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Facultad de Matemáticas
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# Propiedad de punto fijo, normas equivalentes y espacios de funciones no-conmutativos

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# Fixed Point Property, equivalent norms and noncommutative function spaces

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*"Son tantas las cosas que tienen que pasar para que dos personas se encuentren. De eso tratan las matemáticas." (Guillermo Arriaga, del guión de 21 gramos.)*

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# Contents

# Introduction

Let $C$ be a set and $T : C \to C$ a selfmapping. We say that $T$ has a fixed point if there exists $x \in C$ such that $Tx = x$. Some properties about the mapping $T$ and the domain $C$ can assure the existence of fixed points. When $C$ is a closed subset of a Banach space $(X, \| \cdot \|)$, we say that $T$ is Lipschitzian if there is some $K \in \mathbb{R}$ such that

$$\|Tx - Ty\| \leq K\|x - y\| \qquad \forall x, y \in C.$$

If $K < 1$, $T$ is named a contraction mapping. In this case $T$ has a unique fixed point by Banach's Contraction Theorem, which was proved by S. Banach in his Ph. D. Thesis in 1922 [4].

If $K > 1$, general results can not be obtained that guarantee the existence of fixed points. In fact, for every $K > 1$, it is possible to find a Lipsichitzian mapping defined on the unit ball of a Hilbert space, with constant $K$ and without fixed points [48]. Furthermore, P. K. Lin and Y. Sternfeld proved in [58] the following: if $C$ is a convex noncompact subset of a Banach space $X$, then for all $K > 1$ there exists a mapping $T : C \to C$ whose Lipschitz constant is equal to $K$ and such that $T$ is fixed point free.

If $K = 1$, the mapping $T$ is called nonexpansive. A simple translation in $\mathbb{R}^n$ shows that Banach's Contraction Theorem does not extend to the setting of nonexpansive mappings. However, some positive results concerning the existence of fixed points for this class of mappings were found in 1965 by F.E. Browder [11] and D. Göhde [35] for uniformly convex Banach spaces and by W. Kirk [48] for reflexive Banach spaces with normal structure. Since then, many authors have studied the problem of the existence of fixed points for nonexpansive mappings and many positive results have been obtained (see for instance [34, 50] and the references therein). Namely, it is said that a Banach space $X$ has the fixed point property (FPP) if every nonexpansive mapping defined from a closed convex bounded subset into itself has a fixed point. It is well-known that the geometry of the Banach space plays a fundamental role to assure the FPP. In fact, Kirk's result [48] means that a reflexive Banach space with normal structure has the FPP. In particular, uniformly convex or uniformly smooth Banach spaces have the FPP. Many other geometric properties are known to imply the FPP for reflexive Banach spaces (uniform Kadec Klee property, uniform Opial condition, existence of a monotone unconditional basis, etc). On the other hand, the classical nonreflexive Banach spaces $\ell_1$, $c_0$ and $L_1$ do not have the FPP (in fact $L_1$ does not satisfy a stronger condition called the weakly fixed point property [2]). For a long time, it was conjectured that all Banach spaces with the FPP had to be reflexive. In 2008, P.K. Lin [56] gave an unexpected answer to this conjecture: he found the first known nonreflexive Banach space with the FPP. In fact, the Banach space given by P.K. Lin is the sequence space $\ell_1$ endowed with the equivalent norm

$$|||x||| = \sup_k \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|$$

where $x = (x_n) \in \ell_1$.

It was previously known that there exist some Banach spaces that can not be renormed to have the FPP: if $\Gamma$ is uncountable, every renorming of $\ell_1(\Gamma)$ or $c_0(\Gamma)$ contains an asymptotically isometric copy of either $\ell_1$ or $c_0$ respectively and therefore there is no equivalent norm in these spaces with the FPP. Moreover, every renorming of the Banach space $\ell_\infty$ contains $\ell_1(\Gamma)$ for some uncountable $\Gamma$; so $\ell_\infty$ is another Banach space which does not have an equivalent norm satisfying the FPP. On the other hand, T. Domínguez Benavides proved in [17] that every reflexive Banach space can be renormed to have the FPP. This leads us to the following question: Which type of nonreflexive Banach spaces can be renormed to have the FPP?

The main object of this Dissertation is to study new families of nonreflexive Banach spaces which can be renormed to have the FPP. Most of them are closed subspaces of $L_1(\mu)$ of more generally noncommutative $L_1$-spaces. In the last chapter we will go back to the sequence space $\ell_1$ and we will find some new renormings of $\ell_1$, which will let us assure that the set of equivalent norms on $\ell_1$ with the FPP has some "kind" of linear structure.

The Dissertation is based on the four papers [39, 40, 41, 42] and it is divided in five chapters as follows:

In the first chapter we give some definitions and generic results about the FPP and renorming theory. Also we state some basic notation that we will use throughout this memory.

In the second chapter, we find some new classes of nonreflexive Banach spaces which, under an equivalent renorming, satisfy the FPP. Our techniques are inspired by those of P.K. Lin's paper [56] but our applications go beyond the sequence space $\ell_1$ as we will illustrate with many examples. In order to do that, we consider a family of seminorms $\{R_k(\cdot)\}_k$ on certain Banach spaces and a nondecreasing sequence $(\gamma_k) \subset$

$(0, 1)$ such that $\lim_k \gamma_k = 1$. We define a norm in the following way

$$|||x||| = \sup_k \gamma_k R_k(x).$$

We will prove that, under specific conditions verified by the family of seminorms $\{R_k(\cdot)\}_k$, the space $(X, |||\cdot|||)$ has the FPP. As particular cases, we will recover P.K. Lin's result and we will find some new renormings in $\ell_1$ with the FPP since we can consider any non decreasing sequence $(\gamma_k)$ in $(0, 1)$ which has limit equal to 1 instead of the sequence $\frac{8^k}{1+8^k}$. As applications, we will renorm the Fourier-Stieltjes algebra of a separable compact group to have the FPP. Notice that if $G$ is locally compact, its Fourier-Stieltjes algebra $B(G)$ has the FPP if and only if $G$ is finite [52]. We also find some new classes of nonreflexive Banach spaces with the FPP which are nonisomorphic to any subspace of $\ell_1$, as for instance, a renorming of $X := \oplus_1 \sum_n \ell_p^n$.

In the third chapter, we will apply our results to the particular case of some subspaces of $L_1(\mu)$ for a $\sigma$-finite measure. It is known that a closed subspace $X$ of $L_1(\mu)$ has the FPP if and only if $X$ is reflexive [61, 22]. We obtain a sufficient condition to assure that a nonreflexive subspace $X$ of $L_1(\mu)$ can be renormed to have the FPP and we give some new examples of nonreflexive subspaces of $L_1[0, 1]$, not isomorphic to $\ell_1$, that can be renormed with the FPP. In the second section of this chapter, we focus on noncommutative $L_1$-spaces associated to finite von Neumann algebras. These spaces are the preduals of finite von Neumann algebras and they can be considered as an extension of the classical Lebesgue measure spaces (a particular example of a finite von Neumann algebra is $L_\infty[0, 1]$ and its predual is $L_1[0, 1]$). It is known that the predual of a von Neumann algebra $\mathcal{M}$, $L_1(\mathcal{M})$ and all its nonreflexive subspaces contain asymptotically isometric copies of $\ell_1$ [69] and consequently they fail to have the FPP [23]. The main object of this section is to obtain a family of norms, equivalent to the usual norm in $L_1(\mathcal{M})$, which have a better behaviour with respect to the existence of fixed points for nonexpansive mappings defined on a closed convex bounded subset of $L_1(\mathcal{M})$ and we give a sufficient condition (with a topological flavour) so that a nonreflexive subspace of $L_1(\mathcal{M})$ can be renormed with the FPP. As a consequence we will cover the examples given in the previous section for the particular case of the measure space $L_1(\mu)$ and we will deduce that if $\mathcal{M}$ is any atomic finite von Neumann algebra, there exists an equivalent norm in $L_1(\mathcal{M})$ satisfying the fixed point property.

In the fourth chapter, we obtain a renorming result for the fixed point property for affine nonexpansive mappings in the context of the noncommuative $L_1$ spaces generated by a finite von Neumann algebra. We can apply this result to the particular case of $L_1(\mu)$. It is worth mentioning that there exist some affine and $\|\cdot\|_1$-nonexpansive mappings in these spaces which do not have fixed points.

In the fifth chapter, we concentrate on the sequence space $\ell_1$ and on the set of equivalent norms which satisfy the FPP. It is worth saying that in a subsequent paper, P. K. Lin [57] established four conditions which are sufficient to assure that a renorming on $\ell_1$ verifies the FPP. We will check that many of the norms obtained in this chapter with the FPP do not satisfy P. K. Lin's conditions and that we can add two norms together, one with the FPP and the other one failing the FPP and we can still obtain a new renorming which satisfies the FPP. Moreover, we study the stability problem for the fixed point property in the Banach space $\ell_1$. This problem can be formulated as follows: let $X$ be a Banach space with the FPP and $Y$ a Banach space isomorphic to $X$. Does there exist some constant $K = K(X) > 1$ such that $Y$ has the FPP whenever $d(X, Y) < K$? This question has been widely studied by many researchers and many geometric properties have turned to be useful to determinate an upper bound for the Banach-Mazur distance which assures the preservation of the FPP. We will prove that P.K. Lin's norm, along with most of the renormings of $\ell_1$ with the FPP, fail to produce stability of the FPP. This fact contrasts with the case of the classical norms in reflexive Banach spaces with the FPP. In the last section of the chapter we consider the convex cone $\mathcal{P}$ of all norms on $\ell_1$ which are equivalent to $\|\cdot\|_1$ and its subset $\mathcal{A}$ given by the norms of $\mathcal{P}$ satisfying the FPP. We deduce that $\mathcal{A}$ contains rays and we study some properties concerning to the structure of the sets $\mathcal{A}$ and $\mathcal{P} \setminus \mathcal{A}$.

# Resumen

Sean $C$ un conjunto y $T : C \to C$ un operador. Decimos que $T$ tiene un punto fijo si existe $x \in C$ tal que $Tx = x$. Algunas propiedades acerca del operador $T$ y el dominio $C$ pueden asegurar la existencia de puntos fijos. Cuando $C$ es un subconjunto cerrado de un espacio de Banach $(X, \|\cdot\|)$, decimos que $T$ es Lipschitziano si existe $K \in \mathbb{R}$ tal que

$$\|Tx - Ty\| \leq K\|x - y\| \qquad \forall x, y \in C.$$

Si $K < 1$, al operador $T$ se le llama una contracción. En este caso $T$ tiene un único punto fijo por el Principio de Contracción de Banach, el cual fue probado por S. Banach en su tesis doctoral en 1922 [4].

Si $K > 1$, no se pueden obtener resultados generales que garanticen la existencia de puntos fijos. De hecho, para todo $K > 1$, es posible construir un operador Lipsichitziano definido sobre la bola unidad de un espacio de Hilbert, con constante de Lipschitz igual a $K$ y sin puntos fijos [48]. Además, P. K. Lin e Y. Sternfeld demostraron lo siguiente [58]: Si $C$ es un subconjunto convexo y no compacto de un espacio de Banach $X$, entonces para todo $K > 1$ existe un operador $T : C \to C$ cuya constante de Lipschitz es igual a $K$ y tal que $T$ no posee puntos fijos.

Si $K = 1$, se dice que el operador $T$ es no expansivo. Una traslación en $\mathbb{R}^n$ es un ejemplo simple que muestra que el Principio de contracción de Banach no se extiende al marco de los operadores no expansivos. Sin embargo, algunos resultados positivos relativos a la existencia de puntos fijos para esta clase de operadores fueron encontrados en 1965 por F.E. Browder [11] y D. Göhde [35] para espacios de Banach

uniformemente convexos y por W. Kirk [48] para espacios de Banach reflexivos con estructura normal. Desde entonces, diversos autores han estudiado el problema de la existencia de puntos fijos para operadores no expansivos y varios resultados positivos han sido obtenidos (ver por ejemplo [34, 50] y las referencias que en ellos se encuentran). Concretamente, se dice que un espacio de Banach $X$ tiene la propiedad del punto fijo (FPP, por sus siglas en inglés) si todo operador no expansivo definido de un conjunto convexo, cerrado y acotado en sí mismo tiene un punto fijo. Es bien sabido que la geometría de los espacios de Banach juega un papel importante para asegurar la FPP. Efectivamente, el resultado obtenido por Kirk [48] significa que un espacio de Banach reflexivo con estructura normal tiene la FPP. En particular, los espacios de Banach uniformemente convexos o uniformemente suaves tienen la FPP. Se sabe que muchas otras propiedades geométricas implican la FPP para espacios de Banach reflexivos (la propiedad de Kadec Klee uniforme, la condición de Opial uniforme, la existencia de una base monótona e incondicional, etc.). Por otra parte, los espacios de Banach no reflexivos clásicos $\ell_1$, $c_0$ y $L_1$ no tienen la FPP (de hecho, $L_1$ no satisface una condición más fuerte llamada la propiedad débil del punto fijo [2]). Por mucho tiempo, fue un problema abierto saber si todos los espacios de Banach con la FPP eran reflexivos. En 2008, P.K. Lin [56] dio una respuesta inesperada a este problema: encontró el primer espacio de Banach no reflexivo con la FPP. De hecho, el espacio de Banach dado por P. K. Lin es el espacio de sucesiones $\ell_1$ dotado con la norma equivalente

$$|||x||| = \sup_k \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|$$

donde $x = (x_n) \in \ell_1$.

Previamente había sido demostrado que existen espacios de Banach que no pueden renormarse para tener la FPP, por ejemplo, si $\Gamma$ es no numerable, todo renormamiento de $\ell_1(\Gamma)$ o de $c_0(\Gamma)$ contiene una copia asintóticamente isométrica de $\ell_1$ o de $c_0$ respectivamente y por lo tanto no hay normas equivalentes en estos espacios que posean la FPP. Más aún, todo renormamiento de el espacio de Banach $\ell_\infty$ contiene una copia de $\ell_1(\Gamma)$ para algún conjunto $\Gamma$ no numerable por lo cual $\ell_\infty$ es otro ejemplo de un espacio de Banach para el que no existen normas equivalentes que satisfagan la FPP. Por otro lado, T. Domínguez Benavides probó en [17] que todo espacio de Banach reflexivo puede ser renormado para tener la FPP. Ésto sugiere la siguiente pregunta: ¿Qué espacios de Banach no reflexivos pueden ser renormados para tener la FPP?

El principal objetivo de esta Tesis es estudiar nuevas familias de espacios de Banach no reflexivos que puedan ser renormados para tener la FPP. Principalmente obtendremos estos ejemplos entre los subespacios cerrados de $L_1(\mu)$ o más generalmente de los espacios $L_1$ no conmutativos. En el último capítulo retomaremos el espacio de sucesiones $\ell_1$ y encontraremos nuevos renormamientos, los cuáles nos permitirán observar que el conjunto de normas equivalentes sobre $\ell_1$ con la FPP tiene algún "tipo" de estructura lineal.

La Tesis está basada en los cuatro artículos [39, 40, 41, 42] y está dividida en cinco capítulos, como sigue:

En el primer capítulo damos algunas definiciones y resultados generales acerca de la FPP y la teoría de renormamiento. También establecemos parte de la notación básica que emplearemos a lo largo de la presente memoria.

En el segundo capítulo, encontramos nuevas clases de espacios de Banach no reflexivos, los cuales bajo un renormamiento satisfacen la FPP. Nuestras técnicas están inspiradas por la empleada por P.K. Lin en [56], pero nuestras aplicaciones van más allá de el espacio de sucesiones $\ell_1$ como ilustraremos a lo largo de varios ejemplos. Para hacer esto, consideramos una familia de seminormas $\{R_k(\cdot)\}_k$ sobre ciertos espacios de Banach y una sucesión no decreciente $(\gamma_k) \subset (0,1)$ tal que $\lim_k \gamma_k = 1$. Definimos una norma del modo siguiente

$$|||x||| = \sup_k \gamma_k R_k(x).$$

Probaremos que, si la familia de seminormas $\{R_k(\cdot)\}$ satisface algunas condiciones específicas entonces el espacio de Banach $(X, |||\cdot|||)$ tiene la FPP. Como un caso particular, recuperaremos el resultado de P. K. Lin y encontraremos nuevos renormamientos en $\ell_1$ con la FPP ya que podremos considerar cualquier sucesión no decreciente $(\gamma_k)$ en $(0,1)$ con límite igual 1 en lugar de la sucesión $\frac{8^k}{1+8^k}$. Como otras aplicaciones, renormaremos el álgebra de Fourier-Stieltjes de un grupo compacto y separable para que tenga la FPP. Notemos que si $G$ es localmente compacto, su álgebra de Fourier-Stieltjes $B(G)$ tiene la FPP si y sólo si $G$ es finito [52]. También encontramos nuevas clases de espacios de Banach no reflexivos con la FPP que no son isomorfos a ningún subespacio de $\ell_1$, como por ejemplo, un renormamiento de $X := \oplus_1 \sum_n \ell_p^n$.

En el tercer capítulo, aplicaremos nuestro resultado a el caso particular de los subespacios de $L_1(\mu)$ para una medida $\sigma$-finita. Se conoce que un subespacio cerrado $X$ de $L_1(\mu)$ tiene la FPP si y sólo si $X$ es reflexivo [61, 22]. Obtenemos una condición suficiente que asegura que un subespacio no reflexivo $X$ de $L_1(\mu)$ puede ser renormado para tener la FPP y damos algunos ejemplos nuevos de subespacios no reflexivos de $L_1[0,1]$, que no son isomorfos a $\ell_1$, y que pueden ser renormados con la FPP. En la segunda sección de este capítulo, nos enfocamos en los espacios $L_1$ no conmutativos asociados a una álgebra de von Neumann finita. Estos espacios son los preduales de álgebras de von Neumann finitas y pueden ser considerados como una extensión de los espacios de medida clásicos de Lebesgue (un ejemplo particular de una álgebra de von Neumann finita es $L_\infty[0,1]$ y su predual es $L_1[0,1]$). Se sabe que el predual de una álgebra de von Neumann $\mathcal{M}$, $L_1(\mathcal{M})$ y todos sus subespacios no reflexivos contienen copias asintóticamente isometricas de $\ell_1$ [69] y consecuentemente no tiene la FPP [23]. El propósito principal de esta sección es obtener una familia de normas equivalentes a la norma usual en $L_1(\mathcal{M})$, que tengan un mejor comportamiento con respecto a la existencia de puntos fijos para operadores no expansivos definidos sobre un conjunto convexo, cerrado y acotado de $L_1(\mathcal{M})$ y damos una condición suficiente (con un aspecto topológico) de manera que un subespacio no reflexivo de $L_1(\mathcal{M})$ pueda ser renormado para satisfacer la FPP. Como consecuencia recuperamos los ejemplos dados en la sección previa para el caso particular de el espacio $L_1(\mu)$ y deducimos que si $\mathcal{M}$ es un cualquier álgebra de von Neumann finita y atómica, existen normas equivalentes en $L_1(\mathcal{M})$ satisfaciendo la propiedad del punto fijo.

En el cuarto capítulo, obtenemos un resultado acerca de renormamientos para la propiedad del punto fijo para operadores afines y no expansivos en el contexto de los espacios $L_1$ no conmutativos generados por una álgebra de von Neumann finita. Podemos aplicar este resultado a el caso particular de $L_1(\mu)$. Es importante mencionar que existen operadores afines y $\|\cdot\|_1$-no expansivos en estos espacios que no tienen puntos fijos.

En el quinto capítulo, nos concentramos en el espacio de sucesiones $\ell_1$ y en el conjunto de normas equivalentes que no satisfacen la FPP. Cabe decir que en un articulo posterior, P. K. Lin [57] estableció cuatro condiciones las cuales son suficientes para garantizar que un renormamiento en $\ell_1$ verifica la FPP. Comprobaremos que muchas de las normas obtenidas en este capítulo con la FPP no satisfacen las condiciones de P. K. Lin y que podemos sumar dos normas, una con la FPP y la otra sin satisfacer la FPP y seguir obteniendo un renormamiento con la FPP. Aún más, estudiamos el problema de estabilidad de la propiedad del punto fijo en el espacio de Banach $\ell_1$. Este problema puede formularse como sigue: Sea $X$ un espacio de Banach con la FPP e $Y$ un espacio de Banach isomorfo a $X$. ¿Existe alguna constante $K = K(X) > 1$ tal que $Y$ tenga la FPP siempre que $d(X, Y) < K$? Esta pregunta ha sido estudiada ampliamente por un gran número de investigadores y muchas propiedades geométricas han resultado ser útiles para determinar una cota superior para la distancia Banach-Mazur con el fin de asegurar la transmisión de la FPP. Probaremos que la norma de P. K. Lin, así como otros tantos renormamientos de $\ell_1$ con la FPP, fallan para producir estabilidad de la FPP. Este hecho contrasta con el caso de los espacios de Banach reflexivos clásicos con la FPP. En la sección final del capítulo consideramos el cono convexo $\mathcal{P}$ de todas las normas en $\ell_1$ que sean equivalentes a $\|\cdot\|_1$ y su subconjunto $\mathcal{A}$ dado por las normas de $\mathcal{P}$ que satisfagan la FPP. Deducimos que $\mathcal{A}$ contiene rayos y estudiamos algunas propiedades que conciernen a la estructura de los conjuntos $\mathcal{A}$ y $\mathcal{P} \setminus \mathcal{A}$.

# Chapter 1

# Preliminaries

In this Chapter we include some definitions and known facts about the Fixed Point Theory for nonexpansive mappings. In the last few decades a large number of papers and some specific books concerning this subject have appeared. We only consider the ones which are directly related to our work. We will try to state these results showing the evolution of the research in this field.

## 1.1 Fixed Point Property for nonexpansive mappings.

Throughout this Dissertation we will always assume that $(X, \|\cdot\|)$ is a Banach space, that will be denoted by $X$, unless it is necessary.

If $C$ is a subset of $X$ and $T : C \to C$ is a mapping, we say that $x \in C$ is a fixed point of $T$ if $Tx = x$.

**Definition 1.1.1.** Let $C$ be a subset of $X$. A mapping $T : C \to C$ is named a Lipschitzian mapping if there exists $K \geq 0$ such that for every $x, y \in C$, the following is satisfied

$$\|Tx - Ty\| \leq K\|x - y\|.$$

The minimal constant $K$ which satisfies the last inequality is called the Lipschitz constant of $T$.

When $K$ is strictly less than one, $T$ is called a contractive mapping. For this kind of mappings the following result is well-known.

**Theorem 1.1.1** (Banach Contraction Principle)**.** *Consider $C$ a closed subset of a Banach space $X$. If $T : C \to C$ is a contractive mapping then $T$ has a unique fixed point. Moreover, the sequence $(T^n x_0)$ converges to the fixed point for any choice of $x_0 \in C$.*

In fact, the above theorem holds for complete metric spaces and it was proved in Stefan Banach's Thesis in 1922 [4].

A Lipschitzian mapping with Lipschitz constant equal to 1 is called nonexpansive. To be more precise:

**Definition 1.1.2.** A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \forall x, y \in C.$$

Notice that Banach's Contraction Principle does not hold for nonexpansive mappings. For instance, a translation in $\mathbb{R}^n$ is a nonexpansive mapping without fixed points. It was in 1965 when the first positive results concerning the existence of fixed points for nonexpansive mappings appeared under the framework of Banach spaces.

**Definition 1.1.3.** It is said that a Banach space $X$ has the fixed point property (FPP) if every nonexpansive self-mapping defined on a convex closed bounded subset of $X$ has a fixed point.

There exist several Banach spaces failing the FPP as the following examples show:

**Example 1.1.1.** The Banach spaces $\ell_1$ does not have the FPP. Indeed, consider

$$C = \bar{co}(e_n) = \left\{ x = \sum_n \lambda_n e_n : \quad \sum_n \lambda_n = 1, \ \lambda_n \geq 0 \right\}.$$

This set is a closed convex bounded subset of $\ell_1$.

Let $T : C \to C$ be the mapping given by

$$T\left(\sum_n \lambda_n e_n\right) = \sum_n \lambda_n e_{n+1}.$$

Then $T$ is a nonexpansive fixed point free mapping.

**Example 1.1.2.** The Banach space $c_0$ fails to have the FPP. Consider $C$ as the unit ball in the Banach space $c_0$, and define the mapping $T : C \to C$ by

$$T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots).$$

Again, this mapping is nonexpansive and fixed point free.

Some other examples of Banach spaces failing the FPP can be found in [34, Chapter 3]. Obviously, every Banach space which contains an isometric copy of either $\ell_1$ or $c_0$ also fails to have the FPP as for instance the classical nonreflexive spaces $L_1$, $L_\infty$ and $C[0, 1]$.

In 1965, F. Browder [10, 11], D. Göhde [35] and W. Kirk [48] proved independently the following theorems:

**Theorem 1.1.2** (Browder's Theorem [10]). *Every Hilbert space $H$ has the FPP.*

**Theorem 1.1.3** (Browder-Göhde's Theorem [11, 35]). *If $X$ is a uniformly convex Banach space then $X$ has the FPP.*

Recall that a Banach space is called uniformly convex if for every $\varepsilon \in (0, 2]$ there is some $\delta > 0$ so that for any two vectors $x$ and $y$ belonging to unit ball of $X$ we have

$$\|x - y\| > \varepsilon \Rightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Now such classical reflexive Banach spaces as $\ell_p$ or $L_p$ with $1 < p < \infty$ have the FPP, since they are uniformly convex.

Before stating Kirk's Theorem we need some definitions.

**Definition 1.1.4.** Let $X$ be a Banach space and $C$ a nonempty and bounded subset of $X$. We say that $x_0 \in C$ is diametral if

$$\sup_{x \in C} \|x_0 - x\| = \operatorname{diam}(C).$$

The set $C$ is said to be diametral if all its points are diametral points.

**Definition 1.1.5.** A Banach space $X$ has normal structure (NS), if every convex closed bounded subset of $X$ with more than one point is not diametral.

Normal structure was defined by Brodskii and Milman [9] in 1948 and nearly twenty years later W. Kirk proved that this concept can be related with the fixed point property in the following sense:

**Theorem 1.1.4** (Kirk's Theorem [48]). *Let $X$ be a Banach spaces and $C$ a nonempty convex and $w$-compact subset of $X$. If $X$ has NS then every nonexpansive mapping $T : C \to C$ has a fixed point.*

As a consequence we deduce:

**Theorem 1.1.5.** *If a Banach space $X$ has NS and it is reflexive then $X$ has the FPP*

Since every Hilbert space is uniformly convex, Theorem 1.1.2 is included in Theorem 1.1.3. On the other hand, it is well-known that every uniformly convex space is reflexive and it has normal structure [5]. So Browder and Göhde's results can be obtained as particular cases of Kirk's Theorem.

As the previous statements show, the geometry of the Banach space plays an important role to assure the existence of fixed points for nonexpansive mappings. In fact, Kirk's Theorem gave a first boost to the Fixed Point Theory for nonexpansive mappings and after that many researchers have studied new geometric properties which imply the FPP (see for instance [34, 16, 50] and references therein) and many new fixed point theorems for nonexpansives mappings have appeared. We include some of them:

**Theorem 1.1.6** (J. B. Baillon [3])**.** *If $X$ is a uniformly smooth Banach space then $X$ has the FPP.*

It is known that uniformly smoothness implies normal structure.

**Theorem 1.1.7** (D. van Dulst and B. Sims [80])**.** *If $X$ is uniformly Kadec Klee and it is reflexive then $X$ has the FPP.*

**Theorem 1.1.8** (J. P. Gossez and E. Lami Dozo [36])**.** *If $X$ satisfies the uniform Opial condition and it is reflexive then $X$ has the FPP.*

**Theorem 1.1.9** (P. K. Lin [55])**.** *If $X$ has a 1-unconditional basis and is reflexive then $X$ has the FPP.*

There are many other geometric properties in Banach spaces connected to the fixed point property for nonexpansive mappings. However we can separate them in two classes:

**Class 1:**

$$\text{Geometric Property} \Rightarrow \text{Reflexivity and FPP.}$$

**Class 2:**

$$\text{Geometric Property} + \text{Reflexivity} \Rightarrow \text{FPP.}$$

For example, Theorems 1.1.2, 1.1.3 and 1.1.6 belong to Class 1 and Theorems 1.1.5, 1.1.7, 1.1.8 and 1.1.9 belong to Class 2.

On the other hand, there are some other results that show that reflexivity and FPP are equivalent properties in certain classes of Banach spaces.

**Theorem 1.1.10.** *Assume that $X$ is a closed subspace of $L_1[0,1]$. Then $X$ is reflexive if and only if $X$ has the FPP.*

In the last theorem, the statement that all reflexive Banach subspaces of $L_1[0,1]$ verify the FPP was proved by B. Maurey in [62]. The converse was proved by P. N. Dowling, C. J. Lennard and B. Turett in [24].

Other classes of Banach spaces, where FPP and reflexivity are equivalent, are the following:

**Theorem 1.1.11.** *([50, Chapter 9]) Let $(\Omega, \Sigma, \mu)$ be a finite measure space which is not purely atomic. The Orlicz space $L_\phi(\mu)$ endowed with the Orlicz norm has the FPP if and only if it is reflexive.*

**Theorem 1.1.12.** *([26]) Consider the Banach space $K(H)$ of all compact operators on a Hilbert space. A closed subspace $X$ of $K(H)$ has the FPP if and only if it is reflexive.*

In 1996 W. A. Kirk set out the following conjecture [49, Question (I)]:

$$X \text{ is reflexive } \Leftrightarrow X \text{ has the FPP.}$$

For a long time, it was thought that this was true, due to the previous theorems.

Recently in 2008, P. K. Lin proved that the converse of the above equivalence is not true [56], that is, there exists a nonreflexive Banach space with the FPP. However it is still an open problem if reflexivity implies FPP.

A way to show that a Banach space fails to have the FPP is to prove that it contains an asymptotically isometric copy of $\ell_1$ or $c_0$ [22, 23].

**Definition 1.1.6.** We say that a Banach space contains an asymptotically isometric copy of $\ell_1$ if there exist a sequence $(x_n)$ in $X$ and a null-sequence $(\varepsilon_n)$ in $(0,1)$ such that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} |t_n|,$$

for all $(t_n) \in \ell_1$.

Similarly

**Definition 1.1.7.** A Banach space $X$ is said to contain an asymptotically isometric copy of $c_0$ if there is a null sequence $(\varepsilon_n)$ in $(0,1)$ and a sequence $(x_n)$ in $X$ so that

$$\sup_n (1-\varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sup_n |a_n|,$$

for all $(a_n) \in c_0$.

The concept of asymptotically isometric copy of $\ell_1$ was introduced by J. Hagler in his Thesis in 1972 [38] as a way to characterize those Banach spaces whose duals contain an isometric copy of $L_1[0,1]$. This notion was recovered by P. N Dowling and C. J. Lennard in connection to the failure of the FPP as the following result shows [22, 23]:

**Theorem 1.1.13.** *If a Banach space $X$ has an asymptotically isometric copy of $\ell_1$ or $c_0$, then $X$ does not have the FPP.*

Some examples of Banach spaces which contain an asymptotically isometric copy of $\ell_1$ are the following:

- Every nonreflexive subspace of $L_1[0,1]$ contains an asymptotically isometric copy of $\ell_1$ [24]. Every nonreflexive Orlicz space contains an asymptotically isometric copy of $\ell_1$ [23].

- Every renorming of $\ell_1(\Gamma)$, for an uncountable set $\Gamma$, contains an asymptotically isometric copy of $\ell_1$ [24].

- H. Pfitzner proved that every nonreflexive subspace of an $L$-embedded Banach space contains an asymptotically isometric copy of $\ell_1$ [68].

  Recall that if $Y$ is a Banach space and $P$ a projection in $Y$, $P$ is called an $L$-projection if $\|x\| = \|Px\| + \|(I-P)x\|$ for all $x \in Y$. A closed subspace $X \subset Y$ is called an $L$-summand on $Y$ if $X$ is the range of an $L$-projection on $Y$. When $Y = X^{**}$, it is said that $X$ is an $L$-embedded Banach space and there exists a closed subspace $X_s \subset X^{**}$ such that $X^{**} = X \oplus_1 X_s$.

  Particular examples of $L$-embedded Banach spaces are the preduals of von Neumann algebras, in particular $\ell_1$ and $L_1$-spaces, the dual of the disc algebra, etc.

Examples of Banach spaces which contain an asymptotically isometric copy of $c_0$ can be found in [24].

Recall that given $(X, \|\cdot\|)$ a Banach space and $|\cdot|$ a norm on $X$, we say that $\|\cdot\|$ and $|\cdot|$ are equivalent if there exist two positive numbers $m$ and $M$ such that for all $x \in X$

$$m\|x\| \leq |x| \leq M\|x\|.$$

In this case $(X, |\cdot|)$ is called a renorming of $(X, \|\cdot\|)$.

Trying to prove that the previous conjecture was true, P. N. Dowling et al. studied whether every renorming of $\ell_1$ contained an asymptotically isometric copy of $\ell_1$ (remember that every renorming of $\ell_1$ contains an almost isometric copy of $\ell_1$ by James' Theorem) and they found a renorming of $\ell_1$ without such a copy.

**Example 1.1.3.** Consider a nondecreasing sequence $(\gamma_k)$ in $(0, 1)$ such that $\gamma_k \to 1$. Define

$$|||(t_n)||| = \sup_k \gamma_k \sum_{n=k}^{\infty} |t_k|, \qquad \forall (t_n) \in \ell_1.$$

The norm $|||\cdot|||$ is equivalent to the usual one of $\ell_1$ and the Banach space $(\ell_1, |||\cdot|||)$ does not contain an asymptotically isometric copy of $\ell_1$ [21].

Until recently, all the known Banach spaces with the FPP were reflexive. It was in 2008 when P. K. Lin gave the first nonreflexive Banach spaces with the FPP.

**Theorem 1.1.14** ([56]). *Consider the Banach space $\ell_1$ endowed with its usual norm, $\|\cdot\|_1$ and let $(\gamma_k)$ be a nondecreasing sequence in $(0, 1)$. For $(t_k) \in \ell_1$ define*

$$|||(t_k)||| = \sup_k \frac{8^k}{1 + 8^k} \sum_{i=k}^{\infty} |t_i|.$$

*Then $|||\cdot|||$ is an equivalent norm to the usual norm of $\ell_1$ and $(\ell_1, |||\cdot|||)$ has the FPP.*

Actually, H. Fetter proved that it is possible consider $\ell_1$ endowed with the norm given in the Example 1.1.3, with the restriction of $\gamma_1 > \frac{2}{3}$ [30], to have the FPP.

## 1.2 Renormings, stability and FPP

The renorming theory studies the construction of equivalent norms on a Banach space satisfying some specific properties. Also this theory studies properties which are invariant under renormings, or in other words, properties which are preserved under isomorphisms.

After P. K. Lin's example, which is a renorming of $\ell_1$ with the FPP, a natural question is: Can every Banach space be renormed to have the FPP? The answer to this question is no in general, as the following theorem shows:

**Theorem 1.2.1** ([23]). *The Banach spaces $\ell_1(\Gamma)$ and $c_0(\Gamma)$, for $\Gamma$ an uncountable set $\Gamma$, and the Banach space $\ell_\infty$ can not be renormed to have the FPP.*

This is because every renorming of $\ell_1(\Gamma)$ contains an asymptotically isometric copy of $\ell_1$, every renorming of $c_0(\Gamma)$ contains an asymptotically isometric copy of $c_0$ and every renorming of $\ell_\infty$ contains a copy of $\ell_1(\Gamma)$ for some uncountable set $\Gamma$.

However, a partial positive answer was given by T. Domínguez Benavides.

**Theorem 1.2.2** ([17]). *Every reflexive Banach space can be renormed to have the FPP.*

It is unknown if the sequence space $c_0$ can be renormed to have the FPP.

The FPP is not necessarily preserved under isomorphisms: $(\ell_1, |||\cdot|||)$ has the FPP while $(\ell_1, \|\cdot\|_1)$ fails to have the FPP. However for many classical reflexive

Banach spaces, the FPP is preserved under isomorphisms whenever the Banach-Mazur distance between the original space and the isomorphic one is less than a certain constant. Now, we recall the definition of Banach-Mazur distance and we cite some important fixed point results related with the stability.

**Definition 1.2.1.** Let $X$, $Y$ be isomorphic normed spaces. The Banach-Mazur distance between $X$ and $Y$ is defined by

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y\}.$$

The stability problem in fixed point theory can be set out as follows: Let $X$ be a Banach space with the FPP and $Y$ a Banach space isomorphic to $X$. Does there exist some constant $K = K(X) > 1$ such that $Y$ has the FPP whenever $d(X,Y) < K$?. This problem has been widely studied by many researchers, and many geometric properties have turned out to be useful to determine a value for the Banach-Mazur distance which assures the transmission of the FPP (see [50, Chapter 7] and the references therein for a broad exposition about this topic). We include some known facts in this setting:

**Theorem 1.2.3** ([19])**.** *If $X = \ell_p$ with $1 < p < +\infty$, endowed with its usual norm, and $Y$ is isomorphic to $\ell_p$, then $Y$ has the FPP whenever $d(X,Y) < (1 + 2^{\frac{1}{p-1}})^{\frac{p-1}{p}}$.*

**Theorem 1.2.4** ([63])**.** *If $H$ is a Hilbert space and $d(H,Y) < \sqrt{\frac{5+\sqrt{17}}{2}}$, then $Y$ has the FPP.*

In the rest of the chapter we recall some definitions and generic results which will be used in this monograph.

## 1.3 Type and cotype of a Banach space

**Definition 1.3.1.** Consider the family of functions $\{r_n\}$ defined on the unit interval by the formula

$$r_n(x) = (-1)^{i+1}, \text{ if } \frac{i-1}{2^n} \leq x \leq \frac{i}{2^n} \text{ with } i \in \{1, \dots, 2^n\}.$$

for each positive integer $n$. These functions are called Rademacher's functions.

**Definition 1.3.2.** A Banach space $X$ is said to be of type $p$ for some $1 < p \leq 2$, respectively, of cotype $q$ for some $q \geq 2$, if there exists a constant $M < \infty$ so that, for every finite collection of vectors $\{x_j\}_{j=1}^n$ in $X$, we have

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt \leq M \left( \sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}, \tag{1.1}$$

respectively

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt \geq M^{-1} \left( \sum_{j=1}^n \|x_j\|^q \right)^{\frac{1}{q}}. \tag{1.2}$$

We refer to [13, 60] for detailed information about type and cotype. Some useful properties are the following:

1. If a Banach space is of type $p$ and cotype $q$ all its subspaces have the same type and cotype. Actually, the type and cotype constants depend only on the collection of the finite dimensional subspaces.

2. The type and cotype of a Banach space are preserved under isomorphisms.

3. If $X$ is finitely representable in $Y$, then type$(Y) =$ type$(X)$ and cotype$(Y) =$ cotype$(X)$.

   Remember that $X$ is finitely representable in $Y$, if for every finite dimensional subspace $U$ of $X$ and for $\varepsilon > 0$ there exists a subspace $V$ of $Y$ and a linear isomorphism $T : U \to V$ such that $\|T\|\|T^{-1}\| < 1 + \varepsilon$.

## 1.4 Fourier Algebra and Fourier-Stieltjes Algebra of a locally compact group

Everything that we mention about the construction of the Fourier Algebra and the Fourier-Stieltjes Algebra of a locally compact group can be found in [28, 32, 53].

If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, a (Banach algebra) homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a bounded linear map $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a homomorphism $\varphi$ such that $\varphi(x^*) = \varphi(x)^*$ for all $x \in \mathcal{A}$.

A $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\varphi$ from $\mathcal{A}$ to $\mathcal{L}(\mathcal{H})$. In this case the norm-closure $\mathcal{B}$ of $\varphi(\mathcal{A})$ in $\mathcal{L}(\mathcal{H})$ is a $C^*$ subalgebra of $\mathcal{L}(\mathcal{H})$, and we say that $\varphi$ is nondegenerate if $\mathcal{B}$ is nondegenerate, i.e., if there is no $v \in \mathcal{H} \setminus \{0\}$ such that $\varphi(x)v = 0$ for all $x \in \mathcal{A}$.

A topological group $G$ is a group endowed with a topology and such that the group operations

$$G \times G \to G : (x, y) \mapsto xy,$$

and

$$G \to G : x \mapsto x^{-1}$$

are continous with this topology. We say that a topological group $G$ is locally compact if every point in $G$ has a compact neighbourhood.

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. Let $L_1(G)$ denote the equivalence classes of $\lambda$-measurable functions $f : G \mapsto \mathbb{C}$ such that

$$\int |f(x)| d\lambda(x) < \infty$$

with norm $\|f\|_1 = \int |f(x)| d\lambda(x)$. Then $L_1(G)$ is a Banach algebra with convolution product

$$(f * g)(x) = \int f(y) g(y^{-1}x) d\lambda(y)$$

$f, g \in L_1(G)$ (called the group algebra of $G$).

We define $C^*(G)$, the group $C^*$-algebra of $G$, as the completion of $L^1(G)$ with respect to the norm

$$\|f\|_* = \sup \|\pi_f\|,$$

where the supremum is taken over all nondegenerate $*$-representations $\pi$ of $L_1(G)$ as a $*$-algebra of bounded operators on a Hilbert space. Let $C(G)$ be the Banach space of bounded continuous complex-valued functions on $G$ with the supremum norm. Denote the set of continuous positive definite functions on $G$ by $P(G)$, i.e. all continuous functions $\varphi$ on $G$ such that

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \overline{\lambda_j} \varphi(x_i x_j^{-1}) \geq 0$$

for any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in G$.

And denote the set of continuous functions on $G$ with compact support by $C_{00}(G)$. We define the Fourier-Stieltjes algebra of $G$, denoted by $B(G)$, to be the linear span of $P(G)$. $B(G) = C^*(G)$ identifying every $\varphi \in B(G)$ with the functional

$$\langle \varphi, f \rangle = \int f(x)\varphi(x)d\lambda(x)$$

for $f \in L_1(G)$. $B(G)$ with the dual norm and pointwise multiplication is a commutative Banach algebra. The Fourier algebra of $G$, denoted by $A(G)$, is the closure of $B(G) \cap C_{00}(G)$.

It is well known that if $G = \mathbb{T}$, then $B(G)$ is isometric to $\ell_1(\mathbb{Z})$.

## 1.5 Noncommutative $L_1$-spaces

All the definitions and results of this section can be found in [76, 79] and [44, Appendix]. Two important references for noncommutative $L_p$-spaces and noncommutative integration are [65] and [75].

Let $H$ be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. Several important locally convex topologies can be considered on $B(H)$: The uniform topology, given by the operator norm $\|x\|_\infty = \sup\{\|xh\| : h \in H, \|h\| \leq 1\}$; the strong operator topology, defined by the seminorms $x \to \|xh\|$ when $h$ runs over $H$; and the weak operator topology, defined by the family of seminorms $x \to |(xh, g)|$ where $h, g \in H$.

**Definition 1.5.1.** A von Neumann algebra is a subalgebra $\mathcal{M}$ of $B(H)$ which is self-adjoint ($x \in \mathcal{M}$ implies $x^* \in \mathcal{M}$), it contains I (the identity operator) and it is closed in the weak operator topology (or equivalently, it is closed in the strong operator topology).

The von Neumann algebras were introduced by F. J. Murray and J. von Neumann in their famous work [64] motivated by some problems on infinite dimensional representation of groups and the mathematical foundations of quantum mechanics. When $H$ is a separable infinite dimensional Hilbert space and $(e_n)_n$ is an orthonormal basis in $H$, every $T \in B(H)$ has a matrix representation in the form

$$T = ((Te_i, e_j))_{i\geq 1; j\geq 1} .$$

Hence a von Neumann algebra is a unital sub*-algebra of $B(H)$ which is closed in the topology of the convergence coordinate-to-coordinate (which turns out to be the weak operator topology).

A first example of a von Neumann algebra is $B(H)$ itself. If $H$ is a finite dimensional Hilbert space with $\dim(H) = n$, $B(H)$ coincides with $\mathcal{M}_n$, the set of all complex matrices with dimension $n \times n$.

Other simple examples of von Neumann algebras are $\ell_\infty$ and $L_\infty(\mu)$. In fact, fixed an orthonormal basis, $\ell_\infty$ can be seen as the subalgebra of all operators in $B(\ell_2)$ that have a diagonal matrix. If $(\Omega, \Sigma, \mu)$ is a measure space, every $f \in L_\infty(\mu)$ gives an operator on $L_2(\mu)$ given by

$$M_f : L_2(\mu) \to L_2(\mu); \qquad g \in L_2(\mu) \to M_f(g) = f \cdot g \in L_2(\mu)$$

and $L_\infty(\mu)$ is the von Neumann algebra acting in the Hilbert space $L_2(\mu)$ and given by the operators $M_f$ with $f \in L_\infty(\mu)$. In fact, it is a classical result (see for instance [14, Chapter 7]) that every commutative von Neumann algebra $\mathcal{M}$ can be isometrically identified with $L_\infty(\Omega, \mathcal{A}, \mu)$ for some abstract measure space $(\Omega, \mathcal{A}, \mu)$.

An important object in the study of von Neumann algebras is the collection of all orthogonal projections in $\mathcal{M}$, which is denoted by $\mathcal{P}(\mathcal{M})$. It is the analogue in noncommutative integration theory of the underlying $\sigma$-algebra in classical integration theory. If $p, q \in \mathcal{P}(\mathcal{M})$, then $p \leq q$ if and only if $Ran(p) \subset Ran(q)$. For

$p, q \in \mathcal{P}(\mathcal{M})$ the supremum $p \bigvee q \in \mathcal{P}(\mathcal{M})$ and the infimum $p \bigwedge q \in \mathcal{P}(\mathcal{M})$ are given by the orthogonal projections onto $\overline{Ran(p) + Ran(q)}$ and $Ran(p) \cap Ran(q)$ respectively. Actually, $\mathcal{P}(\mathcal{M})$ is a complete lattice, that is, for each collection $\{p_\alpha\}$ in $\mathcal{P}(\mathcal{M})$, the supremum $\bigvee_\alpha p_\alpha$ and infimum $\bigwedge_\alpha p_\alpha$ exist and are given by the orthogonal projections onto $\overline{span}_\alpha\{Ran(p_\alpha)\}$ and $\bigcap_\alpha Ran(p_\alpha)$ respectively. If $p \in \mathcal{P}(\mathcal{M})$, let $p^\perp = I - p \in \mathcal{P}(\mathcal{M})$. A projection $p \in \mathcal{P}(\mathcal{M})$ is called a minimal projection if there exists no $q \in \mathcal{P}(\mathcal{M})$ with $q \leq p$ and $0 \neq q \neq p$. If $x$ is a self-adjoint element of $\mathcal{M}$ and $A$ is a Borel subset of $\mathbb{R}$, we denote by $e^x_A$ the spectral projection of $x$ that corresponds to the subset $A$ (see for instance [76, Chapter 2]).

Two projections $e$ and $f$ in a von Neumann algebras are said to be equivalent if there exists an element $u \in \mathcal{M}$ with $uu^* = e$ and $u^*u = f$. This fact is denoted by $e \sim f$. A projection $e$ is said to be finite if $e \sim f \leq e$ implies $e = f$.

A von Neumann algebra $\mathcal{M}$ is said to be finite if the identity $I$ is a finite projection, which is equivalent to saying that the hypotheses $T \in \mathcal{M}$ and $TT^* = I$, imply that $T^*T = I$ (see [14, Part III, Chapter 8, Theorem 1]). It is clear that every commutative von Neumann algebra is finite.

Let $\mathcal{M}_+$ be the cone of all positive elements of $\mathcal{M}$, that is,

$$\mathcal{M}_+ = \{x \in \mathcal{M} : \langle xh|h\rangle \geq 0, \text{ for all } h \in H\}.$$

**Definition 1.5.2.** Let $\mathcal{M}$ be a von Neumann algebra. A trace on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying:

1) $\tau(x + y) = \tau(x) + \tau(y)$, for all $x, y \in \mathcal{M}_+$.

2) $\tau(\lambda x) = \lambda\tau(x)$; $x \in \mathcal{M}_+$, $\lambda \in [0, +\infty]$ (with the usual convention that $0 \cdot \infty = 0$).

3) $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{M}$.

The trace $\tau$ is said to be

4) normal: if for each $x_\alpha \uparrow x$ in $\mathcal{M}_+$ (in the strong operator topology) we have $\tau(x_\alpha) \uparrow \tau(x)$.

5) faithful: if $\tau(x) = 0$ implies that $x = 0$ for all $x \in \mathcal{M}_+$.

6) semifinite: if for any nonzero $x \in \mathcal{M}_+$ there is a nonzero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < +\infty$.

7) finite: if $\tau(I) < +\infty$.

When the Hilbert space is separable, the von Neumann algebra $\mathcal{M}$ is finite if and only if $\mathcal{M}$ admits a finite normal faithful trace $\tau$ in $\mathcal{M}^+$ (see [76, Corollary 7.14 and Exercise E.7.4]). In case that the von Neumann algebra $\mathcal{M}$ admits a semifinite normal faithful trace $\tau$, $\mathcal{M}$ is said to be a semifinite von Neumann algebra.

If $\mathcal{M} = L_\infty(\Omega, \mathcal{A}, \mu)$ for some measure space $(\Omega, \mathcal{A}, \mu)$, the projection operators are the indicator functions of the sets in $\mathcal{A}$ and a trace $\tau$ is given by

$$\tau(f) = \int f d\mu$$

for all $f \in L_\infty(\Omega, \mathcal{A}, \mu)$. For this reason, the pair $(\mathcal{M}, \tau)$ is in general called a noncommutative measure space and, in case that $\tau(I) = 1$, $(\mathcal{M}, \tau)$ is called a noncommutative (quantum) probability space [72]. See for instance [74] to learn how noncommutative probability spaces can be useful in different aspects of quantum physics.

Also notice that if $(\Omega,\Sigma,\mu)$ is a $\sigma$-finite measure then $L_\infty(\mu)$ is also a finite von Neumann algebra with trace

$$\tau(f)=\sum_{n=1}^{\infty}\frac{1}{2^n\mu(\Omega_n)}\int_{\Omega_n} f d\mu,$$

where $\Omega=\cup\Omega_n$, the colection $\{\Omega_n\}$ is pairwise disjoint and $\mu(\Omega_n)<+\infty$. And $L_1(\mu)$ is isometric to $L_1(\nu)$ with

$$\nu(A)=\sum_{n=1}^{\infty}\frac{1}{2^n\mu(\Omega_n)}\mu(A\cap\Omega_n).$$

The simplest example of semifinite (and not finite) von Neumann algebra is $B(H)$ itself equipped with its usual trace defined by

$$\tau(x)=tr(x)=\sum_{i\in I}\langle xe_i,e_i\rangle$$

for all $x\in B(H)_+$, where $(e_i)_{i\in I}$ denotes an orthonormal basis in $H$. Here the finite projections are the finite rank projections.

Other particular examples of finite von Neumann algebras equipped with a finite faithful normal trace are the following (more examples can be found in the chapter written by G. Pisier and Q. Xu in [47, Chapter 34] and the references therein):

- The hyperfinite $II_1$ factor: Let

  $$(\mathcal{R},\tau)=\bigotimes_{n\geq 1}(\mathcal{M}_2,\tau_2)$$

  be the von Neumann algebra tensor product, where $\mathcal{M}_2$ denotes the complex $2\times 2$ matrices and

  $$\tau_2\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\frac{1}{2}(a+d).$$

Then $\mathcal{R}$ is the hyperfinite $II_1$ factor and $\tau$ is the (unique) normalized trace on $\mathcal{R}$. Thus, $(\mathcal{R}, \tau)$ is a noncommutative probability space.

- Consider a discrete group $\Gamma$. Let $vN(\Gamma) \subset B(\ell_2(\Gamma))$ be the associated von Neumann algebra generated by left translations. Let $\tau_\Gamma$ be the canonical trace on $vN(\Gamma)$ defined as follows: $\tau_\Gamma(x) = \langle x(\delta_e), \delta_e \rangle$ for any $x \in vN(\Gamma)$, where $(\delta_g)_{g\in\Gamma}$ denotes the canonical basis of $\ell_2(\Gamma)$, and where $e$ is the identity of $\Gamma$. This is a normal faithful normalized finite trace on $vN(\Gamma)$ so $(vN(\Gamma), \tau_\Gamma)$ is a noncommutative probability space. An interesting case with many applications in Harmonic Analysis is when $\Gamma = \mathbb{F}_n$, the free group of $n$ generators. We refer to [31] for more information.

Throughout the thesis, we will assume that $\mathcal{M}$ is infinite-dimensional and a finite von Neumann algebra on a separable Hilbert space $H$. Let $\tau$ be a finite normal faithful trace $\tau$ on $\mathcal{M}$ and $p \in [1, \infty)$. For $x \in \mathcal{M}$ define the function $\|\cdot\|_p : \mathcal{M} \to [0, \infty)$ given by

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}.$$

It can be proved that $\|\cdot\|_p$ is a norm on $\mathcal{M}$. The completion of $\mathcal{M}$ for this norm is denoted by $L_p(\mathcal{M}, \tau)$. This is the noncommutative $L_p$-space associated with $(\mathcal{M}, \tau)$ (these spaces can also be defined in case that the trace $\tau$ is semifinite [65]). The particular choice of the trace $\tau$ is unimportant: if $\beta$ is another semifinite normal faithful trace on $\mathcal{M}$, $L_p(\mathcal{M}, \beta)$ is isometric to $L_p(\mathcal{M}, \tau)$ [78, Proposition 1.3], so we denote by $L_p(\mathcal{M})$ this isometrically unique Banach space.

If $1 < p < +\infty$ the dual space of $L_p(\mathcal{M})$ is $L_q(\mathcal{M})$ ($\frac{1}{p} + \frac{1}{q} = 1$) and it is known that $L_p(\mathcal{M})$ is uniformly convex, hence, $L_p(\mathcal{M})$ has the FPP.

In the case $p = 1$, $L_1(\mathcal{M})$ contains an isometric copy of $\ell_1$, so $L_1(\mathcal{M})$ fails to have the FPP. Moreover every nonreflexive space of $L_1(\mathcal{M})$ fails to have the FPP because it contains an asymptotically isometric copy of $\ell_1$ [68].

For the particular case that the von Neumann algebra $\mathcal{M}$ is commutative, it is well known that the Banach space $L_p(\mathcal{M})$ $(1 \leq p < +\infty)$ can be isometrically identified with $L_p(\mu)$ for some measure space and many results valid for the measure spaces $L_p(\mu)$ have been extended for noncommutative $L_p(\mathcal{M})$ spaces. These extensions have required the study of specific new methods, since some classical tools do not work in the noncommutative setting. However, there exist some classical properties in $L_p(\mu)$ which do not hold for noncommutative $L_p$-spaces in general. For more information about similarities and differences between the noncommutative $L_p$-spaces and the classical $L_p(\mu)$ spaces the reader can consult [47, Chapter 34].

For the case that $p = 1$, it can be proved that the dual of $L_1(\mathcal{M})$ can be isometrically identified with $\mathcal{M}$ under the duality: $x \in L_1(\mathcal{M}) \to \tau(xy)$ for every $y \in \mathcal{M}$. Moreover, $\|xy\|_1 \leq \|y\|_\infty \|x\|_1$ and $\|yx\|_1 \leq \|y\|_\infty \|x\|_1$ for all $x \in L_1(\mathcal{M})$, $y \in \mathcal{M}$. Since $L_1(\mathcal{M})^* = \mathcal{M}$ , the von Neumann algebra $\mathcal{M}$ is also denoted by $L_\infty(\mathcal{M})$ and noncommutative $L_1$-spaces are called the preduals of von Neumann algebras.

# Chapter 2

# Renorming and the FPP

In this Chapter we extend P. K. Lin's techniques to a more general setting. In order to do that, we are going to consider a linear topology on some Banach spaces and a family of seminorms $\{R_k(\cdot)\}$ verifying certain hypothesis. In such conditions we will be able to find a renorming with the FPP. As a consequence we will recover P. K. Lin's example given in [56] and we will find several new norms on $\ell_1$ with the FPP. Moreover, our results will let us find some new classes of Banach spaces with the FPP which are nonreflexive and nonisomorphic to any subspace of the sequence space $\ell_1$.

The next observation will be used frequently throughout this Thesis.

Let $C$ be a closed convex bounded subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ a nonexpansive mapping. Fix any $x_0 \in C$. A direct application of Banach's Contraction Principle to the sequence of mappings $T_n : C \to C$ defined by

$$T_n x = \frac{1}{n} x_0 + \left(1 - \frac{1}{n}\right) Tx;$$

provides a sequence $(x_n) \subset C$, where $x_n$ is the unique fixed point of $T_n$, such that

$$\lim_n \|x_n - Tx_n\| = 0.$$

Such sequences are called approximate fixed point sequences (a.f.p.s.).

Moreover, if $d > 0$ and the set

$$D = \left\{ x \in C : \limsup_n \|x_n - x\| \le d \right\}$$

is nonempty, it is easy to check that $D$ is a convex, closed and $T$-invariant subset of $C$. Hence, we can find another a.f.p.s. in $D$.

## 2.1 Main result and first examples

In this section we state the main result of this chapter. As a consequence, we obtain the renorming given in [56] in the sequence space $\ell_1$, which provided the first example of a nonreflexive Banach space with the FPP. Also we give new equivalent norms on $\ell_1$ with the FPP and we obtain some new classes of nonreflexive Banach spaces with the FPP. In particular, we will prove that the Fourier-Stieljtes algebra $B(G)$ of a separable compact group can be renormed to have the FPP. Notice that $B(G)$ itself has the FPP if and only if $G$ is finite [52, Theorem 5.8]. More applications of the main theorem will be studied in the next chapter.

Let $(X, \|\cdot\|)$ be a Banach space endowed with a linear topology $\tau$. Assume that there exists a family of seminorms $R_k : X \to [0, +\infty)$ $(k \ge 1)$ that satisfy the following properties:

I) $R_1(x) = \|x\|$ while for $k \ge 2$, $R_k(x) \le \|x\|$ for all $x \in X$.

II) $\lim_k R_k(x) = 0$ for all $x \in X$.

III) If $x_n \xrightarrow{\tau} 0$ and is norm-bounded, then for all $k \geq 1$ we have

$$\limsup_n R_k(x_n) = \limsup_n \|x_n\|.$$

IV) If $x_n \xrightarrow{\tau} 0$, is norm-bounded and $x \in X$, then

$$\limsup_n R_k(x_n + x) = \limsup_n R_k(x_n) + R_k(x)$$

for all $k \geq 1$.

Then we can state the following:

**Theorem 2.1.1.** *Let $\{\gamma_k\}_k \subset (0,1)$ be any nondecreasing sequence such that $\gamma_k \to 1$ and define*

$$|||x||| = \sup_{k\geq 1} \gamma_k R_k(x); \qquad x \in X.$$

*Then $|||\cdot|||$ is an equivalent norm on $X$ such that $(X, |||\cdot|||)$ satisfies the following property: for every nonempty closed convex bounded subset $C$ which is $\tau$-relatively sequentially compact and for every $T : C \to C$ nonexpansive, there exists a fixed point.*

**Corollary 2.1.2.** *Assume that $X$ is a Banach space with a linear topology $\tau$ and a family of seminorms $\{R_k(\cdot)\}$ satisfying the conditions I), II), III) and IV). If the unit ball of $X$ is $\tau$-relatively sequentially compact then $X$ can be renormed to have the FPP with the equivalent norm given by $|||x||| = \sup_k \gamma_k R_k(x)$.*

Regarding the stability of the FPP, we have the inequalities

$$\gamma_1 \|x\| \leq |||x||| \leq \|x\|$$

for all $x \in X$ and $\gamma_1 \in (0,1)$ can be chosen as close to 1 as we want. This means that, in the Banach-Mazur distance, the norm with the FPP can be considered as close to the original norm $\|\cdot\|$ as we like. This shows that the FPP is not obviously an isomorphic property and that it is very unstable under renormings.

We will prove Theorem 2.1.1 in the next section. Now we give several families of Banach spaces where our results can be applied.

**Example 2.1.1.** A first application of Theorem 2.1.1 is a generalization of P.K. Lin's example given in [56], where he proved that if $\gamma_k = 8^k/(1+8^k)$, then the renorming on $\ell_1$ given by

$$|||x||| = \sup_k \frac{8^k}{1+8^k} \left\| \sum_{n=k}^{\infty} x_n e_n \right\|_1 \qquad \text{for } x = (x_n) \in \ell_1$$

has the FPP. Lin's result can be derived from Theorem 2.1.1 defining the seminorms $R_k(x) = \|\sum_{n=k}^{\infty} x_n e_n\|_1$ and $\tau$ the weak-star topology associated to the duality $\sigma(\ell_1, c_0)$. Notice that if $(x_n)$ is a $w^*$-null sequence, by using the "sliding hump technique" (see [29, p. 173] or [67]) we can obtain properties III) and IV). Since the unit ball is $w^*$-compact and $c_0$ is separable, every closed convex bounded subset is $\sigma(\ell_1, c_0)$-sequentially compact. So every $|||\cdot|||$-nonexpansive mapping defined on a closed convex bounded subset of $\ell_1$ into itself has a fixed point. It is worth saying that using the Theorem 2.1.1 we can replace the condition $\gamma_k = \frac{8^k}{1+8^k}$ and consider any nondecreasing sequence in $(0,1)$ whose limit is equal to 1.

**Example 2.1.2.** Consider again the sequence space $\ell_1$ and change the family of seminorms. Then we can obtain new renormings on $\ell_1$ with the FPP. For instance, let $p > 1$ and for $k \geq 2$ define for $x = (x_n) \in \ell_1$

$$R_k(x) = \sum_{n=2k}^{\infty} |x_n| + \left( \sum_{n=k}^{2k-1} |x_n|^p \right)^{\frac{1}{p}}$$

and $R_1(x) = \|x\|_1$. Using again the "sliding hump technique", it is easy to check that $\{R_k(\cdot)\}_k$ is a family of seminorms that verify properties I), II), III) and IV), so $\ell_1$ with the norm generated by the seminorms $\{R_k(\cdot)\}_k$ satisfies the FPP.

A more general result is the following:

**Corollary 2.1.3.** *Let $\{X_n\}_n$ be a sequence of finite dimensional Banach spaces and consider*

$$X = \oplus_1 \sum_n X_n = \left\{ x = (x_n) : x_n \in X_n, \|x\| = \sum_n \|x_n\|_{X_n} < \infty \right\}.$$

*Then $X$ can be renormed to have the FPP.*

*Proof.* Define the seminorms $R_k(x) = \sum_{n \geq k} \|x_n\|_{X_n}$ and let $\tau$ be the weak star topology where the predual of $X$ is

$$E = \left\{ x = (x_n) : x_n \in X_n, \lim_n \|x_n\| = 0, \|x\| = \sup_n \|x_n\|_{X_n} \right\}.$$

In the topology $\tau$ the null sequences are coordinate-wise convergent to 0. Using this and the "sliding hump technique" is possible to check that the family $\{R_k(\cdot)\}_k$ satisfies properties I), II), III) and IV).

Using the renorming given in Theorem 2.1.1, the Banach space $(X, |||\cdot|||)$ has the FPP. $\square$

A first application of Corollary 2.1.3 is the following example:

**Example 2.1.3.** Consider $X_n = \ell_p^n$ for some $1 < p \leq +\infty$ and

$$X = \oplus_1 \sum_n \ell_p^n.$$

Then we obtain a nonreflexive Banach space that can be renormed to have the FPP and that is not isomorphic to any subspace of $\ell_1$. Indeed, $\ell_p$ is finitely representable in $X$ and the type and the cotype of $X$ are equal to the type and the cotype of $\ell_p$ respectively. Notice that for every $1 < p \leq +\infty$, either the type or the cotype of $\ell_p$ is different from that of $\ell_1$, since $\text{type}(\ell_p) = \min\{2, p\}$ and $\text{cotype}(\ell_p) = \max\{2, p\}$ (see [60, p. 73]). Thus, $X$ is not isomorphic to any subspace of $\ell_1$ and we obtain a new class of nonreflexive Banach spaces with the FPP.

Another application of Corollary 2.1.3 is the following.

**Corollary 2.1.4.** *Let $G$ be a separable compact group and $B(G)$ its Fourier-Stieltjes algebra. Then $B(G)$ can be renormed to have the FPP.*

*Proof.* Using the arguments in the proof of Lemma 3.1 of [53], and having in mind that $B(G)$ is norm separable when $G$ is a separable compact group [37, Corollary 6.9], the Banach space $B(G)$ can be written as

$$B(G) = \oplus_1 \sum_n \mathfrak{T}(H_n)$$

where $H_n$ is a finite dimensional Hilbert space and $\mathfrak{T}(H_n)$ is the set of trace class operators on $H_n$. Applying Corollary 2.1.3 we obtain a renorming of $B(G)$ with the FPP. □

In the particular case that $G = \mathbb{T}$, the circle group, $B(G)$ is isometric to $\ell_1(\mathbb{Z})$ via Bochner's Theorem. Thus, Corollary 2.1.4 includes the sequence space $\ell_1$ and the renorming given by P.K. Lin [56]. Also, recall that $B(G)$ with its usual norm has the FPP if and only if $G$ is finite [52].

## 2.2 Proof of the main result

This section is dedicated to the proof of Theorem 2.1.1. Previously, we need to give a general result for Banach spaces which fail to have the FPP and other two technical lemmas:

**Lemma 2.2.1.** *Let $(X, \|\cdot\|)$ be a Banach space and $C$ a convex, closed, bounded subset of $X$. Let $T : C \to C$ be a nonexpansive mapping and suppose that $T$ is fixed point free. Then there exist some $a > 0$ and a convex closed $T$-invariant subset $D$ of $C$ such that for each approximate fixed point sequence $(x_n)$ in $D$ and for any $z \in D$*

$$\limsup_n \|x_n - z\| \geq a.$$

*Proof.* If the statement is false there exists an a.f.p.s. $(x_n^1)$ in $C$ and $z_1 \in C$ such that

$$\limsup_n \|x_n^1 - z_1\| < \frac{1}{2}.$$

Hence

$$D_1 = \left\{ z \in C : \limsup_n \|x_n^1 - z\| \leq \frac{1}{2} \right\}$$

is a nonempty, convex, closed, $T$-invariant subset of $C$.

With the same argument, we deduce the existence of an approximate fixed point sequence $(x_n^2)$ in $D_1$ and $z_2 \in D_1$ such that

$$\limsup_n \|x_n^2 - z_2\| < \frac{1}{2^2}.$$

Hence the set

$$D_2 = \left\{ z \in D_1 : \limsup_n \|x_n^2 - z\| \leq \frac{1}{2^2} \right\}$$

is again a nonempty, convex, closed, $T$-invariant subset of $D_1$.

In this way we construct a decreasing sequence $(D_n)$ of convex closed bounded $T$-invariant subsets of $C$ such that $\operatorname{diam}(D_n) \leq \frac{1}{2^{n-1}}$. By Cantor's intersection Theorem, $\bigcap_n D_n$ is a singleton. Since each $D_n$ is $T$-invariant this point has to be a fixed point of $T$. Thus, we have obtained a contradiction since $T$ is fixed point free. □

*Remark* 1. Notice that if $X$ is endowed with a topology $\tau$ such that every bounded sequence has a $\tau$-convergent subsequence, then the conditions of Lemma 2.2.1 also imply that

$$\inf \left\{ \limsup_n \|x_n - x\| : (x_n) \subset D, (x_n) \text{ is an a.f.p.s. and } x_n \xrightarrow{\tau} x. \right\} > 0.$$

Indeed, applying the triangular inequality, for every $(x_n) \subset D$ a.f.p.s. such that $(x_n)$ converges to $x$ in the topology $\tau$, we have

$$\limsup_n \|x_n - x\| \geq \frac{1}{2} \limsup_m \limsup_n \|x_n - x_m\| \geq \frac{a}{2}.$$

**Lemma 2.2.2.** *Let $X$ be a Banach space endowed with a linear topology $\tau$ and a family of seminorms $\{R_k(\cdot)_k\}$ satisfying properties I), II), III) and IV) stated above. Define the $|||\cdot|||$ norm as in Theorem 2.1.1 and let $x, y \in X$ and $(x_n)$, $(y_n)$ be two bounded sequences in $X$. Then the following statements are satisfied:*

1. *If $x_n \to 0$ with respect to $\tau$, then*

$$\limsup_n |||x_n||| = \limsup_n \|x_n\|.$$

*2. If $x_n \to x$ and $y_n \to y$ with respect to $\tau$ then*

$$\limsup_m \limsup_n |||x_n - y_m||| \geq \limsup_n |||x_n - x||| + \limsup_m |||y_m - y||| .$$

*Proof.* For every $k \geq 1$, using the definition of the $|||\cdot|||$ norm and property (III), we have

$$\limsup_n \|x_n\| \geq \limsup_n |||x_n||| \geq \gamma_k \limsup_n R_k(x_n) = \gamma_k \limsup_n \|x_n\|$$

Taking the limit as $k$ goes to infinity we deduce the first statement.

In order to get the second statement we do the following. By property (IV) we have

$$\begin{aligned}\limsup_m \limsup_n R_k(x_n - y_m) &= \limsup_m \left[\limsup_n R_k(x_n - x) + R_k(x - y_m)\right] \\ &= \limsup_n R_k(x_n - x) + \limsup_m R_k(y_m - y) + R_k(x - y).\end{aligned}$$

for every $k \geq 1$.

Then, using again the definition of $|||\cdot|||$ and property (III),

$$\begin{aligned}\limsup_m \limsup_n |||x_n - y_m||| &\geq \gamma_k \left[\limsup_n R_k(x_n - x) + \limsup_m R_k(y_m - y) + R_k(x - y)\right] \\ &\geq \gamma_k \left[\limsup_n |||x_n - x||| + \limsup_m |||y_m - y|||\right] .\end{aligned}$$

Taking the limit as $k$ goes to infinity we get the desired result. □

The following lemma is the key for the arguments in the proof of Theorem 2.1.1:

**Lemma 2.2.3.** *Consider the Banach space $(X, |||\cdot|||)$ and let $C$ and $T$ be as in Theorem 2.1.1. If $T$ is fixed point free we find $D$ as in Lemma 2.2.1. Let $K$ be any closed convex $T$-invariant subset of $D$ and denote*

$$\rho = \inf \left\{ \limsup_n |||x_n - x||| : (x_n) \subset K \text{ is an a.f.p.s. and } x_n \xrightarrow{\tau} x \right\} > 0.$$

*Then for every a.f.p.s. $(x_n) \subset K$ which is $\tau$-convergent and for every $z \in K$ we have*

$$\limsup_n |||x_n - z||| \geq 2\rho.$$

*Proof.* Assume that there exist a $\tau$-convergent approximate fixed point sequence $(x_n)$ in $K$ and $z \in K$ such that

$$r = \limsup_n |||x_n - z||| < 2\rho.$$

Then

$$K_1 = \left\{ w \in K : \limsup_n |||x_n - w||| \leq r \right\}$$

is a nonempty, convex, closed, bounded $T$-invariant subset of $K$. Choose an approximate fixed point sequence $(y_n)$ in $K_1$ such that $y_n \xrightarrow{\tau} y$. Denote by $x$ the $\tau$-limit of the sequence $(x_n)$. Then by *2* of Lemma 2.2.2, we have

$$\begin{aligned} r &\geq \limsup_m \limsup_n |||x_n - y_m||| \\ &\geq \limsup_n |||x_n - x||| + \limsup_n |||y_n - y||| \\ &\geq \rho + \rho = 2\rho. \end{aligned}$$

which is a contradiction. □

Now we prove Theorem 2.1.1

*Proof.* Assume the contrary, that $T$ has no fixed point. Let $D$ be as in the conclusion of Lemma 2.2.1. Define

$$c = \inf \left\{ \limsup_n |||x_n - x||| : (x_n) \subset D \text{ is an a.f.p.s. and } x_n \xrightarrow{\tau} x \right\}$$

which is greater than zero by the remark made after Lemma 2.2.1.

Without loss of generality we can assume that $c = 1$. Take $0 < \varepsilon_1 < 1/2$ and an a.f.p.s. $(x_n) \subset D$ such that $x_n \xrightarrow{\tau} x$ and $\limsup_n |||x_n - x||| < 1 + \varepsilon_1$. Again, by translation, we can assume that $x = 0$.

Let us consider now

$$K = \left\{ z \in D : \limsup_n |||x_n - z||| \leq 2 + 2\varepsilon_1 \right\}$$

The set $K$ is closed, convex, $T$-invariant and nonempty. Indeed, we can find $n_0$ such that $x_n \in K$ for all $n \geq n_0$.

Define

$$\rho = \inf \left\{ \limsup_n |||y_n - y||| : (y_n) \subset K \text{ is an a.f.p.s. and } y_n \xrightarrow{\tau} y \right\}$$

It is clear that $1 \leq \rho \leq \limsup_n |||x_n||| < 1 + \varepsilon_1$.

We are going to find an a.f.p.s. $(y_n) \subset K$ and $z \in K$ such that

$$\limsup_n |||y_n - z||| < 2\rho$$

and then we obtain a contradiction according to Lemma 2.2.3.

Notice the following: If $(y_n) \subset K$ is an a.f.p.s. and $y_n \xrightarrow{\tau} y$, then for all $k$,

$$\begin{aligned}
2 + 2\varepsilon_1 &\geq \limsup_m \limsup_n |||x_n - y_m||| = \limsup_m \limsup_n |||x_n - (y_m - y) - y||| \\
&\geq \gamma_k \limsup_m \limsup_n R_k(x_n - (y_m - y) - y) \\
&= \gamma_k \left[ \limsup_n R_k(x_n) + \limsup_m R_k(y_m - y) + R_k(y) \right] \\
&= \gamma_k \left[ \limsup_n |||x_n||| + \limsup_m |||y_m - y||| + R_k(y) \right] \geq \gamma_k \left[ 2 + R_k(y) \right].
\end{aligned}$$

Consequently, if $(y_n) \subset K$ is an a.f.p.s. and $y_n \xrightarrow{\tau} y$, we have

$$R_k(y) \leq 2 \left( \frac{1 + \varepsilon_1}{\gamma_k} - 1 \right)$$

Let

$$p := 1 + \varepsilon_1 + 2\left(\frac{1+\varepsilon_1}{\gamma_1} - 1\right) > \rho, \quad \delta \in \left(\varepsilon_1, \tfrac{1}{2}\right) \quad 0 < \varepsilon_2 < \rho - 2\delta.$$

Since, by Lemma 2.2.2-*1*, $\limsup_n \|x_n\| = \limsup_n |||x_n||| < 1 + \epsilon_1$, we can find $x \in K$ such that $\|x\| < 1 + \epsilon_1$. Also there exists $m \in \mathbb{N}$ such that if $k \geq m$

$$R_k(x) < \varepsilon_2 \qquad \text{(by property II)}$$

and

$$\frac{1+\epsilon_1}{1+\delta} < \gamma_k \qquad (\text{since } \lim_k \gamma_k = 1).$$

We take $\lambda \in (0,1)$ such that

$$\lambda < \frac{\rho(1-\gamma_m)}{\gamma_m(p-\rho)}.$$

Since

$$(2-\lambda)\rho + \lambda(\varepsilon_2 + 2\delta) = 2\rho - \lambda(\rho - (2\delta + \varepsilon_2)) < 2\rho$$

and

$$\gamma_m[(2-\lambda)\rho + \lambda p] < \rho(1+\gamma_m) < 2\rho,$$

we can find $\varepsilon_3 > 0$ such that

$$(2-\lambda)(\rho + \varepsilon_3) + \lambda(\varepsilon_2 + 2\delta) < 2\rho \tag{2.1}$$

and

$$\gamma_m[(2-\lambda)(\rho+\varepsilon_3) + \lambda p] < 2\rho. \tag{2.2}$$

Take $(y_n) \subset K$ to be an a.f.p.s. such that $y_n \xrightarrow{\tau} y$ and

$$\limsup_n \|y_n - y\| = \limsup_n |||y_n - y||| < \rho + \varepsilon_3 \qquad \text{(using Lemma 2.2.2-}\mathit{1}\text{)}.$$

There exists $s \in \mathbb{N}$ such that $\|y_N - y\| < \rho + \varepsilon_3$ for all $N \geq s$ and define

$$z = (1-\lambda)y_s + \lambda x$$

which belongs to $K$ because $K$ is convex.

Let us prove that $\limsup_n |||y_n - z||| < 2\rho$. In order to do this, we will prove that there exists $M > 0$ such that for all $k$ and $N \geq s$ we have

$$\gamma_k R_k\,(y_N - z) < M < 2\rho$$

We split the proof into two cases:

First case: $k \geq m$:

$$\begin{aligned}
\gamma_k R_k(y_N - z) &= \gamma_k R_k(y_N - (1-\lambda)y_s - \lambda x) \\
&\leq R_k(y_N - y - (1-\lambda)(y_s - y) - \lambda(x-y)) \\
&\leq R_k(y_N - y) + (1-\lambda)R_k(y_s - y) + \lambda R_k(x-y) \\
&\leq \|y_N - y\| + (1-\lambda)\|y_s - y\| + \lambda[R_k(x) + R_k(y)] \\
&< (\rho+\varepsilon_3) + (1-\lambda)(\rho+\varepsilon_3) + \lambda[\varepsilon_2 + R_k(y)] \\
&\leq (2-\lambda)(\rho+\varepsilon_3) + \lambda\left[\varepsilon_2 + 2\left(\frac{1+\varepsilon_1}{\gamma_k} - 1\right)\right] \\
&< (2-\lambda)(\rho+\varepsilon_3) + \lambda(\varepsilon_2 + 2\delta) \\
&< 2\rho \qquad \text{(by (2.1))}.
\end{aligned}$$

Second case, $k < m$:

$$\begin{aligned}
\gamma_k R_k(y_N - z) &\leq \gamma_m R_k(y_N - (1-\lambda)y_s - \lambda x) \\
&= \gamma_m\left[R_k(y_N - y - (1-\lambda)(y_s - y) - \lambda(x-y))\right] \\
&\leq \gamma_m\left[R_k(y_N - y) + (1-\lambda)R_k(y_s - y) + \lambda R_k(x-y)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \gamma_m \left[ \|y_N - y\| + (1-\lambda)\|y_s - y\| + \lambda(R_k(x) + R_k(y)) \right] \\
&< \gamma_m \left[ (\rho + \varepsilon_3) + (1-\lambda)(\rho + \varepsilon_3) + \lambda \left(1 + \varepsilon_1 + R_k(y)\right) \right] \\
&\leq \gamma_m \left[ (2-\lambda)(\rho + \varepsilon_3) + \lambda \left( 1 + \varepsilon_1 + 2\left( \frac{1+\varepsilon_1}{\gamma_1} - 1 \right) \right) \right] \\
&= \gamma_m \left[ (2-\lambda)(\rho + \varepsilon_3) + \lambda p \right] < 2\rho \qquad \text{(by (2.2))}
\end{aligned}$$

Take

$$M = \max \left\{ (2-\lambda)(\rho+\varepsilon_3) + \lambda(\varepsilon_2 + 2\delta), \gamma_m \left[ (2-\lambda)(\rho+\varepsilon_3) + \lambda p \right] \right\}.$$

Then, for all $N \geq s$, $|||y_N - z||| < M < 2\rho$. Thus $\limsup_n |||y_n - z||| < 2\rho$ and this finishes the proof. □

# Chapter 3

# Renormings and FPP in $L_1$ spaces

In the first section of this Chapter we are going to consider the Banach space $L_1(\mu)$, where $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space and we will apply Theorem 2.1.1 to this space. As a consequence we will obtain some new results about renorming and FPP for nonreflexive subspaces of $L_1(\mu)$. In the second section we apply Theorem 2.1.1 to the noncommutative $L_1$-space generated by a finite von Neumann algebra.

## 3.1 Applications to the measure function space $L_1(\mu)$

We start defining a family of seminorms $\{R_k(\cdot)\}_{k\geq 1}$ which satisfies properties I),II), III) and IV) stated in Section 2.1.

We denote by $\|\cdot\|_1$ the usual norm on $L_1(\mu)$, that is

$$\|x\|_1 = \int_\Omega |x| d\mu, \qquad \text{for all } x \in L_1(\mu)$$

and $R_1(x) = \|x\|_1$ for all $x \in L_1(\mu)$.

Let $\Omega = \bigcup_n \Omega_n$ be such that $\mu(\Omega_n) < +\infty$ for all $n \in \mathbb{N}$ and denote $A_k = \bigcup_{n=1}^{k} \Omega_n$. For $k \geq 2$ define the seminorms

$$R_k(x) = \sup\left\{\int_{E\cap A_k} |x| d\mu : \mu(E) < \frac{1}{k}\right\} + \left\|x\chi_{A_k^c}\right\|_1 .$$

Let $\tau$ be the topology of the local convergence in measure, which is given by the metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(\Omega_n)} \int_{\Omega_n} \frac{|x-y|}{1+|x-y|} d\mu; \qquad x, y \in L_1(\mu).$$

This topology is related to the convergence almost everywhere in the following way: every sequence that converges almost everywhere also converges locally in measure to the same function. Moreover, if a sequence converges locally in measure, then it has a subsequence that converges almost everywhere [43, pp. 157-158].

Fix any nondecreasing sequence $(\gamma_k)_k \subset (0,1)$ such that $\lim_k \gamma_k = 1$ and define the equivalent norm on $L_1(\mu)$ as

$$|||x||| = \sup_k \gamma_k R_k(x).$$

Now we have all the ingredients to state the following:

**Theorem 3.1.1.** *The seminorms $\{R_k(\cdot)\}_k$ defined above satisfy properties I), II), III) and IV) stated in Chapter 2. Thus the following holds: Let $C$ be a convex bounded closed subset of $L_1(\mu)$ and $T : C \to C$ a $|||\cdot|||$-nonexpansive mapping. If every sequence in $C$ has a subsequence which is almost everywhere convergent, then $T$ has a fixed point.*

Before proving Theorem 3.1.1, we introduce some remarks.

*Remark* 2. Notice that it is not necessary that the limit of the subsequence belongs to the set C in the statement of the previous theorem.

*Remark* 3. The above fixed point result does not hold for the usual norm in $L_1(\mu)$. Indeed, consider $(h_n)$ a disjointly supported normalized sequence in $L_1(\mu)$. Let

$$C = \left\{ \sum_n t_n h_n : t_n \geq 0, \sum_n t_n = 1 \right\} = \overline{\text{conv}}(h_n).$$

The set $C$ is closed convex bounded and every sequence in $C$ has an almost everywhere convergent subsequence. Indeed, consider a sequence $(f_k) \subset C$. Then $f_k = \sum_n t_n(k) h_n$ where $t_n(k) \geq 0$ and $\sum_n t_n(k) = 1$ for every $k$. Define $t^k = (t_n(k))_n$. The sequence $(t^k)_k$ belongs to the unit ball of $\ell_1$ which is $\sigma(\ell_1, c_0)$-compact. So there exists a subsequence, denoted again by $t^k$, which is $\sigma(\ell_1, c_0)$-convergent to some $t = (t_n)_n$ belonging to the unit ball of $\ell_1$. Now we can easily check that $(f_k)$ is pointwise convergent to $f = \sum_n t_n h_n$ (notice that $f$ is not, in general, in $C$).

Define $T : C \to C$ by

$$T\left(\sum_n t_n h_n\right) = \sum_n t_n h_{n+1}.$$

It is easy to check that $T$ is $\|\cdot\|_1$-nonexpansive and has no fixed point in $C$.

*Remark* 4. If the measure space is not $\sigma$-finite and we consider the convergence almost everywhere in $L_1(\mu)$, we can not renorm the space $L_1(\mu)$ so that Theorem 3.1.1 remains true. Indeed, in this case $\ell_1(\Gamma)$ is contained isometrically in $L_1(\mu)$ for some uncountable set $\Gamma$ and every sequence in a bounded subset of $\ell_1(\Gamma)$ has a pointwise convergent subsequence. So if Theorem 3.1.1 holds then we would have a renorming in $\ell_1(\Gamma)$ with the FPP. This is impossible since every renorming of $\ell_1(\Gamma)$ contains an asymptotically isometric copy of $\ell_1$ and then it fails the FPP [23].

As a consequence of this remark we have the following corollary.

**Corollary 3.1.2.** *Let $(\Sigma, \Omega, \mu)$ be a measure space. Then $L_1(\mu)$ can be renormed to satisfy the statements of Theorem 3.1.1 if and only if $\mu$ is $\sigma$-finite.*

Before proving Theorem 3.1.1 we give a simpler definition of the $|||\cdot|||$ norm in two special cases: when $\mu$ is finite and when $\mu$ is purely atomic.

*Remark* 5. Assume that the measure $\mu$ is finite. Consider $A_k = \Omega$ for all $k$. Then $A_k^c = \emptyset$ and $\|x\chi_{A_k^c}\|_1 = 0$ for all $x \in L_1(\mu)$. Therefore

$$R_k(x) = \sup\left\{\int_E |x| d\mu : \mu(E) < \frac{1}{k}\right\}$$

for all $k \in \mathbb{N}$

In this case, the topology of the local convergence in measure is given by the metric

$$d(x,y) = \int_\Omega \frac{|x-y|}{1+|x-y|} d\mu; \qquad x, y \in L_1(\mu).$$

and the convergence with respect to this topology is the convergence in measure.

*Remark* 6. Assume now that $\Omega = \mathbb{N}$ and $\mu$ is the counting measure defined on the subsets of $\mathbb{N}$. Then the space $L_1(\mu)$ becomes the sequence space $\ell_1$. Denote $\Omega_k = \{n\}$ and $A_k = \{1, ..., n\}$. Then

$$R_k(x) = \|x\chi_{A_k^c}\|_1 = \sum_{n=k+1}^{\infty} |x(n)|$$

In this case we recover again the $|||\cdot|||$ norm defined by P.K. Lin in [56] for the particular case $\gamma_k = 8^k/(1+8^k)$. Notice that the convergence almost everywhere in $\ell_1$ is the pointwise convergence and every bounded sequence in $\ell_1$ has a pointwise convergent subsequence because the unit ball of $\ell_1$ is $\sigma(\ell_1, c_0)$-compact.

In general, if the measure space is $\sigma$-finite and purely atomic, $L_1(\mu)$ is isometric to $\ell_1$ [70]. Thus by Theorem 3.1.1, it can be renormed to have the FPP.

Now, we proceed to prove Theorem 3.1.1.

*Proof.* We have to check that properties I), II), III) IV) are fulfilled for the family of seminorms $R_k(\cdot)$ defined on $L_1(\mu)$.

To simplify the notation we let

$$S_k(x) := \sup\left\{\int_E |x|d\mu : \mu(E) < \frac{1}{k}\right\}$$

for $x \in L_1(\mu)$, therefore

$$R_k(x) = S_k(x\chi_{A_k}) + \|x\chi_{A_k^c}\|_1.$$

I) We have $S_k(x\chi_{A_k}) \leq \|x\chi_{A_k}\|$. Therefore $R_k(x) \leq \|x\chi_{A_k}\|_1 + \|x\chi_{A_k^c}\|_1 = \|x\|_1$ for every $x \in L_1(\mu)$.

II) Using the absolute continuity of the norm and that $\lim_k \|x\chi_{A_k^c}\|_1 = 0$ we can check that $\lim_k R_k(x) = 0$ for all $x \in L_1(\mu)$.

III) Fix $k \geq 1$ and let $(x_n)$ be a sequence convergent to the null function locally in measure. Assume the contrary, that is, $\limsup_n R_k(x_n) < \limsup_n \|x_n\|_1$. We can take a sequence, again denoted by $(x_n)$, such that the limits $\lim_n \|x_n\|_1$, $\lim_n S_k(x_n\chi_{A_k})$, $\lim_n \|x_n\chi_{A_k}\|_1$ and $\lim_n \|x_n\chi_{A_k^c}\|_1$ exist, $(x_n)$ converges to the null function almost everywhere and $\lim_n R_k(x_n) < \lim_n \|x_n\|_1$.

Let us prove that $\lim_n S_k(x_n\chi_{A_k}) = \lim_n \|x_n\chi_{A_k}\|_1$:

Using Egoroff's Theorem there exists a measurable set $E \subset A_k$ with $\mu(E) < \frac{1}{k}$ and such that $x_n \to 0$ uniformly on $A_k \setminus E$. In particular $x_n\chi_{A_k\setminus E} \to 0$ in norm and $\lim_n \|x_n\chi_{A_k}\|_1 = \lim_n \|x_n\chi_E\|_1$. Therefore

$$\lim_n \|x_n\chi_{A_k}\|_1 \geq \lim_n S_k(x_n\chi_{A_k}) \geq \lim_n \|x_n\chi_E\|_1 = \lim_n \|x_n\chi_{A_k}\|_1.$$

Now

$$\begin{aligned}\lim_n R_k(x_n) &\geq \lim_n S_k(x_n\chi_{A_k}) + \lim_n \|x_n\chi_{A_k^c}\|_1 \\ &= \lim_n \|x_n\chi_{A_k}\|_1 + \lim_n \|x_n\chi_{A_k^c}\|_1 \\ &= \lim_n \|x_n\|_1\end{aligned}$$

which is a contradiction and property III) holds.

IV) If $k = 1$, since $R_1(\cdot) = \|\cdot\|_1$ and using [8] we obtain

$$\limsup_n \|x_n + x\|_1 = \limsup_n \|x_n\|_1 + \|x\|_1.$$

Assume that $k \geq 2$. Suppose by contradiction that property IV) does not hold. We recall the following lemma for finite measure spaces [1]: Let $(\Omega, \sigma, \mu)$ be a finite measure space and $(h_n)$ be a bounded sequence in $L_1(\mu)$ converging to the null function in measure. Then there exists a subsequence $(h_{n_l})$ and a sequence of pairwise disjoint measurable sets $(E_l)$ such that

$$\lim_l \|h_{n_l} - h_{n_l}\chi_{E_l}\|_1 = 0$$

In particular, for all $k \geq 1$, $\lim_l S_k(h_{n_l} - h_{n_l}\chi_{E_l}) = 0$.

Using the above result we can take a subsequence, again denoted by $(x_n)$, such that there exists a sequence $(h_n)$ of measurable functions defined in $A_k$ which is disjointly supported, $\lim_n S_k((x_n - h_n)\chi_{A_k}) = 0$ and $\limsup_n R_k(x_n + x) < \limsup_n R_k(x_n) + R_k(x)$ for some $x \in L_1(\mu)$. Therefore

$$\begin{aligned}\limsup_n R_k(x_n + x) &= \limsup_n S_k((x_n + x)\chi_{A_k}) + \limsup_n \|(x_n + x)\chi_{A_k^c}\|_1 \\ &= \limsup_n S_k((h_n + x)\chi_{A_k}) + \limsup_n \|x_n\chi_{A_k^c}\|_1 + \|x\chi_{A_k^c}\|_1\end{aligned}$$

Let us prove that $\limsup_n S_k((h_n + x)\chi_{A_k}) = \limsup_n S_k(h_n\chi_{A_k}) + S_k(x\chi_{A_k})$:

Denote by $E_n \subset A_k$ the support of the function $h_n$ and let $\varepsilon > 0$. By the definition of $S_k(\cdot)$, there exists a measurable set $A \subset A_k$ with $\mu(A) < \frac{1}{k}$ such that

$$\int_A |x| d\mu \geq S_k(x\chi_{A_k}) - \varepsilon.$$

Since $\sum_n \mu(E_n) \leq \mu(A_k) < +\infty$ there exists $n_0$ such that

$$\mu(A) + \sum_{n \geq n_0} \mu(E_n) < \frac{1}{k}.$$

Therefore

$$\begin{aligned}
\limsup_n S_k\left((h_n + x)\chi_{A_k}\right) &\geq \limsup_n \int_{A\cup\left(\cup_{n\geq n_0} E_n\right)} |h_n + x|\chi_{A_k} d\mu \\
&= \limsup_n \int_{A\cup\left(\cup_{n\geq n_0} E_n\right)} |h_n|\chi_{A_k} d\mu \\
&\quad + \int_{A\cup\left(\cup_{n\geq n_0} E_n\right)} |x|\chi_{A_k} d\mu \\
&\geq \limsup_n \int_{E_n} |h_n|\chi_{A_k} d\mu + \int_A |x|\chi_{A_k} d\mu \\
&\geq \limsup_n \int_{E_n} |h_n|\chi_{A_k} d\mu + \int_A |x| d\mu \\
&\geq \limsup_n \|h_n\chi_{A_k}\|_1 + S_k(x\chi_{A_k}) - \varepsilon \\
&= \limsup_n S_k(h_n\chi_{A_k}) + S_k(x\chi_{A_k}) - \varepsilon.
\end{aligned}$$

Since $\varepsilon$ is arbitrary we obtain the desired equality. Therefore:

$$\begin{aligned}
\limsup_n R_k(x_n + x) &= \limsup_n S_k(h_n\chi_{A_k}) + S_k(x\chi_{A_k}) \\
&\quad + \limsup_n \|x_n\chi_{A_k^c}\|_1 + \|x\chi_{A_k^c}\|_1 \\
&= \limsup_n S_k(x_n\chi_{A_k}) + S_k(x\chi_{A_k})
\end{aligned}$$

$$+\limsup_n \|x_n\chi_{A_k^c}\|_1 + \|x\chi_{A_k^c}\|_1$$
$$\geq \limsup_n R_k(x_n) + R_k(x)$$

and we obtain IV).

□

It is well-known that the space $L_1(\mu)$ does not have the FPP. In fact, it does not satisfy the weak fixed point property $w$-FPP, D. Alspach showed a nonexpansive mapping from a convex weakly compact subset of $L_1[0,1]$ into itself without fixed points [2]. A Banach space $X$ is said to have the $w$-FPP if every nonexpansive selfmapping defined on a convex $w$-compact subset of $X$ has a fixed point. On the other hand, we know that a subspace $X$ of $L_1[0,1]$ has the FPP if and only if $X$ is reflexive. This leads us to the following question: Can a nonreflexive subspace of $L_1(\mu)$ be renormed to have the FPP? Theorem 3.1.1 lets us give a partial answer to this question:

**Corollary 3.1.3.** *Let $X$ be a closed subspace of $L_1(\mu)$. If the unit ball of $X$ is relatively sequentially compact for the topology of the local convergence in measure, then $(X, |||\cdot|||)$ has the FPP.*

**Corollary 3.1.4.** *Let $X$ be a closed subspace of $L_1(\mu)$. If $X$ is a dual space such that the topology of the local convergence in measure coincides with the weak star topology on the unit ball of $X$, then $(X, |||\cdot|||)$ has the FPP.*

Notice that this is the case of the sequence space $\ell_1$. Here we present another example.

**Example 3.1.1.** Let $\mathbb{D}$ denote the open unit disc. The Bergman space $L_a(\mathbb{D})$ is defined as the subspace of $L_1(\mathbb{D})$ of all analytic functions on $\mathbb{D}$. This space is a dual space and, for bounded sequences, weak* convergence is equivalent to uniform convergence on compact sets [66]. This shows that the weak* topology is finer than the topology of convergence in measure on the unit ball of $L_a(\mathbb{D})$ and consequently these two topologies coincide for $B_{L_a(\mathbb{D})}$. Thus the Bergman space endowed with the $|||\cdot|||$ norm given in this section has the FPP. In fact, J. Lindenstrauss, A. Pelczynski [59] proved that the Bergman space and the sequence space $\ell_1$ are isomorphic, although they did not give an explicit definition of the isomorphism. It turns out to be a difficult problem to find a system of functions which is a basis in $L_a(\mathbb{D})$ equivalent to the unit vector basis in $\ell_1$ (see [81, Notes and Remarks in Chapter III.A] and the references therein). Thus from P.K. Lin's paper [56] we could have deduced that the Bergman space can be renormed to have the FPP. However, using Theorem 3.1.1 we can give explicitly the renorming on the Bergman space with the FPP.

The following example satisfies the hypothesis of Corollary 3.1.3 but does not fit in the scope of Corollary 3.1.4:

**Example 3.1.2.** In [33, Théorème 7] we can find an example of a subspace $X$ of $L_1[0,1]$ such that its unit ball $B_X$ is compact for the topology of convergence in measure but not locally convex for this topology. Then, by Corollary 3.1.3, $(X, |||\cdot|||)$ has the FPP (but the topology of convergence in measure in the unit ball of $L_1[0,1]$ can not coincide with any weak star topology).

To finish this section we show another example of a Banach space which can be renormed to satisfy the FPP:

**Example 3.1.3.** In [7] we can find an example of a Banach space $E$ contained in $L_1$, over a probability space, and such that $E$ fails to have the Radon-Nikodym property and every $L_1$-bounded sequence in $E$ has a subsequence converging in measure. Applying again Corollary 3.1.3, we deduce that $E$ can be renormed to have the FPP and, by the failure of the Radon-Nikodym property, $E$ is not isomorphic to a subspace of $\ell_1$.

## 3.2 FPP in noncommutative $L_1$-spaces

In this section we generalize the renorming given in the last section for $L_1(\mu)$ spaces to the more general case of preduals of von Neumann algebras. Fix an infinite dimensional and finite von Neumann algebra $\mathcal{M}$ with a finite trace $\tau$. Let $L_1(\mathcal{M})$ be the predual of $\mathcal{M}$.

It is known that $L_1(\mathcal{M})$ fails the FPP because it contains an isometric copy of $\ell_1$. Moreover, if there is a nonzero projection $p \in \mathcal{P}(\mathcal{M})$ which dominates no minimal projection, the noncommutative space $L_1(\mathcal{M})$ contains an isometric copy of $L_1[0,1]$ (see [52, Lemma 3.1]) which shows that $L_1(\mathcal{M})$ endowed with its original norm also fails the $w$-FPP [2].

If $X$ is a nonreflexive subspace of $L_1(\mathcal{M})$, H. Pfitzner proved that $X$ contains an asymptotically isometric copy of $\ell_1$ [69] and therefore $X$ also fails to have the FPP.

Our main object is to apply Theorem 2.1.1 to noncommutative $L_1(\mathcal{M})$ spaces. In order to do that we are going to define a family of seminorms $\{R_k(\cdot)\}_{k\geq 1}$ and a linear topology $\mathcal{T}$ on $L_1(\mathcal{M})$ where conditions (I), (II), (III) and (IV) of the previous chapter are fulfilled.

**Definition 3.2.1.** Define a family of seminorms $\{R_k(\cdot)\}_{k\geq 1}$ in $L_1(\mathcal{M})$ in the following way:

If $k=1$, denote $R_1(x) = \|x\|_1 = \tau(|x|)$

If $k \geq 2$ we define

$$R_k(x) := \sup\left\{\|xp\|_1 : \ p \in \mathcal{P}(\mathcal{M}), \tau(p) < \frac{1}{k}\right\}$$

for all $x \in L_1(\mathcal{M})$.

We consider the following topology defined by E. Nelson in [65]:

**Definition 3.2.2.** Let $\mathcal{M}$ be a finite von Neumann algebra and $\tau$ a finite trace on $\mathcal{M}$. The measure topology is defined by the following fundamental system of neighborhoods of zero: For every $\varepsilon > 0$ and $\delta > 0$ let

$$N(\varepsilon, \delta) = \{x \in \mathcal{M} : \exists\, p \in \mathcal{P}(\mathcal{M}) \text{ such that } \|xp\|_\infty \leq \varepsilon \text{ and } \tau(p^\perp) \leq \delta\}.$$

The measure topology is linear and metrizable. We denote the completion of $\mathcal{M}$ with respect to the measure topology by $\widetilde{\mathcal{M}}$. According to [65, Theorem 1] the mapping $(x, y) \mapsto xy$ of $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is continuous and has a unique extension as a mapping of $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$. Also, by [65, Theorem 5] we have $L_p(\mathcal{M}) \subset \widetilde{\mathcal{M}}$ for all $1 \leq p \leq \infty$. The elements of $\widetilde{\mathcal{M}}$ are usually called measurable operators and $\widetilde{\mathcal{M}}$ is also denoted by $L_0(\tau)$. Notice that if $\mathcal{M}$ is $L_\infty[0, 1]$ with trace given by integration with respect to the Lebesgue measure $m$, the convergence with respect to the measure topology coincides with the usual convergence in measure with respect to $m$.

Now we are in condition to state the main result of this section:

**Theorem 3.2.1.** *Let $\{\gamma_k\}_k \subset (0,1)$ be any nondecreasing sequence such that $\gamma_k \to 1$ and define*

$$|||x||| = \sup_{k\geq 1} \gamma_k R_k(x); \qquad x \in L_1(\mathcal{M}).$$

*Then $|||\cdot|||$ is an equivalent norm on $L_1(\mathcal{M})$ such that $(L_1(\mathcal{M}), |||\cdot|||)$ satisfies the following property: for every nonempty closed convex bounded subset $C$ which is relatively compact for the measure topology and for every $T : C \to C$ nonexpansive, there exists a fixed point.*

Before proving the previous theorem we notice the following:

*Remark* 7. The above fixed point result is not valid for the $\|\cdot\|_1$ norm in $L_1(\mathcal{M})$. Let us consider the following example:

Take $x \in L_1(\mathcal{M})$ and take $(A_n)$ a sequence of measurable sets in the spectrum $\sigma(|x|)$ of $|x|$ with $A_n \cap A_m = \emptyset$ for $n \neq m$. For all $n \in \mathbb{N}$ define $p_n = e^{|x|}_{A_n}$, which form a sequence of mutually disjoint projections in $\mathcal{M}$ and let

$$C = \left\{ \sum_n t_n p_n; \quad t_n \geq 0, \sum_n t_n = 1 \right\}$$

The set $C$ is convex, bounded and it is not difficult to check that it is norm closed.

Define $T : C \to C$ by

$$T\left(\sum_n t_n p_n\right) = \sum_n t_n p_{n+1}.$$

Note that $T$ is fixed point free. Moreover, if $(s_n) \in \ell_1$, we have

$$\left\|\sum_n s_n p_n\right\|_1 = \lim_n \left\|\sum_{k=1}^n s_k p_k\right\|_1 = \lim_n \tau\left(\left|\sum_{k=1}^n s_k p_k\right|\right)$$

$$
\begin{aligned}
&= \lim_n \tau\left(\left[\left(\sum_{k=1}^n s_k p_k\right)^* \left(\sum_{k=1}^n s_k p_k\right)\right]^{1/2}\right) \\
&= \lim_n \tau\left(\left[\sum_{k=1}^n |s_k|^2 p_k\right]^{1/2}\right) = \lim_n \tau\left(\sum_{k=1}^n |s_k| p_k\right) \\
&= \lim_n \sum_{k=1}^n |s_k| \tau(p_k) = \sum_n |s_n| \|p_n\|_1
\end{aligned}
$$

The above equality implies that $T$ is a fixed point free $\|\cdot\|_1$-nonexpansive mapping. Let us also check that $C$ is relatively compact for the measure topology in $L_1(\mathcal{M})$. Indeed, let $f_k = \sum_n t_n(k) p_n \in C$ for all $k \in \mathbb{N}$. Define $t_k = (t_n(k))_n$ which belongs to $\ell_1$ for all $k \in \mathbb{N}$. Since the unit ball of $\ell_1$ is $\sigma(\ell_1, c_0)$-compact we can assume, taking a subsequence if necessary, that $(t_k)_k$ tends to some $t = (t_n)_n \in \ell_1$ pointwise. Define $f = \sum_n t_n p_n$. Let us prove that $f_n$ tends to $f$ in measure. Let $\varepsilon, \delta > 0$. Since $(p_n)$ are mutually disjoint,

$$
\tau\left(\sum_n p_n\right) \leq \tau(I) < +\infty.
$$

Take $n_0 \in \mathbb{N}$ such that $\tau(\sum_{n>n_0} p_n) < \delta$. There exists $k_0 \in \mathbb{N}$ such that $\sum_{n=1}^{n_0} |t_n(k) - t(k)| < \varepsilon$ if $k \geq k_0$. Define $q^\perp = \bigvee_{n>n_0} p_n = \sum_{n>n_0} p_n$ and $q = I - q^\perp$. Then $\tau(q^\perp) < \delta$ and for all $k \geq k_0$ we have

$$
\begin{aligned}
\|(f_n - f)q\|_\infty &= \left\|\sum_n (t_n(k) - t_n) p_n q\right\|_\infty = \left\|\sum_{n=1}^{n_0} (t_n(k) - t(k)) p_n\right\|_\infty \\
&\leq \sum_{n=1}^{n_0} |t_n(k) - t(k)| \|p_n\|_\infty < \varepsilon
\end{aligned}
$$

Therefore, for all $k \geq k_0$, $f_k - f \in N(\varepsilon, \delta)$ and $f_k \to f$ in measure (notice that $f$

does not belong to $C$ in general). Therefore, the assertion of Theorem 3.2.1 does not hold for the usual $\|\cdot\|_1$ norm.

*Proof of the Theorem 3.2.1.* We prove that the family of seminorms $\{R_k(\cdot)\}_{k\geq 1}$ satisfies the properties stated in Theorem 2.1.1 when $\mathcal{T}$ is the measure topology defined above:

I) Let $x \in L_1(\mathcal{M})$. For every projection $p \in \mathcal{P}(\mathcal{M})$ we have

$$\|xp\|_1 = \tau(|xp|) \leq \|x\|_1\|p\|_\infty = \|x\|_1,$$

and thus $R_k(x) \leq \|x\|_1$ for all $k \geq 2$.

II) Let us prove that if $x \in L_1(\mathcal{M})$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|xp\|_1 < \varepsilon$ whenever $\tau(p) < \delta$. Consequently, $\lim_k R_k(x) = 0$. By construction of $L_1(\mathcal{M})$, there exists some $x_0 \in \mathcal{M}$ such that $\|x - x_0\|_1 < \varepsilon/2$. Take $0 < \delta < \varepsilon/(2\|x_0\|_\infty + 1)$ and consider $p \in \mathcal{P}(\mathcal{M})$ with $\|p\|_1 = \tau(p) < \delta$. Then

$$\begin{aligned} \|xp\|_1 &\leq \|xp - x_0p\|_1 + \|x_0p\|_1 \\ &\leq \|x - x_0\|_1\|p\|_\infty + \|x_0\|_\infty\|p\|_1 \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

III) It is clear that

$$\limsup_n R_k(x_n) \leq \limsup_n \|x_n\|_1.$$

We have to prove the other inequality. Since $x_n \to 0$ in measure, there exist $n \geq n_0$ and a sequence $(p_n) \subset \mathcal{M}$ of projections such that $x_np_n \in \mathcal{M}$,

$\|x_n p_n\|_\infty < \varepsilon$ and $\tau(p_n^\perp) < \varepsilon$ for $n \geq n_0$. Therefore, for all $n \geq n_0$:

$$\begin{aligned} R_k(x_n) &\geq \|x_n p_n^\perp\|_1 \geq \|x_n\|_1 - \|x_n p_n\|_1 \\ &= \|x_n\|_1 - \|I x_n p_n\|_1 \\ &\geq \|x_n\|_1 - \tau(I)\|x_n p_n\|_\infty \\ &\geq \|x_n\|_1 - \tau(I)\varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary, $\limsup_n R_k(x_n) \geq \limsup_n \|x_n\|_1$ as we wanted to prove.

IV) Notice that, for $k = 1$, this equality can be deduced from the fact that $L_1(\mathcal{M})$ is an $L$-embedded Banach space and the measure topology is the abstract measure topology defined in $L_1(\mathcal{M})$ (see [68, 69, 46]). However, we give a direct proof of this equality using the properties of the von Neumann algebra: Let $x \in L_1(\mathcal{M})$ and $(x_n) \subset L_1(\mathcal{M})$ converging to zero in measure. Fix $\varepsilon > 0$. Using property II), there exists $\delta > 0$ such that if $q \in \mathcal{P}(\mathcal{M})$ with $\tau(q) < \delta$ then $\|xq\|_1 < \varepsilon$. Since $x_n \to 0$ in measure, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exists $p_n \in \mathcal{P}(\mathcal{M})$ such that $\tau(p_n^\perp) < \delta$ and $\|x_n p_n\|_\infty < \varepsilon$. Therefore, for all $n \geq n_0$:

$$\begin{aligned} \|x_n + x\|_1 &= \|x_n p_n + x_n p_n^\perp + x p_n + x p_n^\perp\|_1 \\ &\geq \|x_n p_n^\perp\|_1 - \|x_n p_n\|_1 + \|x p_n\|_1 - \|x p_n^\perp\|_1 \\ &\geq \|x_n p_n^\perp\|_1 - \tau(I)\|x_n p_n\|_\infty + \|x p_n\|_1 - \epsilon \\ &\geq \|x_n\|_1 - \|x_n p_n\|_1 - \tau(I)\varepsilon + \|x\|_1 - \|x p_n^\perp\|_1 - \epsilon \\ &\geq \|x_n\|_1 - 2\tau(I)\varepsilon + \|x\|_1 - 2\varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary, we obtain $\limsup_n \|x_n + x\|_1 \geq \limsup_n \|x_n\|_1 + \|x\|_1$ and the equality holds.

Assume $k \geq 2$: Let $(x_n)$ be a null convergent sequence in measure. Using [77, Proposition 2.2.] and taking a subsequence that we still denote by $(x_n)$, we know that there exists a sequence of projections $(p_n)$ such that $p_n p_m = 0$ if $n \neq m$ and

$$\lim_n \|x_n - x_n p_n\|_1 = 0.$$

Let $x \in L_1(\mathcal{M})$ and $\varepsilon > 0$. Using the definition of $R_k(\cdot)$ there exists a projection $p$ with $\tau(p) < \frac{1}{k}$ and

$$\|xp\|_1 \geq R_k(x) - \varepsilon.$$

Denote by $\bigvee_{n=1}^{\infty} p_n$ the projection onto the subspace $\overline{\sum_n p_n(H)}$.

Since the projections $p_n$ are pairwise disjoint and $\bigvee_{n=1}^{\infty} p_n \leq I$, we have

$$\sum_{n=1}^{\infty} \tau(p_n) \leq \tau\left(\bigvee_{n=1}^{\infty} p_n\right) \leq \tau(I) < \infty.$$

Then there exists $n_0$ such that for all $n \geq n_0$

$$\tau(p) + \sum_{n \geq n_0} \tau(p_n) < \frac{1}{k}.$$

Define $q = p \bigvee \left(\bigvee_{n=n_0}^{\infty} p_n\right)$. Notice that $p_n \leq q$ and $p \leq q$ so $p_n q = p_n$ and $qp = p$. Then

$$\tau(q) \leq \tau(p) + \sum_{n \geq n_0} \tau(p_n) < \frac{1}{k}$$

and

$$\|xp\|_1 = \|xqp\|_1 \leq \|xq\|_1 \|p\|_\infty = \|xq\|_1.$$

On the other hand, since $x_n \to 0$ in measure $x_n p_n \to 0$ in measure. Therefore

$$\begin{aligned}
\limsup_n R_k(x_n + x) &= \limsup_n R_k(x_n p_n + x) \\
&\geq \limsup_n \|(x_n p_n + x)q\|_1 \\
&= \limsup_n \|x_n p_n q + xq\|_1 \\
&= \limsup_n \|x_n p_n\|_1 + \|xq\|_1 \\
&\geq \limsup_n \|x_n p_n\|_1 + \|xp\|_1 \\
&= \limsup_n \|x_n\|_1 + \|xp\|_1 \\
&\geq \limsup_n R(x_n) + R_k(x) - \varepsilon
\end{aligned}$$

Since $\varepsilon$ is arbitrary, we obtain the desired inequality.

□

Now we can state a topological sufficient condition so that a closed subspace of $L_1(\mathcal{M})$ can be renormed to have the FPP:

**Corollary 3.2.2.** *Let $X$ be a closed subspace of $L_1(\mathcal{M})$. If the unit ball of $X$ is relatively compact for the measure topology then $X$ endowed with the $|||\cdot|||$ norm has the FPP.*

This result includes, as a particular case, the condition obtained in Corollary 3.1.3 where subspaces of the Lebesgue measure space $L_1(\mu)$ for $\sigma$-finite measures are considered.

On the other hand, since the unit ball of $L_1(\mathcal{M})$ is always closed in the measure topology [82, Theorem 2.9], from Theorem 3.2.1 we can also deduce the following.

**Corollary 3.2.3.** *Let $\mathcal{M}$ be a finite von Neumann algebra. If the unit ball of $L_1(\mathcal{M})$ is compact for the measure topology, the Banach space $(L_1(\mathcal{M}), |||\cdot|||)$ has the FPP.*

From Theorem 3.2.1 we can deduce the following corollary, which again extends Lin's result in $\ell_1$:

**Corollary 3.2.4.** *Let $\mathcal{M}$ be any finite atomic von Neumann algebra. Then $L_1(\mathcal{M})$ can be renormed to have the FPP.*

*Proof.* According to [78, Proposition 2.2], if $\mathcal{M}$ is a finite atomic von Neumann algebra, then $L_1(\mathcal{M})$ is isomorphic to either $\ell_1$ or $S_1$ where

$$S_1 = (C_1^1 \oplus C_1^2 \oplus \cdots \oplus C_1^n \oplus \cdots)_{\ell_1}$$

and $C_1^n = L_1(B(\ell_2^n), tr^n)$, where $tr^n(x) = \frac{1}{n}\sum_{i=1}^n \langle xe_i, e_i\rangle$ if $x \in B(\ell_2^n)$. Here the von Neumann algebra is $\mathcal{M} = \bigoplus_{n=1}^{\infty} B(\ell_2^n)$ and a finite normal faithful trace on $\mathcal{M}$ can be defined as

$$\tau(x) = \sum_n \frac{1}{2^n} tr^n(x_n)$$

Let us prove that the unit ball of $S_1$ is compact for the measure topology. Consider $(x_k)$ a sequence in the unit ball. Using that $S_1$ is a dual space, we can extract a subsequence, that we still denote by $(x_k)$ such that $x_k \to x \in S_1$ with respect to the weak* topology. Now we prove that $x_k \to x$ w.r.t. the topology of the convergence in measure. Fix $\epsilon, \delta > 0$. Define $p_n = (I^1, I^2, \cdots, I^n, 0, \cdots)$, where $I^n$ denotes the identity operator on $\ell_2^n$. Since $\bigvee_n p_n = I$, $(p_n)_n \to I$ with respect to the strong topology [76, p.-39], we know that $\lim_n \tau(p_n) = \tau(I) < \infty$ since $\tau$ is normal. Take $n_0$ such that $\tau(p_{n_0}^\perp) < \delta$ for all $n \geq n_0$. Using that $C_1^1 \oplus \cdots \oplus C_1^{n_0}$ is finite dimensional and that the convergence on every $C_1^n$ is also in norm we have that

$$\lim_k \|(x_k - x)p_{n_0}\|_\infty = 0.$$

so $x_k \to x \in S_1$ in measure according to Definition 3.2.2. The above shows that $(S_1, |||\cdot|||)$ has the FPP and consequently the corollary is proved. □

We finish this chapter by raising the following two open questions:

1. Can the measure space $L_1[0,1]$ be renormed to have the FPP? Notice that the unit ball of this space is not compact for the measure space (the Rademacher sequence does not have any almost everywhere convergent sequence) so our techniques cannot be applied to deduce that $L_1[0,1]$ is itself FPP "renormable".

   In the next chapter we will prove that we can skip the compactness condition if we assume that the mapping is also affine.

2. Can the trace space $C_1$ be renormed to have the FPP? Here we have some kind of compactness condition, since $C_1$ is a dual Banach space and its unit ball is compact for the $w^*$-topology. However we do not have condition IV), which is required for Theorem 2.1.1, for the norm in $C_1$ an the $w^*$-topology as the following example shows [6]:

   **Example 3.2.1.** Consider $C_1 = C_1(\ell_2, \ell_2)$ defined as the bounded operators from $\ell_2$ into $\ell_2$ with finite trace, and $\|x\|_1 = \sum_n s_n(x)$ where $(s_n(x))_n$ are the singular values of $x$, i.e. the sequence of eigenvalues of $|x| = (x^*x)^{\frac{1}{2}}$.

   For $u, v \in \ell_2$ define the operator $u \otimes v(x) = \langle v, x\rangle u$.

   Consider

$$x = \frac{1}{\alpha} e_1 \otimes e_1$$

and

$$x_n = e_1 \otimes e_n + e_n \otimes e_1 + \alpha e_n \otimes e_n,$$

where $\alpha > 0$ and $\{e_i\}$ is the canonical basis of $\ell_2$. It is not difficult to check that $x_n \overset{w^*}{\to} 0$, $\|x\|_{C_1} = \frac{1}{\alpha}$, $\|x_n\|_{C_1} = \sqrt{\alpha^2 + 4}$ and $\|x_n + x\|_{C_1} = \frac{\alpha^2+1}{\alpha}$ for $n \geq 1$. So,

$$\limsup_n \|x_n + x\|_{C_1} \neq \limsup_n \|x_n\|_{C_1} + \|x\|_{C_1}.$$

# Chapter 4

# A renorming for affine nonexpansive mappings

In the previous chapter we have proved that all closed subspaces of the predual of a finite von Neumann algebra whose unit balls are relatively compact for the measure topology can be renormed to have the fixed point property for nonexpansive mappings. In fact, it is an open problem if the condition about the compactness can be omitted, that is, if the predual of a finite von Neumann algebra (or $L_1[0,1]$ itself) can be renormed to have the FPP. In this chapter we will prove that we do not need the compactness condition if we assume that the mapping is also affine.

## 4.1 Introduction

We will introduce some definitions:

**Definition 4.1.1.** Let $C$ be a convex subset of a Banach space $X$ and $T : C \to C$

a mapping. It is said that $T$ is affine if

$$T(\lambda x + (1-\lambda)y) = \lambda Tx + (1-\lambda)Ty$$

whenever $x, y \in C$ and $\lambda \in [0,1]$

Affine mappings have been useful to characterize weakly compactness in Banach spaces as the following result shows:

**Theorem 4.1.1.** *[18] Let $X$ be a Banach space and $C$ a convex closed bounded subset of $X$. The following are equivalent:*

*i) $C$ is weakly compact.*

*ii) For every closed convex subset $K \subset C$ and for every affine continuous mapping $T : K \to K$, there exists a fixed point.*

In certain Banach spaces the above characterization can be given for a more restrictive class of affine mappings (see for instance [18, Corollary 3.4]).

In case of subsets of $L_1[0,1]$, the following is proved in [25]:

**Theorem 4.1.2.** *[25] Let $C$ be a convex closed bounded subset of $L_1[0,1]$. The following are equivalent:*

*i) $C$ is weakly compact.*

*ii) For every closed convex subset $K \subset C$ and for every affine nonexpansive mapping $T : K \to K$, there exists a fixed point.*

Notice that the affine condition in *ii)* can not be dropped since D. Alspach [2] showed a convex weakly compact subset $C$ of $L_1[0,1]$ and a nonexpansive mapping

$T : C \to C$ without fixed points. A similar characterization was also obtained in case of subsets of a non-commutative $L_1$-space, when $\tau$ is semi-finite trace [25], and for $L$-embedded Banach spaces in [18].

We introduce the following definition:

**Definition 4.1.2.** We say that a Banach space $X$ has the fixed point property for affine nonexpansive mappings (A-FPP) if for every closed convex bounded subset $C \subset X$ and for every affine nonexpansive mapping $T : C \to C$, there exists a fixed point.

The first observation that should be made is that every reflexive Banach space has the A-FPP. This is due to the fact that every affine continuous self-mapping of a closed convex set is weakly continuous and bounded closed subsets are weakly compact whenever $X$ is reflexive. The well-known Schauder-Tychonoff Theorem (see [27, p. 74]) shows therefore that an affine continuous self-mapping of a convex weakly compact subset $C$ of a Banach space $X$ always has a fixed point.

Also notice that the classical nonreflexive sequence spaces $\ell_1$ and $c_0$ fail to have the A-FPP: it was shown in Examples 1.1.1 and 1.1.2 that these spaces fail to have the FPP by using an affine nonexpansive mapping. Again, it is clear that every space which contains an isometric copy of $\ell_1$ or $c_0$ also fails this property. What is more, we can state the following:

**Theorem 4.1.3.** *Let $X$ be a Banach space which contains an asymptotically isometric copy of either $\ell_1$ or $c_0$. Then $X$ fails to have the A-FPP.*

*Proof.* The proof is identical to the ones in [22] and [23] regarding to the failure of the FPP (see also Theorem 2.3 and Theorem 2.4 in [50], Chapter 9). However, we include some details only for the sake of completeness:

If $X$ contains an asymptotically isometric copy of $\ell_1$, there exist a sequence $(x_n) \subset X$ and a sequence $(\epsilon_n) \downarrow 0$ such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} |t_n|$$

for all $(t_n) \in \ell_1$. Take $(\lambda_n)$ a strictly decreasing sequence in $(1, +\infty)$ with $\lim_n \lambda_n = 1$. By passing to subsequences if necessary, we can assume that $\lambda_{n+1} < (1-\epsilon_n)\lambda_n$. Define $y_n := \lambda_n x_n$ for all $n \in \mathbb{N}$ and let

$$C := \left\{ \sum_{n=1}^{\infty} t_n y_n : \; t_n \geq 0, \; \sum_{n=1}^{\infty} t_n = 1 \right\},$$

which is a closed convex bounded subset of $X$. Define $T : C \to C$ by

$$T\left( \sum_{n=1}^{\infty} t_n y_n \right) = \sum_{n=1}^{\infty} t_n y_{n+1}.$$

Then it is not difficult to prove that $T$ is a fixed point free affine nonexpansive mapping.

Assume now that $X$ contains an asymptotically isomorphic copy of $c_0$. There are a null sequence $(\epsilon_n) \in (0,1)$ and a sequence $(x_n)$ in $X$ so that

$$\sup_n (1-\epsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sup_n |t_n|$$

for all $(t_n) \in c_0$.

Take again $(\lambda_n)$ a strictly decreasing sequence in $(1, +\infty)$ with $\lim_n \lambda_n = 1$ and by passing to subsequences if necessary, we can assume that $\lambda_{n+1} < (1-\epsilon_n)\lambda_n$. Define $y_n := \lambda_n x_n$ for all $n \in \mathbb{N}$ and let

$$C := \left\{ \sum_{n=1}^{\infty} t_n y_n : \; (t_n) \in c_0, \; 0 \leq t_n \leq 1 \text{ for all } n \in \mathbb{N} \right\},$$

which is a closed convex bounded subset of $X$. Define $T : C \to C$ by

$$T\left(\sum_{n=1}^{\infty} t_n y_n\right) = y_1 + \sum_{n=1}^{\infty} t_n y_{n+1}.$$

It can be checked that $T$ is an affine nonexpansive mapping without fixed points and the proof is complete. □

Therefore, we can deduce that every nonreflexive subspace of $L_1[0,1]$ or more generally every nonreflexive subspace of $L_1(\mathcal{M})$ fails to have the A-FPP. Also every nonreflexive subspace of an $M$-embedded Banach space (such as the space of the compact operators in a Hilbert space $K(H)$) also fails the A-FPP (see [68]).

We raise the following question: If a Banach space $X$ fails to have the A-FPP, can $X$ still be renormed to have this property?

Again we know that $\ell_1(\Gamma)$, $c_0(\Gamma)$, for $\Gamma$ uncountable, of $\ell_\infty$ can not be renormed to have the A-FPP, since every renorming of such spaces contain an asymptotically isometric copy of either $\ell_1$ or $c_0$.

The main object of this chapter is to prove that for every finite von Neumann algebra, $L_1(\mathcal{M})$ can be renormed to have the A-FPP (and therefore every of its closed subspaces).

## 4.2 A renorming result in $L_1(\mathcal{M})$ with the A-FPP

**Theorem 4.2.1.** *Let $\mathcal{M}$ be a finite von Neumann algebra and $L_1(\mathcal{M})$ the predual of $\mathcal{M}$. Let $|||\cdot|||$ be the renorming in $L_1(\mathcal{M})$ introduced in Chapter 3. Then, $(L_1(\mathcal{M}), |||\cdot|||)$ has the A-FPP.*

Notice that $L_1(\mu)$ spaces, for $\mu$ $\sigma$-finite measures, are also included in the conditions of the above statement.

The proof of the above result mainly follows the steps of the proof of Theorem 2.1.1 in Chapter 2. Notice that if the assertion of the theorem is false we can consider a subset $D$ and an affine nonexpansive mapping $T : D \to D$ as in Lemma 2.2.1 in Chapter 2. The starting point of the proof of Theorem 2.1.1 was to consider the infimum of $\limsup_n |||x_n - x|||$ when $(x_n)$ is an approximate fixed point sequence in $D$ such that $(x_n)$ converges to $x$ with respect to the measure topology. In absence of $\tau$-compactness we can not guarantee that there exists an approximate fixed point sequence which is convergent in measure. The affinity along with the next two results will supply this constraint.

In 1967 Komlós [51] proved the following:

**Theorem 4.2.2.** *Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $(f_n)$ a bounded sequence in $L_1(\mu)$. Then there exists a subsequence $(g_n) \subset (f_n)$ and a function $f \in L_1(\mu)$ such that for every further subsequence $(h_n) \subset (g_n)$,*

$$\frac{1}{n}\sum_{i=1}^{n} h_i \to_n f \qquad \mu - a.e.$$

Komlós Theorem has been extended for more general classes of Banach function spaces (as for instance $L_1(\mu)$ when $\mu$ a $\sigma$-finite measure) [12]. Also Komlós Theorem has been generalized in the setting of non-commutative $L_1$-spaces associated to a finite von Neumann algebra.

**Theorem 4.2.3** (N. Randrianantoanina [73], Proposition 3.11)**.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and suppose that $(x_n)$ is a bounded sequence in $L_1(\mathcal{M})$.*

*Then there exists a subsequence $(g_n) \subset (x_n)$ and a vector $x \in L_1(\mathcal{M})$ such that for every further subsequence $(h_n) \subset (g_n)$,*

$$\frac{1}{n}\sum_{i=1}^{n} h_i \to_n x \qquad \text{with respect to the measure topology.}$$

On the other hand, notice that if $C$ is a closed convex bounded subset of a Banach space, $T : C \to C$ is an affine nonexpansive mapping and $(x_n)$ is an approximate fixed point sequence, then the sequence formed by its arithmetic means is also an approximate fixed point sequence. Indeed,

$$\begin{aligned} \limsup_n \left\| \frac{x_1 + \cdots + x_n}{n} - T\left(\frac{x_1 + \cdots + x_n}{n}\right) \right\| &= \limsup_n \left\| \frac{x_1 - Tx_1 + \cdots + x_n - Tx_n}{n} \right\| \\ &\leq \limsup_n \frac{\|x_1 - Tx_1\| + \cdots + \|x_n - Tx_n\|}{n} \\ &= 0, \end{aligned}$$

since $\limsup_n \|Tx_n - x_n\| = 0$.

Now we are in condition to guarantee that there exist approximate fixed point sequences under the hypotheses of Theorem 4.2.1 which are convergent with respect to the measure topology.

**Proposition 4.2.4.** *Let $D$ be a closed convex bounded subset of $L_1(\mathcal{M})$. Consider $\mathcal{T}$ as the topology of convergence in measure. If $T : D \to D$ is an affine nonexpansive mapping, then the set*

$$A(D) = \{(x_n) \subset D : (x_n) \text{ is an a.f.p.s. and } x_n \to x \text{ w.r.t. } \mathcal{T}\}$$

*is nonempty.*

*Proof.* Since $T$ is nonexpansive, there exists an a.f.p.s. $(x_n)$ in $D$, which is bounded because $D$ is bounded. Using Theorem 4.2.3 we get a subsequence $(y_n)$ of $(x_n)$

and $x \in L_1(\mathcal{M})$ such that $\frac{1}{n}\sum_{i=1}^{n} y_n \to x$ with respect to the measure topology. Moreover $\frac{1}{n}\sum_{i=1}^{n} y_n \in D$ because $D$ is convex and it is also an a.f.p.s. since $T$ is affine. Therefore $A(D)$ is nonempty. □

Now using the properties of the seminorms $\{R_k(\cdot)\}$ stated in Chapter 3, and the same arguments as in the proof Theorem 2.1.1, we finally deduce that $(L_1(\mathcal{M}), |||\cdot|||)$ has the A-FPP and the proof of Theorem 4.2.1 is concluded.

We finish this chapter with the following remarks:

1.- Notice that the renorming result obtained in this chapter shows that the characterization of weakly compact subsets of $L_1(\mathcal{M})$ given in Theorem 4.1.2 is an intrinsic property of the $\|\cdot\|_1$ norm. We could replace the $\|\cdot\|_1$ norm by another equivalent norm (as close as the $\|\cdot\|_1$ norm as we want) and condition *ii)* in Theorem 4.1.2 may hold for every closed convex bounded subset of $L_1(\mathcal{M})$.

2.- We have previously mentioned that it is an open problem if $c_0$ can be renormed to have the FPP. Can $c_0$ be renormed to have the A-FPP ? In Theorem 2.27 of [50, Chapter 9] it is proved that $c_0$ cannot be renormed to have the fixed point property for affine asymptotically nonexpansive mappings (recall that $T : C \to C$ is asymptotically nonexpansive if there exists a sequence $(k_n) \in [1, +\infty)$ with $\lim_n k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$).

# Chapter 5

# Equivalent norms in $\ell_1$ with the FPP

In this chapter we will again focus on the sequence space $\ell_1$. In Chapter 2 we saw that there exist norms on $\ell_1$ equivalent to the usual norm $\|\cdot\|_1$ verifying the FPP. We define $\mathcal{A}$ as the set of all equivalent norms on $\ell_1$ which satisfy the FPP. $\mathcal{A}$ is nonempty and we ask if we can find some topological or linear structure in $\mathcal{A}$. In that way, we prove that $\mathcal{A}$ contains rays. In fact, every renorming in $\ell_1$ which verifies condition (5.1) in Theorem 5.1.1 is the starting point of a (closed or open) ray composed by equivalent norms on $\ell_1$ with the FPP. The standard norm $\|\cdot\|_1$ or P.K. Lin's norm $|||\cdot|||$ are examples of such norms. Moreover, we study some topological properties of the set $\mathcal{A}$ with respect to some equivalent metrics defined on the set of all norms on $\ell_1$ equivalent to $\|\cdot\|_1$.

This chapter is divided in two sections. In Section 1 we will obtain new families of renormings on $\ell_1$ satisfying the FPP. In fact, in a subsequent paper [57] P.K. Lin established four conditions which are sufficient to assure that a renorming on $\ell_1$ verifies the FPP. We will check that many of the norms obtained in our main result do not satisfy P.K. Lin's condition in [57].

Finally, in the second section, we consider the convex cone $\mathcal{P}$ of all norms on $\ell_1$ which are equivalent to $\|\cdot\|_1$ and its subset $\mathcal{A}$ given by the norms of $\mathcal{P}$ satisfying the FPP. We deduce that $\mathcal{A}$ contains rays and we study some properties concerning the structure of the sets $\mathcal{A}$ and $\mathcal{P} \setminus \mathcal{A}$.

## 5.1 Family of norms on $\ell_1$ with the FPP

First of all, we will introduce some notation which will be used throughout this chapter: Let $(\gamma_k)$ be any nondecreasing sequence in $(0,1)$ converging to 1 and denote by $|||\cdot|||$ the renorming on $\ell_1$ given by

$$|||x||| := \sup_k \gamma_k R_k(x),$$

where $R_k(x) := \sum_{n=k}^{\infty} |x(n)|$ for all $x = (x(n)) \in \ell_1$.

The main result of this section is the following:

**Theorem 5.1.1.** *Let $p(\cdot)$ be an equivalent norm to the usual norm on $\ell_1$ such that*

$$\limsup_n p(x_n + x) = \limsup_n p(x_n) + p(x) \tag{5.1}$$

*for every $w^*$-null sequence $(x_n)$ and for all $x \in \ell_1$. Then the norm*

$$|\cdot|_p := p(\cdot) + \lambda\, |||\cdot|||$$

*has the FPP for every $\lambda > 0$.*

Before starting with the proof, notice that we can write

$$|x|_p = \sup_k \rho_k(x)$$

where $\rho_k(x) = p(x) + \lambda\gamma_k R_k(x)$ and that it is not difficult to check that for all weak*-null sequences $(x_n)$ and for all $x \in \ell_1$ the following property holds:

$$\limsup_n \rho_k(x_n + x) = \limsup_n \rho_k(x_n) + \rho_k(x) \tag{5.2}$$

for all $k \in \mathbb{N}$. Moreover, $\limsup_n R_k(x_n) = \limsup_n |||x_n|||$ for all $k \in \mathbb{N}$ if $(x_n)$ is a weak*-null sequence in $\ell_1$.

*Proof.* We proceed by contradiction. So, we assume that condition (5.1) is satisfied and that $(\ell_1, |\cdot|_p)$ fails the FPP. Let $T$ and $D$ be as in Lemma 2.2.1.

We introduce the following notation:

Let $D'$ be any closed convex $T$-invariant subset of $D$. Define

$$A(D') := \{(x_n) \subset D'\}$$

such that $(x_n)$ is an a.f.p.s., is weak*-convergent to some $x \in \ell_1$ and $\lim_n p(x_n - w)$, $\lim_n |||x_n - w|||$ and $\lim_n R_k(x_n - w)$ exist for all $w \in \ell_1$ and for all $k \in \mathbb{N}$. Notice that, from the separability of $\ell_1$, $A(D')$ is always nonempty.

Define

$$s := \inf\left\{\lim |x_n - x|_p : (x_n) \subset A(D),\ x_n \overset{w^*}{\to} x\right\}$$

which is strictly positive by Remark 1 in Chapter 2. Without loss generality we can assume that $s = 1$. Define

$$c := \inf\{|||x||| : p(x) = \lambda\}; \qquad a := \inf\{|||x||| : |x|_p = \lambda\},$$

which are constants strictly greater than zero because the involved norms are equivalent.

We choose $\varepsilon_1 > 0$ such that

$$\frac{1+\varepsilon_1}{1+c} + 2\varepsilon_1 < 1$$

and an a.f.p.s. $(x_n) \subset A(D)$ such that $x_n \stackrel{w^*}{\to} x$ and $\lim_n |x_n - x|_p < 1 + \varepsilon_1$. By translation, we can suppose that $x = 0$.

Define

$$K := \left\{ z \in D : \lim_n |x_n - z|_p \leq 2 + 2\varepsilon_1 \right\}.$$

Notice that $K$ is nonempty, closed, convex, bounded and $T$-invariant. In fact, there exists $n_0 \in \mathbb{N}$ such that $x_n \in K$ for $n \geq n_0$.

Define $r$ by

$$r := \inf \left\{ \limsup_n |y_n - y|_p : (y_n) \subset A(K),\ y_n \stackrel{w^*}{\to} y \right\}.$$

From the definition of $r$ we have

$$1 \leq r \leq \lim_n |x_n|_p < 1 + \varepsilon_1 \tag{5.3}$$

Consider $(y_n) \subset A(K)$ an arbitrary a.f.p.s. such that $y_n \stackrel{w^*}{\to} y$. Then for all $k \in \mathbb{N}$ we have

$$\begin{aligned}
2 + 2\varepsilon_1 &\geq \limsup_m \lim_n |x_n - y_m|_p \\
&\geq \limsup_m \lim_n \rho_k(x_n - y_m) \\
&= \limsup_m \left[ \limsup_n \rho_k(x_n) + \rho_k(y_m) \right] \text{ by (5.2)} \\
&= \limsup_n \rho_k(x_n) + \limsup_m \rho_k(y_m - y) + \rho_k(y) \text{ by (5.2)} \\
&= \lim_n p(x_n) + \lambda\gamma_k \lim_n R_k(x_n) + \lim_m p(y_m - y)
\end{aligned}$$

$$\begin{aligned} & +\lambda\gamma_k \lim_n R_k(y_m - y) + \rho_k(y) \\ = & \lim_n p(x_n) + \lambda\gamma_k \lim_n |||x_n||| + \lim_m p(y_m - y) \\ & +\lambda\gamma_k \lim_n |||y_m - y||| + \rho_k(y) \\ \geq & \gamma_k \left[\lim_n |x_n|_p + \lim_m |y_m - y|_p\right] + \rho_k(y) \\ \geq & 2\gamma_k + \rho_k(y). \end{aligned}$$

Hence, if $(y_n) \subset A(K)$ is an a.f.p.s and $y_n \overset{w^*}{\to} y$, we have

$$\rho_k(y) \leq 2(1-\gamma_k) + 2\varepsilon_1 \tag{5.4}$$

$$< 2 + 2\varepsilon_1. \tag{5.5}$$

Choose $m$ such that

$$\frac{1+\varepsilon_1}{1+c} + 2\varepsilon_1 < m < 1.$$

Notice that $p(w) \leq \frac{|w|_p}{1+c}$ for all $w \in \ell_1$, by definition of the constant $c$. Therefore

$$\lim_n p(x_n) \leq \frac{\lim_n |x_n|_p}{1+c} < \frac{1+\varepsilon_1}{1+c}$$

and there exists $n_0 \in \mathbb{N}$ such that

$$p(x_{n_0}) < \frac{1+\varepsilon_1}{1+c}.$$

On the other hand,

$$\limsup_k \rho_k(x_{n_0}) = p(x_{n_0}) + \lambda \limsup_k \gamma_k R_k(x_{n_0}) = p(x_{n_0}),$$

so we can find $k_0 \in \mathbb{N}$ such that the following hold:

$$\rho_k(x_{n_0}) < \frac{1+\varepsilon_1}{1+c} \tag{5.6}$$

and

$$q_k := \frac{1+\varepsilon_1}{1+c} + 2(1-\gamma_k) + 2\varepsilon_1 < m < 1 \leq r \tag{5.7}$$

for all $k \geq k_0$.

Since the set $K$ is bounded there exists some $H > 0$ such that $|x|_p < H$ for all $x \in K$. Hence

$$\rho_k(x_{n_0}) < H \qquad \text{for all } k \in \mathbb{N}. \tag{5.8}$$

Define $m_0 := 1 - a(1-\gamma_{k_0})$ which is strictly less than 1 and greater than zero, because $a \leq 1$ by definition. Also define

$$h := H + 2 + 2\varepsilon_1 > r > m_0 r \qquad \text{(by 5.3)}. \tag{5.9}$$

We take $\beta \in (0,1)$ such that

$$\beta < \frac{2r(1-m_0)}{h - m_0 r}.$$

Since $\beta$ satisfies the above condition, we have

$$(2-\beta)r + \beta m = 2r - \beta(r-m) < 2r$$

and

$$(2-\beta)m_0 r + \beta h = 2m_0 r + \beta(h - m_0 r) < 2m_0 r + 2r(1-m_0) = 2r.$$

Therefore, we can find $\varepsilon_2 > 0$ such that

$$(2-\beta)(r+\varepsilon_2) + \beta m \ < \ 2r, \tag{5.10}$$

$$(2-\beta)m_0(r+\varepsilon_2) + \beta h \ < \ 2r. \tag{5.11}$$

Set

$$M := \max\{(2-\beta)(r+\varepsilon_2) + \beta m, (2-\beta)m_0(r+\varepsilon_2) + \beta h\}$$

which is strictly less than $2r$. Take $(y_n) \subset A(K)$ an a.f.p.s. such that $y_n \overset{w^*}{\to} y$ and $\lim_n |y_n - y|_p < r + \varepsilon_2$. So, there exists $N_0 \in \mathbb{N}$ such that

$$|y_n - y|_p < r + \varepsilon_2 \tag{5.12}$$

for all $n \geq N_0$.

Moreover,

$$\begin{aligned}
\lim_n \rho_k(y_n - y) &= \lim_n p(y_n - y) + \lambda\gamma_k \lim_n R_k(y_n - y) \\
&= \lim_n p(y_n - y) + \lambda\gamma_k \lim_n |||y_n - y||| \\
&= \lim_n |y_n - y|_p - (1 - \gamma_k)\lambda \lim_n |||y_n - y||| \\
&\leq [1 - (1 - \gamma_k)a] \lim_n |y_n - y|_p \text{ (by definition of } a) \\
&< [1 - (1 - \gamma_k)a](r + \varepsilon_2)
\end{aligned}$$

and we can find $N_1 \geq N_0$ such that

$$\rho_k(y_n - y) < [1 - (1 - \gamma_k)a](r + \varepsilon_2) \leq m_0(r + \varepsilon_2) \tag{5.13}$$

for all $n \geq N_1$ and $k = 1, \cdots, k_0$.

Define now the vector $z_0 \in \ell_1$ as

$$z_0 := \beta x_{n_0} + (1 - \beta) y_{N_1}$$

which belongs to $K$, since $K$ is convex.

Let us now prove that $\lim_n |y_n - z_0|_p \leq M$. In order to do this, we will check that for all $k \in \mathbb{N}$ and $n \geq N_1$ we have

$$\rho_k(y_n - z_0) \leq M.$$

Notice that

$$y_n - z_0 = y_n - y - (1-\beta)(y_{N_1} - y) - \beta(x_{n_0} - y).$$

Fix $n \geq N_1$. We split the proof into two cases:
Case 1: $k > k_0$:

$$\begin{aligned}
\rho_k(y_n - z_0) &\leq \rho_k(y_n - y) + (1-\beta)\rho_k(y_{N_1} - y) + \beta\left[\rho_k(x_{n_0}) + \rho_k(y)\right] \\
&\leq |y_n - y|_p + (1-\beta)|y_{N_1} - y|_p + \beta\left[\rho_k(x_{n_0}) + \rho_k(y)\right] \\
&< (2-\beta)(r+\varepsilon_2) + \beta\left[\rho_k(x_{n_0}) + \rho_k(y)\right] \text{ (by 5.12)} \\
&< (2-\beta)(r+\varepsilon_2) + \beta\left[\frac{1+\varepsilon_1}{1+c} + 2(1-\gamma_k) + 2\varepsilon_1\right] \text{ (by 5.6 and 5.4)} \\
&= (2-\beta)(r+\varepsilon_2) + \beta q_k < (2-\beta)(r+\varepsilon_2) + \beta m \text{ (from 5.7)} \\
&< 2r \text{ (from 5.10)}
\end{aligned}$$

Case 2: $k \leq k_0$:

$$\begin{aligned}
\rho_k(y_n - z_0) &\leq \rho_k(y_n - y) + (1-\beta)\rho_k(y_{N_1} - y) + \beta\left[\rho_k(x_{n_0}) + \rho_k(y)\right] \\
&\leq m_0(2-\beta)(r+\varepsilon_2) + \beta\left[\rho_k(x_{n_0}) + \rho_k(y)\right] \text{ (from 5.13)} \\
&< m_0(2-\beta)(r+\varepsilon_2) + \beta\left[H + 2 + 2\varepsilon_1\right] \text{ (by 5.8 and 5.5)} \\
&\leq m_0(2-\beta)(r+\varepsilon_2) + \beta h \text{ (by 5.9)} \\
&< 2r \text{ (by 5.11)}
\end{aligned}$$

We have showed $\rho_k(y_n - z_0) \leq M$ for all $k \in \mathbb{N}$ and for all $n \geq N_1$. So, $|y_n - z_0|_p \leq M$ for all $n \geq N_1$ and

$$\limsup_n |y_n - z_0|_p \leq M.$$

Define now

$$K_0 := \{w \in K : \limsup_n |y_n - w|_p \leq M\}$$

Notice that the set $K_0$ is nonempty since $z_0 \in K$ and we can find $(w_m) \subset A(K_0)$ such that $w_m \overset{w^*}{\to} w \in \ell_1$. Now, for all $k \in \mathbb{N}$:

$$\begin{aligned} M &\geq \limsup_m \limsup_n |y_n - w_m|_p \\ &\geq \limsup_m \limsup_n \rho_k(y_n - w_m) \\ &= \lim_n \rho_k(y_n - y) + \lim_m \rho_k(w_m - w) + \rho_k(y - w) \\ &\geq \lim_n p(y_n - y) + \lambda\gamma_k \lim_n |||y_n - y||| + \lim_m p(w_m - w) + \lambda\gamma_k \lim_m |||w_m - w||| \end{aligned}$$

Taking limits when $k$ goes to infinity,

$$M \geq \lim_n |y_n - y|_p + \lim_m |w_m - w|_p \geq r + r = 2r,$$

which is a contradiction with the definition of $M$. Therefore $(\ell_1, |\cdot|_p)$ has the FPP as we wanted to prove. □

**Definition 5.1.1.** If $X$ is a Banach space with a basic sequence $(e_n)$, it is said that a sequence of nonzero vectors $(u_j)$ in $X$ is a block basic sequence of $(e_n)$ if there exist an increasing sequence of natural numbers $(p_n)$ and a scalar sequence $(a_j)$ such that $u_j = \sum_{i=p_j+1}^{p_{j+1}} a_j e_j$.

Let $P_k$ denote the natural projection on $\ell_1$. At this point, we should note that P.K. Lin proved the following result in [57]:

**Theorem 5.1.2.** *Let $\|\cdot\|$ be an equivalent norm on $\ell_1$ satisfying the following four properties:*

*(1) there are* $\alpha > 4$ *and a positive (decreasing) sequence* $(\alpha_n)$ *in* $(0,1)$ *such that for any normalized block basis* $\{f_n\}$ *of* $(\ell_1, \|\cdot\|)$ *and* $x \in \ell_1$ *with* $P_{k-1}(x) = x$ *and* $\|x\| < \alpha_k$*,*

$$\limsup_n \|f_n + x\| \leq 1 + \frac{\|x\|}{\alpha}.$$

*(2) there are two strictly decreasing sequences* $\{\beta_k\}$ *and* $\{\gamma_k\}$ *with* $\lim_k \beta_k = 0$ *and* $\lim_k \gamma_k = 1$*, such that for any normalized block basis* $\{f_n\}$ *of* $(\ell_1, \|\cdot\|)$ *and* $x$ *with* $(I - P_k)(x) = x$*,*

$$\liminf_n \|f_n + x\| \geq 1 - \beta_k + \gamma_k^{-1}\|x\|.$$

*(3) for any* $k \in \mathbb{N}$*,* $\|I - P_k\| = 1$*, and*

*(4) the unit ball of* $(\ell_1, \|\cdot\|)$ *is* $\sigma(\ell_1, c_0)$*-closed.*

*Then* $(\ell_1, \|\cdot\|)$ *has the FPP.*

Notice that P.K. Lin's norm $|||\cdot|||$ satisfies the above four conditions. On the other hand, it is obvious that $\|\cdot\|_1$ can not satisfy P.K. Lin's properties, but it is easy to check that $\|\cdot\|_1$ satisfies conditions *(2)*, *(3)* and *(4)* stated above and it only fails condition *(1)*. Therefore, condition *(1)* turned out to be the key for the author in [57] to prove the FPP. A natural question could be if there are equivalent norms on $\ell_1$ having the FPP and failing the sufficient conditions given in Theorem 5.1.2. We answer with the following proposition:

**Proposition 5.1.3.** *The equivalent norm* $|\cdot| = \|\cdot\|_1 + |||\cdot|||$ *does not verify property (1) from Theorem 5.1.2, but it has the FPP.*

*Proof.* Consider the sequence $f_n = \frac{e_n}{1+\gamma_n}$ which is a normalized block basis of $(\ell_1, |\cdot|)$ and let $(\alpha_n)$ be any sequence in $(0,1)$. Let $0 < t_k < \alpha_k$ and $x = \frac{t_k}{1+\gamma_1} e_1$ which verifies $P_{k-1}(x) = x$ for all $k \geq 2$ and $|x| = t_k < \alpha_k$.

Notice that

$$|f_n + x| = \frac{1}{1+\gamma_n} + \frac{t_k}{1+\gamma_1} + \max\left\{\gamma_1\left(\frac{t_k}{1+\gamma_1} + \frac{1}{1+\gamma_n}\right), \gamma_n \frac{1}{1+\gamma_n}\right\}.$$

Assume that there exists some $\alpha > 4$ verifying property (1) of Theorem 5.1.2. Then

$$\begin{aligned} 1 + \frac{t_k}{\alpha} &= 1 + \frac{|x|}{\alpha} \geq \limsup_n |f_n + x| \\ &\geq \limsup_n \left[\frac{1}{1+\gamma_n} + \frac{t_k}{1+\gamma_1} + \gamma_n \frac{1}{1+\gamma_n}\right] = 1 + \frac{t_k}{1+\gamma_1}, \end{aligned}$$

which implies that $4 < \alpha \leq 1 + \gamma_1$, which is a contradiction. Therefore, $\|\cdot\|_1 + |||\cdot|||$ is an equivalent norm failing to satisfy P. K. Lin's sufficient conditions to establish the FPP. □

*Remark* 8. On the other hand, from Theorem 5.1.1 we can deduce that for every linear combination

$$\mu\|x\|_1 + \lambda\,|||x|||\,; \qquad \mu \geq 0 \text{ and } \lambda > 0,$$

the space $\ell_1$ endowed with this norm has the FPP.

But Theorem 5.1.1 can be applied to more general norms on $\ell_1$.

**Proposition 5.1.4.** *Every equivalent norm $p(\cdot)$ on $\ell_1$ which separates disjoint vectors (i.e., $p(x+y) = p(x) + p(y)$ if $supp(x) \cap supp(y) = \emptyset$) satisfies the hypotheses of Theorem 5.1.1.*

The proof of this proposition follows standard arguments so we omit it.

**Example 5.1.1.** Consider

$$\|x\|_\rho := \sum_{n=1}^{\infty} r_n |x_n|.$$

If $x = (x_n)_n \in \ell_1$, where $\rho = (r_n)$ is any bounded sequence on $\mathbb{R}$ such that $\inf_n r_n > 0$. Then $\|\cdot\|_\rho$ is an equivalent norm to $\|\cdot\|_1$, fails to have the FPP (since $(\ell_1, \|\cdot\|_\rho)$ is isometric to $(\ell_1, \|\cdot\|_1)$) and

$$\mu\|x\|_\rho + \lambda\,|||x|||$$

has the FPP for every $\mu \geq 0$ and $\lambda > 0$.

A more general result than Proposition 5.1.4 is the following

**Proposition 5.1.5.** *Let $k \in \mathbb{N}$. Consider $\|\cdot\|_k$ be any norm on $\mathbb{R}^k$ and $p(\cdot)$ be a norm on $\ell_1$ which separates disjoint vectors. Then*

$$\|x\| = \alpha_1 p\left((I - P_k)x\right) + \alpha_2 \|P_k x\|_k$$

*satisfies condition 5.1 in Theorem 5.1.1 for all $\alpha_1, \alpha_2 > 0$.*

**Example 5.1.2.** In the last proposition we can include the norms

$$\|x\| = \alpha_1 \sum_{n=k+1}^{\infty} |x_n| + \alpha_2 \left( \sum_{n=1}^{k} |x_n|^q \right)^{1/q},$$

where $k \in \mathbb{N}$ and $1 \leq q < +\infty$.

## 5.2 On the structure of the set of equivalent norms on $\ell_1$ with the FPP: Stability of the fixed point property.

Let $\mathcal{P}$ be the set of all norms on $\ell_1$ equivalent to $\|\cdot\|_1$ and define

$$\mathcal{A} := \{p(\cdot) : \ p \in \mathcal{P} \text{ and } (\ell_1, p(\cdot)) \text{ has the FPP } \}.$$

Thus, $\mathcal{P} \setminus \mathcal{A}$ is the set of all renormings in $\ell_1$ failing to have the FPP.

In the set $\mathcal{P}$ we can define a natural metric given by

$$h(p,q) := H(B_p, B_q)$$

where $H$ is the Hausdorff distance and $B_p$, $B_q$ are the unit balls of $(\ell_1, p)$, $(\ell_1, q)$ respectively. It is not difficult to check that $(\mathcal{P}, h)$ is a complete metric space.

Another metric defined on $\mathcal{P}$ is the following

$$\rho(p,q) = \sup\{|p(x) - q(x)| : \|x\|_1 \leq 1\}.$$

Then $(\mathcal{P}, \rho)$ is an open set of the complete metric space $(\mathcal{Q}, \rho)$ of all continuous seminorms on $(\ell_1, \|\cdot\|_1)$ endowed with the metric $\rho$ defined as above. Moreover, the metric spaces $(\mathcal{P}, h)$ and $(\mathcal{P}, \rho)$ are equivalent [20].

The fixed point property is an isometric property which implies that $\lambda p \in \mathcal{A}$ (for $\lambda > 0$) whenever $p \in \mathcal{P}$. In this sense, we could restrict the study of the equivalent norms verifying the FPP to the set of normalized norms, that is, equivalent norms which satisfy $\sup_{\|x\|_1 \leq 1} p(x) = 1$. Denote by $\mathcal{E}$ the set of all equivalent normalized norms on $\ell_1$. Following the idea of the Banach-Mazur distance we can also define a metric on $\mathcal{E}$ by

$$d(p,q) = \log \frac{b_{p,q}}{a_{p,q}} = \log \|i\| \|i^{-1}\|$$

where $i$ is the identity operator and

$$a_{p,q} := \inf \left\{ \frac{p(x)}{q(x)} : \ \|x\|_1 = 1 \right\}; \qquad b_{p,q} := \sup \left\{ \frac{p(x)}{q(x)} : \ \|x\|_1 = 1 \right\}.$$

The space $(\mathcal{E}, d)$ is metric and it can be proved that $(\mathcal{E}, \rho)$, $(\mathcal{E}, h)$, $(\mathcal{E}, d)$ are equivalent metric spaces [71] so the three metrics generate the some topology on $\mathcal{E}$.

As we have previously mentioned, until 2008 it was conjectured that the set $\mathcal{A}$ was empty. In that year, P.K. Lin proved that $|||\cdot||| \in \mathcal{A}$ [56], where $|||\cdot|||$ is the norm introduced in Chapter 1. It could be interesting to know some topological or linear properties properties of the sets $\mathcal{A}$ and $\mathcal{P} \setminus \mathcal{A}$.

Notice that $\gamma_1 \|x\|_1 \leq |||x||| \leq \|x\|_1$ for all $x \in \ell_1$, which implies that

$$\rho(|||\cdot|||, \|\cdot\|_1) \leq 1 - \gamma_1.$$

From this inequality we deduce that the set $\mathcal{A}$ is not closed in $(\mathcal{P}, \rho)$, since $\|\cdot\|_1 \in \mathcal{P} \setminus \mathcal{A}$ and for all $\epsilon > 0$ we can find $p \in B_\rho(\|\cdot\|_1, \epsilon)$ such that $p \in \mathcal{A}$ (recall that $\gamma_1$ can be chosen as close to one as we like).

Let us check that P.K. Lin's norm in $\ell_1$ does not have stability for the FPP.

**Proposition 5.2.1.** *The norm $|||\cdot|||$ does not have stability for the FPP.*

*Proof.* We will check that for every $K > 1$ there is an equivalent norm on $\ell_1$, whose Banach-Mazur distance with $(\ell_1, |||\cdot|||)$ is less than $K$ but failing the FPP.

Take $k_0$ such that $1/\gamma_{k_0} < K$ and define the norm

$$|||x|||_{k_0} = \sup_{1\leq k\leq k_0} \gamma_k \sum_{n=k}^{\infty} |x_n|; \qquad \text{for all } x = (x_n)_n \in \ell_1.$$

It is not difficult to check that $|||x|||_{k_0} \leq |||x||| \leq \frac{1}{\gamma_{k_0}} |||x|||_{k_0}$ for all $x \in \ell_1$ and therefore

$$d((\ell_1, |||\cdot|||), (\ell_1, |||\cdot|||_{k_0}) \leq \frac{1}{\gamma_{k_0}} < K.$$

Define the linear mapping $T : (\ell_1, \|\cdot\|_1) \to (\ell_1, |||\cdot|||_{k_0})$ given by

$$Tx = \frac{1}{\gamma_{k_0}}(\underbrace{0, \cdots, 0}_{k_0-1}, x_1, x_2, \cdots) \qquad \text{for all } x = (x_n)_n \in \ell_1.$$

Since $T$ is an isometry the space $(\ell_1, |||\cdot|||_{k_0})$ contains an isometric copy of $\ell_1$ and hence it fails the FPP. □

Notice that the same happens for $\|\cdot\|_1 + \lambda\, |||\cdot|||$ for every $\lambda > 0$. Indeed, fix $\lambda > 0$ and consider the linear mapping $S$ from $(\ell_1, \|\cdot\|_1)$ to $(\ell_1, \|\cdot\|_1 + \lambda\, |||\cdot|||_{k_0})$ given by

$$S(x) = \frac{1}{1+\lambda\gamma_{k_0}}(\underbrace{0, \cdots, 0}_{k_0-1}, x_1, x_2, \cdots) \qquad \text{for all } x = (x_n)_n \in \ell_1.$$

This mapping is again an isometry which shows that $(\ell_1, \|\cdot\|_1 + \lambda\, |||\cdot|||_{k_0})$ fails the FPP and we can easily check that $d((\ell_1, \|\cdot\|_1 + \lambda\, |||\cdot|||), (\ell_1, \|\cdot\|_1 + \lambda\, |||\cdot|||_{k_0})) \leq \frac{1}{\gamma_{k_0}}$. Since we can choose $\gamma_{k_0}$ as close as 1 as we want, we deduce that neither of these norms have stability for the fixed point property.

The "bad" behaviour of the renormings in $\ell_1$ regarding the stability of the FPP contrasts with the fact that, for the classical reflexive Banach spaces, the FPP is preserved under renormings when the Banach-Mazur distance is close enough.

Moreover, from the previous instability result we can deduce that $\mathcal{A}$ is not an $\rho$-open set either. In fact $|||\cdot||| \in \mathcal{A}$, $|||\cdot|||_{k_0} \notin \mathcal{A}$ and $\rho(|||\cdot|||, |||\cdot|||_{k_0}) \le 1 - \gamma_{k_0}$, where $\gamma_{k_0}$ can again be chosen as close to 1 as we like.

By the equivalence among the metrics, $\mathcal{A}$ is neither closed nor open in $(\mathcal{P}, h)$ and the some holds for the set $\mathcal{A} \cap \mathcal{E}$ in the metric space $(\mathcal{E}, d)$.

We would like to point out that from Theorem 5.1.1 we can also deduce the following: as close as we like, in the Banach-Mazur sense, of a renorming $p(\cdot)$ with condition (5.1) and failing the FPP, we can find equivalent norms which verify the FPP. So the property of failing the FPP is not stable for norms satisfying condition (5.1) either.

In what follows we show many examples of norms in the set $\mathcal{P} \setminus \mathcal{A}$:

Let $C := \{(x_n) \in \ell_1 : x_n \ge 0, \sum_{n=1}^{\infty} x_n = 1\}$. Of course, the set $C$ is closed convex and bounded. If $S : \ell_1 \to \ell_1$ is the right shift mapping defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$, then for every positive integer $k$, the set $C$ is invariant under the mapping $S^k$. Moreover, $S^k$ is fixed point free on $C$.

Therefore, every norm $q$ equivalent to the standard one belongs to $\mathcal{P} \setminus \mathcal{A}$ whenever one of the mappings $S^k$ is $q$-nonexpansive, that is, if for some positive integer $k$,

$$q(0, \overset{(k)}{\ldots}, 0, v_1, v_2, \ldots) \le q(v_1, v_2, \ldots) \tag{5.14}$$

holds for every vector $v = (v_1, v_2, \ldots) \in \ell_1$. There are many well known norms on $\ell_1$ which satisfy Condition (5.14) as, for instance, the following.

1. The standard one $\|\cdot\|_1$.

2. The norm $\|\cdot\|_c$ defined by $\|(x_n)\|_c = \sum_{n=1}^{\infty} |x_n| + |\sum_{n=1}^{\infty} x_n|$.

3. The family of norms defined by

$$\|(x_n)\|_\rho := \sum_{n=1}^{\infty} r_n |x_n|$$

where $\rho = (r_n)$ is a decreasing sequence on the closed interval $[1, b]$.

4. For a positive integer $k$ and $a > 0$, let $\|\cdot\|_{a,k}$ be the norm defined on $\ell_1$ as

$$\|x\|_{a,k} = a|x_1 + \ldots + x_k| + \|x\|_1.$$

For each $x \in \ell_1$ one has that $\|x\|_1 \leq \|x\|_{a,k} \leq (1+a)\|x\|_1$, and it is obvious that

$$\|(0, \overset{(k)}{\ldots}, 0, v_1, v_2, \ldots)\|_{a,k} = \|(v_1, v_2, \ldots)\|_1 \leq \|(v_1, v_2, \ldots)\|_{a,k},$$

that is, $S^k$ is $\|\cdot\|_{a,k}$-nonexpansive.

5. The (dual) norm $\|x\|_l := \max\{\|x^+\|_1, \|x^-\|_1\}$, where $x^+$ and $x^-$ stand respectively for the positive an the negative parts of $x = (x_n) \in \ell_1$. Notice that in [54] was given a fixed point free self-mapping of the weak*-compact convex set $K := \{(x_n) \in \ell_1 : x_n \geq 0, \sum_{n=1}^{\infty} x_n \leq 1\}$, namely $T(x) = (1 - \sum_{n=1}^{\infty} x_n, x_1, x_2, \ldots)$. This mapping $T$ is 2 Lipschitzian with respect to the norm $\|\cdot\|_1$ but it is $\|\cdot\|_l$-nonexpansive on $K$.

6. Of course, every positive linear combination of a finite number of norms satisfying Condition (5.14) again satisfy such condition.

Notice that $\|\cdot\|_c$ and $\|\cdot\|_l$ fail to satisfy condition (5.1) in Theorem 5.1.1.

The set of all of norms satisfying Condition (5.14) is indeed large, but is far from be equal to the set $\mathcal{P} \setminus \mathcal{A}$: Let $U : \ell_1 \to \ell_1$, be the mapping defined by

$$U(x) = (0, x_1, 0, x_2, 0, x_3, 0, x_4, \ldots).$$

Of course, $\|U(x)\|_1 = \|x\|_1$, and $U$ maps $C$ into $C$. If $x = U(x)$ for some $x \in \ell_1$, then $x = 0_{\ell_1} \notin C$, and therefore $U$ is a fixed point free self-mapping of $C$.

Let $\|\cdot\|_U$ be the norm on $\ell_1$ defined by

$$\|x\|_U = \sum_{i=1}^{\infty} |x_{2i}| + \sum_{i=1}^{\infty} a|x_{2i-1}|$$

where $a > 1$ is previously given. It turns out that, for every $x \in \ell_1$

$$\|U(x)\|_U = \|x\|_1 \leq \|x\|_U$$

which implies that $U$ is $\|\cdot\|_U$-nonexpansive on $C$. However, $\|\cdot\|_U$ fails (5.14), in spite of the fact that it belongs to $\mathcal{P} \setminus \mathcal{A}$.

We can define $\mathcal{N}_{U^k}$ as the set of all the equivalent renormings of $\ell_1$ for which $U^k$ is nonexpansive. Even more, for any operator $V$ on $\ell_1$ which leaves invariant the set $C$ (that is, with $V(C) \subset C$) and which has no fixed points on $C$ we can consider a subset $\mathcal{N}_V$ of $\mathcal{P} \setminus \mathcal{A}$, namely the set of those norms, say $q$, for which $V$ is $q$-nonexpansive. In any case we have seen that the set $\mathcal{P} \setminus \mathcal{A}$ is, in some sense, quite complex.

These examples raise the problem (seemingly hard) of giving a characterization of the equivalent renormings of $(\ell_1, \|\cdot\|_1)$ lacking the FPP. Recently, T. Domínguez Benavides [15] has proved the following interesting result:

**Theorem 5.2.2.** *Let $X$ be a Banach space which contains an isomorphic copy of either $\ell_1$ or $c_0$. Then, the set of renormings of $X$ failing the FPP is dense in the set of renormings of $X$ with respect to the Banach-Mazur distance.*

The proof of the above statement is based on James' theorem [45, Lemma 2.1.]. We include some details from [15] for completeness: Assume for instance that $X$

contains an isomorphic copy of $\ell_1$ and let $|\cdot|$ be an equivalent norm on $X$. For every $\epsilon > 0$ there exists a subspace $Y$ of $X$ such that $(1-\epsilon)||Ty||_1 \leq |y| \leq ||Ty||_1$ for all $y \in Y$, where $T$ is an isomorphism from $Y$ onto $\ell_1$. Define $p(y) := ||Ty||_1$, which is an equivalent norm on $Y$, satisfying $(1-\epsilon)p(y) \leq |y| \leq p(y)$ for all $y \in Y$ and $(Y, p(\cdot))$ fails to have the FPP (since it is isometric to $(\ell_1, \|\cdot\|_1)$). It is proved in [15] that there exists a norm $q(\cdot)$ on $X$ such that $q(y) = p(y)$ for all $y \in Y$ and $(1-\epsilon)q(x) \leq |x| \leq q(x)$ for all $x \in X$. Hence $(X, q(\cdot))$ fails to have the FPP (since it contains a subspace failing the FPP) and the Banach-Mazur distance beetween $(X, |\cdot|)$ and $(X, q(\cdot))$ can be choosen as close as one as we like.

The same arguments holds for the case of $c_0$ and more generally, in [15] it is proved the some result for all Banach spaces which contains isomorphically a non-distortable subspace.

Notice that Theorem 5.2.2 shows that $\mathcal{P} \setminus A$ is dense in $\mathcal{P}$ and that no norm in $\ell_1$ produces stability of the FPP, which is surprising if we compare with the case of classical reflexive Banach spaces. Furthermore, in reflexive Banach spaces, even in Hilbert spaces, is not known whether or not they can be equivalently renormed to fail the FPP.

Another natural question is: Is $\mathcal{A}$ also dense in $\mathcal{P}$? From the results given in the previous section it can be deduced that $\mathcal{A}$ is dense in the set of all renormings of $\ell_1$ satisfying condition 5.1 of Theorem 5.1.1.

The examples given in this section and the results concerning the unstability of the FPP let us suggest that the set $\mathcal{A}$ should be negligible, or, in other words, that the property of failing FPP is generic for the elements of $\mathcal{P}$.

To finish, notice that $\mathcal{P}$ is a convex cone, that is, if $p_1, p_2 \in \mathcal{P}$ and $\alpha_1, \alpha_2 \geq 0$, then $\alpha_1 p_1 + \alpha_2 p_2 \in \mathcal{P}$. Is the set $\mathcal{A}$ also a convex cone? From Theorem 5.1.1 we can deduce that $\mathcal{A}$ does have some kind of linear structure. In fact, it contains rays: if $p(\cdot)$ is a norm in $\mathcal{P}$ satisfying property (5.1) of Theorem 5.1.1, then $p(\cdot)$ is the starting point of an (open or closed) ray of norms contained in $\mathcal{A}$.

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