The three dimensional globally modified Navier-Stokes equations: Recent developments

T. Caraballo and P.E. Kloeden

Abstract The globally modified Navier-Stokes equations (GMNSE) were introduced by Caraballo, Kloeden & Real [1] in 2006 and have been investigated in a number of papers since then, both for their own sake and as a means of obtaining results about the 3-dimensional Navier-Stokes equations. These results were reviewed by Kloeden *et al* [11], which was published in 2009, but there have been some important developments since then, which will be reviewed here.

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1 Introduction

The 3-dimensional Navier-Stokes equations (NSE) are an intriguing system of partial differential equations. They have been intensively investigated for many years, but some very basic issues on their solvability remain unresolved. For example, although weak solutions are known to exist for all future time for each initial condition in the function space H, it is not known if there is a unique weak solution. Nor is it known if a strong solution for each initial condition in the function space V can exist for more than a short time.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary Γ. The system of Navier-Stokes equations (NSE) on Ω with a homogeneous Dirichlet boundary condition is given by

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T. Caraballo
Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, ES-41080 Sevilla, Spain, e-mail: caraball@us.es

P.E. Kloeden
Institut für Mathematik, Goethe-Universität, D-60054 Frankfurt am Main, Germany, e-mail: kloeden@math.uni-frankfurt.de

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) \quad \text{in } \ (\tau, +\infty) \times \Omega, \\ \nabla \cdot u \ = 0 \quad \text{in } \ (\tau, +\infty) \times \Omega, \\ u = 0 \ \text{ on } (\tau, +\infty) \times \Gamma, \\ u(\tau, x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, $\tau \in \mathbb{R}$ the initial time, u_0 the initial velocity field, and $f(t)$ a given external force field.

There have been many modifications of the Navier-Stokes equations, starting with Leray and mostly involving the nonlinear term, see the review paper of Constantin [6]. Another modification, called the globally modified Navier-Stokes equations (GMNSE), was introduced by Caraballo, Kloeden & Real [1] in 2006.

Fix $N \in \mathbb{R}^+$ and define $F_N : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r \in \mathbb{R}^+.$$

The system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|)\left[(u \cdot \nabla) u\right] + \nabla p = f(t) \quad \text{in } \ (\tau, +\infty) \times \Omega, \\ \nabla \cdot u \ = 0 \quad \text{in } \ (\tau, +\infty) \times \Omega, \\ u = 0 \ \text{ on } (\tau, +\infty) \times \Gamma, \\ u(\tau, x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2}$$

is called the *globally modified Navier-Stokes equations (GMNSE)* with parameter N.

The GMNSE (2) are indeed *globally* modified since the modifying factor $F_N(\|u\|)$ depends on the norm $\|u\| = \|\nabla u\|_{(L^2(\Omega))^{3\times 3}}$, which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x \in \Omega$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions. It is worth mentioning that, for a different purpose, Flandoli & Maslowski [9] used a similar global cut off function involving the $D(A^{1/4})$ norm for the two-dimensional stochastic Navier-Stokes equations.

The GMNSE (2) violate the basic laws of mechanics, but mathematically they are a well defined system of equations, just like the modified versions of the NSE of Leray and others with other mollifications of the nonlinear term. They are nevertheless interesting mathematically in their own right, but are also useful for obtaining new results about the 3-dimensional Navier-Stokes equations, which will be briefly discussed below.

1.1 Notation

The usual notation and abstract framework for the Navier-Stokes equations of Lions [17] and Temam [24]) is used with H denoting the closure of

$$\mathscr{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u \ = 0 \right\},$$

in $(L^2(\Omega))^3$ with inner product $(u,v) = \sum_{j=1}^3 \int_\Omega u_j(x) v_j(x)\, \mathrm{d}x$, for $u, v \in (L^2(\Omega))^3$, with associated norm $|\cdot|$, and V denoting the closure of $\mathscr{V}$ in $(H_0^1(\Omega))^3$ with inner product $((u,v)) = \sum_{i,j=1}^3 \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, \mathrm{d}x$, for $u, v \in (H_0^1(\Omega))^3$, with associated norm $\|\cdot\|$. In addition, b_N and B_N are defined by

$$b_N(u,v,w) = F_N(\|v\|) b(u,v,w), \quad \forall u,v,w \in V.$$

and

$$\langle B_N(u,v), w \rangle = b_N(u,v,w), \quad \forall u,v,w \in V,$$

respectively, where b is the trilinear form on $V \times V \times V$ given by

$$b(u,v,w) = \sum_{i,j=1}^3 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x, \quad \forall u,v,w \in V.$$

Finally, define $A : V \to V'$ by $\langle Au, v \rangle = ((u,v))$. Then $Au = -P\Delta u, \forall u \in D(A)$, where $D(A) = (H^2(\Omega))^3 \cap V$ and P is the orthonormal projector from $(L^2(\Omega))^3$ onto H.

2 Existence and regularity of solutions

The existence, uniqueness and regularity theory of strong and weak solutions of the 3-dimensional GMNSE is closer to that of the two-dimensional than the three-dimensional NSE due to the special properties of b_N, which is linear in u and w, but nonlinear in v, and satisfies $b_N(u,v,v) = 0$ for all $u, v \in V$ as well as the estimate

$$|b_N(u,v,w)| = F_N(\|v\|) |b(u,v,w)| \leq N C_1 \|u\| \|w\| \quad \forall u,v,w \in V, \tag{3}$$

This and many other estimates, that can be found in Caraballo, Kloeden & Real [1], are very similar to those for the two-dimensional NSE and lead to similar results. In particular, the GMNSE have a unique global strong solution for each initial condition in the function space V as well as global weak solutions for each initial condition in the function space H, which instantaneously become strong solutions. Originally, in [1] it was not known if the weak solutions were unique, but this was later established by Romito [23] and thus allowed a number of proofs that had appeared in some papers published in the period between these two to be simplified.

2.1 Weak solutions

Let $u_0 \in H$ and $f \in L^2(\tau,T;(L^2(\Omega))^3)$ for all $T > \tau$ be given. A weak solution of (2) is any $u \in L^2(\tau,T;V)$ for all $T > \tau$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t) + \nu Au(t) + B_N(u(t),u(t)) = f(t) \text{ in } \mathscr{D}'(\tau,+\infty;V'), \\ u(\tau) = u_0, \end{cases} \tag{4}$$

or equivalently

$$(u(t),w) + \nu \int_\tau^t ((u(s),w))\, ds + \int_\tau^t b_N\,(u(s),u(s),w)\; ds = (u_0,w) + \int_\tau^t (f(s),w)\; ds, \tag{5}$$

for all $t \geq \tau$ and all $w \in V$.

Due to (3), unlike the 3-dimensional NSE, any weak solution $u(t)$ of GMNSE belongs to $C([\tau,+\infty);H)$ and satisfies (see Remark 1 in [1]) the energy equality

$$|u(t)|^2 - |u(s)|^2 + 2\nu \int_s^t \|u(r)\|^2\, dr = 2\int_s^t (f(r),u(r))\, dr \quad \text{for all } \tau \leq s \leq t. \tag{6}$$

The existence of weak solutions of the GMNSE is contained in Theorem 1 below which also considers the existence of strong solutions. The following result is the counterpart of Serrin's classical theorem on the 3-dimensional NSE which says that a strong solution, if it exists, is unique in the class of weak solutions. Strong solutions (to be defined below) for the GMNSE are examples of the weak solutions in the next theorem.

Theorem 1. *([1] Theorem 3) If there exists a weak solution u of (2) such that* $u \in L^2(\tau,T;D(A))$ *for all* $T > \tau$*, then u is the unique weak solution of (2).*

This result is not as important as originally thought since the weak solutions of the GMNSE have been shown to be unique. The proof is similar to the NSE case and depends on the following result.

Lemma 1. *([1] Lemma 6) For all M, N, p, r* $\in \mathbb{R}^+$ *it holds*

$$|F_M(p) - F_N(r)| \leq \frac{|M-N|}{r} + \frac{|p-r|}{r}.$$

2.2 Strong solutions

The following theorem is the basic existence and regularity result for strong and also weak solutions of the GMNSE.

Theorem 2. *([2]) Suppose $f \in L^2(\tau,T;(L^2(\Omega))^3)$ for all $T > \tau$, and let $u_0 \in H$ be given. Then, there exists a unique weak solution u of (2), which is, in fact, a strong solution in the sense that*

$$u \in C([\tau+\varepsilon,T];V) \cap L^2(\tau+\varepsilon,T;D(A)), \tag{7}$$

for all $T > \tau+\varepsilon > \tau$.

Moreover, if $u_0 \in V$, then

$$u \in C([\tau,T];V) \cap L^2(\tau,T;D(A)), \tag{8}$$

for all $T > \tau$.

The first statement in Theorem 2 was originally given as "there exists *at least one* weak solution u of the GMNSE" in Theorem 7 of [1], but takes its present form after Romito showed that "there exists *at most one* weak solution u of the GMNSE" in Theorem 1.1 of [23]. Romito used the estimate

$$|\,(w,B(u,v))\,| \leq \|u\|_{L^6}\|\nabla v\|_{L^2}\|w\|_{L^2}^{\frac{1}{2}}\|w\|_{L^6}^{\frac{1}{2}} \leq c_0\langle u,v\rangle|w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}, \tag{9}$$

for u, v, $w \in V$, since $\|u\|_{L^6} \leq c\|u\|$. He used this to show that the nonlinear term $\mathscr{N}\mathscr{L}(u,v) := F_N(\|u\|)B(u,u) - F_N(\|v\|)B(v,v)$ could be estimated by

$$(w,\mathscr{N}\mathscr{L}(u,v)) \leq \nu\|w\|^2 + C(c_0,\nu)N^4|w|^2,$$

where $w = u - v$, the difference of two weak solutions.

2.2.1 Continuity of strong solutions on data

Strong solutions $u^{(N)}(t,\tau,u_0)$ of the GMNSE (2) with parameter N depend continuously on the parameter N as well as on the initial value u_0.

Theorem 3. *([13] Theorem 8) Suppose that $f \in L^2(\tau,T;(L^2(\Omega))^3)$ for all $T > \tau$, and let $N,M > 0$, and $u_0, v_0 \in V$ be given. Denote by $u^{(N)}(t) = u^{(N)}(t,\tau,u_0)$ (respectively, $v^{(M)}(t) = u^{(M)}(t,\tau,v_0)$) the solution of the GMNSE (2) corresponding to the parameter N and the initial value u_0 (respectively, to the parameter M and the initial condition v_0). Then, there exists a positive constant $C > 0$ depending only on Ω and ν such that for all $t \geq \tau$*

$$||v^{(M)}(t) - u^{(N)}(t)||^2 \leq [||v_0 - u_0||^2 + C(M-N)^2\int_\tau^t |Au^{(N)}(s)|^2 ds]\times \tag{10}$$
$$\times \exp\left(C\left(M^4(t-\tau) + \int_\tau^t |Au^{(N)}(s)|^2 ds\right)\right)$$

and

$$\nu\int_\tau^t |Av^{(M)}(s)-Au^{(N)}(s)|^2ds \le [||v_0-u_0||^2+C(M-N)^2\int_\tau^t|Au^{(N)}(s)|^2ds]\times \tag{11}$$

$$\left[1+\left(C\int_\tau^t\left(|Au^{(N)}(s)|^2+M^4\right)ds\right)\times\exp\left(C\left(M^4(t-\tau)+\int_\tau^t|Au^{(N)}(s)|^2ds\right)\right)\right].$$

As a consequence of the previous theorem, we obtain

Theorem 4. *Suppose that $f\in L^2(\tau,T;(L^2(\Omega))^3)$ for all $T>\tau$. Then, for any $u_0\in V$ and $N>0$ given,*

$$u^{(M)}(\cdot,\tau,v_0)\to u^{(N)}(\cdot,\tau,u_0)\ \text{ in }\ C([\tau,T];V)\cap L^2(\tau,T;D(A))$$

as $(M,v_0)\to(N,u_0)$ in $\mathbb{R}^+\times V$ for all $T>\tau$.

2.2.2 Estimates of strong solutions in $D(A)$.

With stronger assumptions on the external forcing term f, estimates of the solution in the norm of $D(A)$ can be obtained.

Theorem 5. *([3] Proposition 4) Suppose that $f\in W^{1,\infty}(\tau,+\infty;H)$, and let $u^{(N)}(t)$ be a solution of the GMNSE (2) with parameter N. Then*

$$u^{(N)}(t)\in D(A),\quad \forall t>\tau, \tag{12}$$

and there exist two positive constants $K_f^{(N)}$ and $M_f^{(N)}$, independent of ε, τ, u_0 and t, and increasing with $|f|_\infty$ and $|f'|_\infty$, such that

a) if $u(\tau)=u_0\in V$, then

$$|Au^{(N)}(t)|^2\le(1+\varepsilon^{-1})\left[R_f^{(N)}+M_f^{(N)}(1+t-\tau)\|u_0\|^2e^{-\nu\lambda_1(t-\tau)}\right], \tag{13}$$

for all $t\ge\tau+\varepsilon$, $\varepsilon\in(0,1]$;

b) in general, if $u(\tau)=u_0\in H$, then

$$|Au^{(N)}(t)|^2\le(1+\varepsilon^{-1})R_f^{(N)}+\varepsilon^{-1}(1+\varepsilon^{-1})M_f^{(N)}(1+t-\tau)(1+|u_0|^2)e^{-\nu\lambda_1(t-\tau)}, \tag{14}$$

for all $t\ge\tau+2\varepsilon$, $0<\varepsilon\le1$. In particular, there exists a $T_0=T_0(|u_0|)$ depending only on $|u_0|$, $K_f^{(N)}$ and $M_f^{(N)}$, such that

$$|Au^{(N)}(t)|^2\le 2R_f^{(N)},\quad \forall t\ge\tau+T_0(|u_0|). \tag{15}$$

Remark 1. Observe that (14) implies that if $f\in W^{1,\infty}(\tau,+\infty;H)$, then every solution of GMNSE belongs to $L^\infty(\tau+\varepsilon,+\infty;D(A))$ for all $\varepsilon>0$. If, moreover, the initial datum $u_0\in D(A)$, then it can be proved that the corresponding solution $u=u^{(N)}(t)$ of GMNSE belongs to $L^\infty(\tau,+\infty;D(A))$, and more exactly,

$$\sup_{t \geq \tau} |Au(t)| < +\infty.$$

3 Global attractor in V: Existence and dimension estimate

3.1 Autonomous case

Assume now that the forcing term f does not depend on time and for each $u_0 \in V$ define $S^{(N)}(t)u_0 := u^{(N)}(t,u_0)$, where $u^{(N)}(t,u_0)$ is the unique strong solution $u^{(N)}(t)$ of (2) with initial time $\tau = 0$. ¿From Theorems 2 and 4, it follows that $\{S^{(N)}(t)\}_{t \geq 0}$ is a C^0 semigroup in V. Let $u^{(N)}(t) = S^{(N)}(t)u_0$ with $u_0 \in V$. The same arguments as for the NSE give the inequality

$$\frac{d}{dt}|u^{(N)}|^2 + \nu\lambda_1 |u^{(N)}|^2 \leq \frac{1}{\nu\lambda_1}|f|^2 \tag{16}$$

and hence the estimate

$$|u^{(N)}(t)|^2 \leq |u_0|^2 e^{-\nu\lambda_1 t} + \frac{1}{\nu^2\lambda_1^2}|f|^2 \left(1 - e^{-\nu\lambda_1 t}\right),$$

from which it follows that $S^{(N)}(t)$ possesses a set $\mathscr{B}_H$ in H which absorbs bounded sets of V, and which is given by $\mathscr{B}_H := \{u \in H \ : |u|^2 \leq 1 + \frac{1}{\nu^2\lambda_1^2}|f|^2 \ \}$. Similarly, but more complicatedly, $S^{(N)}(t)$ has an absorbing set $\mathscr{B}_V^{(N)}$ in V (i.e., which absorbs bounded sets of V) given by

$$\mathscr{B}_V^{(N)} := \left\{ u \in V \ : \|u\|^2 \leq 1 + \frac{|f|^2}{\nu^2\lambda_1}\left(2 + \frac{C^{(N)}}{\nu\lambda_1^2}\right)\right\}. \tag{17}$$

Note that $\mathscr{B}_V^{(N)} \subset \mathscr{B}_V^{(N^*)}$ for $N \leq N^*$ in view of the definition of the constant $C^{(N)}$ (see [1] for details).

Moreover, the semigroup $S^{(N)}(t)$ in V is asymptotically compact since it satisfies the flattening property ([12], see also [22]): "*For any bounded set B of V and for any $\varepsilon > 0$, there exists $T_\varepsilon(B) > 0$ and a finite dimensional subspace V_ε of V, such that $\left\{P_\varepsilon S^{(N)}(t)B, t \geq T_\varepsilon(B)\right\}$ is bounded and*

$$\left\|(I - P_\varepsilon)S^{(N)}(t)u_0\right\| < \varepsilon \quad for \quad t \geq T_\varepsilon(B), u_0 \in B, \tag{18}$$

where $P_\varepsilon : V \to V_\varepsilon$ is the projection operator." It thus follows that the GMNSE (2) has a global attractor $\mathscr{A}_N$ in V for each N. In particular, $\mathscr{A}_N \subset \mathscr{B}_V^{(N)}$ for each N and

Theorem 6. *([1] Theorem 10) If $f \in (L^2(\Omega))^3$, then the GMNSE (2) has a global attractor $\mathscr{A}_N$ in V for each $N > 0$. Moreover the set-valued mapping $N \mapsto \mathscr{A}_N$ is upper semi continuous, i.e.*

$$dist_V(\mathscr{A}_M, \mathscr{A}_N) \to 0 \quad as \quad M \to N, \tag{19}$$

where $dist_V$ is the Hausdorff semi distance on V.

The upper semi continuous dependence of the global attractors $\mathscr{A}_N$ in N follows by standard theorems in dynamical systems theory in view of the continuity of the semigroups $S^{(N)}$ in N established in Theorem 4.

3.1.1 Global attractor in $D(A)$.

With time-independent forcing (so that the stronger assumption of Theorem 5 is satisfied) it is possible to obtain an absorbing set $\mathscr{B}_N := \{v \in D(A) : |Av|^2 \le 2R_f^{(N)}\}$ in $D(A)$ for the semigroup $\{S^{(N)}(t)\}_{t\ge 0}$ and hence the above global attractor $\mathscr{A}_N$ actually belongs to $D(A)$. In fact

Corollary 1. *([3] Corollary 7) The global attractor $\mathscr{A}_N$ of the GMNSE is a bounded subset of $D(A)$.*

3.2 Nonautonomous case

In the nonautonomous case, when f depends on time, the counterpart of a semigroup is a 2-parameter semigroup of operators $U^{(N)}(t,\tau)$, with $U^{(N)}(t,\tau)u_0 = u^{(N)}(t,\tau,u_0)$ the solution of (2) for $u_0 \in V$. In addition, the counterpart of an attractor is a pullback attractor, i.e., a family of nonempty compact subsets $\{\mathscr{A}_N(t), t \in \mathbb{R}\})$ in V, which is invariant in the sense that $S^{(N)}(t,\tau)\mathscr{A}_N(\tau) = \mathscr{A}_N(t)$ for all $t \ge \tau$ and is pullback attracting in V, see [13]. Supposing that f belongs to $L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$ and satisfies

$$\int_{-\infty}^{t} e^{\nu\lambda_1 s}|f(s)|^2\,ds < +\infty \quad \text{for all } t \in \mathbb{R}, \tag{20}$$

where λ_1 is the first eigenvalue of A, the existence of a pullback attractor in V for the GMNSE was established in [13] Theorem 13. Among other properties for the pullback attractor in V, a finite bound on the fractal dimension, which could increase with increasing time, was also obtained in [13].

Theorem 7. *([13] Theorem 22) Suppose that $f \in W^{1,2}_{loc}(\mathbb{R};L^2(\Omega)^3)$ satisfies*

$$f \in L^\infty(-\infty,t_0;L^2(\Omega)^3), \text{ and } \sup_{r\le t_0}\int_r^{r+1} |f'(s)|^2\,ds < +\infty, \text{ for all } t_0 \in \mathbb{R}. \tag{21}$$

Then, for each $N > 0$ *and each* $t_0 \in \mathbb{R}$ *there exists a* $d^{(N)}(t_0) \in [0,+\infty)$ *such that the fractal dimension of the pullback attractor* $\{\mathscr{A}_N(t), t \in \mathbb{R}\})$ *of the GMNSE (2) satisfies the bound*

$$d_F^V(\mathscr{A}_N(t)) \leq d^{(N)}(t_0) \quad \text{for all } t \leq t_0. \tag{22}$$

Recall that the fractal dimension of a nonempty subset C of a metric space (X, d_X) is given by

$$\mathrm{d}_F^X(C) := \limsup_{\varepsilon \downarrow 0} \frac{\log(N_\varepsilon(C))}{\log(1/\varepsilon)}, \tag{23}$$

where $N_\varepsilon(C)$ denotes the minimum number of balls in X with radius ε which are required to cover C.

4 Globally modified NSE with delays

There are many real situations in which one can consider that a model is better described if we allow some delay in the equations. These situations may appear, for instance, when we want to control the system by applying a force which takes into account not only the present state of the system but the history of the solutions. Therefore, it is interesting to consider the following version of GMNSE (we will refer to it as GMNSED):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|u\|)[(u\cdot\nabla)u] + \nabla p = g(t,u_t) \;\text{ in } (\tau,+\infty)\times\Omega, \\ \nabla\cdot u = 0 \;\text{ in }\; (\tau,+\infty)\times\Omega, \\ u = 0 \;\text{ on } (\tau,+\infty)\times\Gamma, \\ u(\tau,x) = u^0(x), \quad x\in\Omega, \\ u(\tau+s,x) = \phi(s,x),\; s\leq 0, x\in\Omega, \end{cases} \tag{24}$$

where $\tau \in \mathbb{R}$ is an initial time, the term $g(t,u_t)$ is an external force depending eventually on the history of the solution, where u_t denotes the segment of solution up to time t (in other words, $u_t : s \in (-\infty,0] \mapsto u_t(s) := u(t+s)$) and ϕ is a given velocity field defined for $s \leq 0$.

This is a general formulation when the delay is allowed to be infinite. But on some occasions it can be finite or bounded. In these cases, we consider the initial vector field ϕ defined in a bounded interval $[-h,0]$ and the segment solution u_t is also defined in the same interval.

Some examples for the delay external force will be given below, but first, it is important to note that the function g is not defined directly on the phase space but on some class of continuous functions: either in $\mathbb{R}\times C([-h,0];H)$ with the sup norm (for finite delays), or $\mathbb{R}\times C_\gamma((-\infty,0];H)$ (in the infinite delay case) where the space

$C_\gamma(H) := C_\gamma((-\infty,0];H)$, defined as

$$C_\gamma((-\infty,0];H) := \left\{ \varphi \in C((-\infty,0];H) : \exists \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \in H \right\},$$

is a Banach space for the norm

$$\|\varphi\|_\gamma := \sup_{s\in(-\infty,0]} e^{\gamma s}|\varphi(s)|.$$

1. **Constant delay**. Consider $g(t,u_t) := G_1(u(t-h))$ where $G_1 : H \to H$ is a suitable function and $h > 0$ is the constant delay. Here the function $g : \mathbb{R} \times C([-h,0];H) \to H$ is defined as:

$$g(t,\xi) = G_1(\xi(-h)), \;\; \xi \in C([-h,0];H).$$

   Notice that the time variable t does not play any role, so we are in an autonomous situation.
2. **Variable delay**. In this case, the delay term is given by $g(t,u_t) := G_2(t,u(t-\rho(t)))$, where $\rho(t) \in [-h,0]$ is a delay function. Now, the function g is given by

$$g(t,\xi) = G_2(\xi(-\rho(t))), \;\; \xi \in C([-h,0];H),$$

   where it is clear that the time variable t is necessary for this case. So, we are in a non-autonomous model.
3. **Distributed infinite delay**. (cf. [19]) Let us consider the operator $g : \mathbb{R} \times C_\gamma(H) \to (L^2(\Omega))^3$ defined as

$$g(t,\xi) = \int_{-\infty}^0 G_3(t,s,\xi(s))ds, \;\; t \in \mathbb{R}, \; \xi \in C_\gamma(H),$$

   where the function $G_3 : \mathbb{R} \times (-\infty,0) \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies suitable assumptions. This situation corresponds to the case

$$g(t,u_t) = \int_{-\infty}^0 G_3(t,s,u(t+s))ds,$$

   which is also non-autonomous.

On the one hand, the two first cases (constant and variable delay) have been analyzed in [5] where the authors proved existence and uniqueness of weak solutions, existence and asymptotic behaviour of stationary solutions, and the existence of pullback attractor (which becomes the global attractor in the autonomous case). On the other hand, the infinite delay case is studied in [19], where the existence and uniqueness of solutions, and the existence and asymptotic behaviour of stationary solutions is proved.

We will only include below some representative results from the paper [5], so we consider g to be defined as in case (2).

Assume $G_2 : \mathbb{R} \times H \longrightarrow H$ is such that

c1) $G_2(\cdot, u) : \mathbb{R} \longrightarrow H$ is measurable, $\forall u \in H$,
c2) there exists nonnegative function $m \in L^p_{loc}(\mathbb{R})$ for some $1 \leq p \leq +\infty$, and a non-decreasing function $L : (0,\infty) \to (0,\infty)$, such that for all $R > 0$ if $|u|, |v| \leq R$, then

$$|G_2(t,u) - G_2(t,v)| \leq L(R) m^{1/2}(t) |u - v|,$$

for all $t \in \mathbb{R}$, and
c3) there exists a nonnegative function $f \in L^1_{loc}(\mathbb{R})$, such that for any $u \in H$,

$$|G_2(t,u)|^2 \leq m(t)|u|^2 + f(t), \quad \forall t \in \mathbb{R}.$$

Finally, we suppose $\phi \in L^{2p'}(-h,0;H)$ and $u^0 \in H$, where $\frac{1}{p} + \frac{1}{p'} = 1$.
In this situation, we consider a delay function $\rho \in C^1(\mathbb{R})$ such that $0 \leq \rho(t) \leq h$ for all $t \in \mathbb{R}$, and there exists a constant ρ_* satisfying

$$\rho'(t) \leq \rho_* < 1 \quad \forall t \in \mathbb{R}. \tag{25}$$

Definition 1. Let $\tau \in \mathbb{R}$, $u^0 \in H$ and $\phi \in L^{2p'}(-h,0;H)$ be given. A weak solution of (24) is a function

$$u \in L^{2p'}(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$$

for all $T > \tau$, such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t) + \nu A u(t) + B_N(u(t), u(t)) = G(t, u(t - \rho(t))) \text{ in } \mathscr{D}'(\tau, +\infty; V'), \\ u(\tau) = u^0, \\ u(t) = \phi(t - \tau) \quad t \in (\tau - h, \tau), \end{cases}$$

or equivalently

$$(u(t), w) + \nu \int_\tau^t ((u(s), w))\, ds + \int_\tau^t b_N(u(s), u(s), w)\, ds = \left(u^0, w\right) \tag{26}$$
$$+ \int_\tau^t (G(s, u(s - \rho(s))), w)\, ds,$$

for all $t \geq \tau$ and all $w \in V$, and coincides with $\phi(t)$ in $(\tau - h, \tau)$.

The existence and uniqueness of weak (and strong) solutions of our problem is established in a similar way as we did in the non-delay case, but with necessary changes due to the delay term.

Theorem 8. *([5], Theorem 3.1) Under the conditions c1)-c3), assume that $\tau \in \mathbb{R}$, $u^0 \in H$ and $\phi \in L^{2p'}(-h,0;H)$ are given. Then, there exists a unique weak solution u of (24) which is, in fact, a strong solution in the sense that*

$$u \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; D(A)), \tag{27}$$

for all $T - \tau > \varepsilon > 0$.

Moreover, if $u^0 \in V$*, then* $u \in C([\tau, T]; V) \cap L^2(\tau, T; D(A))$*, for all* $T > \tau$.

Next, we state a result about the asymptotic behavior of the solutions of problem (24) when t goes to $+\infty$.

Let us suppose that c1)-c3) hold with $m \in L^\infty(\mathbb{R})$, assume also that

$$\nu^2 \lambda_1^2 (1-\rho_*) > |m|_\infty,$$

where $|m|_\infty := \|m\|_{L^\infty(\mathbb{R})}$,
and let us denote by $\varepsilon > 0$ the unique solution of

$$\varepsilon - \nu\lambda_1 + \frac{|m|_\infty e^{\varepsilon h}}{\nu\lambda_1(1-\rho_*)} = 0. \tag{28}$$

We can now formulate the following result (see also [19] for a similar result in the infinite delay case).

Theorem 9. *([5], Theorem 4.1) Under the previous assumptions, for any* $(u^0, \phi) \in H \times L^2(-h, 0; H)$*, and any* $\tau \in \mathbb{R}$*, the corresponding solution* $u(t; \tau, u^0, \phi)$ *of problem (24) satisfies*

$$\begin{aligned} &\left|u(t;\tau,u^0,\phi)\right|^2 \\ \le &\left(\left|u^0\right|^2 + \frac{|m|_\infty e^{\varepsilon h}}{\nu\lambda_1(1-\rho_*)} \int_{-h}^0 e^{\varepsilon s} |\phi(s)|^2 \, ds\right) e^{\varepsilon(\tau - t)} \\ &+ \frac{e^{-\varepsilon t}}{\nu\lambda_1} \int_\tau^t e^{\varepsilon s} f(s)\, ds, \end{aligned} \tag{29}$$

for all $t \ge \tau$.

In particular, if $\int_\tau^\infty e^{\varepsilon s} f(s)\, ds < \infty$*, then every solution* $u(t; \tau, u^0, \phi)$ *of (24) converges exponentially to* 0 *as* $t \to +\infty$.

Finally, the existence of pullback attractor is also proved in [5] by following a similar scheme to the one used in [4] for the two-dimensional Navier-Stokes equations with delay.

5 Statistical solutions of GMNSE

The autonomous GMNSE with $\tau = 0$ and $f \in H$, i.e. f is independent of time t, are considered in this section and N is held fixed here. Let $S^{(N)}$ be the semigroup in V generated by the autonomous GMNSE and let $\mathscr{A}_N$ be its global attractor in V. Probability measures on H here are with respect to the σ-algebra of Borel subsets of H.

Definition 2. A probability measure on H is said to be $S^{(N)}$-invariant if

$$\mu(V) = 1 \quad \textit{and} \quad \mu(E) = \mu\left(S^{(N)}(t)^{-1}E\right), \quad \forall t \geq 0, \tag{30}$$

for every Borel subset E of V (recall that a Borel set in V is a Borel set in H).

Theorem 10. *([3] Theorem 10) The support of any $S^{(N)}$-invariant measure on H is included in the global attractor $\mathscr{A}_N$.*

The existence of such measures is obtained by time averaging. The results below generalize those of Foias *et al*. [10] (see also Lukaszewicz [18]) for the two-dimensional NSE to the GMNSE.

5.1 Time-averages solutions in the autonomous case

Let LIM denote a generalized limit on $\mathscr{B}([0,\infty))$, the space of all bounded real-valued functions on $[0,\infty)$ (see [3] for the definition).

Definition 3. A time-average measure of the solution $u(t)$ of the autonomous GMNSE is a probability measure μ on H such that $C(H) \subset L^1(H,\mu)$ and

$$\mathrm{LIM}_{T\to\infty} \frac{1}{T}\int_0^T \varphi(u(t))\,dt = \int_H \varphi(v)\,d\mu(v), \quad \forall \varphi \in C(H). \tag{31}$$

Proposition 1. *([3] Proposition 13) Any time-average measure μ of a solution $u(t)$ of the autonomous GMNSE is carried by $D(A)$, i.e., $\mu(D(A)) = 1$.*

Proposition 2. *([3] Proposition 14) For any solution $u(t)$ of the autonomous GMNSE such that $u(0) \in V$ there exists a time-average measure μ of this solution such that moreover $C(V) \subset L^1(H,\mu)$ and*

$$\mathrm{LIM}_{T\to\infty} \frac{1}{T}\int_0^T \varphi(u(t))\,dt = \int_H \varphi(v)\,d\mu(v) \quad \forall \varphi \in C(V). \tag{32}$$

Proposition 3. *([3] Proposition 16) Let $u(t)$ be the solution of the autonomous GMNSE corresponding to $u_0 \in V$ and let μ be a time-average measure of $u(t)$ such that $C(V) \subset L^1(H,\mu)$ and (32) is satisfied for all $\varphi \in C(V)$. Then μ is an $S^{(N)}$-invariant measure.*

5.2 Stationary statistical solutions of the autonomous GMNSE

Define

$$G_N(v) = -\nu Av - B_N(v,v) + f, \quad \forall v \in V, \tag{33}$$

and let $\mathscr{T}$ be the set of real valued functionals $\Phi = \Phi(v)$ on H such that

(i) $c_r := \sup_{|v| \leq r} |\Phi(v)| < +\infty$ for all $r > 0$;

(ii)for any $v \in V$ there exists $\Phi'(v) \in V$ such that

$$\frac{|\Phi(v+w) - \Phi(v) - (\Phi'(v), w)|}{|w|} \to 0 \quad \text{as } |w| \to 0 \text{ with } w \in V; \tag{34}$$

(iii)the mapping $v \mapsto \Phi'(v)$ is continuous and bounded as function from V into V.

Definition 4. A stationary statistical solution of the GMNSE is a probability measure μ on H such that

(i) $\int_H \|v\|^2 \, d\mu(v) < +\infty$;

(ii) $\int_H \langle G_N(v), \Phi'(v) \rangle \, d\mu(v) = 0$ for any $\Phi \in \mathscr{T}$;

(iii) $\int_{\{a \leq |v|^2 < b\}} \{\nu \|v\|^2 - (f, v)\} \, d\mu(v) \leq 0$ for any $0 \leq a < b \leq +\infty$.

The following results were proved in [3].

Theorem 11. *([3] Theorem 19) Any $S^{(N)}$-invariant probability measure on H is a stationary statistical solution of the autonomous GMNSE.*

Corollary 2. *([3] Corollary 20) Let μ be a time-average measure of a solution $u(t)$ of the GMNSE such that $C(V) \subset L^1(H, \mu)$ holds and (32) is satisfied for all $\varphi \in C(V)$. Then μ is a stationary statistical solution of the autonomous GMNSE.*

As partial counterpart of Theorem 11 was given in [19].

Theorem 12. *([19], Theorem 15) Let μ be a stationary statistical solution of GMNSE such that there exists a bounded and measurable subset $\mathscr{B}_N$ of $D(A)$ satisfying $\mu(H \setminus \mathscr{B}_N) = 0$. Then μ is an S_N-invariant probability measure on H.*

6 Numerical solution of the globally modified NSE

There is an extensive literature on the numerical analysis of the 3-dimensional Navier-Stokes equations, much of which is based on the pioneering ideas of Temam [24]. In this spirit Deugoue & Djoko [8] investigated the implicit Euler scheme applied to the GMNSE, specifically

$$\frac{u^{m+1} - u^m}{k} + \nu A u^{m+1} + B_N(u^{m+1}, u^{m+1}) = f^{m+1} \tag{35}$$

with time stepsize k, where

$$f^{m+1} = \frac{1}{k} \int_{mk}^{(m+1)k} f(t) \, dt.$$

They establish uniform bounds on u^m (with respect to m) and its temporal difference quotient in different function spaces and find conditions under which u^m is continuous in N and u^0 and for which (35) is uniquely solvable. They also establish the existence of absorbing sets in both H and V spaces, which is the first step in showing the existence of attractors. Finally they consider the limit as $N \to \infty$ and prove the following theorem.

Theorem 13. *([8], Theorem 6.11) Let $f, f' \in L_\infty(\mathbb{R}^+, H)$ and $v^0 \in D(A)$ with k sufficently small. Then the sequence $\{u^{m,N}\}_N$ of solutions of the implicit Euler scheme* (35) *converges to a weak solution of the following time discrete* 3*-dimensional Navier-Stokes equations*

$$\frac{1}{k}(u^{m+1} - u^m, w) + \nu(\nabla u^m, \nabla w) + b(u^m, u^m, w) = (f^m, w), \quad \text{for all } w \in V, \tag{36}$$

as $N \to \infty$.

7 Weak solutions of the 3-dimensional Navier-Stokes Equations

Useful results about the 3-dimensional Navier-Stokes equations can be obtained from the GMNSE.

7.1 Weak Kneser property of the attainability set of weak solutions

The Kneser property for ordinary differential equations says that the attainability set of the solutions emanating from a given initial value is compact and connected. This property was shown by Kloeden & Valero [15] in a combination of Corollary 3.2 and Theorem 3.3 to hold for the weak solutions of the GMNSE in the strong topology of space H before it was known that the weak solutions of the GMNSE for a given initial value were unique, which makes the result trivial. This result was then used in [15] to show that the attainability set of the weak solutions of the 3-dimensional Navier-Stokes equations satisfying an energy inequality are weakly compact and weakly connected. A simplified proof, also using properties of the GMNSE, was later given in [16].

More precisely, for every initial datum $u_0 \in H$ it is well known that at least one weak solution of (1) exists such that

$$V_\tau(u(t)) \le V_\tau(u(s)) \quad \text{for all } t \ge s, \text{ a.a. } s > \tau \text{ and } s = \tau, \tag{37}$$

where $V_\tau(u(t)) := \frac{1}{2}|u(t)|^2 + \nu \int_\tau^t \|u(r)\|^2 dr - \int_\tau^t (f(r), u(r))\, dr$. Denote the corresponding attainability set for $t \ge \tau$ by

$$K_t(u_0) = \{u(t) \, : \, u(\cdot) \text{ is a weak solution of (1) satisfying (37)}\}.$$

We have:

Theorem 14. *([15] Theorem 2.1) Let* $f \in L^{\infty}(\tau,T;H)$ *for all* $T > \tau$*. Then, for all* $t \geq 0$ *and* $u_0 \in H$*, the attainability set* $K_t(u_0)$ *is compact and connected with respect to the weak topology on* H*.*

7.2 Convergence to weak solutions of the 3-dimensional NSE

Theorem 15. *([1] Theorem 13) Suppose that* $f \in L^2(\tau,T;(L^2(\Omega))^3)$ *for each* $T > \tau$ *and let* $u^{(N)}(t)$ *be a weak solution of the GMNSE (2) with the initial value* $u_0^{(N)} \in H$*, where* $u_0^{(N)} \rightharpoonup u_0$ *weakly in* H *as* $N \to \infty$*.*

Then, there exists a subsequence $\left\{u^{(N_j)}(t)\right\}$ *which converges as* $N_j \to \infty$*, weak-star in* $L^{\infty}(\tau,T;H)$*, weakly in* $L^2(\tau,T;V)$ *and strongly in* $L^2(\tau,T;H)$*, to a weak solution* $u(t)$ *on the interval* $[\tau,T]$ *of the NSE (1) with initial condition* u_0*, for every* $T > \tau$*.*

The proof is based on the fact that a weak solution of the GMNSE (2) with the initial value $u_0^{(N)} \in H$, where $u_0^{(N)} \rightharpoonup u_0$ weakly in H as $N \to \infty$, satisfies the energy inequality

$$\frac{d}{dt}|u^{(N)}|^2 + \nu\|u^{(N)}\|^2 \leq \frac{1}{\nu\lambda_1}|f|^2 \tag{38}$$

uniformly in $N > 0$. One easily obtains a convergent subsequence. The main difficulty is to show that limiting function is a weak solution of the NSE (1) for the given initial condition u_0, i.e satisfies the variational equation (4) with b_N replaced by b. The following lemma is required here.

Lemma 2. *([1] Lemma 12) For each* $p \geq 1$*, it follows that*

$$F_N\left(\|u^{(N)}(s)\|\right) \to 1 \qquad in \quad L^p(\tau,T;\mathbb{R}), \qquad as \quad N \to \infty.$$

7.3 Existence of bounded entire weak solutions of 3-dimensional NSE

When the forcing term $f \in (L^2(\Omega))^3$ is independent of time, Theorem 15 and the existence of a global attractor $\mathscr{A}_N$ of the GMNSE (2) for each N can be used to show that the NSE (1) have bounded entire weak solutions, that is, weak solutions which exist and are bounded for all $t \in \mathbb{R}$. Such solutions are interesting as they would belong to a global attractor of the 3-dimensional NSE, if such an attractor were to exist.

Theorem 16. *([1] Theorem 11) Suppose that $f \in (L^2(\Omega))^3$. Then there exists a bounded entire weak solution of the NSE (1). More exactly, there exists a bounded entire weak solution of the NSE (1) with initial value u_0 for each $u_0 \in \mathscr{U}_0$, where $\mathscr{U}_0$ is the subset in H consisting of the weak H-cluster points of sequences $u_0^{(N)} \in \mathscr{A}_N$ for $N \to \infty$.*

The set $\mathscr{U}_0$ here is obviously a non-empty subset of the closed and bounded subset $\mathscr{B}_H$ of H. A similar result holds with essentially the same proof in the nonautonomous case, as well as for the GMNED analyzed in Section 4 (see [21] for more details).

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