BREATHERS IN FPU SYSTEMS, NEAR AND FAR FROM THE PHONON BAND

B. SÁNCHEZ-REY†, JFR. ARCHILLA†, G JAMES‡, AND J. CUEVAS†
†Nonlinear Physics Group, University of Sevilla, Spain
‡Département de Génie Mathématique, INSA de Toulouse, France
Email: bernardo@us.es

Introduction. This work is motivated by a recent breathers existence proof in the one dimensional FPU system, given by the equations:

\[
\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},
\]

where \(V\) is a smooth interaction potential satisfying \(V(0) = 0\) and \(V''(0) > 0\). Using a center manifold technique, one can prove the existence of small amplitude breathers (SAB) with frequencies \(\omega_b\) slightly above the phonon band if \(B = \frac{1}{2} V''(0)V''(4)(0) - (V''(3)(0))^2 > 0\), and their non-existence for \(B < 0\). Our aim is to test numerically the range of validity of this theoretical result and to explore new phenomena. For this purpose we shall fix \(V(u) = u^2/2 + a u^3 + \frac{1}{4} u^4\), which yields \(B = 3(1 - 12a^2)\).

We work with the difference variables \(u_n = x_n - x_{n-1}\) more suitable for the use of our numerical method. We also use periodic boundary conditions \(u_{n+2p}(t) = u_n(t)\) so that the maximum frequency of the linear phonons is exactly 2 as in the infinite lattice. Our computations are performed using a numerical scheme based on the anti-continuous limit and Newton method.

Test and range of validity. First, we have computed numerically SAB (i.e. breathers whose amplitudes go to zero when \(w_b \to 2^+\)) in the case when \(B > 0\). We have obtained breathers with symmetries \(u_n(t) = u_{-n}(t)\) (Page mode) and \(u_n(t) = u_{-n-1}(t + T_b/2)\) (Sievers-Takeno mode), where \(T_b = 2\pi/\omega_b\) is the breather period. The force \(y_n = V'(u_n)\) is the variable used in reference. In Fig.1 (left) it is shown that the maxima of the force are of order \(\mu^{1/2}\) when \(\mu = w_b - 2 \to 0^+\), as predicted by the theory, up to relatively large values. Thus if \(B > 0\) breathers exist for any small value of energy in our FPU system (1).

Another property of these SAB is that their width diverges when \(w_b \to 2^+\). More precisely the theory predicts that their spatial extend is of order \(\mu^{-1/2}\), which is in accordance with our numerical observations.
Other numerical observations. For $B > 0$, we have numerically continued the SAB as $\omega_b$ goes away from the phonon band. We have found that the maxima amplitudes of the oscillations, $\sup |u_n|$, are also approximately linear functions of $\mu^{1/2}$. This is expected for small $\mu$, since $u_n = y_n + O(y_n^2)$, but it occurs surprisingly far from the phonon band, at least until values of $\mu \approx 1$ (see fig.1, left). We have also checked that the Page mode fits very well to the NLS soliton $u_n(t) = \alpha \sqrt{\mu} (-1)^n \cos(\omega_b t) \cosh(\beta \sqrt{\mu} n)^{-1}$, even far from the top of the phonon band.

![Figure 1. Left: Force (squares) and amplitudes (circles) maxima versus $\mu^{1/2}$. The cubic coefficient in $V$ is $a = -0.1$ ($B = 2.64$). Right: Comparison between a SAB (full circles) for $a = -0.1$ and a LAB (blank squares) for $a = -1/3$ ($B = -1$) having the same frequency $w_b = 2.01$. The dashed line represents the linear phonon with frequency 2.](image)

For $B < 0$ and $V$ strictly convex ($-\frac{1}{12} < |a| < \frac{1}{\sqrt{5}}$), breathers exist near the top of the phonon band but they are large amplitude breathers$^1$ (LAB), i.e. their amplitudes do not go to zero when $w_b \to 2^+$. As a consequence there is an energy gap for breathers creation in these FPU systems. In figure 1 (right) we compare a SAB and a LAB having the same frequency $w_b = 2.01$. We have found LAB with the same symmetries as SAB (Page and Sievers-Takeno modes). The Page mode fits very well to an exponential profile having the form $u_n(t) = \alpha(\omega_b) (-1)^n \cos(\omega_b t) |\sigma(\omega_b)|^{1/|n|}$ where $\sigma(\omega_b) = 1 - (\omega_b^2)/2 + (\omega_b/2)(\omega_b^2 - 4)^{1/2} \in (-1, 0)$. As (1) is formulated as a mapping in a loop space$^2$ and $\omega_b > 2$, the linearized operator has a purely hyperbolic spectrum and the constant $\sigma(\omega_b)$ is the closest eigenvalue to $-1$ (with $\sigma(2) = -1$). Consequently, for $\omega_b \approx 2$ one can ask if the iterated map admits a global center manifold containing these LAB.

References