On geodesic and monophonic convexity [1]

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Abstract

In this paper we deal with two types of graph convexities, which are the most natural *path convexities* in a graph and which are defined by a system $\mathcal{P}$ of paths in a connected graph G: the *geodesic convexity* (also called metric convexity) which arises when we consider shortest paths, and the *monophonic convexity* (also called minimal path convexity) when we consider chordless paths. First, we present a realization theorem proving, that there is no general relationship between monophonic and geodetic hull sets. Second, we study the contour of a graph, showing that the contour must be monophonic. Finally, we consider the so-called edge Steiner sets. We prove that every edge Steiner set is edge monophonic.

1. Introduction

A *convexity* on a finite set V is a family $\mathcal{C}$ of subsets of V, to be regarded as *convex sets*, which is closed under intersection and which contains both V and the empty set. The pair $(V, \mathcal{C})$ is called a *convexity space.* A finite *graph-convexity* space is a pair $(G, \mathcal{C})$, formed by a finite connected graph $G = (V, E)$ and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space satisfying that every member of $\mathcal{C}$ induces a connected subgraph of G [4,5]. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of $\mathbb{R}^n$ is convex if every segment joining two points of X is entirely contained in it. Similarly, a vertex set W of a finite connected graph G is said to be a convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W.

In this paper we deal with two types of graph convexities, which are the most natural *path convexities* in a graph and which are defined by a system $\mathcal{P}$ of paths in G: the *geodesic convexity* (also called metric convexity) [5,6,7,11] which arises when we consider shortest paths, and the *monophonic convexity* (also called minimal path convexity) [4,5] when we consider chordless paths.

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In what follows, $G = (V, E)$ denotes a finite connected graph with no loops or multiple edges. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. A *chord* of a path $u_0u_1 \dots u_h$ is an edge u_iu_j, with $j \geq i + 2$. A $u - v$ path ρ is called *monophonic* if it is a chordless path, and *geodesic* if it is a shortest $u - v$ path, that is, if $|E(\rho)| = d(u, v)$.

The *geodesic closed interval* $I[u, v]$ is the set of vertices of all $u - v$ geodesics. Similarly, the *monophonic closed interval* $J[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $W \subseteq V$, the *geodesic closure* $I[W]$ of W is defined as the union of all geodesic closed intervals $I[u, v]$ over all pairs $u, v \in W$. The *monophonic closure* $J[W]$ is the set formed by the union of all monophonic closed intervals $J[u, v]$.

A vertex set $W \subseteq V$ is called *geodesically convex* (or simply *g-convex*) if $I[W] = W$, while it is said to be *geodetic* if $I[W] = V$. Likewise, W is called *monophonically convex* (or simply *m-convex*) if $J[W] = W$, and is called *monophonic* if $J[W] = V$. The smallest g-convex set containing W is denoted $[W]_g$ and is called the *g-convex hull* of W. Similarly, the *m-convex hull* $[W]_m$ of W is defined as the minimum m-convex set containing W. Observe that $J[W] \subseteq [W]_m$, $I[W] \subseteq [W]_g$ and $[W]_g \subseteq [W]_m$. A *g-hull* (*m-hull*) set of G is a vertex set W satisfying $[W]_g = V$ ($[W]_m = V$).

For a nonempty set $W \subseteq V$, a connected subgraph of G with the minimum number of edges that contains all of W must be clearly a tree. Such a tree is called a *Steiner W-tree*. The *Steiner interval* $S(W)$ of W consists of all vertices that lie on some Steiner W-tree. If $S(W) = V$, then W is called a *Steiner set* for G [3].

The *monophonic* (*m-hull*, *geodetic*, *g-hull*, *Steiner*, respectively) number of G, denoted by $mn(G)$ ($mhn(G)$, $gn(G)$, $ghn(G)$, $st(G)$, respectively) is the minimum cardinality of a monophonic (m-hull, geodetic, g-hull, Steiner, respectively) set in G. Clearly, $ghn(G) \leq gn(G)$, since every geodetic set is a g-hull set. In [7] the authors showed that, apart from the previous one, no other general relationship among the parameters $ghn(G)$, $gn(G)$ and $st(G)$ exists. In Section 2, we approach the same problem by replacing the parameter $st(G)$ by both $mhn(G)$ and $mn(G)$.

In Section 3, we examine a number of monophonic convexity issues involving three types of vertices: contour, peripheral and extreme vertices [2]. We prove, among other facts, that the contour of a graph is monophonic. It is interesting to notice that this kind of results are closely related to the graph reconstruction problem, in the sense that we want to obtain all the vertices of a graph by considering a certain kind of paths joining vertices of a fixed set W.

In [3], it was shown that every Steiner set in G is also geodetic. Unfortunately, this particular result turned out to be wrong and was disproved by Pelayo [10]. In [7], the authors proved that every Steiner set is monophonic. As a consequence, they immediately derived that, in the class of distance-hereditary graphs (i.e., those graphs for which every monophonic path is a geodesic [8]), every Steiner set is geodetic. They also approached the problem of determining for which classes of chordal graphs (i.e., without induced cycles of length greater than 3) every Steiner set is geodetic, proving this statement to be true both for Ptolemaic graphs (i.e, distance-hereditary chordal graphs [5]) and interval graphs (i.e., chordal graphs without induced asteroid triples [9]). In Section 4, we focus our attention on the edges of geodesic and monophonic paths, approaching the same problems and obtaining similar results.

2. Monophonic and geodetic parameters

Let us review the main definitions involved in this section. A vertex set $W \subseteq V$ is a g-hull set if its g-convex hull $[W]_g$ covers all the graph, i.e., if $[W]_g = V$. Moreover, W is called geodetic if $I[W] = V$. The g-hull number $ghn(G)$ of G is defined as the minimum cardinality of a hull set. The geodetic number $gn(G)$ of G is the minimum cardinality of a geodetic set [6]. Certainly, $ghn(G) \leq gn(G)$.

Although it has been shown that determining the geodetic number and the hull number of a graph is a NP-hard problem [6], it is rather simple to obtain these two parameters for a wide range of classes of graphs as paths, cycles, trees, (bipartite) complete graphs, wheels and hypercubes.

A vertex set $W \subseteq V$ is a m-hull set if $[W]_m = V$. Moreover, W is called monophonic if $J[W] = V$. The m-hull number $mhn(G)$ of G is the minimum cardinality of a m-hull set. The monophonic number $mn(G)$ of G is the minimum cardinality of a monophonic set. Certainly, $mhn(G) \leq mn(G) \leq gn(G)$ and $mhn(G) \leq ghn(G)$, since every monophonic set is a m-hull set, every geodetic set is monophonic, and every g-hull set is a m-hull set. Nevertheless, it is not true that every g-hull set be monophonic. For example, if we consider the complete bipartite graph $K_{3,3}$, with $V_1 = \{a, b, c\}$ and $V_2 = \{e, f, g\}$, it is easy to see that the set $W = \{a, b\}$ satisfies $[W]_g = V$ and $J[W] = V \setminus \{c\}$.

At this point, what remains to be done is to ask the following question: *Is there any other general relationship among the parameters $mhn(G)$, $mn(G)$, $ghn(G)$ and $gn(G)$, apart from the previous known inequalities*?

The next realization theorem shows that, unless we restrict ourselves to a specific class of graphs, the answer is negative.

Theorem 1 *For any integers a, b, c, d such that $3 \leq a \leq b \leq c \leq d$, there exists a connected graph $G = (V, E)$ satisfying one of the following conditions:*

(i) $a = mhn(G)$, $b = mn(G)$, $c = ghn(G)$ and $d = gn(G)$,

(ii) $a = mnh(G)$, $b = ghn(G)$, $c = mn(G)$ and $d = gn(G)$.

3. Contour, peripheral and extreme vertices

Given a connected graph $G = (V, E)$, the *eccentricity* of a vertex $u \in V$ is defined as $ecc_G(u) = ecc(u) = \max\{d(u, v) | v \in V\}$. Hence, the diameter D of G can be defined as the maximum eccentricity of the vertices in G. The *periphery* of G, denoted $Per(G)$, is the set of vertices that have maximum eccentricity, i.e., the set of the so-called *peripheral* vertices. A vertex v is said to be *simplicial* in G if the subgraph induced by its neighborhood $N(v)$ is a clique. The *extreme set* of G, denoted $Ext(G)$, is the set of all its simplicial vertices. With the aim of generalizing these two definitions, the so-called *contour* of G was introduced in [2] as follows. A vertex $u \in W$ is said to be a *contour vertex* of W if $ecc_W(u) \geq ecc_W(v)$, for all $v \in N(u) \cap W$. The contour $Ct(W)$ of W is the set formed by all the contour vertices of W. If $W = V$, this set is called the contour of G and it is denoted $Ct(G)$. Notice that $Per(G) \cup Ext(G) \subseteq Ct(G)$.

Remark. We have examples of graphs showing that there is no general relationship between peripheral and extreme vertices.

Let $W \subseteq V$ be a m-convex (g-convex) set and let $F = \langle W \rangle_G$ be the subgraph of G induced by W. A vertex $v \in W$ is called an *m-extreme vertex* (*g-extreme vertex*) of W if $W \setminus \{v\}$ is a m-convex (g-convex) set. A vertex v of a m-convex (g-convex) set W is a m-extreme (g-extreme) vertex of W if and only if v is simplicial in F [5].

A convexity space $(V, \mathcal{C})$ is a *convex geometry* if it satisfies the so-called *Minkowsky-Krein-Milman* property: *Every convex set is the convex hull of its extreme vertices.* Notice that this condition allows us to rebuild every convex set from its extreme vertices, by using the convex hull operator. Farber and Jamison [5] proved that the monophonic (geodesic) convexity of a graph G is a convex geometry if and only if G is chordal (Ptolemaic). Cáceres et al. [2] obtained a similar property to the previous one, valid for every graph, by considering, instead of the extreme vertices, the contour vertices.

Theorem 2 *[2] Let $G = (V, E)$ be a connected graph and $W \subseteq V$ a g-convex set. Then, W is the g-convex hull of its contour vertices.*

As was pointed out in [2], the contour of a graph needs not to be geodetic. Nevertheless, we prove this assertion to be true in the following case.

Proposition 3 *If $Ct(G) = Per(G)$, then $Ct(G)$ is a geodetic set.*

PROOF. Let x be a vertex of $V(G) \setminus Ct(G)$. Since the eccentricities of two adjacent vertices differ by at most one unit, if x is not a contour vertex, then there exists a vertex $y \in V$, adjacent to x, such that its eccentricity satisfies $ecc(y) = ecc(x) + 1$. This fact implies the existence of a path $\rho(x) = x_0 x_1 x_2 \ldots x_r$, such that $x = x_0$, $x_i \notin Ct(G)$ for $i \in \{0, \ldots, r-1\}$, $x_r \in Ct(G)$, and $ecc(x_i) = ecc(x_{i-1}) + 1 = l + i$ for $i \in \{1, \ldots, r\}$, where $l = ecc(x)$. Moreover, $\rho(x)$ is a shortest $x - x_r$ path, since otherwise, the eccentricity of x_r would be less than $l + r$. But $x_r \in Ct(G) = Per(G)$ implies that $ecc(x_r) = D$ and $D = l + r$. Thus, there exists a vertex $z \in Per(G)$ such that $D = d(z, x_r) \leq d(z, x) + d(x, x_r) \leq ecc(x) + r = l + r = D$, that is, $d(z, x_r) = d(z, x) + d(x, x_r)$. Hence, x is on a shortest path between the vertices $z, x_r \in Per(G) = Ct(G)$.

As a consequence of this result, we have the following corollary.

Corollary 4 *If $Ct(G)$ has exactly two vertices, $Ct(G)$ is a geodetic set.*

Now, we approach the same issues by considering the monophonic convexity.

Theorem 5 *The contour of any connected graph G is a monophonic set.*

Corollary 6 *Let G be a connected graph and let $W \subseteq V$ be a m-convex set. Then, every vertex of W lies on a monophonic path joining contour vertices of W.*

PROOF. Let $F = \langle W \rangle_G$ be the subgraph of G induced by W, and let Ω be the set of contour vertices of W. Certainly, $\Omega = Ct(F)$. Hence, the previous statement is equivalent to saying that the contour of F is a monophonic set.

Corollary 7 *Let G be a connected graph and let $W \subseteq V$ be a m-convex set. Then, W is the m-convex hull of its contour vertices.*

Finally, we show another consequence of Theorem 5, which was directly proved in [2].

Corollary 8 *The contour of a distance-hereditary graph is a geodetic set.*

4. The edge Steiner problem

In this section, we focus our attention on the edges that lie in paths joining two vertices of $G = (V, E)$. We define the *edge intervals* of a graph as follows. The *edge geodetic closed interval* $I_e[u, v]$ is the set of edges of all $u - v$ geodesics. Similarly, the *edge monophonic closed interval* $J_e[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $W \subseteq V$, the *edge geodetic closure* $I_e[W]$ of W is the union of all edge closed intervals $I_e[u, v]$ over all pairs $u, v \in W$. The *edge monophonic closure*, $J_e[W]$, is defined as the union of all edge closed monophonic intervals over all pairs $u, v \in W$. In other words, we have

$$I_e[W] = \bigcup_{u,v\in W} I_e[u, v], \quad J_e[W] = \bigcup_{u,v\in W} J_e[u, v].$$

A vertex set W for which $J_e[W] = E$ is called an *edge monophonic set*. Similarly, W is an *edge geodetic set* if $I_e[W] = E$ [1]. A set $W \subseteq V$ is an *edge Steiner set* if the edges lying in some Steiner W-tree cover E. Notice that: (1) every edge Steiner set is a Steiner set, (2) every edge geodetic set is geodetic, (3) every edge monophonic set is monophonic, and (4) every edge geodetic set is an edge monophonic set. It is easy to find examples where the converses of these statements are not true. We have obtained the following results.

Theorem 9 *Every edge Steiner set of a connected graph is an edge monophonic set.*

Corollary 10 *In the class of connected distance-hereditary graph, every edge Steiner set is an edge geodetic set.*

Analogously to the vertex case, this last result also holds for interval graphs. First, we need to prove the following lemma.

Lemma 11 *(i) If P is a $x-y$ walk in G, any vertex of W is adjacent to at least a vertex of $V(P)$. (ii) If T_W is a Steiner W-tree, for any $u \in V(T_W) \setminus W$, u lies in the unique $x - y$ path of T_W. Moreover, there exists a Steiner W-tree, T^*_W, formed by the unique $x-y$ path of T_W and vertices of W adjacent to vertices of that path.*

Theorem 12 *In the class of connected interval graphs, every edge Steiner set is an edge geodetic set.*

In the preceding section we have seen that the contour of a graph is a monophonic set. Nevertheless, it is quite easy to find graphs whose contour is not an edge monophonic set.

It remains an open question the problem of characterizing those classes of chordal graphs for which *every edge Steiner set is edge geodetic*. We know this statement to be true for interval and Ptolemaic graphs, and false for split graphs (i.e., those chordal graphs whose complementary is also chordal).

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