EXISTENCE OF WEAK-RENORMALIZED SOLUTION FOR A NONLINEAR SYSTEM

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Abstract

We prove an existence result for a coupled system of the reaction-diffusion kind. The fact that no growth condition is assumed on some nonlinear terms motivates the search of a weak-renormalized solution.

1 Introduction. Description of the problem

This paper investigates the existence of a solution for the nonlinear system

\[
\begin{align*}
-\Delta u - \nabla \cdot (\beta(v)X(u)) &= f \quad \text{in } \Omega, \\
-\Delta v - \nabla \cdot (\beta'(v)X(u)) &= g \quad \text{in } \Omega, \\
u &= 0, \quad v = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) denotes a bounded open subset of \( \mathbb{R}^N \), \( X \) is a \( C^1 \) bounded \( \mathbb{R}^N \)-valued function on \( \mathbb{R} \), i.e.

\[
X \in (C^1(\mathbb{R}))^N \cap (C^0(\mathbb{R}))^N,
\]

\( \beta \) is a function whose second derivatives are bounded, i.e.

\[
\beta \in W^{2,\infty}(\mathbb{R})
\]

and

\[
f, g \in H^{-1}(\Omega).
\]

Here, the main difficulty to find a solution is that no growth restrictions are assumed on \( X' \). Since \( f \) and \( g \) belong to \( H^{-1}(\Omega) \), it is natural to look for solutions \( u \) and \( v \) belonging to \( H_0^1(\Omega) \). Thus, it is not clear how

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to give a sense to $\nabla \cdot (\beta(v)X'(u))$. This inconvenient can be overcome by introducing a weak-renormalized formulation of this problem, essentially obtained through pointwise multiplication of the first equation of (1) by $h(u)$, where $h \in C^1(\mathbb{R})$ and its support is compact.

**Remark.** We can view this system as a simplified model of a nonlinear elasticity problem characterized by a constitutive law of the form

$$\sigma = \sigma_l + Y(u),$$

where

$$(\sigma_l)_{ij} = \sum a_{ijkl} \varepsilon_{kl}(u), \quad \varepsilon_{kl}(u) = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right), \quad Y_{ij} \in C^0(\mathbb{R}^2).$$

Indeed, the conservation of momentum reads

$$\nabla \cdot \sigma = F$$

($F$ is given), which is in some sense a generalization of (1). In this paper, we study the case in which

$$Y(u) = \begin{pmatrix} \beta(u_2)X_1'(u_1) & \beta'(u_2)X_1(u_1) \\ \beta(u_2)X_2'(u_1) & \beta'(u_2)X_2(u_1) \end{pmatrix}.$$ 

2 The main result

**Theorem 2.1.** Under the assumptions (2), (3), (4), there exists $\{u, v\}$, with $u, v \in H^1_0(\Omega)$, such that the second equation in (1) is satisfied in the usual weak or distributional sense and the first equation holds in the following sense:

$$\begin{cases} -\nabla \cdot (h(u)\nabla u) + \nabla u \cdot \nabla h(u) - \nabla \cdot (\beta(v)h(u)X'(u)) \\ + \beta(v)X'(u) \cdot \nabla h(u) = fh(u) \text{ in } D'(\Omega) \forall h \in C^1_0(\mathbb{R}). \end{cases}$$

A couple $\{u, v\}$ as above will be called a weak-renormalized solution to (1).
Remark. In (5), every term belongs to $\mathcal{D}'(\Omega)$. Indeed, $h(u)$ belongs to $H^1_0(\Omega)$, the first term is in $H^{-1}(\Omega)$. The second one is in $L^1(\Omega)$. For instance, since $h$ has a compact support, we can put

$$h(u)X'(u) = h(u)X'(T_M(u)) \quad \text{and} \quad h'(u)X'(u) = h'(u)X'(T_M(u))$$

for some $M > 0$, where $T_M$ is the usual truncation at level $M$. Thus, we see that the third term in the left belongs to $W^{-1,\infty}(\Omega)$ and the fourth term belongs to $L^2(\Omega)$.

Remark. Renormalized solutions to PDE’s were introduced by R. DíPerna and P.L. Lions in [4] in the framework of the Boltzmann equation. They have been used in connection with various nonlinear elliptic equations by P. Benilan et al. [2], L. Boccardo et al. [3] and P.L. Lions and F. Murat [6] (see also [7]). In the analysis of existence results for systems, weak-renormalized solutions were first considered by R. Lewandowski [5] (see also [1]).

In this paper, in order to solve (1), we will extend the techniques used in [3] in the context of a single equation.

Remark. With regard to uniqueness, it is an open problem. If we follow the classical argument of considering two solutions $u^i, v^i$ for $i = 1, 2$ of (1), and we compute the difference of (5) written for $u^1, v^1$ and for $u^2, v^2$, we find expressions with terms of the form $X'(\cdot)u$ that we are not able to estimate. There is another argument, due to P. L. Lions and F. Murat [7], which leads to the uniqueness of renormalized solutions, but it cannot be applied here.

3 The proof of theorem 2.1

First step. The introduction of a family of approximations.

For each $\varepsilon > 0$, let us put $X^\varepsilon(s) = X(T_{1/\varepsilon}(s))$ for all $s \in \mathbb{R}$. We will introduce the following approximation to (1):

$$\begin{align*}
-\Delta u^\varepsilon - \nabla \cdot (\beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon)) &= f \quad \text{in } \Omega, \\
-\Delta v^\varepsilon - \nabla \cdot (\beta'(v^\varepsilon)X(u^\varepsilon)) &= g \quad \text{in } \Omega, \\
u^\varepsilon, v^\varepsilon &\in H^1_0(\Omega),
\end{align*}$$

(6)
In order to solve (6), we will apply Schauder’s theorem. Thus, for any given $\varepsilon$ and $(u, v) \in L^2(\Omega) \times L^2(\Omega)$, we set $R^\varepsilon\{u, v\} = \{u^\varepsilon, v^\varepsilon\}$, with $\{u^\varepsilon, v^\varepsilon\}$ being the unique solution to the linear system

$$
\begin{align*}
-\Delta u^\varepsilon &= f + \nabla \cdot (\beta(v)(X^\varepsilon)'(u)) \quad \text{in } \Omega, \\
-\Delta v^\varepsilon &= g + \nabla \cdot (\beta'(v)X(u)) \quad \text{in } \Omega,
\end{align*}
$$

(7)

Obviously, $R^\varepsilon = R_3 \circ R_2 \circ R_1^\varepsilon$, where

1. $R_1^\varepsilon : L^2(\Omega) \times L^2(\Omega) \mapsto H^{-1}(\Omega) \times H^{-1}(\Omega)$ is the nonlinear continuous mapping given by

$$
R_1^\varepsilon\{u, v\} = \{f + \nabla \cdot (\beta(v)(X^\varepsilon)'(u)), g + \nabla \cdot (\beta'(v)X(u))\} \quad \forall (u, v) \in L^2(\Omega) \times L^2(\Omega),
$$

2. $R_2 : H^{-1}(\Omega) \times H^{-1}(\Omega) \mapsto H^1_0(\Omega) \times H^1_0(\Omega)$ associates to each $(f, g) \in H^{-1}(\Omega) \times H^{-1}(\Omega)$ the unique solution $(w, z)$ of the following linear system

$$
\begin{align*}
-\Delta w &= f \quad \text{in } \Omega, \\
-\Delta z &= g \quad \text{in } \Omega, \\
w, z &= H^1_0(\Omega),
\end{align*}
$$

3. $R_3$ is the compact embedding of $H^1_0(\Omega) \times H^1_0(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$.

Since $R_1^\varepsilon$ maps the whole space $L^2(\Omega) \times L^2(\Omega)$ inside a ball, Schauder’s theorem can be applied and (6) possesses at least one solution $\{u^\varepsilon, v^\varepsilon\}$.

**Second step.** A priori estimates and weak convergence. Choosing $u^\varepsilon$ and $v^\varepsilon$ as test functions in the first and second equation in (6) respectively, one finds:

$$
\int_\Omega \nabla u^\varepsilon \nabla u^\varepsilon + \int_\Omega \beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon = \langle f, u^\varepsilon \rangle_{H^{-1}, H^1_0}. 
$$

(8)

$$
\int_\Omega \nabla v^\varepsilon \nabla v^\varepsilon + \int_\Omega \beta'(v^\varepsilon)X(u^\varepsilon) \cdot \nabla v^\varepsilon = \langle g, v^\varepsilon \rangle_{H^{-1}, H^1_0}. 
$$

(9)
For $\varepsilon$ sufficiently small, $X = X \circ T_{1/\varepsilon} = X^\varepsilon$, whence we can replace $X(u^\varepsilon)$ by $X^\varepsilon(u^\varepsilon)$ in (9).

Let us introduce the function $H = (H_1, H_2, \ldots, H_n)$, with

$$H_i(t, s) = \int_0^s \beta(0)(X^{\varepsilon}_i)'(\theta) d\theta + \int_0^t \beta'(\theta)X^{\varepsilon}_i(s) d\theta.$$  

Then,

$$\int_{\Omega} \beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon + \int_{\Omega} \beta'(v^\varepsilon)X^\varepsilon(u^\varepsilon) \cdot \nabla v^\varepsilon = \int_{\Omega} \nabla \cdot H(u^\varepsilon, v^\varepsilon) = 0,$$

thanks to Stokes' theorem. Summing (8) and (9), we obtain

$$\int_{\Omega} |\nabla u^\varepsilon|^2 + \int_{\Omega} |\nabla v^\varepsilon|^2 = \langle f, u^\varepsilon \rangle_{H^{-1}, H^1_0} + \langle g, v^\varepsilon \rangle_{H^{-1}, H^1_0}$$

and

$$\|u^\varepsilon\|_{H^1_0}^2 + \|v^\varepsilon\|_{H^1_0}^2 \leq \|f\|_{H^{-1}}^2 + \|g\|_{H^{-1}}^2.$$

Consequently, at least for a subsequence, still indexed by $\varepsilon$, we can conclude that

$$u^\varepsilon \to u, \quad v^\varepsilon \to v \quad \text{weakly in } H^1_0(\Omega),$$

$$u^\varepsilon \to u, \quad v^\varepsilon \to v \quad \text{strongly in } L^p(\Omega) \quad \forall p \in [1, 2^*) \text{ and a.e.} \quad \text{(10)}$$

Here, we have denoted by $2^*$ the exponent furnished by the Sobolev embedding theorem, that is

$$\begin{cases} 2^* = \frac{2N}{N-2} & \text{if } N \geq 3, \\ 2^* < +\infty \text{ arbitrarily large if } N = 2. \end{cases}$$

**Third step.** The strong convergence of $v^\varepsilon$ in $H^1_0$.

It is easy to see that $v$ is a weak solution to the problem

$$\begin{cases} -\Delta v - \nabla \cdot (\beta'(v)X(u)) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \quad \text{(11)} \end{cases}$$

Indeed, since $\beta'$ and $X$ are continuous and bounded, it is clear that $\beta'(v^\varepsilon) \to \beta'(v)$ strongly in $L^p$ for all $p \in [1, 2^*)$ and $X(u^\varepsilon) \to X(u)$.
strongly in $L^r$ for all $r \in [1, +\infty)$. This enables us to pass to the limit in the second equation in (6).

From (11), we also see that
\[
\int_{\Omega} |\nabla v|^2 = -\int_{\Omega} \beta'(v)X(u) \cdot \nabla v + \int_{\Omega} gv.
\] (12)

Let us use $v^\varepsilon$ as a test function in the second equation in (6). We find:
\[
\int_{\Omega} |\nabla v^\varepsilon|^2 = -\int_{\Omega} \beta'(v^\varepsilon)X(u^\varepsilon) \cdot \nabla v^\varepsilon + \int_{\Omega} gv^\varepsilon.
\] (13)

Arguing as above, we can pass to the limit in the right hand side in (13). Accordingly, we have:
\[
\int_{\Omega} |\nabla v^\varepsilon|^2 \to -\int_{\Omega} \beta'(v)X(u) \cdot \nabla v + \int_{\Omega} gv.
\]

This, combined with (12), gives the convergence in norm in $H^1_0$ for $v^\varepsilon$ and, consequently,
\[
v^\varepsilon \rightharpoonup v \text{ strongly in } H^1_0.
\] (14)

**Fourth step.** The strong convergence of $u^\varepsilon$ in $H^1_0$.

We will first prove that
\[
\lim_{K \to +\infty} \left( \limsup_{\varepsilon \to 0} \int_{\{|u^\varepsilon| > K\}} |\nabla u^\varepsilon|^2 \right) = 0
\] (15)

Thus, let us consider the test functions $u^\varepsilon - T_K(u^\varepsilon)$ in the first equation in (6). Notice that
\[
\nabla (u^\varepsilon - T_K(u^\varepsilon)) = \begin{cases} 
\nabla u^\varepsilon & \text{if } |u^\varepsilon| \geq K, \\
0 & \text{if } |u^\varepsilon| < K.
\end{cases}
\]

Hence,
\[
\int_{\{|u^\varepsilon| \geq K\}} |\nabla u^\varepsilon|^2 + \int_{\Omega} \beta(v^\varepsilon)(1 - T_K'(u^\varepsilon))(X^\varepsilon)'(u^\varepsilon) \cdot \nabla u^\varepsilon
\] \[
= \langle f, u^\varepsilon - T_K(u^\varepsilon) \rangle.
\] (16)
We can put 
\[(Y^\varepsilon_K)^i(t) = \int_0^t (1 - T^\varepsilon_K(\theta))(X^\varepsilon)'(\theta) d\theta,\]
Thus, the second term in the left hand side of (16) can be written in the form
\[\int_\Omega (\nabla \cdot Y^\varepsilon_K(u^\varepsilon)) \beta(v^\varepsilon) = - \int \nabla \cdot Y^\varepsilon_K(u^\varepsilon) \cdot \nabla \beta(v^\varepsilon).\]
Moreover,
\[Y^\varepsilon_K(s) = \begin{cases} X^\varepsilon(s) - X^\varepsilon(K) & \text{if } s > K, \\ 0 & \text{if } \vert u^\varepsilon \vert \leq K, \\ X^\varepsilon(s) - X^\varepsilon(-K) & \text{if } s < -K. \end{cases}\]
Since \(X \in C^0_b(\mathbb{R})^N\), for \(\varepsilon > 0\) sufficiently small, \(Y^\varepsilon_K\) is independent of \(\varepsilon\) and \(X^\varepsilon_K(u^\varepsilon)\) is bounded by a constant independent of \(\varepsilon\). We also have
\[\limsup_{\varepsilon \to 0} \vert Y^\varepsilon_K(u^\varepsilon) \vert \leq \vert X(u) - X(K)\vert \mathbb{1}_{\{u > K\}} + \vert X(u) - X(-K)\vert \mathbb{1}_{\{u < -K\}}\]
for all \(K > 0\). Therefore,
\[
\begin{cases}
\limsup_{\varepsilon \to 0} \int_{\vert u^\varepsilon \vert > K} \vert \nabla u^\varepsilon \vert^2 \\
+ \int_\Omega \vert X(u) - X(K)\vert \cdot \vert \nabla \beta(v) \vert \mathbb{1}_{\{u > K\}} + \langle f, u - T_K(u) \rangle,
\end{cases}
\]
whence
\[
\begin{cases}
\lim_{K \to +\infty} \left( \limsup_{\varepsilon \to 0} \int_{\vert u^\varepsilon \vert > K} \vert \nabla u^\varepsilon \vert^2 \right) \\
\leq \lim_{K \to +\infty} \left[ \int_\Omega \vert X(u) - X(K)\vert \cdot \vert \nabla \beta(v) \vert \mathbb{1}_{\{u > K\}} \\
+ \int_\Omega \vert X(u) - X(-K)\vert \cdot \vert \nabla \beta(v) \vert \mathbb{1}_{\{u < -K\}} \\
+ \lim_{K \to +\infty} \langle f, u - T_K(u) \rangle = 0.\right)
\end{cases}
\]
This proves (15). Let us introduce the sets $F_{i,j}^\varepsilon$,

$$F_{i,j}^\varepsilon = \{ x \in \Omega : |u^\varepsilon - T^j_j(u)| \leq i \}.$$ 

We are now going to prove that

$$\lim_{j \to +\infty} \left( \limsup_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} |
\nabla (u^\varepsilon - T^j_j(u))|^2 \right) = 0 \quad \forall i \geq 1. \quad (19)$$

Thus, let us use $T_i^j(u^\varepsilon - T^j_j(u))$ as test function in the first equation of (6). We obtain

$$\int_{\Omega} \nabla u^\varepsilon \cdot \nabla T_i^j(u^\varepsilon - T^j_j(u)) + \int_{\Omega} \beta(u^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla T_i^j(u^\varepsilon - T^j_j(u))
= \langle f, T_i^j(u^\varepsilon - T^j_j(u)) \rangle. \quad (20)$$

Let us notice that

$$\nabla T_i^j(u^\varepsilon - T^j_j(u)) = 0 \text{ in } \Omega \setminus F_{i,j}^\varepsilon.$$ 

We can then write (20) in the form

$$\int_{F_{i,j}^\varepsilon} \nabla u^\varepsilon \cdot \nabla T_i^j(u^\varepsilon - T^j_j(u)) + \int_{F_{i,j}^\varepsilon} \beta(u^\varepsilon)(X^\varepsilon)'(u^\varepsilon) \cdot \nabla T_i^j(u^\varepsilon - T^j_j(u))
= \langle f, T_i^j(u^\varepsilon - T^j_j(u)) \rangle. \quad (21)$$

Since

$$|u^\varepsilon| \leq |u^\varepsilon - T^j_j(u)| + |T^j_j(u)| \leq i + j \quad \text{if } x \in F_{i,j}^\varepsilon,$$

we can write $T_{1/\varepsilon}(u^\varepsilon) = T_{i+j}(u^\varepsilon)$ for all $x \in F_{i,j}^\varepsilon$ whenever $\varepsilon$ is sufficiently small. This gives:

$$(X^\varepsilon)'(u^\varepsilon) = X'(T_{i+j}(u^\varepsilon))T'_{i+j}(u^\varepsilon) = X'(T_{i+j}(u^\varepsilon)) \text{ in } F_{i,j}^\varepsilon.$$ 

Thus, for small $\varepsilon > 0$, the second term in the left in (21) is

$$\int_{F_{i,j}^\varepsilon} \beta(u^\varepsilon)X'(T_{i+j}(u^\varepsilon)) \cdot \nabla T_i^j(u^\varepsilon - T^j_j(u))$$
and converges to

$$\int_{\Omega} \beta(v)X'(T_{i+j}(u)) \cdot \nabla T_i^j(u - T^j_j(u)) \quad (22)$$
as \( \varepsilon \to 0 \), since

\[
T_i(u^\varepsilon - T_j(u)) \to T_i(u - T_j(u)) \text{ weakly in } H_0^1
\]

and \( \beta(v^\varepsilon)X'(T_{i+j}(u^\varepsilon)) \) is bounded in \((L^\infty(\Omega))^N\) and converges a.e. to \( \beta(v)X'(T_{i+j}(u)) \).

Let us introduce \( H_{i,j} = (H_{i,j}^1, H_{i,j}^2, ..., H_{i,j}^N) \), with

\[
H_{i,j}^1(s) = \int_0^s T_i'(\theta - T_j(\theta))(1 - T_j'(\theta))X'(T_{i+j}(\theta)) \, d\theta.
\]

Then (22) can be rewritten in the form

\[
\int_\Omega (\nabla \cdot H_{i,j}^1(u)) \beta(v) = - \int_\Omega H_{i,j}^1(u) \cdot \nabla \beta(v)
\]

Moreover, it is not difficult to check that

\[
H_{i,j}^1(u) = \begin{cases} 
X(i + j) - X(j) & \text{if } j < |u| < i + j, \\
0 & \text{otherwise.}
\end{cases}
\]

For any \( i \), we have \( H_{i,j}^1(u) \to 0 \) a.e. as \( j \to +\infty \). Since \( X \) is bounded, \( H_{i,j}^1(u) \) is also bounded. Thus, we obtain from Lebesgue’s theorem that

\[
\int_\Omega H_{i,j}^1(u) \cdot \nabla \beta(v) \to 0 \quad \text{as } j \to \infty.
\]

for all \( i \geq 1 \). Recalling (20) we see we have proved the following:

\[
\lim_{j \to +\infty} \left( \lim_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} \nabla u^\varepsilon \cdot \nabla T_i(u^\varepsilon - T_j(u)) \right) = \lim_{j \to +\infty} \langle f, T_i(u - T_j(u)) \rangle.
\]

(23)

On the other hand,

\[
\lim_{j \to +\infty} \left( \lim_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} \nabla T_j(u) \cdot \nabla T_i(u^\varepsilon - T_j(u)) \right) = \lim_{j \to +\infty} \int_\Omega \nabla T_j(u) \cdot \nabla T_i(u - T_j(u)).
\]
Consequently,
\[
\lim_{j \to +\infty} \left( \lim_{\varepsilon \to 0} \int_{F_{i,j}^\varepsilon} |\nabla (u^\varepsilon - T_j(u))|^2 \right) = \lim_{j \to +\infty} \left( \langle f, T_i(u - T_j(u)) \rangle - \int_\Omega \nabla T_j(u) \cdot \nabla T_i(u - T_j(u)) \right),
\]
(24)
Notice that, the terms on the right hand side of (24) can be bounded as follows:
\[
\langle f, T_i(u - T_j(u)) \rangle - \int_\Omega \nabla T_j(u) \cdot \nabla T_i(u - T_j(u)) \\
\leq (\| f \|_{H^{-1}} + \| u \|) \| u - T_j(u) \|
\]
and this converges to 0 as \( j \to +\infty \). Therefore, (19) is satisfied.

We can now prove that \( u^\varepsilon \) converges strongly in \( H_0^1 \). Indeed, observe that, if \( x \in \Omega \setminus F_{i,j}^\varepsilon \), then
\[
|u^\varepsilon| \geq |u^\varepsilon - T_j(u)| - |T_j(u)| \geq i - j,
\]
so that \( \Omega \setminus F_{i,j}^\varepsilon \subset E_{i-j}^\varepsilon \), with
\[
E_{i-j}^\varepsilon = \{ x \in \Omega : |u^\varepsilon(x)| \geq i - j \}.
\]
Therefore,
\[
\frac{1}{2} \int_\Omega |\nabla (u^\varepsilon - u)|^2 \leq \frac{1}{2} \int_{F_{i,j}^\varepsilon} |\nabla (u^\varepsilon - u)|^2 + \frac{1}{2} \int_{E_{i-j}^\varepsilon} |\nabla (u^\varepsilon - u)|^2 \\
\leq \int_{F_{i,j}^\varepsilon} |\nabla (u^\varepsilon - T_j(u))|^2 + \int_{F_{i,j}^\varepsilon} |\nabla (T_j(u) - u)|^2 + \int_{E_{i-j}^\varepsilon} |\nabla u^\varepsilon|^2 + \int_{E_{i-j}^\varepsilon} |\nabla u|^2 \leq 2(A_{ij}^\varepsilon + B_{ij}^\varepsilon + C_{ij}^\varepsilon + D_{ij}^\varepsilon).
\]
(25)
We have seen in (19) that
\[
\lim_{j \to +\infty} \limsup_{\varepsilon \to 0} A_{ij}^\varepsilon = 0 \quad \forall i \geq 1
\]
(26)
The second term \( B_{ij}^\varepsilon \) satisfies
\[
\limsup_{\varepsilon \to 0} B_{ij}^\varepsilon \leq \int_\Omega |\nabla (T_j(u) - u)|^2.
\]
whence we also have

$$
\lim_{j \to +\infty} \lim_{\varepsilon \to 0} B_{ij}^\varepsilon = 0 \quad \forall i \geq 1 \tag{27}
$$

From (15) we know that

$$
\lim_{j \to +\infty} \lim_{\varepsilon \to 0} C_{ij}^\varepsilon = 0 \quad \text{as } i, j \to +\infty, \ i - j \to +\infty. \tag{28}
$$

Finally, this is also true for $D_{ij}^\varepsilon$, since $u \in H^1_0$:

$$
\lim_{j \to +\infty} \lim_{\varepsilon \to 0} D_{ij}^\varepsilon = 0 \quad \text{as } i, j \to +\infty, \ i - j \to +\infty. \tag{29}
$$

From (25) and (26)–(29), we deduce at once that $u_\varepsilon \to u$ strongly in $H^1_0$ as $\varepsilon \to 0$.

**Fifth step.** End of the proof of theorem 1.1.

Let us chose $h \in C^1_c(\mathbb{R})$ and $\varphi, \psi \in D$. Multiplying the first equation in (6) by $h(u^\varepsilon)\varphi$ and the second one by $\psi$ and integrating by parts, we obtain:

$$
\begin{aligned}
\int_{\Omega} (\nabla u^\varepsilon + \beta(v^\varepsilon)(X^\varepsilon)'(u^\varepsilon)) \cdot \nabla (h(u^\varepsilon)\varphi) &= \langle f, h(u^\varepsilon)\varphi \rangle \\
\int_{\Omega} (\nabla v^\varepsilon + \beta'(v^\varepsilon)X^\varepsilon(u^\varepsilon)) \cdot \nabla \psi &= \langle g, \psi \rangle.
\end{aligned} \tag{30}
$$

Since $h$ and $h'$ have compact support on $\mathbb{R}$, for $\varepsilon$ sufficiently small we have

$$
(X^\varepsilon)'(t)h(t) = X'(t)h(t), \quad (X^\varepsilon)'(t)h'(t) = X'(t)h'(t).
$$

Both functions belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$. Thus, we can write (30) as follows

$$
\begin{aligned}
\int_{\Omega} h(u^\varepsilon) \nabla u^\varepsilon \cdot \nabla \varphi + \int_{\Omega} h'(u^\varepsilon)|\nabla u^\varepsilon|^2 \varphi + \int_{\Omega} \beta(v^\varepsilon)h(u^\varepsilon)X'(u^\varepsilon) \cdot \nabla \varphi \\
+ \int_{\Omega} \beta'(v^\varepsilon)h'(u^\varepsilon)(X'(u^\varepsilon) \cdot \nabla u^\varepsilon) \varphi &= \langle f, h(u^\varepsilon)\varphi \rangle \\
\int_{\Omega} \nabla v^\varepsilon \nabla \psi + \int_{\Omega} \beta'(v^\varepsilon)X(u^\varepsilon) \cdot \nabla \psi &= \langle g, \psi \rangle.
\end{aligned} \tag{31}
$$

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Now, using the strong convergence of $u^\varepsilon$ to $u$ in $H^1_0(\Omega)$, it is easy to pass to the limit in each term of (31); this yields

$$\begin{cases}
\int_\Omega h(u)\nabla u \cdot \nabla \varphi + \int_\Omega h'(u)|\nabla u|^2 \varphi + \int_\Omega \beta(v)h(u)X'(u) \cdot \nabla \varphi \\
+ \int_\Omega \beta(v)h'(u)(X'(u) \cdot \nabla u) \varphi = \langle f, h(u) \varphi \rangle \\
\int_\Omega \nabla v \cdot \nabla \psi + \int_\Omega \beta'(v)X(u) \cdot \nabla \psi = \langle g, \psi \rangle.
\end{cases}$$

This completes the proof.

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