# DECISION ANALYSIS: VECTOR OPTIMIZATION THEORY

**(Vector optimizacion theory/general/red convexity)**

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## 1. INTRODUCTION

Decision making is an integral part of our daily lives. It ranges in scope from the individual to the largest groups and societies, including nations and, ultimately, organization at the global level. It considers situations ranging in complexity from the simple to the most complex involving multiple objetives.

Because of the diversity of situations in which multi-objective decision problems can arise and because of the multiplicity of factors that are involved, the literature on the subject produced since the early 1960s is large, as well as diverse in emphasis and style of treatment, and the general indication is that this trend will continue. Theoretical and methodological developments have been based on a number of different viewpoints, reflecting the breadth of disciplines involved.

The term *multiobjective decision-making process* refers to the entire process of problem solving, consisting essentially of the five steps.

1.  Initiation Step: Recognition of the need for change and diagnosis of the system.

2.  Problem-Formulation Step: Problem definition with specification of objectives and identification of attributes or objectives measures.

3.  System-Modeling Step: Construction of model together with paramenter estimation.

4.  Analysis-Evaluation Step: Generation of the set of alternatives and estimation of the values of attributes. Decision.

5.  Implementation Step: Implementation and reevaluation.

An important class of multiobjective decision problems is the vector optimization problems VOP. From a methodological viewpoint, these are mathematical programming problems with a vector-valued objective function. From the decision-making viewpoint, this class of multiobjective decision problems arises when the decision rule implies that each attribute (or objective function) is to be kept as extreme, *i.e.* as high or as low as possible.

A Vector Optimization Problem VOP is formulated as

$$\text{Min} \quad [f_1(x), f_2(x), ..., f_p(x)]$$
$$\text{subject to} \quad x \in X,$$

where $X \subseteq \mathbb{R}^n$ is nonempty, and $f_j$ denotes a real-valued function defined on $X$ for $j = 1, 2, ..., p$.

Needless to say, a maximization nonlinear programming problem can be trated as a minimization problem, because maximize $f(x)$ is equal to minimize $-f(x)$.

It is known that for this problem, the concept of optimal solution found in single-objective optimization problem is not valid. The concept of an ideal point —one that minimizes each objective— is in general not feasible. On the other hand, multiple objectives are usually non-commensurable and cannot be combined into a single objective. Moreover, the objectives often conflict with each other. Consequently, the concept of optimality for single-objective optimization problems cannot be directly applied to VOP. The concept of Pareto optimality, characterizing an efficient solution, has been introduced for VOP. The solutions of a VOP are referred to in the literature in general as noninferior, efficient, Pareto-optimal, or nondominated solutions. Other variants include weakly efficient solutions, local efficient solutions, etc.

The concept of solution for a VOP was introduced at the turn of the century (1896) by Pareto, a prominent

economist, but it is only since 1951, when Kuhn and Tucker published necessary and sufficient conditions for (proper) noninferiority, that considerable effort has been devoted to developing procedures for generating noninferior solutions to a VOP.

The above problem VOP generalizes the following classic Scalar Optimization Problem that is usually formulated as follows

$$\text{SOP} \quad \text{Min} \qquad \theta(x)$$
$$\text{subject to:} \quad x \in X \subseteq \mathbb{R}^n,$$

where $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$.

The study of the solutions of a multiobjective programming problem may be approached from two aspects; one, trying to relate them with the solutions to the scalar problems, whose resolutions has been studied extensively and another, trying to locate conditions which are easier to deal with computationally and which guarantes efficiency. As much in one case as in the other, convexity concept plays an important role, as a fundamental condition in order to obtain the desired results.

In the past few years, extensive literature the other families of more general functions to substitute the convex functions in the mathematical programming has grown immensely. Such functions are called generalized convex functions. Within these and because of their importance, we point out the invex and preinvex functions, defined be Hanson [2], Craven [3] and Elster and Nehse [4] and studied extensively by other authors [18], [19].

In this paper we utilize the properties of generalized convexity functions to obtain results which will allow us to characterize the solutions of the multiobjective programming problems.

This paper consists of seven parts. In Section 1 and 2 we define a general vector optimization problem and introduce some basic definitions and properties for these problems. Section 3 and 6 discuss relations between vector optimization problem and some scalar problems associated. Section 4 presents a collection of the most important definitions of generalized convexity appeared in the last few years. Section 5 study local efficiency under generalized convexity assumption. Finally, Section 7 we examine necessary and sufficient conditions of optimality for vector optimization problems, with and without differentiability and analyze the weakest convexity assumptions necessary for stablish these conditions.

## 2.  DEFINITIONS AND PROPERTIES

The following convention for equalities and inequalities will be used. If $x, y \in \mathbb{R}^n$, then

- $x = y$ iff $x_i = y_i, \quad i = 1, ..., p$;

- $x \mathrm{D} \leqq y$ iff $x_i \leqslant y_i, \quad i = 1, ..., p$;

- $x \leqslant y$ iff $x_i \leqslant y_i, \quad i = 1, ..., p$ with strict inequality holding for at least one $i$;

- $x < y$ iff $x_i < y_i, \quad i = 1, ..., p$;

Let $X$ be a monempty subset of $\mathbb{R}^n$.

**Definition 2.1.**   *A point, $\bar{x} \in X$, is said to be an **Efficient Solution** of VOP if there is no $x \in X$ such that $f(x) \leqslant f(\bar{x})$.*

The set of efficient solutions will be denoted by $\mathcal{E}(X)$.

**Definition 2.2.**   *A point, $\bar{x} \in X$, is said to be a **Weakly Efficient Solution** of VOP if there is no $x \in X$ such that $f(x) < f(\bar{x})$.*

*The set of weakly efficient solutions will be denoted by $\mathcal{WE}(X)$.*

**Definition 2.3.**   *A point, $\bar{x} \in X$, is said to be a **Properly Efficient Solution** of VOP if there exists a scalar $M > 0$ such that, for each $i$, we have*

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqslant \mathrm{M},$$

*for some $j$ such that $f_j(x) > f_j(\bar{x})$ whenever $x \in X$ and $f_i(x) < f_i(\bar{x})$.*

*The set of properly efficient solutions will be denoted by $\mathcal{PE}(X)$.*

It is easy to verify that

$$\mathcal{PE}(X) \subseteq \mathcal{E}(X) \subseteq \mathcal{WE}(X).$$

**Definition 2.4.**   *A point, $\bar{x} \in X$ is said to be a **Local Efficient Solution** of VOP if there exists $\delta > 0$, such that $\bar{x}$ is an efficient solution of VOP in $X \cap B(\bar{x}, \delta)$, where $B(\bar{x}, \delta)$ is a $\delta$-neighborhood around $\bar{x}$.*

*The set of local efficient solutions will be denoted by $\mathcal{LE}(X)$.*

Again, it is easy to see that

$$\mathcal{LE}(X) \subseteq \mathcal{E}(X).$$

Locally properly efficient and locally weakly efficient solutions can be defined in a similar way.

## 3. COMMON APROACHES TO CHARACTERIZING EFFICIENT SOLUTIONS

In order to operate the notion of efficient solutions, we relate it to a familiar concept. The most common strategy is to characterize efficient solutions *in terms of optimal solutions of appropiate scalar optimization problems.* Among the many possible ways of obtaining a scalar problem from VOP, the following are the most commonly used.

(i) *The weighting problem.* Let $W = \{w/w \in \mathbb{R}^n$, $w_j \geqslant 0$, and $\sum_{j=1}^{n} w_j = 1\}$ be a set of nonnegative weights. The weighting problem is defined for some $w \in W$ as

$$(P(w)) \quad \text{Minimize} \quad w^T f(x)$$
$$\text{subject to} \quad x \in X \subseteq \mathbb{R}^n.$$

(ii) *The kth-objective ε-constraint problem*

$$(P_k(\varepsilon)) \quad \text{Minimize} \quad f_k(x)$$
$$\text{subject to} \quad f_j(x) \leqslant \varepsilon_j, \ j = 1, ..., p, \ j \neq k,$$
$$x \in X \subseteq \mathbb{R}^n,$$

where $\varepsilon = (\varepsilon_1, ..., \varepsilon_{k-1}, \varepsilon_{k+1}, ..., \varepsilon_p)^T \in \mathbb{R}^{p-1}$

For a given point $\bar{x}$, we shall use the symbol $P_k(\bar{x})$ to represent the problem $P_k(\varepsilon)$, where $\varepsilon_j = f_j(\bar{x})$, $j \neq k$.

Now, we describe fundamental resuls concerning the characterization of an efficient solution of VOP in terms of solutions of the SOPs above, [8].

• *Result 1*, $\bar{x}$ is an efficient solution of VOP iff $\bar{x}$ solves $P_k(\bar{x})$ for every $k = 1, 2, ..., p$.

• *Result 2.* If $\bar{x}$ solves $P_k(\bar{x})$ for some $k$ and if the solution is unique, then $\bar{x}$ is an efficient solution of VOP.

• *Result 3.* $\bar{x}$ is an efficient solution of VOP if there exists $w \in W$ such that $\bar{x}$ solves $P(w)$ and if either one of the following two conditions holds:

(i) $w_j > 0$ for all $j = 1, ..., p$, or

(ii) $\bar{x}$ is the unique solution of $P(w)$.

• *Result 4.* Let $w \in W$ with $w > 0$ be fixed. If $\bar{x}$ solves $P(w)$, then $\bar{x}$ is properly efficient for VOP.

• *Result 5.* Let $w \in W$ be fixed. If $\bar{x}$ solves $P(w)$, then $\bar{x}$ is weakly efficient for VOP.

• *Result 6.* If $\bar{x}$ solves $P_k(\bar{x})$ for some $k$, then $\bar{x}$ is weakly efficient for VOP.

In order to give necessary conditions for Results 4, 5 and 6, we will introduce some useful concepts.

## 4. GENERALIZED CONVEXITY

In this section, we present various types of functions that generalize to convex and concave functions, but that share only some of their desirable properties. As we will see, many of the results presented later in this paper do not require the restrictive assumption of convexity but rather the less restrictive assumptions.

**Definition 4.1.** *Let $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be. Then $\theta$ is a*

• *quasiconvex funtion in the convex set $X$ if for each $x_1, x_2 \in X$, the following inequality is true,*

$$\theta(\lambda x_1 + (1 - \lambda)x_2) \leqslant max\{\theta(x_1), \theta(x_2)\} \quad \forall \lambda \in [0, 1].$$

• *strictly quasiconvex function in the convex set $X$ if for each $x_1, x_2 \in X$ with $\theta(x_1) \neq \theta(x_2)$, we have*

$$\theta(\lambda x_1 + (1 - \lambda)x_2) \leqslant max\{\theta(x_1), \theta(x_2)\} \quad \forall \lambda \in (0, 1).$$

*Every strictly quasiconvex function is not quasiconvex. To illustrate this assertion, let us consider the following function given by Karamardian.*

$$\theta(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x \neq 0 \end{cases}$$

*By definition, f is strictly quasiconvex, however, f is not quasiconvex, since for all $\hat{x} \neq 0$, we have that $0 = \frac{1}{2}\hat{x} + \frac{1}{2}(-\hat{x})$, $\theta(\hat{x}) = \theta(-\hat{x}) = 0$ and $\theta(0) > \theta(\hat{x})$. So, we shall define another version of quasiconvexity that contains to quasiconvex and strictly quasiconvex funtions.*

• *explicity quasiconvex function in the convex set $X$ if $\theta$ is a quasiconvex and strictly quasiconvex function*

*The class of explicity quasiconvex fuctions contains strictly the class of convex functions. Because, a function can be explicity quasiconvex and strictly concave. The function $\theta : \mathbb{R}^+ \to \mathbb{R}$, $\theta(x) = +\sqrt{x}$ illustrates the above assertion.*

• *preinvex function on $X$ if $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0, 1]$ there exists a vector $\eta(x_1, x_2) \in \mathbb{R}^n$ such that*

$$\theta(x_2 + \lambda\eta(x_1, x_2)) \leqslant \lambda\theta(x_1) + (1 - \lambda)\theta(x_2).$$

*Zang, Choo and Avriel [17] studied fuctions whose local minima are global minima. They noted that the*

*vector $x_1 - x_2$ does not play an essential role in the caracterization of global minima in the case of convex and quasiconvex functions. Therefore, it can be replaced by a general vector $\eta(x_1, x_2)$. In fact, a function strictly concave can be preinvex, as it can seen in the following example, where $\theta : \mathbb{R} \to \mathbb{R}$, $\theta(x) = -|x|$*

$$\eta(x_1, x_2) = \begin{cases} x_1 - x_2 & if \quad x_2 \leqslant 0 \ \ and \ \ x_1 \leqslant 0 \\ x_1 - x_2 & if \quad x_2 \geqslant 0 \ \ and \ \ x_1 \geqslant 0 \\ x_2 - x_1 & if \quad x_2 > 0 \ \ and \ \ x_1 < 0 \\ x_2 - x_1 & if \quad x_2 < 0 \ \ and \ \ x_1 > 0 \end{cases}$$

It is clear that

convexity $\Rightarrow$ explicity quasiconvexity $\Rightarrow$ quasiconvexity

and that taking $\eta(x_1, x_2) = x_1 - x_2$, we also have that

convexity $\Rightarrow$ pre-invexity

However, it is not possible to stablish a general relation between quasiconvex and preinvex functions, because there exist quasiconvex functions which are not preinvex functions, and there exist preinvex functions which are not are quasiconvex functions.

In the above definitions we do not suppose any differentiability assumptions on the functions. Many programming problems involve differentiable functions. It is important then to take advantage of this property. So, we present some definitions where we suppose that functions are differentiable.

**Definition 4.2.** *Let $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on the open set X. Then $\theta$ is a*

- *pseudoconvex function in the convex set X if $\forall x_1, x_2 \in X$ have that*

$$\theta(x_1) - \theta(x_2) < 0 \Rightarrow \nabla\theta(x_2)^T (x_1 - x_2) < 0.$$

*For differentiable functions, it is also possible to replace the vector $x_1 - x_2$ by a general vectorial function $\eta(x_1, x_2)$, in the same way that scalar case.*

- *invex function on X if for all $x_1, x_2 \in X$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that*

$$\theta(x_1) - \theta(x_2) \geqslant \eta(x_1, x_2)^T \nabla\theta(x_2).$$

- *pseudoinvex function on X if for all $x_1, x_2 \in X$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that*

$$\eta(x_1, x_2)^T \nabla\theta(x_2) \geqslant 0 \Rightarrow \theta(x_1) - \theta(x_2) \geqslant 0.$$

- *quasiinvex function on X if for all $x_1, x_2 \in X$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that*

$$\theta(x_1) - \theta(x_2) \leqslant 0 \Rightarrow \eta(x_1, x_2)^T \nabla\theta(x_2) \leqslant 0.$$

It is clear that

convexity $\Rightarrow$ pseudoconverxity $\Rightarrow$
    invexity $\Rightarrow$ pseudoinvexity.

For a vectorial function $f(x)$ the same concepts apply in the sense that the above definitions hold for each component of $f$. For preinvex, invex and pseudoinvex functions, it is required that the vector $\eta$ be the same for each component of $f$.

For the scalar functions, the class of invex functions and the class of pseudoinvex functions coincide, as it is shown in [6]. This does not happen in the vectorial case.

## 5. LOCAL EFFICIENCY

We shall study local efficient solutions under generalized convexity assumptions for the objetive function. The following results are generalizations of the ones in the scalar case for the minima points.

**Theorem 5.1.** *Let X be a nonempty convex subset of $\mathbb{R}^n$ and let f be an explicity quasiconvex function defined on X. If $\bar{x}$ is a local efficient solution of VOP, then $\bar{x}$ is a global efficient solution.*

**Proof.** Let $\bar{x}$ be such that $\bar{x} \in \mathcal{LE}(X)$ and $\bar{x} \notin \mathcal{E}(X)$, then there exists $y \in X$ such that $f(y) \leqslant f(\bar{x})$.

Since $\bar{x} \in \mathcal{LE}(X)$, there exits $\delta > 0$ such that $x \in \mathcal{E}(B(\bar{x}, \delta) \cap X)$. Let $z \in (\bar{x}, y) \cap B(\bar{x}, \delta)$, where $(\bar{x}, y)$ denotes the line joining $\bar{x}$, and $y$, defined as

$$(\bar{x}, y) = \{x \in X/x = \lambda\bar{x} + (1 - \lambda)y, \ \lambda \in (0,1)\}.$$

For each index $k$ such that $f_k(y) = f_k(\bar{x})$ we have that

$$f_k(z) \leqslant max\{f_k(y), f_k(\bar{x})\} = f_k(\bar{x}). \tag{1}$$

For each index $i$ such that $f_i(y) < f_i(\bar{x})$ we have that

$$f_i(z) < max\{f_i(\bar{x}), f_i(y)\} = f_i(\bar{x}). \tag{2}$$

Combining (1) and (2), it is obvious that each $\delta > 0$, there exists $z \in X \cap B(\bar{x}, \delta)$ such that $f(z) \leqslant f(\bar{x})$. Therefore $\bar{x} \notin \mathcal{LE}(X)$ and this contradicts the hypothesis.    $\square$

The result is also true for preinvex functions as shown the following theorem.

**Theorem 5.2.** *Let f be pre-invex on X. If $\bar{x}$ is a locally (weakly) efficient solution of VOP, then $\bar{x}$ is an (weakly) efficient solution of VOP.*

**Proof.** Let $\bar{x}$ be a locally efficient solution for VOP, then there exists a $\delta$ such that $\bar{x} \in \mathcal{E}(B(\bar{x}, \delta) \cap X)$. Let us suppose that there exists another $y \in X$ such that

$f(y) \leqslant f(\bar{x})$. As $f$ is a pre-invex function on $X$, there exists a vector $\eta(\bar{x}, y)$, such that

$$f(\bar{x} + \lambda\eta(\bar{x}, y)) \leqq \lambda f(\bar{x}) + (1 - \lambda)f(y), \quad \forall \lambda \in [0,1].$$

Let $z(\lambda) = \bar{x} + \lambda\eta(\bar{x}, y)$. For a suitable $\lambda_0$, $z(\lambda_0) \in B(\bar{x}, \delta) \cap X$, and so $f(z(\lambda_0)) \leqslant f(\bar{x})$. And this is a contradiction.

For weakly efficient solutions the proof is similar. □

For properly efficient solutions we have next result, whose proof can be found in [16].

**Theorem 5.3.**  *Let $f$ be pre-invex on $X$. If $\bar{x}$ is a locally properly efficient solution of VOP then $\bar{x}$ is a properly efficient solution of VOP.*

## 6.  SCALAR PROBLEMS

In this Section, we give results that characterize the VOP solutions by solving some scalar optimization problems. In fact, these results provide necessary conditions for the ones studied in Section 3. To do it, we use the concepts introduced in Section 4.

Beato et al. proved next result involving explicity quasiconvex function [19].

**Theorem 6.1.**  *If $X$ is a convex subset of $\mathbb{R}^n$ and $f$ is an explicity quasiconvex function defined on $X$, then $\bar{x} \in \mathcal{WE}(X)$ if and only if $\bar{x}$ solves $P_k(\bar{x})$ for some $k$.*

The result in Theorem 6.1 is stronger than those considered in Section 3, because the only required condition is that $\bar{x}$ solves $P_k(\bar{x})$ for some $k$ and no uniqueness of solutions is required.

Next theorem characterize properly and weakly efficient solutions for VOP using weighting scalar problems [16], [20].

**Theorem 6.2.**  *Let $f$ be pre-invex on $X$. Then $\bar{x}$ is properly (weakly) efficient in VOP if and only if $\bar{x}$ solves $P(w)$, with $w \in W$ and $w > 0$ ($w \geqslant 0$).*

## 7.  OPTIMALITY CONDITIONS

In the scalar optimization problem whose objetive function in convex and differentiable, it is well known that stationary points coincide with optimal points, as it is stated in the following result.

**Theorem 7.1.**  *Let $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a diferentiable and convex function on the convex and open set $X$, then*

$$\bar{x} \text{ is a global minimun of } \theta \text{ in } X \Leftrightarrow \nabla\theta(\bar{x}) = 0$$

Some authors have shown that the hypothesis of convexity is rather restrictive, giving results for more general functions.

In fact, Hanson [2] proved that the theorem holds for invex functions. Craven and Glober [21] also proved that this characterizes invex functions.

**Theorem 7.2.**  *A function, $\theta$, is invex in $X$ if and only if every critical point of $\theta$ is a global minimizer of $\theta$ in $X$.*

Problems so far considered have no restrictions. Now, we study the case where the problem includes some constraints. We first define them.

The Constrained Scalar Optimization Problem is formulated as,

$$
\begin{array}{lll}
\text{(CSOP)} & \text{Min} & \theta(x) \\
& \text{subject to:} & g(x) \leqq 0, \\
& & x \in X \subseteq \mathbb{R}^n,
\end{array}
$$

where $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

For constrained problems the invexity defined by Hanson [2] is a sufficient condition, but not a necessary condition for every Kuhn-Tucker point to be a global minimizer. Martin [14] defined a weaker invexity notion, called Kuhn-Tucker invexity or KT-invexity, whick is both necessary and sufficient to establish the Kuhn-Tuker conditions.

**Definition 7.1.**  *The scalar programming problem CSOP, is said to be KT-invex on the feasible set with respect to $\eta$, if for any $x_1$, $x_2 \in X$ with $g(x_1) \leqq 0$ and $g(x_2) \leqslant \leqq 0$, there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that*

$$
\begin{aligned}
\theta(x_1) - \theta(x_2) &\geqslant \eta(x_1, x_2)^T \nabla\theta(x_2), \\
-\eta(x_1, x_2)^T \nabla g_i(x_2) &\geqslant 0, \quad \forall_i \in I(x_2),
\end{aligned}
$$

*where $I(x_2) = \{i : i = 1, ..., m \text{ such that } g_i(x_2) = 0\}$.*

For scalar constrained problems, Martin [14] gave the following result.

**Theorem 7.3.**  *Every Kuhn-Tuker stationary point of problem CSOP is a global minimizer if and only if CSOP is KT-invex.*

### 7.1.  Unconstrained Multiobjective Programming Problems

In this section, we characterize the solutions for an unconstrained multiobjective programming problem. As in the scalar case, the concept of invexity function plays an important role. Next definition generalizes the concept of invexity for the p-dimensional case.

**Definition 7.2.**   *Let* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *be a differentiable function on the open set X. Then f is a* **vector invex function** *on X, with respect to* $\eta$, *if for all* $x_1, x_2 \in X$, *there exists* $\eta(x_1, x_2) \in \mathbb{R}^n$ *such that,*

$$f(x_1) - f(x_2) \geq \nabla f(x_2)\eta(x_1, x_2),$$

*where* $\nabla f(x_2)$ *is a matrix of dimensions* $p \times n$, *whose rows are the gradient vectors of each component of the function valued at the point* $x_2$.

Since our purpose is to establish conditions for multi-objective problems, similar to those given by Kuhn-Tucker for the scalar problems, we need to define an analogous concept to the stationary point or critical point for the scalar function.

**Definition 7.3.**   *A feasible point,* $\bar{x} \in X$, *is said to be a* **Vector Critical Point** *(VCP) to VOP if there exists a vector* $\lambda \in \mathbb{R}^p$ *with* $\lambda \geqslant 0$ *such that* $\lambda^T \nabla f(\bar{x}) = 0$.

Scalar stationary points are those whose vector gradients are zero. For vector problems, the vector critical points are those such that there exists a non negative linear combination of the gradient vectors of each component of the objective function, valued at that point, equal to zero. In the case $p = 1$ it is easy to see that both concepts coincide.

For properly efficient solutions, Geoffrion [22] proved next result.

**Theorem 7.4.**   *Let* $\bar{x}$ *be a properly efficient solution for VOP. Then there exists* $\bar{\lambda} > 0$ *such that* $\bar{\lambda}^T \nabla f(\bar{x}) = 0$.

Later, Craven [3] established the following theorem for weakly efficient solutions of VOP.

**Theorem 7.5.**   *Let* $\bar{x}$ *be a weakly efficient solution for problem VOP. Then there exists* $\bar{\lambda} \geqslant 0$ *such that* $\bar{\lambda}^T \nabla f(\bar{x}) = 0$.

Then, every weakly efficient solution is a vector critical point, but to establish reciprocal we need some convexity hypotheses.

For properly efficient solutions Weir [23] proved.

**Theorem 7.6.**   *Let* $\bar{x} \in X$ *be a feasible point and suppose that there exists* $\bar{\lambda} > 0$ *such that* $\bar{\lambda}^T \nabla f(\bar{x}) = 0$ *and f is invex function with respect to* $\eta$ *in X, then* $\bar{x}$ *is a properly efficient solution for VOP.*

A similar result for weakly efficient solutions was proved by Osuna, Beato and Rufián [24].

**Theorem 7.7.**   *Let* $\bar{x}$ *be a vector critical point to problem VOP and let f be an invex function at* $\bar{x}$ *with respect to* $\eta$, *then* $\bar{x}$ *is a weakly efficient solution for VOP.*

We have proved that if the vector objetive function is invex, then all vector critical points are weakly efficient solutions. That equivalence is true under weaker conditions. To prove this assertion, we first define the pseudoinvexity concept for vector functions.

**Definition 7.4.**   *Let* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *be a differentiable function on the open set X. Then, f is a* **vector pseudoinvex function** *on X, with respect to* $\eta$, *if for all* $x_1, x_2 \in X$ *there exists* $\eta(x_1, x_2) \in \mathbb{R}^n$, *such that,*

$$f(x_1) - f(x_2) < 0 \Rightarrow \nabla f(x_2)\eta(x_1, x_2) < 0.$$

It is clear if $f$ is invex, $f$ is pseudoinvex too.

Now, we prove that Theorem 7.7 is also true if the objective function is pseudoinvex on X. Moreover, this characterizes the class of vector pseudoinvex functions.

**Theorem 7.8.**   *All vector critical points are weakly efficient solutions if and only if f is a vector pseudoinvex function on X.*

**Proof.**   Let us suppose that all vector critical points are weakly efficient solutions and let $\bar{x} \in \mathcal{WE}(X)$, then the system

$$f_i(x) - f_i(\bar{x}) < 0 \qquad i = 1, ..., p,$$

has no solution in $x \in X$.

On the other hand, if $\bar{x}$ is a VCP then $\exists \lambda$ such that $\lambda^T \nabla f(\bar{x}) = 0$. Applying Gordan's theorem [25] the next system has no solution at $u \in \mathbb{R}^n$

$$\nabla f(\bar{x})^T u < 0 \qquad i = 1, ..., p$$

and the reciprocals are also true. Thus, if there exists $x \in X$ such that $f(x) < f(\bar{x})$, then there exists $\eta(x, \bar{x}) \in \mathbb{R}^n$ such that $\nabla f(\bar{x})\eta(x, \bar{x}) < 0$, and so, $f$ is pseudoinvex on X.

Now, let us assume that $f$ is pseudoinvex on X and suppose that $\bar{x}$ is a vector critical point but that it is not a weakly efficient solution. Then there exists another point $x \in X$ such that $f(x) < f(\bar{x})$, thus $\nabla f(\bar{x})\eta(x, \bar{x}) < 0$.

On the other hand, there exists $\lambda \in \mathbb{R}^p$, $\lambda \geqslant 0$, such that $\lambda^T \nabla f(\bar{x}) = 0$. And this is a contradition to Gordan's alternative theorem.

Last theorem coincides with the one proved by Martin [14] for the scalar case, since in this case the invex and pseudoinvex functions coincide.

## 7.2.   Constrained Multiobjective Programming Problems

Consider the following Constrained Vector Optimization Problem CVOP.

CVOP    Minimize    $f(x) = (f_1(x), ..., f_p(x))$
        subject to: $g(x) \leqslant 0$,
                    $x \in X \subseteq \mathbb{R}^n$,

where $g : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Now, we give results that extend the well known ones due to Kuhn-Tucker and Fritz-John for scalar programming problems, where they established relationships between the solutions of a constrained scalar programming problem and the points which fulfill certain conditions known as the saddle point optimality criteria [26]. So, we first give new definitions of saddle points for the vectorial case.

**Definition 7.5.** $(\bar{x}, \bar{r}, \bar{v}) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$ *is said to be a* **Vector Fritz-John Saddle Point** *for Problem CVOP if*

$$\bar{r}^T f(\bar{x}) + v^T g(\bar{x}) \leqslant \bar{r}^T f(\bar{x}) + \bar{v}^T g(\bar{x}) \leqslant \bar{r}^T f(x) + \bar{v}^T g(\text{x}),$$

$$\forall v \geqq 0, \quad \forall x \in X,$$

$$(\bar{r}, \bar{v}) \geqslant 0.$$

*If* $\bar{r} \neq 0$ *then* $(\bar{x}, \bar{r}, \bar{v})$ *is said to be a* **Vector Kuhn-Tucker Saddle Point** *for Problem CVOP.*

The advantages of these saddle point conditions for CVOP in contrast to those already existing in the literature, [16], [27], [28], [29], [30], [31], is that the multiplier for the restrictions is a vector and not a function or a matrix, but above all that the vector saddle point conditions are scalar conditions, not vector conditions, due to which it is not necessary to solve any vector problem in order to find the vector saddle points, which simplifies the task. Let us note that Definition 7.5 coincides with Fritz-John and Kuhn-Tucker definitions if $f$ is a scalar function.

Next result (Osuna, Rufián and Ruíz, [20]) is similar to the well known one in the scalar case.

**Lemma 7.1.** *If* $(\bar{x}, \bar{r}, \bar{v})$ *is a vector Fritz-John saddle point and there exists a point* $\hat{x} \in X$ *such that* $g(\hat{x}) < 0$ *(Slater's constraint qualification) then* $(\bar{x}, \bar{r}, \bar{v})$ *is a vector Kuhn-Tucker saddle point.*

The following theorem [16], [20] shows that under certain convexity conditions, all weakly and properly efficient points are vector saddle points. This result is similar to the one for the scalar case that relates the optimal solutions and the Kuhn-Tucker or Fritz-John saddle points.

**Theorem 7.9.** *Let* $(f, g)$ *be a pre-invex function on X and let* $\bar{x}$ *be a weakly efficient in CVOP, then there exists* $(\bar{r}, \bar{v}) \geqslant 0$ *such that* $(\bar{x}, \bar{r}, \bar{v})$ *is a vector Fritz-John saddle point for CVOP.*

*Moreover, if* $\bar{x}$ *is properly efficient in CVOP and the Slater's constraint qualification is satisfied, then* $(\bar{x}, \bar{r}, \bar{v})$ *is a vector Kuhn-Tucker saddle point with* $\bar{r} > 0$.

Just as in the scalar case, the sufficient condition of optimality does not require any convexity hypotheses, as stated next theorem [20], [22].

**Theorem 7.10.** *If* $(\bar{x}, \bar{r}, \bar{v})$ *is a vector Kuhn-Tucker saddle point, then* $\bar{x}$ *is weakly efficient for CVOP.*

*Moreover, if* $\bar{r} > 0$ *then* $\bar{x}$ *is properly efficient for CVOP.*

Let us note that, for weakly efficient solution, it is not necessary that each $\bar{r}_i$ be strictly positive, being enough $\bar{r} \neq 0$.

Now, we characterize weakly efficient solutions for the CVOP using concepts similar to Fritz-John and Kuhn-Tucker optimality condition concepts, and assuming that $f$ and $g$ are differentiable functions on the open set $X$.

We denote by $\nabla f(x) \in N^{m \times n}$ whose rows are the gradient vectors of each component of $f$, $\nabla f$, $\nabla f_j(\bar{x})$ $j = 1...p$ and we denote by $\nabla g(x) \in N^{m \times n}$ the gradient matrix of $g$.

**Definition 7.6.** *A feasible point,* $\bar{x} \in X$, *is said to be a* **Vector Fritz-John Point** *(VFJP) to the problem CVOP, if there exists a vector* $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^{p+m}$, *with* $(\bar{\lambda}, \bar{\mu}) \geqslant 0$ *such that*

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0, \qquad (2a)$$

$$\bar{\mu}^t g(\bar{x}) = 0. \qquad (2b)$$

**Definition 7.7.** *A feasible point,* $\bar{x} \in X$ *is said to be a* **Vector Kuhn-Tucker Point** *(VKTP) to the problem CVOP, if there exists a vector* $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^{p+m}$, *with* $(\bar{\lambda}, \bar{\mu}) \geqslant 0$ *and* $\bar{\lambda} \neq 0$ *such that*

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0, \qquad (3a)$$

$$\bar{\mu}^t g(\bar{x}) = 0. \qquad (3b)$$

Observe that in Definition 7.7 it is not necessary $\bar{\lambda}$ to be strictly positive; it is sufficient that $\bar{\lambda} \neq 0$.

The following results, due to Osuna, Beato and Rufián [33], extend the scalar case in a natural way. In fact, the above definitions coincide with the Fritz-John and Kuhn-Tucker conditions when $f$ is a numerical function.

**Theorem 7.11.** *Let* $\bar{x}$ *be a weakly efficient solution for problem CVOP, then there exist* $\bar{\lambda}$ *and* $\bar{\mu}$ *such that* $\bar{x}$ *is a vector Fritz-John point for CVOP.*

If we add a constraint qualification, we can be sure that $\bar{\lambda}$ is not equal to zero.

**Theorem 7.12.** *Let supposse that the Kuhn-Tucker constraint qualification is satisfied at $\bar{x}$ for problem CVOP. Let $\bar{x}$ be a weakly efficient solution for this problem, then there exist $\bar{\lambda}$ and $\bar{\mu}$ such that $\bar{x}$ is a vector Kuhn-Tucker point for problem CVOP.*

*If $\bar{x}$ is a properly efficient solution, then $\bar{\lambda} > 0$.*

To establish reciprocal of the Theorem 7.12, we need a generalized convexity hypothesis. As in the scalar case, we prove that KT-invexity for the optimization problem is sufficient for all vector Kuhn-Tucker points to be weakly efficient solutions.

We first define a KT-invex multiobjective programming problem.

**Definition 7.8.** *The problem CVOP is said to be a* **vector KT-invex problem** *on the feasible set with respect to $\eta$, if for any $x_1$, $x_2 \in X$, with $g(x_1) \leqq 0$ and $g(x_2) \leqq 0$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$, such that*

$$f(x_1) - f(x_2) \geqq \geqslant \nabla f(x_2)\eta(x_1, x_2),$$
$$-\nabla g_i(x_2)\eta(x_1, x_2) \geqslant 0 \quad \forall i \in I(x_2).$$

Now, we give the following theorem for vector KT-invex problems, whose proof can be found in [33].

**Theorem 7.13.** *Every vector Kuhn-Tucker point is a weakly efficient solution if problem CVOP is KT-invex.*

As for the unconstrained problems, to ensure that weakly efficient solution for CVOP to be a vector Kuhn-Tucker point for problem CVOP, the KT-invexity condition can be relaxed. So, we define a weaker generalized convex condition for a vector programming problem.

**Definition 7.9.** *The problem CVOP is said to be a* **vector KT-pseudoinvex problem** *whith respect to $\eta$, if for any $x_1$, $x_2 \in X$, with $g(x_1) \leqq \geqq$ and $g(x_2) \leqq \geqq$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$, such that*

$$f(x) - f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x}) < 0,$$
$$-\nabla g_i(\bar{x})\eta(x, \bar{x}) \geqslant 0 \quad \forall i \in I(\bar{x}).$$

It is easy to see that if CVOP is KT-invex then CVOP is KT-pseudoinvex, too.

Now we prove that this condition is necessary and sufficient for the set of vector Kuhn-Tucker points and the set of weakly efficient point to be the same.

**Theorem 7.14.** *Every vector Kuhn-Tucker point is weakly efficient for CVOP if and only if problem CVOP is a KT-pseudoinvex problem.*

**Proof.** Let $\bar{x}$ be a vector Kuhn-Tucker point for CVOP and let us suppose that this problem is KT-

pseudoinvex. We will see that $\bar{x}$ is weakly efficient for CVOP.

If there exists another feasible point $x$ such that $f(x) < f(\bar{x})$, then

$$\nabla f(\bar{x})\eta(x, \bar{x}) > 0 \Rightarrow \bar{\lambda}^T \nabla f(\bar{x})\eta(x, \bar{x}) < 0 \quad \forall \bar{\lambda} \geqslant 0. \quad (4)$$

Since $\bar{x}$ was assumed a VKTP, it verifies that

$$\bar{\lambda}^T \nabla f(\bar{x})\eta(x, \bar{x}) + \sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) = 0. \quad (5)$$

From (4) and (5), we have that

$$\sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) > 0. \quad (6)$$

CVOP is KT-pseudoinvex, then

$$-\nabla g_j(\bar{x})\eta(x, \bar{x}) \geqslant 0, \quad \forall; \in I(\bar{x}).$$

Since by hypothesis $\bar{\mu}_j \geqslant 0$, we have that

$$-\bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) \geqslant 0 \quad \forall; \in I(\bar{x}),$$

and therefore,

$$\sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) \leqslant 0,$$

which contradicts (6).

To show the reverse, let us suppose that every vector Kuhn-Tucker point is a weakly efficient solution. If $\bar{x}$ is a vector Kuhn-Tucker point, then there exists $\bar{\lambda} \geqslant 0$ and $\bar{\mu} \geqq 0$ such that,

$$\bar{\lambda}^T \nabla f(\bar{x}) + \sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0. \quad (7)$$

By Motzkin's alternative theorem [26], the following system does not have any solution

$$\nabla f_i(\bar{x})u < 0, \quad i = 1, ..., p,$$
$$\nabla g_j(\bar{x})u \leqslant 0, \quad j \in I(\bar{x}). \quad (8)$$

If $\bar{x}$ is a VKTP, then $\bar{x}$ is weakly efficient and, therefore, the system

$$f_i(x) - f_i(\bar{x}) < 0, \quad i = 1, ..., p,$$
$$g(x) \leqq 0, \quad (9)$$

does not have any solution.

If $\bar{x}$ is not a VKTP, then (7) does not have any solution. This implies that (8) has a solution.

If $\bar{x}$ is not a VKTP then $\bar{x}$ is not a weakly efficient point and (9) does not have any solution.

Then, for all $x \in X$ with $g(x) \leqq 0$, if $f_i(x) < fi(\bar{x})$, there exists $\eta(x, \bar{x})$ such that

$$\nabla f_i(\bar{x})\eta(x, \bar{x}) < 0, \qquad i = 1, ..., p,$$

$$-\nabla g_j(\bar{x})\eta(x, \bar{x}) \geqslant 0, \qquad \forall_j \in I(\bar{x}).$$

The $\forall x$, $\bar{x} \in X$ verifying $g(x) \leqslant 0$, $g(\bar{x}) \leqslant 0$, we have that

$$f(x) - f(\bar{x}) < 0 \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x}) < 0,$$

and

$$-\nabla g_j(\bar{x})\eta(x, \bar{x}) \geqslant 0, \qquad \forall_j \in I(\bar{x}).$$

Therefore, CVOP is a KT-pseudoinvex problem.    □

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