$H^2$-boundedness of the pullback attractors for non-autonomous 2D-Navier-Stokes equations in bounded domains

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Abstract

We prove some regularity results for the pullback attractors of a non-autonomous 2D-Navier-Stokes model in a bounded domain $\Omega$ of $\mathbb{R}^2$. We establish a general result about $(H^2(\Omega))^2 \cap V$-boundedness of invariant sets for the associate evolution process. Then, as a consequence, we deduce that, under adequate assumptions, the pullback attractors of the non-autonomous 2D-Navier-Stokes equations are bounded not only in $V$ but also in $(H^2(\Omega))^2$.

Key words: 2D-Navier-Stokes equations, pullback attractors, invariant sets, $H^2$-regularity.
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1 Introduction and setting of the problem

Let us consider the following problem for a non-autonomous 2D-Navier-Stokes system:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f(t) \quad \text{in} \quad \Omega \times (\tau, +\infty), \\
\nabla \cdot u &= 0 \quad \text{in} \quad \Omega \times (\tau, +\infty), \\
u > 0 \quad \text{and} \quad \Omega \subset \mathbb{R}^2 \quad \text{is a bounded open set}, \quad \text{with regular boundary} \quad \partial \Omega, \\
u > 0 \quad \text{is the kinematic viscosity,} \quad u \quad \text{is the velocity field of the fluid,} \\
p \quad \text{the pressure,} \quad \tau \in \mathbb{R} \quad \text{is a given initial time,} \\
\Omega \subset \mathbb{R}^2 \quad \text{is a given external force field.}
\end{aligned}
\]  

(1)

where \( \Omega \subset \mathbb{R}^2 \) is a bounded open set, with regular boundary \( \partial \Omega \), the number \( \nu > 0 \) is the kinematic viscosity, \( u \) is the velocity field of the fluid, \( p \) the pressure, \( \tau \in \mathbb{R} \) is a given initial time, \( u_\tau \) is the initial velocity field, and \( f(t) \) a given external force field.

To set our problem in the abstract framework, we consider the following usual abstract spaces (see [1] and [2–4]):

\[ V = \{ u \in (C^\infty_0(\Omega))^2 : \text{div} \ u = 0 \}, \]

\[ H = \text{the closure of} \ V \ \text{in} \ (L^2(\Omega))^2 \ \text{with inner product} \ (\cdot, \cdot) \ \text{and associate norm} \ ||\cdot||, \ \text{where for} \ u, v \in (L^2(\Omega))^2, \]

\[ (u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x)v_j(x)dx, \]

\[ V = \text{the closure of} \ V \ \text{in} \ (H^1_0(\Omega))^2 \ \text{with scalar product} \ ((\cdot, \cdot)) \ \text{and associate norm} \ ||\cdot||, \ \text{where for} \ u, v \in (H^1_0(\Omega))^2, \]

\[ ((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx. \]

We also consider the operator \( A : V \to V' \) defined by \( \langle Au, v \rangle = ((u, v)) \). Denoting \( D(A) = (H^2(\Omega))^2 \cap V \), then \( Au = -P\Delta u, \forall u \in D(A) \), is the Stokes operator \( (P \text{ is the ortho-projector from} \ (L^2(\Omega))^2 \ \text{onto} \ H) \).

Now we define the continuous trilinear form \( b \) on \( V \times V \times V \) by

\[ b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u, v, w \in V. \]
It is well known that
\[ b(u, v, v) = 0 \quad \text{for all} \ u, v \in V. \quad (2) \]

We remember (see [2] or [3]) that there exists a constant \( C_1 > 0 \) only dependent on \( \Omega \) such that
\[ |b(u, v, w)| \leq C_1 |u|^{1/2} |v|^{1/2} |w|^{1/2} |w|^{1/2}, \quad \forall u, v, w \in V, \quad (3) \]
and
\[ |b(u, v, w)| \leq C_1 |Au| |v| |w|, \quad \forall u \in D(A), v \in V, w \in H, \quad (4) \]
Assume that \( u_\tau \in H \) and \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \).

**Definition 1.1** A solution of \((1)\) is a function \( u \in C([\tau, T]; H) \cap L^2(\tau, T; V) \) for all \( T > \tau \), with \( u(\tau) = u_\tau \), such that for all \( v \in V \),
\[
\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = (f(t), v),
\]
where the equation must be understood in the sense of \( D'(\tau, +\infty) \).

Under the conditions above (e.g. cf. [2] or [3]), there exists a unique solution \( u(\cdot) = u(\cdot; \tau, u_\tau) \) of \((1)\). Moreover, this solution \( u \) satisfies that \( u \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; (H^2(\Omega))^2) \) for every \( \varepsilon > 0 \) and \( T > \tau + \varepsilon \). In fact, if \( u_\tau \in V \), then \( u \in C([\tau, T]; V) \cap L^2(\tau, T; (H^2(\Omega))^2) \) for every \( T > \tau \).

Therefore, we can define a process \( U = \{U(t, \tau), \ \tau \leq t\} \) in \( H \) as
\[
U(t, \tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in H, \quad \forall \tau \leq t, \quad (6)
\]
and the restriction of this process to \( V \) is a process in \( V \).

A pullback attractor for the process \( U \) defined by \((6)\) (cf. [5–7]) is a family \( \mathcal{A} = \{A(t) : t \in \mathbb{R}\} \) of compact subsets of \( H \) such that
\begin{enumerate}
  \item [(invariance)] \( U(t, \tau)A(\tau) = A(t) \) for all \( \tau \leq t \),
  \item [(pullback attraction)] \( \lim_{\tau \to -\infty} \sup_{u_\tau \in H} \inf_{v \in A(t)} |U(t, \tau)u_\tau - v| = 0 \), for all \( t \in \mathbb{R} \), for any bounded subset \( B \subset H \).
\end{enumerate}
It can be proved (see [9]) that, under the above conditions, if in addition \( f \) satisfies
\[
\int_{-\infty}^{0} e^{\mu r} |f(r)|^2 dr < +\infty,
\]
for some \( 0 < \mu < 2\nu\lambda_1 \), where \( \lambda_1 \) denotes the first eigenvalue of the Stokes operator \( A \), then there exists a pullback attractor for the process \( U \) defined by (6).

Several studies on this model have already been published (cf. [5], [8,9]). However, as far as we know, no one refers to the \( H^2 \)-regularity we will consider in this paper.

In the next section we prove some results which, in particular, imply that, under suitable assumptions, any pullback attractor \( \mathcal{A} \) for \( U \) satisfies that, for every \( t \in \mathbb{R} \) (for similar results for reaction-diffusion equations see [10], and for related results for Navier-Stokes equations see [11]).

2 \( H^2 \)-boundedness of invariant sets

In this section we prove that, under suitable assumptions, any family of bounded subsets of \( H \) which is invariant for the process \( U \), is in fact bounded in \( (H^2(\Omega))^2 \cap V \).

First, we recall a result (cf. [2]) which will be used below.

**Lemma 2.1** Let \( X, Y \) be Banach spaces such that \( X \) is reflexive, and the inclusion \( X \subset Y \) is continuous. Assume that \( \{u_n\} \) is a bounded sequence in \( L^\infty(t_0,T;X) \) such that \( u_n \rightharpoonup u \) weakly in \( L^q(t_0,T;X) \) for some \( q \in [1, +\infty) \) and \( u \in C^0([t_0,T];Y) \).

Then, \( u(t) \in X \) and \( \|u(t)\|_X \leq \liminf_{n \to 1} \|u_n\|_{L^\infty(t_0,T;X)} \), for all \( t \in [t_0,T] \).

For each integer \( n \geq 1 \), we denote by \( u_n(t) = u_n(t; \tau, u_\tau) \) the Galerkin approximation of the solution \( u(t; \tau, u_\tau) \) of (1), which is given by
\[
u_{nj}(t)w_j, \quad j = 1, \ldots, n,
\]
and is the solution of
\[
\begin{cases}
\frac{d}{dt} (u_n(t), w_j) + \nu((u_n(t), w_j)) + b(u_n(t), u_n(t), w_j) = (f(t), w_j), \\
(u_n(\tau), w_j) = (u_\tau, w_j) \quad j = 1, \ldots, n,
\end{cases}
\]
where \( \{w_j : j \geq 1\} \subset V \) is the Hilbert basis of \( H \) formed by the eigenvectors of the Stokes operator \( A \). Observe that by the regularity of \( \Omega \), all the \( w_j \) belong to \((H^2(\Omega))^2\).

We first prove the following result.

**Proposition 2.2** Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \). Then, for any bounded set \( B \subset H \), any \( \tau \in \mathbb{R} \), any \( \varepsilon > 0 \) and any \( t > \tau + \varepsilon \), the following three properties are satisfied:

i) The set \( \{u_n(r; \tau, u_\tau) : r \in [\tau + \varepsilon, t], u_\tau \in B, n \geq 1\} \), is a bounded subset of \( V \).

ii) The set of functions \( \{u_n(\cdot; \tau, u_\tau) : u_\tau \in B, n \geq 1\} \), is a bounded subset of \( L^2(\tau + \varepsilon, t; D(A)) \).

iii) The set of time derivatives functions \( \{u_n'(\cdot; \tau, u_\tau) : u_\tau \in B, n \geq 1\} \), is a bounded subset of \( L^2(\tau + \varepsilon, t; H) \).

**Proof.**

Let us fix a bounded set \( B \subset H \), \( \tau \in \mathbb{R} \), \( \varepsilon > 0 \), \( t > \tau + \varepsilon \), and \( u_\tau \in B \).

Multiplying by \( \gamma_{nj}(t) \) in (7), summing from \( j = 1 \) to \( n \), and using (2), we obtain

\[
\frac{1}{2} \frac{d}{d\theta} |u_n(\theta)|^2 + \nu \|u_n(\theta)\|^2 = (f(\theta), u_n(\theta)) \quad \text{a.e. } \theta > \tau. \tag{8}
\]

Observing that

\[
|(f(\theta), u_n(\theta))| \leq \frac{1}{2\nu\lambda_1} |f(\theta)|^2 + \frac{\nu\lambda_1}{2} |u_n(\theta)|^2
\]

\[
\leq \frac{1}{2\nu\lambda_1} |f(\theta)|^2 + \frac{\nu}{2} \|u_n(\theta)\|^2,
\]

from (8) we deduce

\[
\frac{d}{d\theta} |u_n(\theta)|^2 + \nu \|u_n(\theta)\|^2 \leq \frac{1}{\nu\lambda_1} |f(\theta)|^2,
\]

and integrating between \( \tau \) and \( r \),

\[
|u_n(r)|^2 + \nu \int_{\tau}^{r} \|u_n(\theta)\|^2 d\theta \tag{9}
\]

\[
\leq |u_\tau|^2 + \frac{1}{\nu\lambda_1} \int_{\tau}^{t} |f(\theta)|^2 d\theta, \quad \forall r \in [\tau, t], \forall n \geq 1.
\]

Now, multiplying in (7) by \( \lambda_j \gamma_{nj}(t) \), where \( \lambda_j \) is the eigenvalue associated to the eigenvector \( w_j \), and summing from \( j = 1 \) to \( n \), we obtain
\[
\frac{1}{2} \frac{d}{d\theta} \|u_n(\theta)\|^2 + \nu |Au_n(\theta)|^2 + b(u_n(\theta), u_n(\theta), Au_n(\theta)) = (f(\theta), Au_n(\theta)),
\] (10)
a.e. \(\theta > \tau\). Observe that
\[
| (f(\theta), Au_n(\theta)) | \leq \frac{1}{\nu} |f(\theta)|^2 + \frac{\nu}{4} |Au_n(\theta)|^2,
\]
and by (5) and Young’s inequality,
\[
|b(u_n(\theta), u_n(\theta), Au_n(\theta))| \leq C_1|u_n(\theta)|^{1/2}\|u_n(\theta)\||Au_n(\theta)|^{3/2}
\leq \frac{\nu}{4} |Au_n(\theta)|^2 + C^{(\nu)}|u_n(\theta)|^2\|u_n(\theta)\|^4,
\] (11)
where \(C^{(\nu)} = 27C_4^1(4\nu^3)^{-1}\).

Thus, from (10) we deduce
\[
\frac{d}{d\theta} \|u_n(\theta)\|^2 + \nu |Au_n(\theta)|^2 \leq \frac{2}{\nu} |f(\theta)|^2 + 2C^{(\nu)}|u_n(\theta)|^2\|u_n(\theta)\|^4,
\] (12)
a.e. \(\theta > \tau\).

From this inequality, in particular we deduce
\[
\|u_n(r)\|^2 \leq \|u_n(s)\|^2 + \frac{2}{\nu} \int_{\tau}^{r} |f(\theta)|^2 d\theta + 2C^{(\nu)} \int_{s}^{r} |u_n(\theta)|^2\|u_n(\theta)\|^4 d\theta
\]
for all \(\tau \leq s \leq r \leq t\), and therefore, by Gronwall’s lemma,
\[
\|u_n(r)\|^2 \leq \left(\|u_n(s)\|^2 + \frac{2}{\nu} \int_{\tau}^{r} |f(\theta)|^2 d\theta\right)\exp \left(2C^{(\nu)} \int_{\tau}^{r} |u_n(\theta)|^2\|u_n(\theta)\|^2 d\theta\right)
\]
for all \(\tau \leq s \leq r \leq t\).

Integrating this last inequality for \(s\) between \(\tau + \varepsilon/2\) and \(r\), we obtain
\[
(r - \tau - \frac{\varepsilon}{2}) \|u_n(r)\|^2 \leq \left(\int_{\tau}^{r} \|u_n(s)\|^2 ds + \frac{2(t - \tau)}{\nu} \int_{\tau}^{r} |f(\theta)|^2 d\theta\right)
\times \exp \left(2C^{(\nu)} \int_{\tau}^{r} |u_n(\theta)|^2\|u_n(\theta)\|^2 d\theta\right)
\]
for all \(\tau + \varepsilon/2 \leq r \leq t\), and in particular,
\[
\|u_n(r)\|^2 \leq \frac{2}{\varepsilon} \left( \int_{\tau}^{t} \|u_n(s)\|^2 \, ds + \frac{2(t - \tau)}{\nu} \int_{\tau}^{t} |f(\theta)|^2 \, d\theta \right) \exp \left( 2C(\nu) \int_{\tau}^{t} |u_n(\theta)|^2 \|u_n(\theta)\|^2 \, d\theta \right)
\] (13)

for all \( \tau + \varepsilon \leq r \leq t \), for any \( n \geq 1 \).

From (9) and (13), the assertion in i) holds. Moreover, by (12),

\[
\nu \int_{\tau+\varepsilon}^{t} |Au_n(\theta)|^2 \, d\theta \leq \|u_n(\tau + \varepsilon)\|^2 + \frac{2}{\nu} \int_{\tau}^{t} |f(\theta)|^2 \, d\theta + 2C(\nu) \int_{\tau+\varepsilon}^{t} |u_n(\theta)|^2 \|u_n(\theta)\|^2 \, d\theta,
\]

and therefore, by i), the assertion in ii) holds.

On the other hand, multiplying by the derivative \( \gamma'_{nj}(t) \) in (7), and summing from \( j = 1 \) till \( n \), we obtain

\[
|u'_n(\theta)|^2 + \frac{\nu}{2} \frac{d}{d\theta} \|u_n(\theta)\|^2 + b(u_n(\theta), u_n(\theta), u'_n(\theta)) = (f(\theta), u'_n(\theta)),
\] (14)
a.e. \( \theta > \tau \).

Observing that

\[
| (f(\theta), u'_n(\theta)) | \leq \frac{1}{4} |u'_n(\theta)|^2 + |f(\theta)|^2,
\]

and by (4)

\[
|b(u_n(\theta), u_n(\theta), u'_n(\theta))| \leq C_1 |Au_n(\theta)\| u_n(\theta) \||u'_n(\theta)|
\leq \frac{1}{4} |u'_n(\theta)|^2 + C_1^2 |Au_n(\theta)|^2 \|u_n(\theta)\|^2,
\]

we obtain from (14)

\[
|u'_n(\theta)|^2 + \nu \frac{d}{d\theta} \|u_n(\theta)\|^2 \leq 2|f(\theta)|^2 + 2C_1^2 |Au_n(\theta)|^2 \|u_n(\theta)\|^2.
\]

Integrating this last inequality, we deduce that

\[
\int_{\tau+\varepsilon}^{t} |u'_n(\theta)|^2 \, d\theta \leq \nu \|u_n(\tau + \varepsilon)\|^2 + 2 \int_{\tau}^{t} |f(\theta)|^2 \, d\theta + 2C_1^2 \sup_{\theta \in [\tau + \varepsilon, t]} \|u_n(\theta)\|^2 \int_{\tau+\varepsilon}^{t} |Au_n(\theta)|^2 \, d\theta,
\]

and therefore iii) follows from i) and ii).
Corollary 2.3 Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \). Then, for any bounded set \( B \subset H \), any \( \tau \in \mathbb{R} \), any \( \varepsilon > 0 \), and any \( t > \tau + \varepsilon \), the set \( \bigcup_{r \in [\tau + \varepsilon, t]} U(r, \tau)B \) is a bounded subset of \( V \).

Proof. This is a straightforward consequence of Lemma 2.1, assertion i) in Proposition 2.2, and the well known fact (e.g. cf. [1–4]) that for all \( u_\tau \in B \) the Galerkin approximations \( u_n(\cdot; \tau, u_\tau) \) converge weakly to \( u(\cdot; \tau, u_\tau) \) in \( L^2(\tau, t; V) \), and \( u(\cdot; \tau, u_\tau) \in C([\tau, t]; H) \).

Assuming additional regularity for the time derivative of \( f \), we can improve the above results.

Proposition 2.4 Assume that \( f \in W^{1,2}_{\text{loc}}(\mathbb{R}; H) \). Then, for any bounded set \( B \subset H \), any \( \tau \in \mathbb{R} \), any \( \varepsilon > 0 \), and any \( t > \tau + \varepsilon \), the following two properties are satisfied:

iv) The set of time derivatives \( \{ u'_n(r; \tau, u_\tau) : r \in [\tau + \varepsilon, t], u_\tau \in B, n \geq 1 \} \) is a bounded subset of \( H \).

v) The set \( \{ u_n(r; \tau, u_\tau) : r \in [\tau + \varepsilon, t], u_\tau \in B, n \geq 1 \} \) is a bounded subset of \( D(A) \).

Proof. Let us fix a bounded set \( B \subset H \), \( \tau \in \mathbb{R} \), \( \varepsilon > 0 \), \( t > \tau + \varepsilon \), and \( u_\tau \in B \).

As we are assuming that \( f \in W^{1,2}_{\text{loc}}(\mathbb{R}; H) \), we can differentiate with respect to time in (7), and then, multiplying by \( \gamma'_{nj}(t) \), and summing from \( j = 1 \) to \( n \), we obtain

\[
\frac{1}{2} \frac{d}{d\theta} |u'_n(\theta)|^2 + \nu |u'_n(\theta)|^2 + b(u'_n(\theta), u_n(\theta), u'_n(\theta)) = (f'(\theta), u'_n(\theta))
\]
a.e. \( \theta > \tau \).

From this inequality, taking into account that

\[
| (f'(\theta), u'_n(\theta)) | \leq \frac{\nu}{2} |u'_n(\theta)|^2 + \frac{1}{2\nu \lambda_1} |f'(\theta)|^2,
\]

and by (3)

\[
|b(u'_n(\theta), u_n(\theta), u'_n(\theta))| \leq C_1 |u'_n(\theta)||u_n(\theta)||u'_n(\theta)|
\leq \frac{\nu}{2} |u'_n(\theta)|^2 + \frac{C_2}{2\nu} |u'_n(\theta)|^2 |u_n(\theta)|^2,
\]

we deduce

\[
\frac{d}{d\theta} |u'_n(\theta)|^2 \leq \frac{1}{\nu \lambda_1} |f'(\theta)|^2 + \frac{C_2}{\nu} |u'_n(\theta)|^2 |u_n(\theta)|^2.
\]
Integrating in the last inequality,

$$|u'_n(r)|^2 \leq |u'_n(s)|^2 + \frac{1}{\nu \lambda_1} \int^t_\tau |f'(\theta)|^2 \, d\theta + \frac{C^2_1}{\nu} \int^s_\tau |u'_n(\theta)|^2 \|u_n(\theta)\|^2 \, d\theta,$$

for all $\tau \leq s \leq r \leq t$.

Thus, by Gronwall’s inequality,

$$|u'_n(r)|^2 \leq \left(|u'_n(s)|^2 + \frac{1}{\nu \lambda_1} \int^t_\tau |f'(\theta)|^2 \, d\theta\right) \exp \left(\frac{C^2_1}{\nu} \int^t_{\tau + \varepsilon/2} \|u_n(\theta)\|^2 \, d\theta\right),$$

for all $\tau + \varepsilon/2 \leq s \leq r \leq t$.

Now, integrating this inequality with respect to $s$ between $\tau + \varepsilon/2$ and $r$, we obtain

$$(r - \tau - \varepsilon/2) |u'_n(r)|^2 \leq \left(\int^t_{\tau + \varepsilon/2} |u'_n(s)|^2 \, ds + \frac{t - \tau}{\nu \lambda_1} \int^t_\tau |f'(\theta)|^2 \, d\theta\right) 
\times \exp \left(\frac{C^2_1}{\nu} \int^t_{\tau + \varepsilon/2} \|u_n(\theta)\|^2 \, d\theta\right),$$

for all $\tau + \varepsilon/2 \leq r \leq t$, and any $n \geq 1$. In particular, thus,

$$|u'_n(r)|^2 \leq 2 \left(\int^t_{\tau + \varepsilon/2} |u'_n(s)|^2 \, ds + \frac{t - \tau}{\nu \lambda_1} \int^t_\tau |f'(\theta)|^2 \, d\theta\right) 
\times \exp \left(\frac{C^2_1}{\nu} \int^t_{\tau + \varepsilon/2} \|u_n(\theta)\|^2 \, d\theta\right),$$

for all $\tau + \varepsilon \leq r \leq t$, and any $n \geq 1$.

From this inequality and properties i) and iii) in Proposition 2.2, we obtain iv).

On the other hand, multiplying again in (7) by $\lambda_j \gamma_{nj}(t)$, and summing once more from $j = 1$ to $n$, we obtain

$$(u'_n(r), Au_n(r)) + \nu |Au_n(r)|^2 + b(u_n(r), u_n(r), Au_n(r)) = (f(r), Au_n(r)), \quad (15)$$

for all $r \geq \tau$. But

$$|(u'_n(r), Au_n(r))| \leq \frac{2}{\nu} |u'_n(r)|^2 + \frac{\nu}{8} |Au_n(r)|^2,$$
Therefore, taking into account (11), we deduce from (15) that
\[
\frac{\nu}{2} |Au_n(r)|^2 \leq \frac{2}{\nu} (|u'_n(r)|^2 + |f(r)|^2) + C'(\nu) |u_n(r)|^2 \|u_n(r)\|^4 \tag{16}
\]
for all \( r \geq \tau \).

Thus, since in particular \( f \in C(\mathbb{R}; H) \), from i) in Proposition 2.2, iv) and inequality (16), we deduce v).

As a direct consequence of the above, we can now establish our main results.

**Theorem 2.5** Assume that \( f \in W^{1,2}_{\text{loc}}(\mathbb{R}; H) \). Then, for any bounded set \( B \subset H \), any \( \tau \in \mathbb{R} \), any \( \varepsilon > 0 \), and any \( t > \tau + \varepsilon \), the set \( \bigcup_{r \in [\tau+\varepsilon,t]} U(r,\tau)B \) is a bounded subset of \( D(A) = (H^2(\Omega))^2 \cap V \).

**Proof.** This follows from Lemma 2.1, Proposition 2.4, and the well known facts that \( u_n(\cdot; \tau, u_\tau) \) converges weakly to \( u(\cdot; \tau, u_\tau) \) in \( L^2(\tau,t; V) \), and \( u(\cdot; \tau, u_\tau) \) belongs to \( C([\tau+\varepsilon,t]; V) \).

**Theorem 2.6** Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \), and \( \hat{A} = \{ A(t) : t \in \mathbb{R} \} \) is a family of bounded subsets of \( H \), such that \( U(t,\tau)A(\tau) = A(t) \) for any \( \tau \leq t \). Then:

i) For any \( T_1 < T_2 \), the set \( \bigcup_{t \in [T_1,T_2]} A(t) \) is a bounded subset of \( V \).

ii) If moreover \( f' \in L^2_{\text{loc}}(\mathbb{R}; H) \), then for any \( T_1 < T_2 \), the set \( \bigcup_{t \in [T_1,T_2]} A(t) \) is a bounded subset of \( (H^2(\Omega))^2 \cap V \).

**Proof.** It is enough to observe that if \( \tau < T_1 - 1 \) is fixed, then
\[
\bigcup_{t \in [T_1,T_2]} A(t) \subset \bigcup_{t \in [\tau+1,T_2]} U(t,\tau)A(\tau).
\]
Now, apply Corollary 2.3 and Theorem 2.5.
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References


