cdot \nabla \theta_m), \nabla \Delta \theta_m) = 0 \tag{15}
\]

Notice that if Dirichlet boundary condition is imposed for the temperature \(\theta\), the boundary terms do not vanish in the integration by parts and we can not obtain the previous inequalities.

Integrating by parts the first term of (15) (again the boundary term vanishes since \((\partial_t \nabla \theta_m \cdot n)|_{\partial \Omega} = 0\), the term remains \(\frac{1}{2} \frac{d}{dt} |\Delta \theta_m|^2\). The second term is

\[
(\nabla [\nabla \cdot (k(\theta_m)\nabla \theta_m)], \nabla \Delta \theta_m) = (k''(\theta_m)(\nabla \theta_m)^2, \nabla \Delta \theta_m) \\
+ 2(k'(\theta_m)\nabla^2 \theta_m \nabla \theta_m, \nabla \Delta \theta_m) \\
+ (k'(\theta_m)\nabla \theta_m \Delta \theta_m, \nabla \Delta \theta_m) + (k(\theta_m)\nabla \Delta \theta_m, \nabla \Delta \theta_m). \tag{15}
\]

Hence, we deduce of (15) that \((|k''(\theta_m)| \leq k''_{\text{max}} = \max |k''|)\)

\[
\frac{1}{2} \frac{d}{dt} |\Delta \theta_m|^2 + k_0|\Delta \theta_m|^2 \leq k''_{\text{max}} [[(\nabla \theta_m)^2, \nabla \Delta \theta_m]] \\
+ 2k''_{\text{max}} [(\nabla^2 \theta_m \nabla \theta_m, \nabla \Delta \theta_m)] + k''_{\text{max}} [(\nabla \theta_m \Delta \theta_m, \nabla \Delta \theta_m)] \\
+ |(\nabla u_m \nabla \theta_m, \nabla \Delta \theta_m)| + |(u_m \nabla^2 \theta_m, \nabla \Delta \theta_m)| \\
:= L_1 + L_2 + L_3 + L_4 + L_5.
\]

Replacing in the above inequality the following estimations

\[
L_1 \leq C |\nabla \theta_m|^6 |\nabla \Delta \theta_m| \leq \delta \|\theta_m\|^6_3 + C_\delta \|\theta_m\|^6_2
\]

\[
L_2 \leq C |\nabla^2 \theta_m|^3 |\nabla \theta_m| \leq C \|\theta_m\|^3_2 \|\theta_m\|^3_3 \leq \delta \|\theta_m\|^3_3 + C_\delta \|\theta_m\|^3_2
\]

\[
L_3 \leq C |\nabla \theta_m| |\Delta \theta_m| |\nabla \Delta \theta_m| \leq C \|\theta_m\|^3_2 \|\theta_m\|^3_3 \leq \delta \|\theta_m\|^3_3 + C_\delta \|\theta_m\|^3_2
\]

\[
L_4 \leq C |\nabla u_m|^2 |\nabla \theta_m| \leq C \|\nabla \theta_m\|^2_1 \|\theta_m\|^2_3 \leq \delta \|\theta_m\|^2_3 + C_\delta \|\theta_m\|^2_2
\]

\[
L_5 \leq C |u_m|^2 |\nabla \theta_m| |\nabla \Delta \theta_m| \leq C \|\nabla \theta_m\|^2_1 \|\theta_m\|^2_3 \leq \delta \|\theta_m\|^2_3 + C_\delta \|\theta_m\|^2_2
\]
we get, taking δ small enough,

\[
\frac{d}{dt} \|\theta_m\|^2 + k_{\text{min}} \|\theta_m\|^2 \leq C_S (\|\theta_m\|^2 + \|u_m\|^2 \|\theta_m\|^2)
\]

Finally, (14) added to (16) proves the Lemma.

5. PROOF OF THEOREM 2.2

If we denote

\[
\Phi_m(t) = \int_\Omega (\nu(\theta_m) + 1) |\nabla u_m|^2 + \|\theta_m\|^2 + |\partial_t \theta_m|^2
\]

\[
\Psi_m(t) = \|u_m\|^2 + |\partial_t u_m|^2 + \|\theta_m\|^2 + |\partial_t \theta_m|^2
\]

taking an adequate balance between inequalities from Lemmas 2 and 3 in order to vanish the term $\|g\|^2_{L^2(I^L)} \|\theta_m\|^2$ at the right hand-side, one has

\[
\begin{cases}
\Phi'_m + C \Psi_m \leq \varepsilon \Psi_m \Phi_m^{1/2} + C_0(t) + D \Phi_m^3 \\
\Phi_m(0) = \Phi_{m0}
\end{cases}
\]

where $C, D > 0$ are constant and $C_0(t)$ is a positive function depending on data $f$. Concretely, $\Phi_m(0) = C_0 \|f\|^2_2$.

Let $\Phi_m(0) \leq \delta$ for $\delta$ a small enough constant (that we will specify latter).

First step: If $\Phi_m(0) \leq \delta$ and $\|f\|_{L^2(I^L)} \leq \delta$, then $\Phi_m(t) \leq 2\delta \forall t \in [0, T]$.

Indeed, by an absurd argument, let $T^*$ be the first value in $[0, T]$ such that $\Phi_m(T^*) = 2\delta$, hence

$\Phi_m(T^*) = 2\delta$ and $\Phi_m(s) < 2\delta \ \forall s \in [0, T^*)$.

Moreover, there exists a Poincaré constant $C_p > 0$ such that $\Phi_m(t) \leq C_p \Psi_m(t)$. Then for $\varepsilon$ small enough we have

$C \Psi_m - \varepsilon \Psi_m \Phi_m^{1/2} \geq C \Psi_m - \varepsilon \Psi_m (2\delta)^{1/2} \geq \tilde{C} \Phi_m \geq \frac{\tilde{C}}{C_p} \Phi_m \equiv \tilde{C} \Phi_m$.

The above inequality together (17) lead:

\[
\begin{cases}
\Phi'_m + \tilde{C} \Phi_m \leq C_0(t) + D \Phi_m^3 \\
\Phi_m(0) = \Phi_{m0}.
\end{cases}
\]

in $[0, T^*]$. Then, $\Phi'_m + \tilde{C} \Phi_m \leq C_0(t) + 4\delta^2 D \Phi_m$ in $[0, T^*]$. We can find $\delta$ such that $\tilde{C} - 4\delta^2 D \geq \tilde{C}$ being $\tilde{C}$ a positive constant. Therefore,

$\Phi'_m + \tilde{C} \Phi_m \leq C_0(t)$ in $[0, T^*]$

hence

\[
(e^{\tilde{C} t} \Phi_m)' \leq e^{\tilde{C} t} C_0(t) \quad \text{in } [0, T^*].
\]

Integrating in $[0, T^*]$ one finds:

$e^{\tilde{C} T^*} \Phi_m(T^*) \leq \Phi_m(0) + \int_0^{T^*} e^{\tilde{C} t} C_0(t),$

hence

$\Phi_m(T^*) \leq \delta e^{-\tilde{C} T^*} + \int_0^{T^*} C_0(t).$
We can choose \( \int_0^T C_0(t) \) small enough such that the right side is smaller that 2\( \delta \) (for example \( \int_0^T C_0(t) \leq \delta \)). Thus, we arrive at a contradiction.

**Second step:** If \( \Phi_m(0) \) and \( \|f\|_{L^2(L^2)} \) are small enough, then \( \Phi_m(T) \leq \Phi_m(0) \)

Now, as \( \Phi_m(t) < 2\delta \forall t \in [0,T] \), we can repeat the above argument and to obtain (18) in \([0,T]\). Therefore, integrating (19) in \([0,T]\) we arrive at

\[
\Phi_m(T) \leq \Phi_m(0)e^{-CT} + \int_0^T C_0(s).
\]

Again, for \( \int_0^T C_0(t) \) small enough (for example \( \int_0^T C_0(t) \leq \delta (1 - e^{-CT}) \)) one obtains that \( \Phi_m(T) \leq \Phi_m(0) \).

**Third step:** Existence of approximate reproductive solution

Given \( (u_{m0}, \theta_{n0}) \in V^m \times W^m \), we define the map

\[
L^m : [0, T] \rightarrow \mathbb{R}^m \times \mathbb{R}^m
\]

where \( \Phi_m(t) \) is the (unique) approximate solution of problem (5) corresponding to the initial data \( (u_{m0}, \theta_{n0}) \).

Now, varying the initial data \( (u_{m0}, \theta_{n0}) \), we are going to define a new map

\[
\mathcal{R}^m : \mathcal{B} \subset \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m \times \mathbb{R}^m
\]

as follows: given \( L_0^m \in \mathbb{R}^m \times \mathbb{R}^m \), we define \( \mathcal{R}^m(L_0^m) = L^m(T) \), where \( L^m(t) \) is related to the solution of problem (5) with initial data \( L_0^m = L^m(0) \) and

\[
\mathcal{B} = \{ (\xi_{1m}, ..., \xi_{mm}, \zeta_{1m}, ..., \zeta_{mm}) : L_0^m : \Phi_m(0) \leq \delta \}.
\]

By uniqueness of approximate solution of problem (5), this map is well-defined. Moreover, using regularity of the corresponding ordinary differential system (equivalent to (5)), this map is continuous. By the second step, \( \mathcal{R}^m \) apply \( \mathcal{B} \) into \( \mathcal{B} \) and is a closed, convex and compact set. Consequently, Brouwer Theorem implies the existence of fixed point of \( \mathcal{R}^m \), which give us existence of reproductive Galerkin solution.

**Fourth step:** Pass to the limit in reproductive approximate solutions

If the data of the problem are small, thanks to the first step we have

\[
\Phi_m(t) = \int_\Omega (\nu(\theta_m) + 1)|\nabla u_m|^2 + ||\theta_m||_2^2 + |\partial_t \theta_m|_2^2 \leq 2\delta
\]

and (19). Therefore, the following uniformly bounds hold:

\[
\begin{align*}
(u_m) & \in L^\infty(H^1) \cap L^2(H^2), \\
(\theta_m) & \in L^\infty(H^3_N) \cap L^2(H^3_N), \\
(\partial_t u_m) & \in L^2(L^2), \\
(\partial_t \theta_m) & \in L^\infty(L^2) \cap L^2(H^1).
\end{align*}
\]
Using compactness results for time spaces with values in Banach spaces with the triplet \( H^2 \hookrightarrow H^1 \hookrightarrow L^2 \) and \( H^3 \hookrightarrow H^2 \hookrightarrow H^1 \), one has

\[
(u_m) \text{ is relatively compact in } L^2(H^1)
\]

and

\[
(\theta_m) \text{ is relatively compact in } L^2(H^2).
\]

In fact, this compactness is sufficient in the pass to the limit in (5) in order to control the nonlinear terms.

Now, we go to pass to the limit in reproductive conditions. From estimations of \( \theta_m \) in \( L^\infty(H^2) \) and \( (\partial_t \theta_m) \) in \( L^2(H^1) \) and using the triplet of spaces \( H^2 \hookrightarrow H^1 \hookrightarrow H^1 \), one has that \( \theta_m \) is relatively compact in \( C([0,T];H^1) \), hence \( \theta_m(T) \to \theta(T) \) and \( \theta_m(0) \to \theta(0) \) strongly in \( H^1(\Omega) \). Since \( \theta_m(T) = \theta_m(0) \), then \( \theta(T) = \theta(0) \) in \( H^1(\Omega) \). Finally, since \( \theta_m(T) \) and \( \theta_m(0) \) are bounded in \( H^2(\Omega) \), we have that \( \theta(T) = \theta(0) \) in \( H^2(\Omega) \).

The argument for \( u \) is similar, hence one deduces \( u(T) = u(0) \) in \( H^1(\Omega) \).

To prove \( \partial_t \theta(0) = \partial_t \theta(T) \) we go to consider the orthogonal projector \( P_m : H^1 \to W^m \)

\[
P_m(g) = \sum_{k=1}^{m} (\nabla g, \nabla \varphi_i) \varphi_i, \quad \forall g \in H^1
\]

then \( \|P_m\|_{L(H^1,H^1)} \leq 1 \) and \( \|P_m\|_{L(H^1),L(H^1)} \leq 1 \) (see [4]). Since

\[
P_m(g) = \sum_{k=1}^{m} \mu_k(g, \varphi_i) \varphi_i, \quad \forall g \in H^1
\]
then \( P_m \) is also the orthogonal projector from \( L^2 \) to \( W^m \) respect to the \( L^2 \) scalar product. Therefore

\[
(P_m(g), \varphi_i) = (g, \varphi_i) \quad \forall g \in H^1(\Omega),
\]
hence the Galerkin equation for \( \theta_m \) can be written as

\[
\partial_t \theta_m = P_m (-u_m \cdot \nabla \theta_m + \nabla \cdot (k(\theta_m)\nabla \theta_m))
\]

Differenting respect to the time the \( \theta_m \)-equation in (5),

\[
\partial_{tt} \theta_m = P_m (-\partial_t u_m \cdot \nabla \theta_m - u_m \cdot \nabla \partial_t \theta_m + \nabla \cdot (k'(\theta_m)\partial_t \theta_m \nabla \theta_m + k(\theta_m)\partial_t \nabla \theta_m))
\]
In particular

\[
||\partial_{tt} \theta_m||_{L^2(\Omega)} \leq ||-\partial_t u_m \cdot \nabla \theta_m - u_m \cdot \nabla \partial_t \theta_m||_{L^2(\Omega)}
\]

\[
+ ||\nabla \cdot (k'(\theta_m)\partial_t \theta_m \nabla \theta_m + k(\theta_m)\partial_t \nabla \theta_m)||_{L^2(\Omega)}
\]

\[
\leq |\partial_t u_m|_2 |\nabla \theta_m|_3 + |u_m|_3 |\nabla \partial_t \theta_m|_2 + |k'(\theta_m)|_2 |\partial_t \theta_m \nabla \theta_m|_2 + |k(\theta_m)|_2 |\partial_t \nabla \theta_m|_2
\]
The terms on the right hand-side of previous inequality will be bounded in \( L^2(0,T) \).

Indeed, \( \partial_t u_m \) is bounded in \( L^2(L^2) \) and \( \nabla \theta_m \) in \( L^\infty(H^1) \), hence

\[
|\partial_t u_m|_2 |\nabla \theta_m|_3 \quad \text{is bounded in } L^2(0,T)
\]

Using that \( u_m \) is bounded in \( L^\infty(H^1) \) and \( \partial_t \nabla \theta_m \) in \( L^2(L^2) \), one has

\[
|u_m|_3 |\nabla \partial_t \theta_m|_2 \quad \text{is bounded in } L^2(0,T)
\]

Using that \( \partial_t \theta_m \) and \( \nabla \theta_m \) are bounded in \( L^4(L^3) \) and \( L^\infty(H^1) \) respectively, one has that \( k'(\theta_m)\partial_t \theta_m \nabla \theta_m \) is bounded in \( L^2(L^2) \). Finally, \( k(\theta_m)\partial_t \nabla \theta_m \) is bounded in \( L^2(L^2) \).
Consequently, $\|\partial_{tt} \theta_m\|_{(H^1)'}$ is uniformly bounded in $L^2(0,T)$. This along with $\partial_t \theta_m$ is uniformly bounded in $L^\infty(L^2)$ gives that $\partial_t \theta_m$ is relatively compact in $C([0,T]; (H^1)')$, which is sufficient to prove $\partial_t \theta(0) = \partial_t \theta(T)$.

References


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