

B.-Y. CHEN'S INEQUALITY FOR SUBMANIFOLDS OF GENERALIZED SPACE FORMS

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In this article, we investigate sharp inequalities involving δ -invariants for submanifolds in both generalized complex space forms and generalized Sasakian space forms, with arbitrary codimension.

Key Words: δ -invariant; Scalar Curvature; Mean Curvature; Generalized Complex Space Form; Generalized Sasakian Space Form

1. INTRODUCTION

To study submanifolds of a Riemannian manifold, we must consider some intrinsic invariants and the extrinsic ones as well. Among invariants, Riemannian invariants are the intrinsic characteristics of the Riemannian manifold. In fact, curvature is known as the most naturally important intrinsic invariant according to Berger in [4]. In this regard, sectional, scalar and Ricci curvatures are mostly concerned.

In [10] B.-Y. Chen introduced new types of curvature invariants (called the δ -invariants), by defining two strings of scalar-valued Riemannian curvature functions, namely $\delta(n_1, \dots, n_k)$ and

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$\hat{\delta}(n_1, \dots, n_k)$ for every (n_1, \dots, n_k) satisfying $n_1 < n, n_j \geq 2$ and $n_1 + \dots + n_k \leq n$. The first string of δ -invariants, $\delta(n_1, \dots, n_k)$, extend naturally the Riemannian invariant introduced in [6, 7].

There are many papers studying δ -invariants. For instance, in [8] B.-Y. Chen established sharp inequalities for submanifolds of a complex space form, while in [15], Oiaga and Mihai investigated δ -invariants for slant submanifolds in this kind of ambient spaces. In [12], Kim *et al.* studied them in generalized complex space forms. We can also refer to some papers on Chen’s inequalities in Sasakian space forms (see [5, 11, 14]) and Arslan *et al.*, ([2, 3]) studied δ -invariants for submanifolds in locally conformal almost cosymplectic manifolds and in (κ, μ) -contact space forms. Recently, the fourth named author ([13, 18]) investigated the sharp inequalities involving δ -invariant for CR -submanifolds and totally real submanifolds in locally conformal Kaehler space forms, respectively.

In this paper, we study δ -invariants for any kind of submanifolds of either generalized complex space forms or generalized Sasakian space forms, with arbitrary codimension.

2. PRELIMINARIES

An almost Hermitian manifold (\tilde{M}, J, g) is said to be a generalized complex space form if there exist two functions f_1 and f_2 on \tilde{M} such that

$$\begin{aligned} \tilde{R}(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + f_2\{g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\}, \end{aligned}$$

for any vector fields X, Y, Z on \tilde{M} , where \tilde{R} denotes the curvature tensor of \tilde{M} (see [17]). In such a case, we will write $\tilde{M}(f_1, f_2)$. Many authors have studied these manifolds and their submanifolds. For example, one main reference concerning these spaces is [17], in which Tricerri and Vanhecke established an important obstruction for their existence in dimensions greater than or equal to 6. In fact, in these dimensions a generalized complex space form reduces to a complex space form. Nevertheless, Olszak provided some interesting examples of 4-dimensional generalized complex space forms with non-constant functions in [16].

On the other hand, a generalized Sasakian space form is an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ such that its Riemannian curvature tensor is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned}$$

f_1, f_2, f_3 being differentiable functions on \tilde{M} . We will write $\tilde{M}(f_1, f_2, f_3)$. These spaces were defined and studied by the first two named authors and Blair in [1]. In that paper, they also gave

some procedures to construct interesting examples by using warped products and conformal changes of metric.

Given a submanifold M of a generalized (either complex or Sasakian) space form \tilde{M} , we also use g for the induced Riemannian metric on M . We denote by $\tilde{\nabla}$ the Levi-Civita connection on \tilde{M} and by ∇ the induced Levi-Civita connection of M . Then the Gauss and Weingarten formulas are given respectively by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \tilde{\nabla}_X V &= -A_V X + D_X V\end{aligned}$$

for vector fields X, Y tangent to M and a vector field V normal to M , where h denotes the second fundamental form, D the normal connection and A_V the shape operator in the direction of V . The second fundamental form and the shape operator are related by

$$g(h(X, Y), V) = g(A_V X, Y).$$

Moreover, the mean curvature vector H on an n -dimensional submanifold M is defined by $H = (1/n)$ trace h .

For $p \in M$ and any $X \in T_p M$, we write either $JX = PX + FX$ or $\phi X = TX + NX$, where $PX, TX \in T_p M$, $FX, NX \in T_p^\perp M$, depending on the ambient space being a generalized complex space form or a generalized Sasakian space form. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_p M$, we put either

$$\|P\|^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j),$$

or

$$\|T\|^2 = \sum_{i,j=1}^n g^2(Te_i, e_j), \quad \|N\|^2 = \sum_{i=1}^n |Ne_i|^2. \tag{2.1}$$

Moreover, if $\pi \subset T_p M$ is a plane section at $p \in M$ it is easy to see that

$$\Theta(\pi) = g^2(Pe_1, e_2)$$

(or, similarly, $\Theta(\pi) = g^2(Te_1, e_2)$ in the almost contact metric case) is a real number in $[0, 1]$ which is independent on the choice of the orthonormal basis $\{e_1, e_2\}$ of π .

For an n -dimensional Riemannian manifold M , we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M, p \in M$. For any orthonormal basis e_1, \dots, e_n of the tangent space $T_p M$, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i < j} K(e_i \wedge e_j).$$

If we put $(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_pM\}$, then the Riemannian invariant δ_M introduced by B.-Y. Chen in [6, 7] is given by:

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$

If L is a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \dots, e_r\}$ is an orthonormal basis of L , then the scalar curvature $\tau(L)$ of the r -plane section L is defined by

$$\tau(L) = \sum_{\alpha < \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{2.2}$$

If L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of L . As B.-Y. Chen points out in [9], geometrically, $\tau(L)$ is nothing but the scalar curvature of the image $\exp_p(L)$ of L at p under the exponential map at p . We set either

$$\Psi(L) = \sum_{1 \leq i < j \leq r} g^2(Pe_i, e_j) \quad \text{or} \quad \Psi(L) = \sum_{1 \leq i < j \leq r} g^2(Te_i, e_j).$$

For an integer $k \geq 0$, $\mathcal{S}(n, k)$ denotes the finite set consisting of unordered k -tuples (n_1, \dots, n_k) of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \dots + n_k \leq n$, while $\mathcal{S}(n)$ denotes the set of unordered k -tuples with $k \geq 0$ for a fixed n . For each k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the two sequences of Riemannian invariants $\mathcal{S}(n_1, \dots, n_k)(p)$ and $\hat{\mathcal{S}}(n_1, \dots, n_k)(p)$ are defined respectively by

$$\begin{aligned} \mathcal{S}(n_1, \dots, n_k)(p) &= \inf\{\tau(L_1) + \dots + \tau(L_k)\}, \\ \hat{\mathcal{S}}(n_1, \dots, n_k)(p) &= \sup\{\tau(L_1) + \dots + \tau(L_k)\}, \end{aligned}$$

where L_1, \dots, L_k run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \dots, k$. The two strings of Riemannian curvature invariants $\delta(n_1, \dots, n_k)(p)$ and $\hat{\delta}(n_1, \dots, n_k)(p)$ introduced by B.-Y. Chen in [10] are given by

$$\begin{aligned} \delta(n_1, \dots, n_k)(p) &= \tau(p) - \mathcal{S}(n_1, \dots, n_k)(p), \\ \hat{\delta}(n_1, \dots, n_k)(p) &= \tau(p) - \hat{\mathcal{S}}(n_1, \dots, n_k)(p). \end{aligned} \tag{2.3}$$

Clearly, $\delta(n_1, \dots, n_k) \geq \hat{\delta}(n_1, \dots, n_k)$ for any k -tuple (n_1, \dots, n_k) in $\mathcal{S}(n)$. For more explanations about these δ -invariants and their relationship with that introduced in [6, 7], we refer to [9].

For each $(n_1, \dots, n_k) \in \mathcal{S}(n)$, let $c(n_1, \dots, n_k)$ and $b(n_1, \dots, n_k)$ denote the positive constants given by

$$\begin{aligned} c(n_1, \dots, n_k) &= \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}, \\ b(n_1, \dots, n_k) &= \frac{1}{2} \left(n(n-1) - \sum_{j=1}^k n_j(n_j-1) \right). \end{aligned}$$

We give the following lemma for later use.

Lemma 2.1 — ([6]) Let a_1, \dots, a_n, b be $n + 1$ ($n \geq 2$) real numbers such that

$$\left(\sum_{i=1}^n a_i\right)^2 = (n - 1) \left(\sum_{i=1}^n a_i^2 + b\right).$$

Then, $2a_1a_2 \geq b$, with the equality holding if and only if $a_1 + a_2 = a_3 = \dots = a_n$.

3. δ -INVARIANTS OF SUBMANIFOLDS OF GENERALIZED COMPLEX SPACE FORMS

Let M be an n -dimensional submanifold isometrically immersed in a generalized complex space form $\tilde{M}(f_1, f_2)$ of complex dimension m . Then, the Gauss' equation is given by

$$\begin{aligned} R(X, Y, Z, W) = & f_1\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ & + f_2\{g(X, JZ)g(JY, W) \\ & - g(Y, JZ)g(JX, W) + 2g(X, JY)g(JZ, W)\} \\ & + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned} \tag{3.1}$$

where R is the Riemannian curvature tensor of M and $R(X, Y, Z, W) = g(R(X, Y)Z, W)$. Hence, it is easily seen that the scalar curvature τ of M at p is obtained by

$$2\tau(p) = n^2\|H\|^2 - \|h\|^2 + n(n - 1)f_1 + 3f_2\|P\|^2, \tag{3.2}$$

where $\|H\|^2$ and $\|h\|^2$ are the squared mean curvature and the squared norm of the second fundamental form.

The following result was obtained as Theorem 3.3 of [12]:

Theorem 3.1 — ([12]) *Let M be an n -dimensional submanifold of an $m(\geq 3)$ -dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then, for any point $p \in M$ and any plane section $\pi \subset T_pM$, we have*

$$\tau - K(\pi) \leq \frac{n - 2}{2} \left(\frac{n^2}{n - 1}\|H\|^2 + (n + 1)f_1\right) + 3 \left(\frac{\|P\|^2}{2} - \Theta(\pi)\right) f_2. \tag{3.3}$$

The equality holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ for T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}\}$ for $T_p^\perp M$ such that (a) $\pi = \text{Span}\{e_1, e_2\}$ (b) the shape operators $A_r = A_{e_r}$, $r = n + 1, \dots, 2m$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c \end{pmatrix}, \tag{3.4}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 & \dots & 0 \\ d_r & -c_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad r = n + 2, \dots, 2m \tag{3.5}$$

where $a + b = c$ and $c_r, d_r \in \mathbb{R}$.

Let us point out that the normal vector e_{n+1} appearing in the above theorem is in the direction of the mean curvature vector H .

As an application of this result, the authors of [12] also obtained some B.-Y. Chen inequalities for θ -slant submanifolds of a generalized complex space form. Now, we can prove some more general results, depending on the function f_2 being negative or positive:

Theorem 3.2 — Let M be an n -dimensional submanifold of an $m(\geq 3)$ -dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. If $f_2 \leq 0$, then we have

$$\delta_M \leq \frac{n-2}{2} \left(\frac{n^2}{n-1} \|H\|^2 + (n+1)f_1 \right). \tag{3.6}$$

The equality holds at a point p of M if and only if there exist an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M$ and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}\}$ of $T_p^\perp M$ such that (a) the subspace spanned by e_3, \dots, e_n is totally real, (b) $K(e_1 \wedge e_2) = \inf K$ at p , and (c) the shape operators $A_r = A_{e_r}$, $r = n + 1 \dots, 2m$ take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c \end{pmatrix}, \tag{3.7}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 & \dots & 0 \\ d_r & -c_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad r = n + 2, \dots, 2m \tag{3.8}$$

where $a + b = c$ and $c_r, d_r \in \mathbb{R}$.

PROOF : By Theorem 3.1, we have (3.3) which implies

$$\begin{aligned} \delta_M \leq & \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)f_1 \\ & + 3f_2 \left(\sum_{j=3}^n g^2(Pe_1, e_j) + \sum_{j=3}^n g^2(Pe_2, e_j) + \frac{1}{2} \sum_{i,j=3}^n g^2(Pe_i, e_j) \right) \end{aligned}$$

$$\leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)f_1. \tag{3.9}$$

If the equality in (3.6) holds, then both inequalities in (3.9) become equalities. Clearly, the second inequality in (3.9) is an equality if and only if $\text{Span}\{e_3, \dots, e_n\}$ is totally real. Thus, the equality in (3.6) implies condition (a) of the theorem. Moreover, it is clear that we also have $K(e_1 \wedge e_2) = \inf K$ at p . The remaining part of the theorem follows from Theorem 3.1. \square

Theorem 3.3 — *Let M be an n -dimensional submanifold of an $m(\geq 3)$ -dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. If $f_2 \geq 0$, then we have*

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)f_1 + \frac{3}{2}nf_2. \tag{3.10}$$

The equality in (3.10) holds identically if and only if n is even and M is holomorphic.

PROOF : For the case of $f_2 \geq 0$, we must maximize the term $\|P\|^2 - 2\Theta(\pi)$ in (3.3). The maximum value is reached for $\|P\|^2 = n$ and $\Theta(\pi) = 0$, that is, M is holomorphic. So, n is even. Hence, (3.10) is obtained with equality holding if and only if n is even and M is holomorphic. \square

Concerning the strings of invariants $\delta(n_1, \dots, n_k)$, a result for θ -slant submanifolds was obtained in [12]. Now, we can also prove two more general results for any kind of submanifolds, depending again on function f_2 . We also study the equality cases.

Theorem 3.4 — *Let M be an n -dimensional submanifold of an m -dimensional generalized complex space form $\tilde{M}(f_1, f_2)$ satisfying $f_2 \leq 0$. Then we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)\|H\|^2 + b(n_1, \dots, n_k)f_1 \tag{3.11}$$

for any k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$. The equality case of inequality (3.11) holds at a point $p \in M$ if and only if there exists an orthonormal basis e_1, \dots, e_{2m} at p such that the shape operators of M in $\tilde{M}(f_1, f_2)$ at p take the following forms:

$$A_r = \begin{pmatrix} A_1^r & \dots & 0 & & \\ \vdots & \ddots & \vdots & & \mathbf{0} \\ 0 & \dots & A_k^r & & \\ & & \mathbf{0} & & \mu_r I \end{pmatrix}, \quad r = n+1, \dots, 2m, \tag{3.12}$$

where I is an identity matrix and A_j^r are symmetric $n_j \times n_j$ submatrices such that

$$\text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = \mu_r. \tag{3.13}$$

PROOF : Let M be a submanifold of a generalized complex space form $\tilde{M}(f_1, f_2)$.

Let $(n_1, \dots, n_k) \in \mathcal{S}(n)$. Put

$$\eta = 2\tau - \frac{n^2(n+k-1-\sum n_j)}{(n+k-\sum n_j)}\|H\|^2 - n(n-1)f_1 - 3f_2\|P\|^2. \tag{3.14}$$

Substituting (3.2) in (3.14), we have

$$n^2 \|H\|^2 = \gamma(\eta + \|h\|^2), \quad \gamma = n + k - \sum n_j. \tag{3.15}$$

Let L_1, \dots, L_k be mutually orthogonal subspaces of T_pM with $\dim L_j = n_j, j = 1, \dots, k$. By choosing an orthonormal basis e_1, \dots, e_{2m} at p such that

$$L_j = \text{Span}\{e_{n_1+\dots+n_{j-1}+1}, \dots, e_{n_1+\dots+n_j}\}, \quad j = 1, \dots, k$$

and e_{n+1} is in the direction of the mean curvature vector, we obtain from (3.15) that

$$\left(\sum_{i=1}^n a_i\right)^2 = \gamma \left(\eta + \sum_{i=1}^n a_i^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right), \tag{3.16}$$

where $a_i = h_{ii}^{n+1}, i = 1, \dots, n$.

We set

$$\Delta_1 = \{1, \dots, n_1\}, \dots, \Delta_k = \{n_1 + \dots + n_{k-1} + 1, \dots, n_1 + \dots + n_k\}.$$

In other words, the equation (3.16) can be rewritten in the form

$$\begin{aligned} \left(\sum_{i=1}^{\gamma+1} \bar{a}_i\right)^2 &= \gamma \left(\eta + \sum_{i=1}^{\gamma+1} (\bar{a}_i)^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right. \\ &\quad \left. - \sum_{2 \leq \alpha_1 \neq \beta_1 \leq n_1} a_{\alpha_1} a_{\beta_1} - \sum_{\alpha_2 \neq \beta_2} a_{\alpha_2} a_{\beta_2} - \dots - \sum_{\alpha_k \neq \beta_k} a_{\alpha_k} a_{\beta_k} \right), \\ \alpha_2, \beta_2 \in \Delta_2, \dots, \alpha_k, \beta_k \in \Delta_k \end{aligned} \tag{3.17}$$

where we put

$$\begin{aligned} \bar{a}_1 &= a_1, \bar{a}_2 = a_2 + \dots + a_{n_1}, \\ \bar{a}_3 &= a_{n_1+1} + \dots + a_{n_1+n_2}, \dots, \bar{a}_{k+1} = a_{n_1+\dots+n_{k-1}+1} + \dots + a_{n_1+\dots+n_k}, \\ \bar{a}_{k+2} &= a_{n_1+\dots+n_k+1}, \dots, \bar{a}_{\gamma+1} = a_n. \end{aligned}$$

Applying Lemma 2.1 to (3.17), we can obtain the following inequality

$$\begin{aligned} &\sum_{\alpha_1 < \beta_1} a_{\alpha_1} a_{\beta_1} + \sum_{\alpha_2 < \beta_2} a_{\alpha_2} a_{\beta_2} + \dots + \sum_{\alpha_k < \beta_k} a_{\alpha_k} a_{\beta_k} \\ &\geq \frac{\eta}{2} + \sum_{i < j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2, \\ &\alpha_j, \beta_j \in \Delta_j, \quad j = 1, \dots, k. \end{aligned} \tag{3.18}$$

Furthermore, from (2.2) and Gauss' equation we see that

$$\begin{aligned} \tau(L_j) &= \frac{n_j(n_j - 1)}{2} f_1 + 3f_2 \sum_{\alpha_j < \beta_j} g^2(e_{\alpha_j}, P e_{\beta_j}) \\ &\quad + \sum_{r=n+1}^{2m} \sum_{\alpha_j < \beta_j} (h_{\alpha_j \alpha_j}^r h_{\beta_j \beta_j}^r - (h_{\alpha_j \beta_j}^r)^2), \quad \alpha_j, \beta_j \in \Delta_j, \quad j = 1, \dots, k. \end{aligned} \quad (3.19)$$

Thus, combining (3.18) and (3.19) we get

$$\begin{aligned} \tau(L_1) + \dots + \tau(L_k) &\geq \frac{\eta}{2} + \sum_{j=1}^k \left(\frac{n_j(n_j - 1)}{2} f_1 + 3f_2 \Psi(L_j) \right) \\ &\quad + \frac{1}{2} \sum_{r=n+1}^{2m} \sum_{(\alpha, \beta) \notin \Delta^2} (h_{\alpha \beta}^r)^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{j=1}^k \left(\sum_{\alpha_j \in \Delta_j} h_{\alpha_j \alpha_j}^r \right)^2 \\ &\geq \frac{\eta}{2} + \sum_{j=1}^k \left(\frac{n_j(n_j - 1)}{2} f_1 + 3f_2 \Psi(L_j) \right), \end{aligned} \quad (3.20)$$

where $\Delta = \Delta_1 \cup \dots \cup \Delta_k$, $\Delta^2 = (\Delta_1 \times \Delta_1) \cup \dots \cup (\Delta_k \times \Delta_k)$. Consequently, from (2.3), (3.14) and (3.20) we can obtain (3.11). If the equality in (3.11) holds at a point p , then the inequalities in (3.18) and (3.20) are actually equalities at p . In this case, by applying Lemma 2.1, (3.17), (3.18), (3.19) and (3.20), we also obtain (3.12) and (3.13). The converse can be verified by a straightforward computation. \square

Theorem 3.5 — *Let M be an n -dimensional submanifold of an m -dimensional generalized complex space form $\tilde{M}(f_1, f_2)$ satisfying $f_2 > 0$. Then we have*

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k) \|H\|^2 + b(n_1, \dots, n_k) f_1 + \frac{3}{2} f_2 \|P\|^2 \quad (3.21)$$

for any k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$. Moreover, the equality case of inequality (3.21) holds at a point $p \in M$ if and only if there exists an orthonormal basis e_1, \dots, e_{2m} at p such that the shape operators of M in $\tilde{M}(f_1, f_2)$ at p take the forms (3.12).

PROOF : By using (3.20) and $f_2 > 0$, one gets (3.21). \square

Let us point out that, according to the result of [17] which we recalled in Section 2, if f_2 is not identically zero, Theorems 3.2 and 3.3 reduce just to Theorems 3 and 4 of [8], and Theorem 3.4 is just Theorem 8.1 of [10]. Nevertheless, all these results would be useful if $f_2 = 0$, and the last one is also true in dimension 4.

4. δ -INVARIANTS OF SUBMANIFOLDS OF GENERALIZED SASAKIAN SPACE FORMS

Now, let M be an $(n+1)$ -dimensional submanifold isometrically immersed in a $(2m+1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$, such that M is tangent to the structure vector field ξ of \tilde{M} . Then, the Riemannian curvature tensor \tilde{R} on $\tilde{M}(f_1, f_2, f_3)$ is given by

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = & f_1\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ & + f_2\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) \\ & + 2g(X, \phi Y)g(\phi Z, W)\} \\ & + f_3\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ & + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z)\}, \end{aligned} \tag{4.1}$$

and the scalar curvature τ of M at p can be obtained by

$$2\tau(p) = (n + 1)^2\|H\|^2 - \|h\|^2 + (n + 1)nf_1 + 3f_2\|T\|^2 - 2nf_3, \tag{4.2}$$

where $\|H\|^2$ and $\|h\|^2$ are the squared mean curvature and the squared norm of the second fundamental form.

To state a result similar to Theorem 3.1, given a point $p \in M$ and a plane section $\pi \subset T_pM$, we need to recall from [14] the function $\Phi(\pi) = (\eta(X))^2 + (\eta(Y))^2$, where X, Y are any orthonormal vectors spanning π . Then, we obtain:

Theorem 4.1 — *Let M be an $(n + 1)$ -dimensional submanifold of a $(2m + 1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ such that M is tangent to the structure vector field ξ of \tilde{M} . Then, for any point $p \in M$ and any plane section $\pi \subset T_pM$, we have*

$$\begin{aligned} \tau - K(\pi) \leq & \frac{n - 1}{2} \left(\frac{(n + 1)^2}{n} \|H\|^2 + (n + 2)f_1 \right) \\ & + 3 \left(\frac{\|T\|^2}{2} - \Theta(\pi) \right) f_2 - (n - \Phi(\pi))f_3. \end{aligned} \tag{4.3}$$

The equality holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ for T_pM and an orthonormal basis $\{e_{n+2}, \dots, e_{2m+1}\}$ for $T_p^\perp M$ such that (a) $\pi = \text{Span}\{e_1, e_2\}$ (b) the shape operators $A_r = A_{e_r}, r = n + 2, \dots, 2m + 1$, take the following forms:

$$A_{n+2} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c \end{pmatrix}, \tag{4.4}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 & \dots & 0 \\ d_r & -c_r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad r = n + 3, \dots, 2m + 1 \tag{4.5}$$

where $a + b = c$ and $c_r, d_r \in \mathbb{R}$.

PROOF : If we put

$$\rho = 2\tau - \frac{(n + 1)^2(n - 1)}{n} \|H\|^2 - (n + 1)nf_1 - 3\|T\|^2f_2 + 2nf_3 \tag{4.6}$$

and we substitute (4.2) into (4.6), we have

$$(n + 1)^2 \|H\|^2 = n(\|h\|^2 + \rho). \tag{4.7}$$

Let $\pi \subset T_pM$ be a plane section. We choose an orthonormal frame $\{e_1, \dots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \dots, e_{2m+1}\}$ of $T_p^\perp M$ such that π is spanned by e_1, e_2 and e_{n+2} is in the direction of the mean curvature vector H . Hence, (4.7) gives

$$\left(\sum_{i=1}^{n+1} h_{ii}^{n+2} \right)^2 = n \left\{ \sum_{i=1}^{n+1} (h_{ii}^{n+2})^2 + \sum_{i \neq j} (h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j} (h_{ij}^r)^2 + \rho \right\},$$

and so, by applying Lemma 2.1, we obtain:

$$2h_{11}^{n+2}h_{22}^{n+2} \geq \sum_{i \neq j} (h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j} (h_{ij}^r)^2 + \rho. \tag{4.8}$$

On the other hand, from (4.1) and the Gauss equation we find:

$$\begin{aligned} K(\pi) &= f_1 + 3\Theta(\pi)f_2 - \Phi(\pi)f_3 \\ &\quad + h_{11}^{n+2}h_{22}^{n+2} - (h_{12}^{n+2})^2 + \sum_{r=n+3}^{2m+1} (h_{11}^r h_{22}^r - (h_{12}^r)^2). \end{aligned} \tag{4.9}$$

Then, from (4.8) and (4.9) we get:

$$\begin{aligned} K(\pi) &\geq f_1 + 3\Theta(\pi)f_2 - \Phi(\pi)f_3 + \frac{\rho}{2} + \sum_{r=n+2}^{2m+1} \sum_{j>2} \{(h_{1j}^r)^2 + (h_{2j}^r)^2\} \\ &\quad + \frac{1}{2} \sum_{i \neq j > 2} (h_{ij}^{n+2})^2 + \frac{1}{2} \sum_{r=n+3}^{2m+1} \sum_{i,j > 2} (h_{ij}^r)^2 + \frac{1}{2} \sum_{r=n+3}^{2m+1} (h_{11}^r + h_{22}^r)^2. \end{aligned} \tag{4.10}$$

Finally, combining (4.6) and (4.10), we obtain (4.3).

If the equality in (4.3) holds, then the inequalities in (4.8) and (4.10) become equalities. Thus, we have:

$$\begin{aligned} h_{1j}^{n+2} &= h_{2j}^{n+2} = h_{ij}^{n+2} = 0, \quad i \neq j > 2; \\ h_{1j}^r &= h_{2j}^r = h_{ij}^r = 0, \quad r = n + 3, \dots, m; \quad i, j = 3, \dots, n + 1; \\ h_{11}^{n+3} + h_{22}^{n+3} &= \dots = h_{11}^{2m+1} + h_{22}^{2m+1} = 0. \end{aligned}$$

Furthermore, we may choose e_1, e_2 such that $h_{12}^{n+2} = 0$. Moreover, by applying Lemma 2.1, we also have:

$$h_{11}^{n+2} + h_{22}^{n+2} = h_{33}^{n+2} = \dots = h_{n+1, n+1}^{n+2}.$$

Therefore, with respect to the chosen orthonormal basis $\{e_1, \dots, e_{2m+1}\}$, the shape operators of M take the forms (4.4) and (4.5).

The converse follows from a direct calculation. □

As in the previous section, we can obtain some Chen inequalities for any kind of submanifolds as an application of this result. Now, they depend on the signs of both f_2 and f_3 and they appear in the following corollary whose proof follows directly from Theorem 4.1.

Corollary 4.1 — Let M be an $(n + 1)$ -dimensional submanifold of a $(2m + 1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ such that M is tangent to the structure vector field ξ of \tilde{M} . Then, the following inequalities are satisfied:

i) If $f_2 \leq 0$ and $f_3 \leq 0$,

$$\delta_M \leq \frac{(n + 1)^2(n - 1)}{2n} |H|^2 + \frac{(n - 1)(n + 2)}{2} f_1 - n f_3.$$

ii) If $f_2 \leq 0$ and $f_3 > 0$,

$$\delta_M \leq \frac{(n + 1)^2(n - 1)}{2n} |H|^2 + \frac{(n - 1)(n + 2)}{2} f_1 - (n - 1) f_3.$$

iii) If $f_2 > 0$ and $f_3 \leq 0$,

$$\delta_M \leq \frac{(n + 1)^2(n - 1)}{2n} |H|^2 + \frac{(n - 1)(n + 2)}{2} f_1 + 3 \frac{\|T\|^2}{2} f_2 - n f_3.$$

iv) If $f_2 > 0$ and $f_3 > 0$,

$$\delta_M \leq \frac{(n + 1)^2(n - 1)}{2n} |H|^2 + \frac{(n - 1)(n + 2)}{2} f_1 + 3 \frac{\|T\|^2}{2} f_2 - (n - 1) f_3.$$

Moreover, we can improve Theorem 4.1 by working with generalized Sasakian space forms endowed with an (α, β) trans-Sasakian structure, i.e., such that

$$(\tilde{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{4.11}$$

for any vector fields X, Y, α, β being two differentiable functions on the ambient manifold. In such a case, it is easy to see from (4.11) that

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi)$$

and hence

$$h(X, \xi) = -\alpha NX \tag{4.12}$$

for any tangent vector field X . Actually, some important examples of trans-Sasakian generalized Sasakian space forms were obtained in [1]. Then, we can prove the following theorem for plane sections orthogonal to the structure vector field:

Theorem 4.2 — *Let M be an $(n + 1)$ -dimensional submanifold of a $(2m + 1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ such that M is tangent to the structure vector field ξ of \tilde{M} . If \tilde{M} has an (α, β) trans-Sasakian structure, then for any point $p \in M$ and any plane section $\pi \subset T_p M$, orthogonal to ξ_p , we have*

$$\begin{aligned} \tau - K(\pi) &\leq \frac{n-1}{2} \left(\frac{(n+1)^2}{n} \|H\|^2 + (n+2)f_1 \right) \\ &+ 3 \left(\frac{\|T\|^2}{2} - \Theta(\pi) \right) f_2 - nf_3 - \alpha^2 \|N\|^2. \end{aligned} \tag{4.13}$$

The equality holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ for $T_p M$ and an orthonormal basis $\{e_{n+2}, \dots, e_{2m+1}\}$ for $T_p^\perp M$ such that (a) $e_{n+1} = \xi_p$ (b) $\pi = \text{Span}\{e_1, e_2\}$ (c) the shape operators $A_r = A_{e_r}, r = n+2, \dots, 2m+1$, take the following forms:

$$A_{n+2} = \begin{pmatrix} a & 0 & 0 & \mu_1^{n+2} \\ 0 & -a & 0 & \vdots \\ 0 & 0 & 0_{n-2} & \mu_n^{n+2} \\ \mu_1^{n+2} & \dots & \mu_n^{n+2} & 0 \end{pmatrix}, \tag{4.14}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 & \mu_1^r \\ d_r & -c_r & 0 & \vdots \\ 0 & 0 & 0_{n-2} & \mu_n^r \\ \mu_1^r & \dots & \mu_n^r & 0 \end{pmatrix}, \quad r = n+3, \dots, 2m+1 \tag{4.15}$$

where $c_r, d_r \in \mathbb{R}$.

PROOF : Let us consider $\{e_1, \dots, e_{n+1}\}$ an orthonormal basis for $T_p M$ and $\{e_{n+2}, \dots, e_{2m+1}\}$ an orthonormal basis for $T_p^\perp M$ such that $e_{n+1} = \xi_p, \pi = \text{Span}\{e_1, e_2\}$ and e_{n+2} has the direction of the mean curvature vector H . Then, by following the same steps as in the proof of Theorem 4.1, we obtain the inequality

$$\begin{aligned} \tau - K(\pi) &\leq \frac{n-1}{2} \left(\frac{(n+1)^2}{n} \|H\|^2 + (n+2)f_1 \right) + 3 \left(\frac{\|T\|^2}{2} - \Theta(\pi) \right) f_2 - nf_3 \\ &\quad - \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n+1} (h_{i,n+1}^r)^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} (h_{n+1,n+1}^r)^2, \end{aligned} \tag{4.16}$$

in which we have preserved the terms related to the structure vector field. But, as \tilde{M} is an (α, β) trans-Sasakian manifold, it follows from (2.1) and (4.12) that

$$\sum_{r=n+2}^{2m+1} \sum_{i=1}^{n+1} (h_{i,n+1}^r)^2 = \alpha^2 \|N\|^2, \quad \sum_{r=n+2}^{2m+1} (h_{n+1,n+1}^r)^2 = 0.$$

Therefore, we obtain (4.13). The study of the equality case can be done in a similar way of that of Theorem 4.1, by taking now into account that

$$h_{11}^{n+2} + h_{22}^{n+2} = h_{33}^{n+2} = \dots = h_{n+1,n+1}^{n+2} = 0$$

by virtue of (4.12). □

From the above theorem, we can also state some general inequalities, but now we have to consider the invariant $\delta_M^{\mathcal{D}}$ defined by the second named author in [5] by

$$\delta_M^{\mathcal{D}}(p) = \tau(p) - \inf_{\mathcal{D}} K(p),$$

for any $p \in M$, where

$$(\inf_{\mathcal{D}} K)(p) = \inf \{ K(\pi) : \text{plane sections } \pi \text{ orthogonal to } \xi_p \}.$$

It is obvious that $\delta_M^{\mathcal{D}} \leq \delta_M$.

In fact, we obtain the following corollary whose proof follows directly from Theorem 4.2:

Corollary 4.2 — Let M be an $(n+1)$ -dimensional submanifold of a $(2m+1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ such that M is tangent to the structure vector field ξ of \tilde{M} . If \tilde{M} has an (α, β) trans-Sasakian structure, then the following inequalities are satisfied:

i) If $f_2 \leq 0$,

$$\delta_M^{\mathcal{D}} \leq \frac{(n+1)^2(n-1)}{2n} |H|^2 + \frac{(n-1)(n+2)}{2} f_1 - nf_3 - \alpha^2 \|N\|^2.$$

ii) If $f_2 > 0$,

$$\delta_M^{\mathcal{D}} \leq \frac{(n+1)^2(n-1)}{2n} |H|^2 + \frac{(n-1)(n+2)}{2} f_1 + 3 \frac{\|T\|^2}{2} f_2 - nf_3 - \alpha^2 \|N\|^2.$$

On the other hand, concerning the strings of invariants $\delta(n_1, \dots, n_k)$, we can obtain for a generalized Sasakian space form some results similar to those of the previous section for a complex space form. To do so, we just have to introduce the function

$$\Upsilon(L) = \sum_{1 \leq i < j \leq r} ((\eta(e_i))^2 + (\eta(e_j))^2),$$

L being a subspace of $T_p M$ of dimension $r \geq 2$ spanned by an orthonormal basis $\{e_1, \dots, e_r\}$. Then, by following similar steps to those in the proof of (3.20), we can state the following inequality:

Theorem 4.3 — *Let M be an $(n + 1)$ -dimensional submanifold of a $(2m + 1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ such that M is tangent to the structure vector field ξ of \tilde{M} . Given $(n_1, \dots, n_k) \in \mathcal{S}(n)$, for $p \in M$ let L_j be an n_j -plane section of $T_p M$, $j = 1, \dots, k$. Then, we have:*

$$\begin{aligned} \tau - \sum_{j=1}^k \tau(L_j) &\leq c(n_1, \dots, n_k) \|H\|^2 + b(n_1, \dots, n_k) f_1 \\ &+ 3 \left(\frac{\|T\|^2}{2} - \sum_{j=1}^k \Psi(L_j) \right) f_2 - (n - \sum_{j=1}^k \Upsilon(L_j)) f_3. \end{aligned} \quad (4.17)$$

From Theorem 4.3 we directly obtain:

Corollary 4.3 — *Let M be an $(n + 1)$ -dimensional submanifold of a $(2m + 1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ such that M is tangent to the structure vector field ξ of \tilde{M} . Then, for any k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the following inequalities are satisfied:*

i) If $f_2 \leq 0$ and $f_3 \leq 0$,

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k) \|H\|^2 + b(n_1, \dots, n_k) f_1 - n f_3.$$

ii) If $f_2 \leq 0$ and $f_3 > 0$,

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k) \|H\|^2 + b(n_1, \dots, n_k) f_1 - (n - k) f_3.$$

iii) If $f_2 > 0$ and $f_3 \leq 0$,

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k) \|H\|^2 + b(n_1, \dots, n_k) f_1 + 3 \frac{\|T\|^2}{2} f_2 - n f_3.$$

iv) If $f_2 > 0$ and $f_3 > 0$,

$$\delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k) \|H\|^2 + b(n_1, \dots, n_k) f_1 + 3 \frac{\|T\|^2}{2} f_2 - (n - k) f_3.$$

Finally, let us give some information about Chen inequalities for an n -dimensional submanifold M which is normal to the structure vector field ξ of a $(2m + 1)$ -dimensional generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$. In such a case, the Gauss' equation is given from (4.1) by

$$\begin{aligned} R(X, Y, Z, W) = & f_1\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ & + f_2\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) + 2g(X, \phi Y)g(\phi Z, W)\} \\ & + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \end{aligned} \quad (4.18)$$

and we can calculate the scalar curvature τ of M at a point p as

$$2\tau(p) = n^2\|H\|^2 - \|h\|^2 + n(n-1)f_1 + 3f_2\|T\|^2, \quad (4.19)$$

which look like equations (3.1) and (3.2), respectively. Therefore, working from (4.18) and (4.19) with the same techniques, we would obtain the similar results corresponding to Theorems 3.1–3.5.

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