$H^2$-boundedness of the pullback attractor for a non-autonomous reaction-diffusion equation

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Abstract

We prove some regularity results for the pullback attractor of a reaction-diffusion model. First we establish a general result about $H^2$-boundedness of invariant sets for an evolution process. Then, as a consequence, we deduce that the pullback attractor of a non-autonomous reaction-diffusion equation is bounded not only in $L^2(\Omega) \cap H^1_0(\Omega)$ but in $H^2(\Omega)$.

Key words: reaction-diffusion equations, non-autonomous (pullback) attractors, invariant sets, $H^2$-regularity.

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1 Introduction and setting of the problem

Let us consider the following problem for a non-autonomous reaction-diffusion equation:

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = f(u) + h(t) \quad \text{in} \quad \Omega \times (\tau, +\infty), \\
u = 0 \quad \text{on} \quad \partial\Omega \times (\tau, +\infty), \\
u(x, \tau) = u_\tau(x), \quad x \in \Omega,
\end{cases}$$

(1)
where \( \Omega \subset \mathbb{R}^N \) is a bounded open set, \( \tau \in \mathbb{R}, u_\tau \in L^2(\Omega), f \in C^1(\mathbb{R}) \) and \( h \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \). We assume that there exist positive constants \( \alpha_1, \alpha_2, k, l, \text{ and } p > 2 \) such that
\[
-k - \alpha_1 |s|^p \leq f(s)s \leq k - \alpha_2 |s|^p, \quad \forall s \in \mathbb{R},
\]
(2)
\[
f'(s) \leq l, \quad \forall s \in \mathbb{R}.
\]
(3)
Let us denote
\[
\mathcal{F}(s) := \int_0^s f(r)dr.
\]
Then, there exist positive constants \( \bar{\alpha}_1, \bar{\alpha}_2 \) and \( \bar{k} \) such that
\[
-k - \bar{\alpha}_1 |s|^p \leq \mathcal{F}(s) \leq \bar{k} - \bar{\alpha}_2 |s|^p, \quad \forall s \in \mathbb{R}.
\]
(4)
It is well-known (see, e.g. [5] or [8]) that under the conditions above, for any initial condition \( u_\tau \in L^2(\Omega) \), there exists a unique solution \( u(\cdot) = u(\cdot; \tau, u_\tau) \) of (1), i.e., a unique function \( u \in L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C^0([\tau, T]; L^2(\Omega)) \) for all \( T > \tau \), such that
\[
 u(t) - \int_\tau^t \Delta u(s) ds = u_\tau + \int_\tau^t (f(u(s)) + h(s)) ds \quad \forall t \geq \tau,
\]
where the equality must be understood in the sense of the dual of \( H^1_0(\Omega) \cap L^p(\Omega) \).

Therefore, we can define a process \( U = \{U(t, \tau), \quad \tau \leq t \} \) in \( L^2(\Omega) \) as
\[
 U(t, \tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in L^2(\Omega), \quad \forall \tau \leq t.
\]
(5)
A pullback attractor for the process \( U \) defined by (5) (cf. [1], [2], [3]) is a family \( \mathcal{A} = \{A(t) : t \in \mathbb{R}\} \) of compact subsets of \( L^2(\Omega) \) such that
a) (invariance) \( U(t, \tau)A(\tau) = A(t) \) for all \( \tau \leq t \),
b) (pullback attraction) \( \lim_{\tau \to -\infty} \sup_{u_\tau \in B} \inf_{v \in A(t)} |U(t, \tau)u_\tau - v| = 0 \), for all \( t \in \mathbb{R} \), for any bounded subset \( B \subset L^2(\Omega) \),
where \( \cdot \) denotes the norm in \( L^2(\Omega) \).

It can be proved that, under the above conditions, if in addition \( f \) satisfies
\[
 \int_{-\infty}^{\infty} e^{\lambda_1 r} |h(r)|^2 dr < +\infty \quad \forall t \in \mathbb{R},
\]
where \( \lambda_1 \) denotes the first eigenvalue of the negative Laplacian with zero Dirichlet boundary condition in \( \Omega \), then there exists a pullback attractor for the process \( U \) defined by (5).
Several studies on this model have already been published (see [4], [6], [7], [9]). However, as far as we know, no one refers to the $H^2$-regularity we will consider in this paper.

In the next section we prove some results which, in particular, imply that, under suitable assumptions, any pullback attractor $A$ for $U$ satisfies that $A(t)$ is a bounded subset of $H^2(\Omega) \cap H^1_0(\Omega) \cap L^p(\Omega)$, for every $t \in \mathbb{R}$.

2 $H^2$-boundedness of invariants sets

In this section we prove that, under suitable assumptions, every family of bounded subsets of $L^2(\Omega)$ which is invariant for the process $U$, is in fact bounded in $H^2(\Omega)$.

First, we recall a lemma (see [5]) which is necessary for the proof of our result.

**Lemma 2.1** Let $X, Y$ be Banach spaces such that $X$ is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(t_0, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(t_0, T; X)$ for some $q \in [1, +\infty)$ and $u \in C^0([t_0, T]; Y)$.

Then, $u(t) \in X$ for all $t \in [t_0, T]$ and

$$\|u(t)\|_X \leq \sup_{n \geq 1} \|u_n\|_{L^\infty(t_0, T; X)} \quad \forall t \in [t_0, T].$$

We will denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)$, by $\|\cdot\| = |\nabla \cdot|$ the norm in $H^1_0(\Omega)$, and by $\langle \cdot, \cdot \rangle$ the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

For each integer $n \geq 1$, we denote by $u_n(t) = u_n(t; \tau, u_\tau)$ the Galerkin approximation of the solution $u(t; \tau, u_\tau)$ of (1), which is given by

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t)w_j, \quad (6)$$

and is the solution of

$$\begin{cases}
\frac{d}{dt}(u_n(t), w_j) = \langle \Delta u_n(t), w_j \rangle + (f(u_n(t)), w_j) + (h(t), w_j), \\
(u_n(\tau), w_j) = (u_\tau, w_j) \quad j = 1, \ldots, n,
\end{cases} \quad (7)$$

where $\{w_j : j \geq 1\}$ is the Hilbert basis of $L^2(\Omega)$ formed by the eigenfunctions associated to $-\Delta$ in $H^1_0(\Omega)$.

We first prove the following result.
Proposition 2.2 Assume that $f \in C^{1}(\mathbb{R})$ satisfies (2) and (3). Suppose moreover that $\Omega \subset \mathbb{R}^N$ is a bounded $C^s$ domain, with $s \geq \max(2, N(p-2)/2p)$, and $h \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$. Then, for any bounded set $B \subset L^2(\Omega)$, any $r \in \mathbb{R}$, any $\varepsilon > 0$ and any $t > \tau + \varepsilon$, the set $\{u(r; \tau, u_r) : r \in [\tau + \varepsilon, t], u_r \in B, n \geq 1\}$, is a bounded subset of $H^1_0(\Omega) \cap L^p(\Omega)$.

**Proof.** Observe that by the regularity of $\Omega$, all the eigenfunctions $w_j$ associated to $-\Delta$ in $H^1_0(\Omega)$ belong to $H^2(\Omega) \cap H^1_0(\Omega) \cap L^p(\Omega)$.

Let us fix a bounded set $B \subset L^2(\Omega)$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$, and $u_r \in B$.

Multiplying by $\gamma_{nj}$ in (7), and summing from $j = 1$ to $n$, we obtain

$$
\frac{1}{2} \frac{d}{dr} |u_n(r)|^2 + \|u_n(r)\|^2 = (f(u_n(r)), u_n(r)) + (h(r), u_n(r)).
$$

(8)

Using (2),

$$(f(u_n(r)), u_n(r)) \leq \int_{\Omega} (k - \alpha_2 |u_n(x, r)|^p) \, dx$$

$$= k |\Omega| - \alpha_2 \|u_n(r)\|_{L^p(\Omega)}^p.$$

On the other hand,

$$(h(r), u_n(r)) \leq \frac{1}{2\lambda_1} |h(r)|^2 + \frac{\lambda_1}{2} |u_n(r)|^2$$

$$\leq \frac{1}{2\lambda_1} |h(r)|^2 + \frac{1}{2} \|u_n(r)\|^2.$$

Thus, from (8) we deduce

$$\frac{d}{dr} |u_n(r)|^2 + \|u_n(r)\|^2 + 2\alpha_2 \|u_n(r)\|_{L^p(\Omega)}^p \leq \frac{1}{\lambda_1} |h(r)|^2 + 2k |\Omega|,$$

and integrating between $\tau$ and $r$

$$|u_n(r)|^2 + \int_{\tau}^r \|u_n(s)\|^2 \, ds + 2\alpha_2 \int_{\tau}^r \|u_n(s)\|_{L^p(\Omega)}^p \, ds$$

$$\leq |u_{\tau}|^2 + \frac{1}{\lambda_1} \int_{\tau}^t |h(s)|^2 \, ds + 2k |\Omega| (t - \tau), \ \forall \tau \in [\tau, t], \ \forall n \geq 1.$$

(9)

Now, multiplying by the derivative $\gamma'_{nj}$ in (7), and summing from $j = 1$ to $n$,

$$|u_n'(r)|^2 + \frac{1}{2} \frac{d}{dr} |u_n(r)|^2 = (f(u_n(r)), u'_n(r)) + (h(r), u'_n(r))$$

$$\leq \frac{1}{2} |h(r)|^2 + \frac{1}{2} |u_n'(r)|^2 + \frac{d}{dr} \int_{\Omega} \mathcal{F}(u_n(x, r)) \, dx.$$
Integrating now between \( s \in [\tau, r] \) and \( r \leq t \), we obtain

\[
\int_s^r |u'_n(\theta)|^2 d\theta + \|u_n(r)\|^2 \leq \|u_n(s)\|^2 + \int_\tau^t |h(\theta)|^2 d\theta + 2 \int_\Omega \mathcal{F}(u_n(x, r)) \, dx - 2 \int_\Omega \mathcal{F}(u_n(x, s)) \, dx,
\]

which, jointly with (4), yields that

\[
\int_s^r |u'_n(\theta)|^2 d\theta + \|u_n(r)\|^2 \leq \|u_n(s)\|^2 + \int_\tau^t |h(\theta)|^2 d\theta + 4k_1 \|u_n(s)\|^p_{L^p(\Omega)},
\]

for all \( s \in [\tau, r] \), and any \( r \in [\tau, t] \).

Integrating in this last inequality with respect to \( s \) from \( \tau \) to \( r \), we in particular obtain

\[
(r - \tau) \left( \|u_n(r)\|^2 + 2\tilde{\alpha}_2 \|u_n(r)\|^p_{L^p(\Omega)} \right) \leq \int_\tau^t \|u_n(s)\|^2 ds + (t - \tau) \int_\tau^t |h(s)|^2 ds + 4k_1 \|u_n(s)\|^p_{L^p(\Omega)} ds,
\]

for all \( r \in [\tau, t] \), and for any \( n \geq 1 \). From this inequality and (9), our result holds.

**Corollary 2.3** Under the assumptions in Proposition 2.2, for any bounded set \( B \subseteq L^2(\Omega) \), any \( \tau \in \mathbb{R} \), any \( \varepsilon > 0 \), and any \( t > \tau + \varepsilon \), the set \( \bigcup_{r \in [\tau + \varepsilon, t]} U(r, \tau)B \) is a bounded subset of \( H^1_0(\Omega) \cap L^p(\Omega) \).

**Proof.** This is a straightforward consequence of Lemma 2.1, Proposition 2.2, and the well known fact that \( u_n(\cdot, \tau, u_\tau) \) converges weakly to \( u(\cdot, \tau, u_\tau) \) in \( L^2(\tau, t; H^1_0(\Omega)) \cap L^p(\tau, t; L^p(\Omega)) \).

**Proposition 2.4** In addition to the assumptions in Proposition 2.2, assume that \( h \in W^{1,2}_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \). Then, for any bounded set \( B \subseteq L^2(\Omega) \), any \( \tau \in \mathbb{R} \), any \( \varepsilon > 0 \), and any \( t > \tau + \varepsilon \), the set \( \{u_n(r; \tau, u_\tau): r \in [\tau + \varepsilon, t], u_\tau \in B, n \geq 1\} \) is a bounded subset of \( H^2(\Omega) \).

**Proof.** Let us fix a bounded set \( B \subseteq L^2(\Omega), \tau \in \mathbb{R}, \varepsilon > 0, t > \tau + \varepsilon \), and \( u_\tau \in B \).

As we are assuming that \( h \in W^{1,2}_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \), we can differentiate with respect to time in (7), and then, multiplying by \( \gamma'_{nj} \), and summing from \( j = 1 \) to \( n \),
we obtain
\[
\frac{1}{2} \frac{d}{dr} |u'_n(r)|^2 + \|u'_n(r)\|^2 = (f'(u_n(r))u'_n(r), u'_n(r)) + (h'(r), u'_n(r)) \\
\leq l |u'_n(r)|^2 + \frac{1}{2} |u'_n(r)|^2 + \frac{1}{2} |h'(r)|^2.
\]

In particular, integrating in the last inequality,
\[
|u'_n(r)|^2 \leq |u'_n(s)|^2 + (2l + 1) \int_{\tau+\varepsilon/2}^t |u'_n(\theta)|^2 d\theta + \int_{\tau+\varepsilon/2}^t |h'(\theta)|^2 d\theta,
\]
for all \(\tau + \varepsilon/2 \leq s \leq r \leq t\). Now, integrating with respect to \(s\) between \(\tau + \varepsilon/2\) and \(r\),
\[
(r - \tau - \varepsilon/2) |u'_n(r)|^2 \leq [(2l + 1)(t - \tau - \varepsilon/2) + 1] \int_{\tau+\varepsilon/2}^t |u'_n(\theta)|^2 d\theta \\
+ (r - \tau - \varepsilon/2) \int_{\tau+\varepsilon/2}^t |h'(\theta)|^2 d\theta,
\]
for all \(\tau + \varepsilon/2 \leq r \leq t\), and, in particular,
\[
|u'_n(r)|^2 \leq 2\varepsilon^{-1}[(2l + 1)(t - \tau - \varepsilon/2) + 1] \int_{\tau+\varepsilon/2}^t |u'_n(\theta)|^2 d\theta \\
+ \int_{\tau+\varepsilon/2}^t |h'(\theta)|^2 d\theta,
\]
for all \(r \in [\tau + \varepsilon, t]\).

On the other hand, multiplying in (7) by \(\lambda_j \gamma_{nj}\), where \(\lambda_j\) is the eigenvalue associated to the eigenfunction \(w_j\), and summing once more from \(j = 1\) to \(n\), we obtain
\[
(u'_n(r), \Delta u_n(r)) = |\Delta u_n(r)|^2 + (f(u_n(r)), \Delta u_n(r)) + (h(r), \Delta u_n(r)).
\]

But, it follows from (3) that
\[
- (f(u_n(r)), \Delta u_n(r)) = - \int_{\Omega} (f(u_n(x, r)) - f(0)) \Delta u_n(x, r)dx \\
- f(0) \int_{\Omega} \Delta u_n(x, r)dx \\
\leq l \|u_n(r)\|^2 + \frac{1}{4} |\Delta u_n(r)|^2 + (f(0))^2 |\Omega| \\
= l (u_n(r), -\Delta u_n(r)) + \frac{1}{4} |\Delta u_n(r)|^2 + (f(0))^2 |\Omega| \\
\leq l^2 |u_n(r)|^2 + \frac{1}{2} |\Delta u_n(r)|^2 + (f(0))^2 |\Omega|,
\]
and thus, from (12) we obtain
\[ |\Delta u_n(r)|^2 \leq 8 |u_n'(r)|^2 + 8 |h(r)|^2 + 4l^2 |u_n(r)|^2 + 4 (f(0))^2 |\Omega|, \]  
for all \( r \geq \tau \).

Finally, observe that by (10)
\[ \int_{\tau + \varepsilon/2}^{t} |u_n'(\theta)|^2 d\theta \leq \|u_n(\tau + \varepsilon/2)\|^2 + \int_{\tau}^{t} |h(\theta)|^2 d\theta + 4\bar{k} |\Omega| + 2\tilde{a}_1 \|u_n(\tau + \varepsilon/2)\|_{L^p(\Omega)}, \]  
Taking into account that, in particular, \( h \in C^0([\tau, t]; L^2(\Omega)) \), the result is a direct consequence of Proposition 2.2 and estimates (11), (13) and (14).

**Corollary 2.5**  
Under the assumptions of Proposition 2.4, for any bounded set \( B \subset L^2(\Omega) \), any \( \tau \in \mathbb{R} \), any \( \varepsilon > 0 \), and any \( t > \tau + \varepsilon \), the set \( \bigcup_{r \in [\tau + \varepsilon, t]} U(r, \tau)B \) is a bounded subset of \( H^2(\Omega) \).

**Proof.** This follows from Lemma 2.1, propositions 2.2 and 2.4, and the well known facts that \( u_n(\cdot; \tau, u_\tau) \) converges weakly to \( u(\cdot; \tau, u_\tau) \) in \( L^2(\tau, t; H^1_0(\Omega)) \), and \( u(\cdot; \tau, u_\tau) \in C^0([\tau + \varepsilon, t]; H^1_0(\Omega)) \).

As a direct consequence of the above results, we can now establish our main results.

**Theorem 2.6**  
Under the assumptions in Proposition 2.4, if \( \mathcal{A} = \{ \mathcal{A}(t) : t \in \mathbb{R} \} \) is a family of bounded subsets of \( L^2(\Omega) \), such that \( U(t, \tau) \mathcal{A}(\tau) = \mathcal{A}(t) \) for any \( \tau \leq t \), then for any \( T_1 < T_2 \), the set \( \bigcup_{t \in [T_1, T_2]} \mathcal{A}(t) \) is a bounded subset of \( H^2(\Omega) \cap H^1_0(\Omega) \cap L^p(\Omega) \).

In particular, we have the following result for pullback attractors.

**Corollary 2.7**  
Under the assumptions in Proposition 2.4, if \( \mathcal{A} = \{ \mathcal{A}(t) : t \in \mathbb{R} \} \) is a pullback attractor for the process defined by (5), then for any \( T_1 < T_2 \), the set \( \bigcup_{t \in [T_1, T_2]} \mathcal{A}(t) \) is a bounded subset of \( H^2(\Omega) \cap H^1_0(\Omega) \cap L^p(\Omega) \).

**References**


