LACUNARY NON–CONTINUABLE BOUNDARY–REGULAR HOLOMORPHIC FUNCTIONS WITH UNIVERSAL PROPERTIES

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Abstract. A holomorphic function φ in a Jordan domain G in the complex plane is constructed with all its derivatives extending continuously up to the boundary ∂G that happens to be a natural boundary of φ. In addition, the action of a certain class of operators on φ presents some universal properties related to the overconvergence phenomenon.

§ 1. INTRODUCTION AND NOTATION

In this paper, we are concerned with the problem of the existence of holomorphic functions defined on a Jordan domain G of the complex plane that enjoy **simultaneously** several properties, namely :

- The boundary of G is the natural boundary of those functions.
- They are boundary-regular, that is, their derivatives of all orders extend continuously up to the boundary of G.
- The power series expansion of each such a function around a prefixed point of G presents gaps outside a prescribed sequence S of integers with upper density $\overline{d}(S) = 1$.
- The action of a certain class of operators –including, for instance, the identity and the differentiation operators of all orders– on the partial sums of their Taylor

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expansions satisfy some kind of "external universality" which is, in fact, a strong version of overconvergence.

The aim of this note is the construction of a function with all above properties. The precise statement together with its proof will be postponed till Section 3. In the remainder of this section, the pertinent terminology will be fixed, and some historical or bibliographical notes will be pointed out. A number of preparatory results will be stated in Section 2, where we also introduce a new class of operators, which are rather "natural" to our goal.

As usual, by $\mathbb{C}, \mathbb{D}, \mathbb{Q}, \mathbb{N}, \mathbb{N}_0$ we denote the complex plane, the open unit disk, the set of rational numbers, the set of positive integers and $\mathbb{N} \cup \{0\}$, respectively. A subsequence $\{n_j\}_{j \geq 1}$ in $\mathbb{N}$ or $\mathbb{N}_0$ will always mean a strictly increasing sequence $n_1 < n_2 < \cdots$. If $M \subset \mathbb{C}$ then $M^0, \overline{M}, \partial M$ will stand for the interior, the closure and the boundary, respectively, of M in $\mathbb{C}$. If G is a domain (i.e. a nonempty, connected open subset) of $\mathbb{C}$, then $H(G)$ represents the set of holomorphic functions on G. Let be given a function $f \in H(G)$, then we say that f is holomorphic exactly on G (or G is the domain of holomorphy of f, or ∂G is the natural boundary of f) if f is analytically noncontinuable across any point of ∂G or, more precisely, for every $a \in G$, the radius of convergence of the Taylor series of f with center at a equals the Euclidean distance between a and ∂G. By $H_e(G)$ we abbreviate the class of all functions which are exactly holomorphic on G. Mittag-Leffler discovered in 1884 that $H_e(G) \neq \emptyset$ for all domains G, see [15, Chapter 10]. It is clear that if $f \in H_e(G)$ then f has no holomorphic extension to any domain containing G strictly.

Let $G \subset \mathbb{C}$ be a domain. Then $A^\infty(G)$ denotes the class of holomorphic functions in G with very regular behavior at the boundary, that is,

$$A^\infty(G) = \left\{ f \in H(G) : f^{(\ell)} \text{ has a continuous extension to } \overline{G} \quad \text{for all } \ell \in \mathbb{N}_0 \right\}.$$

Notice that while $H(G)$ is a Fréchet space (i.e. a completely metrizable locally convex space) when endowed with the topology of uniform convergence on compacta. In the case that G is boundes then the class $A^\infty(G)$ also becomes a Fréchet space under the metric topology defined by $f_n \to f$ in $A^\infty(G)$ if and only if $f_n^{(\ell)} \to f^{(\ell)}$ uniformly in $\overline{G}$ for every $\ell \in \mathbb{N}_0$.

For a domain $G \subset \mathbb{C}$ (a compact set $L \subset \mathbb{C}$, respectively) we denote by $\mathcal{M}(G)$ ($\mathcal{M}(L)$, respectively) the collection of all compact sets $K \subset \overline{G}^c$ ($K \subset L^c$, respectively) with connected complement in $\mathbb{C}$. If $K \subset \mathbb{C}$ is compact, then by $A(K)$ we mean the family of all functions which are continuous on K and holomorphic in its interior K^0. The class $A(K)$ becomes a Banach space under the maximun norm.

Note that $A^\infty(G) \cap H_e(G)$ may well be empty or not. For instance, the function φ with $\varphi(z) := \sum_{n=0}^\infty \exp(-2^{n/2}) z^{2^n}$ belongs to $A^\infty(\mathbb{D}) \cap H_e(\mathbb{D})$ (see [30, Chapter 16]), but $A(\overline{G}) \cap H_e(G) = \emptyset$ if $G := \mathbb{D} \setminus [0, 1)$. Another interesting example of a function $\varphi \in A^\infty(\mathbb{D}) \cap H_e(\mathbb{D})$ is given by

$$\varphi(z) := 2z + \sum_{n=0}^\infty \frac{z^{2^n}}{2^{n^2}}.$$

It turns out that φ is one-to-one on $\overline{\mathbb{D}}$ and hence mapping $\mathbb{D}$ conformally onto a Jordan domain G whose boundary ∂G is a C^∞-curve which is nowhere analytic.

Let us recall that J. Siciak proved in [31] a strong statement about noncontinuability in a N-dimensional setting (his proof leans on typical methods of several complex variables) whose one-dimensional instance asserts that if $G \subset \mathbb{C}$ is a bounded domain such that $G = \overline{G}^0$ and $\overline{G}^c$ is connected then $H_e(G) \cap A^\infty(G) \neq \emptyset$.

Suppose that $S = \{s_j\}_{j \geq 1}$ be a subsequence of $\mathbb{N}_0$ and let $\nu_S(n)$ be the number of $m \in S$ with $m \leq n$. Then the upper and lower density of S are defined as

$$\overline{d}(S) := \limsup_{n \to \infty} \frac{\nu_S(n)}{n}, \qquad \underline{d}(S) := \liminf_{n \to \infty} \frac{\nu_S(n)}{n}.$$

If $\overline{d}(S) = \underline{d}(S) =: d(S)$ then S is said to have the density $d(S)$.

If now $G \subset \mathbb{C}$ is a domain and $z_0 \in G$ then by $H_{S,z_0}(G)$ we mean the class of holomorphic functions in G whose power series expansion around z_0 presents gaps outside S or, equivalently,

$$H_{S,z_0}(G) = \{f \in H(G) : \ f^{(n)}(z_0) = 0 \ \ \text{for all} \ \ n \notin S\}.$$

Therefore if $f \in H_{S,z_0}(G)$ we have in a neighborhood of z_0 that

$$f(z) = \sum_{\nu=0}^\infty a_\nu (z - z_0)^\nu \qquad \text{with} \ \ a_\nu = 0 \ \ \text{for all} \ \ \nu \notin S.$$

For the sake of simplicity, we set $H_{S,0}(G) = H_S(G)$. Moreover, $\mathcal{P}_S$ will stand for the family of lacunary polynomials $P(z) = \sum_{\nu=0, \ \nu \in S}^\infty c_\nu z^\nu$ with gaps outside S.

If $f \in H(G)$, $z_0 \in G$ and $n \in \mathbb{N}_0$, then we denote by $S(f, z_0, n)$ the partial sum of order n of the Taylor expansion $f(z) = \sum_{\nu=0}^\infty \frac{f^{(\nu)}(z_0)}{\nu!} (z - z_0)^\nu$ of f around z_0, that is, $S(f, z_0, n)(z) := \sum_{\nu=0}^n \frac{f^{(\nu)}(z_0)}{\nu!} (z - z_0)^\nu$.

A century ago Porter discovered that certain Taylor series with radius of convergence 1 enjoy the property that some subsequences of their sequences of partial sums (with

$z_0 = 0$) converge at some points outside the closed unit disk $\overline{\mathbb{D}}$. This phenomenon is called overconvergence. Starting from 1970, this idea has been largely developed and strengthened along various ways, as for instance : the partial sums have been replaced by the action of certain infinite matrices –with constant or non-constant entries– ; the overconvergence has been reinforced to the universal property of uniform approximation to any function $f \in A(K)$ for certain compact sets K –with $K \cap \overline{G} = \emptyset$ or even $K \cap G = \emptyset$, where G is a domain– ; the Taylor series have been generalized to Laurent series or Faber series ; and some properties have been shown to be generic in the space $X \subset H(G)$ where they are studied (that is, the subset of functions of X satisfying each of such properties is residual in X). These improvements are contained in a number of papers by Chui-Parnes, Melas, Nestoridis, Costakis, Katsoprinakis, Papadoperakis, Vlachou, Gehlen, Müller and the authors, among others (see [18, 7, 19, 27, 20, 13, 16, 24, 25, 32, 33, 2, 3, 28, 9, 4] and the references contained in them). Finally holomorphic functions satisfying both properties of universality-overconvergence and lacunarity have been found by Gharibyan, Müller and the third author in [14].

§ 2. PRELIMINARIES AND A NEW CLASS OF OPERATORS

This section is devoted to state several auxiliary results to be used later, and to consider certain classes of operators which are adequate for the statement of our main result.

Let be given a fixed $\alpha \in \mathbb{R}$ and consider the logarithmic α-spirals

$$L_\alpha := \{z = e^{(1+i\alpha)t},\ t \in \mathbb{R}\} \cup \{0\}\,.$$

Then a set $M \subset \mathbb{C}$ is called α-starlike with respect to $z_0 = 0$ if

$$M \cdot (L_\alpha \cap \overline{\mathbb{D}}) := \{z = \zeta w,\ \zeta \in M,\ w \in L_\alpha \cap \overline{\mathbb{D}}\} = M$$

and M is called α-starlike with respect to $z_0 \in M$ if $M_{z_0} := \{z = \zeta - z_0,\ \zeta \in M\}$ is α-starlike with respect to the origin. If $\alpha = 0$ then M is starlike in the traditional sense.

The content of the following lemma can be found in [13] and [20].

Lemma 2.1. *Let S be a subsequence of $\mathbb{N}_0$ with $\overline{d}(S) = 1$ and suppose that K is a compact set with connected complement and $0 \in K^0$. Assume that f is holomorphic on K with*

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu \quad \textbf{where } f_\nu = 0 \textbf{ for all } \nu \notin S$$

near the origin. Suppose in addition that one of the two conditions is satisfied :

(a) $d(S) = 1$,

(b) $\overline{d}(S) = 1$ *and the component of K which contains the origin is α-starlike with respect to the origin.*

Then for every $\varepsilon > 0$ there exists a lacunary polynomial $P \in \mathcal{P}_S$ such that

$$\max_{z \in K} |f(z) - P(z)| < \varepsilon.$$

Recall that a power series $\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ is said to have Ostrowski gaps (p_k, q_k) $(k \in \mathbb{N})$ if p_k, q_k are positive integers such that

$$p_1 < q_1 \leq p_2 < q_2 \leq \cdots, \quad \lim_{k \to \infty} \frac{q_k}{p_k} = \infty,$$

$$\text{and} \quad \lim_{\substack{\nu \to \infty \\ \nu \in I}} |a_\nu|^{1/\nu} = 0, \quad \text{where} \quad I = \bigcup_{k \in \mathbb{N}} (p_k, q_k).$$

Lemma 2.2. *Assume that G is a domain. Let $z_0 \in G$ and let $f \in H(G)$ such that the Taylor expansion of f around z_0 has Ostrowski gaps (p_k, q_k) $(k \in \mathbb{N})$. Then*

$$\sup_{\zeta \in L} \sup_{z \in K} |S(f, z_0, p_k)(z) - S(f, \zeta, p_k)(z)| \to 0 \qquad (k \to \infty) \tag{1}$$

for every pair K, L of compact sets with $K \subset \mathbb{C}$, $L \subset G$.

Proof : In [19, Theorem 1] it is shown that the expression in (1) without "$\sup_{\zeta \in L}$" tends to zero for each compact set K. But its proof reveals in fact that such convergence to zero holds uniformly with respect to ζ whenever ζ belongs to a compact subset of G.

Next we are going to consider two kinds of operators (i.e. continuous linear self-mappings) on the space $\mathcal{E} := H(\mathbb{C})$ of entire functions. This first kind is that of operators $T : \mathcal{E} \to \mathcal{E}$ having dense range. For instance, if $T(\mathcal{E}) \supset \{\text{polynomials}\}$ then T has dense range. Trivially, T has dense range if it is surjective. The second kind of operators is less usual, and it is fixed in the following definition.

Definition 2.1. Let $L \subset \mathbb{C}$ be a compact set and T be an operator on $\mathcal{E}$. Then we say that T is **compactly L-externally controlled** if the following property is satisfied :

Given $\varepsilon > 0$ and a compact set $K \subset \mathcal{M}(L)$, there are $\delta > 0$ and $M \in \mathcal{M}(L)$ such that

$$\left[h \in \mathcal{E} \text{ and } \sup_{z \in K} |h(z)| < \delta \right] \quad \textbf{implies} \quad \sup_{z \in K} |(Th)(z)| < \varepsilon.$$

Examples 2.3. 1. Let $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then Φ is said to be of exponential type provided that there are positive constants A, B such that $|\Phi(z)| \leq A \exp(B|z|)$ for all $z \in \mathbb{C}$. Consider its associated formal linear (in general, infinite order) differential operator $\Phi(D) = \sum_{n=0}^{\infty} a_n D^n$ defined as $\Phi(D)f = \sum_{n=0}^{\infty} a_n f^{(n)}$ ($f \in \mathcal{E}$). Then $\Phi(D)$ is in fact a well-defined operator on $\mathcal{E}$. This is easy to see just by taking into account the Cauchy estimates as well as the fact that Φ is of exponential type if and only if the sequence $\{(n!|a_n|)^{1/n}\}_{n \geq 1}$ is bounded. By the Malgrange-Ehrenpreis theorem (see [11] or [23]) we have that $\Phi(D)$ is surjective (so it has dense range) as soon as $\Phi \not\equiv 0$.

Assume now that Φ is of subexponential type, that is, for given $\varepsilon > 0$ there is a positive constant A such that $|\Phi(z)| \leq A \exp(\varepsilon|z|)$ for all $z \in \mathbb{C}$; equivalently, $\lim_{n \to \infty} (n!|a_n|)^{1/n} = 0$ (see for instance [6]; see also [1] for a good exposition about the corresponding operators $\Phi(D)$). Then $\Phi(D) := T$ is compactly L-externally controlled for every compact set $L \subset \mathbb{C}$. Indeed, if $\varepsilon > 0$ and $K \in \mathcal{M}(L)$ are fixed, we can choose a Jordan domain J such that $K \subset J^0$, $L \cap \overline{J} = \emptyset$ and $\gamma := \partial J$ is rectifiable. Recall that $(n!|a_n|)^{1/n} \to 0$ ($n \to \infty$). Therefore given $\widetilde{\varepsilon} := \frac{\mathrm{dist}(K,\gamma)}{2}$ there is a constant $A \in (0, +\infty)$ such that $n!|a_n| \leq A\widetilde{\varepsilon}^n$ ($n \in \mathbb{N}_0$). Let us define

$$M := \overline{J} \qquad \text{and} \qquad \delta := \frac{\varepsilon \cdot \mathrm{dist}(K,\gamma)}{A \cdot \mathrm{length}(\gamma)}.$$

Then $M \in \mathcal{M}(L)$ and $\delta > 0$. Now, if we make γ oriented counterclockwise, we get from the Cauchy integral formula for derivatives that for every $z \in K$ and every $h \in \mathcal{E}$ one has

$$|(Th)(z)| = \left| \sum_{n=0}^{\infty} a_n h^{(n)}(z) \right| =$$

$$= \left| \sum_{n=0}^{\infty} \frac{a_n \cdot n!}{2\pi i} \oint_{\gamma} \frac{h(t)}{(t-z)^{n+1}} \, dt \right| \leq \sum_{n=0}^{\infty} \frac{A\widetilde{\varepsilon}^n}{2\pi} \cdot \frac{\sup_{t \in \gamma} |h(t)| \cdot \mathrm{length}(\gamma)}{(\mathrm{dist}(K,\gamma)^{n+1}} \leq$$

$$\leq \frac{A \cdot \mathrm{length}(\gamma) \sup_{z \in \gamma} |h(t)|}{2 \cdot \mathrm{dist}(K,\gamma)} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{A \cdot \mathrm{length}(\gamma)}{\mathrm{dist}(K,\gamma)} \cdot \sup_{w \in M} |h(w)|.$$

Hence $\sup_{z \in K} |(Th)(z)| < \varepsilon$ whenever $\sup_{z \in M} |h(z)| < \delta$, as required.

2. The second part of the above example covers the cases $T = D^n$ ($n \in \mathbb{N}_0$), where $D^0 := I =$ the identity operator. Indeed, just take $\Phi(z) := z^n$. However, if Φ is of exponential type then $\Phi(D)$ is not always controlled in the sense of Definition 2.1. For instance, if we take $\Phi(z) := e^z$ then $\Phi(D)$ is the translation operator that takes a function $h \in \mathcal{E}$ to the function $z \mapsto h(z+1)$, which is not controlled for some compact set $L \subset \mathbb{C}$. In fact, more is true : If $\varphi \in \mathcal{E}$ is not the identity then the

composition operator $C_\varphi : \mathcal{E} \to \mathcal{E}$ defined as $C_\varphi h = h \circ \varphi$ is not compactly L-externally controlled for some compact set L. Indeed, fix $\alpha \in \mathbb{C}$ such that $\beta := \varphi(\alpha) \neq \alpha$ and choose $L := \{\beta\}$, $\varepsilon_0 := 1$, $K = \{\alpha\}$. Observe that $K \in \mathcal{M}(L)$. Now fix $\delta > 0$ and $M \in \mathcal{M}(L)$. By the Runge approximation theorem (see [12]) –applied to the compact set $M \cup \{\beta\}$– we can find a polynomial h such that $|h(z) - 0| < \delta$ $(z \in M)$ and $|h(\beta) - 2| < 1$. Hence $\sup_{z \in M} |h(z)| < \delta$ but $\sup_{z \in K} |(C_\varphi h)(z)| = |h(\beta)| > \varepsilon_0$, as required. It is clear that C_φ is L-controlled for all compact sets if φ is the identity. If φ is not the identity but it is a nonexpansive similarity –that is, $\varphi(z) \equiv a(z - b) + b$, where $|a| \leq 1$ and b is the (unique) finite fixed point of φ– then C_φ is compactly L-externally controlled, where L is any closed ball with center at b.

As for the density of the range, we claim that if $\varphi \in \mathcal{E}$ then C_φ has dense range if and only if φ is a similarity $\varphi(z) \equiv az + b$ $(a, b \in \mathbb{C}$ with $a \neq 0)$. Indeed, the part "if" is evident because C_φ would be surjective. Finally, suppose that C_φ has dense range and that, by the way of contradiction, φ is not one-to-one. Then there are points $a, b \in \mathbb{C}$ with $a \neq b$ such that $\varphi(a) = \varphi(b)$. By density, there is sequence $\{f_n\}_{n \geq 1} \subset \mathcal{E}$ for which $f_n \circ \varphi \to g$ $(n \to \infty)$ in $\mathcal{E}$, where $g(z) \equiv z$. In particular, $\lim_{n \to \infty} f_n(\varphi(a)) = a$ and $\lim_{n \to \infty} f_n(\varphi(b)) = b$, which is absurd because $\varphi(a) = \varphi(b)$. Therefore φ is an injective entire function, so it is a similarity, which proves the claim.

3. Let $\alpha \in \mathcal{E}$ and consider the multiplication operator $M_\alpha : f \in \mathcal{E} \mapsto \alpha f \in \mathcal{E}$. It is easy to see that M_α is always compactly L-externally controlled for all compact sets $L \subset \mathbb{C}$ and that, in addition, M_α has dense range if and only if α has no zeros.

4. Given a compact set $L \subset \mathbb{C}$, the family $\mathcal{A}$ of compactly L-externally controlled operators is a vector algebra in the space of all operators on $\mathcal{E}$, that is, if α, β are complex numbers and T_1, T_2 are in $\mathcal{A}$, then the operators $\alpha T_1 + \beta T_2$ and $T_1 \circ T_2$ are in $\mathcal{A}$ too. Indeed, this is evident for $\alpha T_1 + \beta T_2$. As for the composition $T_1 \circ T_2$, fix a number $\varepsilon > 0$ together with a compact set $K \in \mathcal{M}(L)$. Then there are $\delta_1 > 0$ and $M_1 \in \mathcal{M}(L)$ such that $\|T_1 f\|_K < \varepsilon$ whenever $f \in \mathcal{E}$ and $\|f\|_{M_1} < \delta_1$. By using now that T_2 is controlled, there are $\delta > 0$ and $M \in \mathcal{M}(L)$ such that $[h \in \mathcal{E}$ and $\|h\|_S < \delta]$ implies $\|T_2 h\|_{M_1} < \delta_1$. Then if $\|h\|_M < \delta$ we obtain $\|T_1 T_2 h\|_K < \varepsilon$, and we are done.

§ 3. CONSTRUCTION OF A UNIVERSAL FUNCTION

We are now ready to construct the promised universal function with respect to overconvergence having moreover additional properties of lacunarity, boundary-regular behavior and non-continuability.

Theorem 3.1. *Suppose that G is a Jordan domain, that $z_0 \in G$ and that S is a subsequence of $\mathbb{N}_0$ satisfying at least one of the following conditions :*

(a) $d(S) = 1$,

(b) $\overline{d}(S) = 1$ and G is α-starlike with respect to $z_0 \in G$.

Then there exist a function $\varphi \in A^\infty(G) \cap H_e(G) \cap H_{S,z_0}(G)$ and a subsequence $\{p_k\}_{k\geq 1} \subset \mathbb{N}_0$ for which the following properties hold :

(A) For each compact set $L \subset G$ we have $S(\varphi, \zeta, p_k) \to \varphi$ $(k \to \infty)$ in $A^\infty(G)$ uniformly for all $\zeta \in L$.

(B) For each compact set $K \in \mathcal{M}(\overline{G})$, each compactly $\overline{G}$-externally controlled operator T on $\mathcal{E}$ with dense range, and each $f \in A(K)$, there exists a subsequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{N}_0$ such that

$$\limsup_{j\to\infty} \sup_{\zeta\in L} \sup_{z\in K} |(TS(\varphi,\zeta,p_{k_j}))(z) - f(z)| = 0$$

for every compact set $L \subset G$.

Proof : 1. Without loss of generality, we can assume that $z_0 = 0$. Let $\{K_\nu^\star\}_{\nu\geq 1}$ be an exhausting sequence for $\mathcal{M}(G)$, that is, $K_\nu^\star \in \mathcal{M}(G)$ for each ν and, given $K \in \mathcal{M}(G)$, there is $\nu \in \mathbb{N}$ depending on K such that $K \subset K_\nu$ (see for instance [Lemma 2.9][5]). Let $\{\Pi_\nu^\star\}_{\nu\geq 1}$ be an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Suppose that $\{(K_n, \Pi_n)\}_{n\geq 1}$ is an arrangement of all $K_\nu^\star$ and $\Pi_\nu^\star$ in which any combination $(K_\nu^\star, \Pi_\mu^\star)$ occurs infinitely many often.

We choose a sequence of Jordan domains G_n with rectifiable boundary satisfying

$$\overline{G} \subset G_{n+1} \subset \overline{G_{n+1}} \subset G_n \qquad (n \in \mathbb{N}),$$

$$\overline{G_n} \cap K_n = \emptyset \qquad (n \in \mathbb{N}) \qquad \textbf{and} \qquad \bigcap_{n=1}^\infty G_n = \overline{G}.$$

In the case that G is α-starlike with respect to $z_0 = 0$ then we assume in addition that all G_n are α-starlike also (see for instance Duren [10], Theorem 2.19).

2. We construct sequences $\{p_n\}_{n\geq 1}$, $\{q_n\}_{n\geq 1} \subset \mathbb{N}_0$ and a sequence $\{P_n\}_{n\geq 1}$ of polynomials by induction. First, we define

$$\delta_n := \text{dist}(\overline{G}, \partial G_n), \quad \lambda_n := \text{length } (\partial G_n), \quad \varepsilon_n := \frac{\delta_n^n}{n! n^2 \lambda_n} \qquad (n \in \mathbb{N}).$$

Without loss of generality we may assume $\delta_n < 1$ $(n \in \mathbb{N})$.

By Lemma 2.1 there exists a polynomial

$$P_1(z) = \sum_{\nu=0}^{p_1} a_\nu z^\nu \quad \textbf{with} \quad a_\nu = 0 \quad \textbf{for} \quad \nu \notin S$$

which satisfies

$$\max_{z \in \overline{G_1}} |P_1(z)| < \varepsilon_1 \quad \textbf{and} \quad \max_{z \in K_1} |P_1(z) - \Pi_1(z)| < 1.$$

We assume that $P_1, \ldots, P_n$ have already been determined and that P_n has the form

$$P_n(z) = \sum_{\nu = q_{n-1}}^{p_n} a_\nu z^\nu \quad \text{with} \quad a_\nu = 0 \quad \textbf{for} \quad \nu \notin S.$$

We have set $q_0 := 0$. Choose $q_n \in \mathbb{N}$ with $q_n > np_n$. Observing that $S_n := \{t \in S : t \geq q_n\}$ also satisfies $d(S_n) = 1$ if (a) holds and $\overline{d}(S_n) = 1$ if (b) holds we can find by Lemma 2.1 again a polynomial

$$P_{n+1}(z) = \sum_{\nu = q_n}^{p_{n+1}} a_\nu z^\nu \quad \textbf{with} \quad a_\nu = 0 \quad \textbf{for} \quad \nu \notin S \tag{1}$$

which satisfies

$$\max_{z \in \overline{G_{n+1}}} |P_{n+1}(z)| < \varepsilon_{n+1} \tag{2}$$

and

$$\max_{z \in K_{n+1}} \left| P_{n+1}(z) - \left\{ \Pi_{n+1}(z) - \sum_{\nu=1}^{n} P_\nu(z) \right\} \right| < \frac{1}{n+1}. \tag{3}$$

By induction we get $\{p_n\}_{n \geq 1}$, $\{q_n\}_{n \geq 1}$ and $\{P_n\}_{n \geq 1}$.

3. For fixed $l \in \mathbb{N}_0$ and $n > l$ we obtain from the Cauchy integral formula for derivatives (we can assume that ∂G_n is oriented counterclockwise) that

$$\max_{z \in \overline{G}} |P_n^{(l)}(z)| = \max_{z \in \overline{G}} \left| \frac{l!}{2\pi i} \oint_{\partial G_n} \frac{P_n(\zeta)}{(\zeta - z)^{l+1}} \, d\zeta \right| \leq \frac{l!}{2\pi} \cdot \lambda_n \cdot \frac{\varepsilon_n}{(\delta_n)^{l+1}} < n! \lambda_n \cdot \frac{\varepsilon_n}{\delta_n^n} = \frac{1}{n^2}.$$

Therefore the series $\sum_{n=1}^{\infty} P_n^{(l)}(z)$ converges for each $l \in \mathbb{N}_0$ uniformly on $\overline{G}$, and it follows that the function φ, which is defined by

$$\varphi(z) := \sum_{n=1}^{\infty} P_n(z),$$

is holomorphic on G and that each derivative $\varphi^{(l)}$ has a continuous extension to $\overline{G}$. In other words, $\varphi \in A^\infty(G)$.

4. We consider the power series of φ around the origin. By the special form (1) of the polynomials P_n and by the property $q_n > np_n$ $(n \in \mathbb{N})$, the powers in P_n and P_m do not overlap if $n \neq m$ and therefore the power series of φ is given by

$$\varphi(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \quad \text{with} \quad a_\nu = 0 \quad \textbf{for} \quad \nu \notin S. \tag{4}$$

Thus, $\varphi \in H_S(G)$. For its partial sums $S(\varphi, 0, n)$ we obtain especially

$$S(\varphi, 0, p_k)(z) = \sum_{\nu=0}^{p_k} a_\nu z^\nu = \sum_{\nu=1}^{k} P_\nu(z),$$

and we get for each $l \in \mathbb{N}_0$

$$(S(\varphi, 0, p_k))^{(l)}(z) = \sum_{\nu=1}^{k} P_\nu^{(l)}(z) \to \varphi^{(l)}(z) \qquad (k \to \infty)$$

uniformly on $\overline{G}$. Hence

$$S(\varphi, 0, p_k) \to \varphi \qquad \text{in} \quad A^\infty(G). \tag{5}$$

Let us define the functions of two complex variables $F_k : G \times \mathbb{C} \to \mathbb{C}$ $(k \in \mathbb{N})$ by

$$F_k(\zeta, z) := S(\varphi, \zeta, p_k)(z) - S(\varphi, 0, p_k)(z) = \sum_{\nu=0}^{p_k} \left\{ \frac{\varphi^{(\nu)}(\zeta)}{\nu!}(z - \zeta)^\nu - \frac{\varphi^{(\nu)}(0)}{\nu!} z^\nu \right\}.$$

Then every F_k is separately analytic with respect to ζ, z; whence it is analytic in $G \times \mathbb{C}$ by Hartog's theorem [pages 25 and 93–95][17]. Note that, due to Lemma 2.2, F_k tends to zero compactly in $G \times \mathbb{C}$. Therefore the Weierstrass convergence theorem for several variables (see [page 154][26]) guarantees that $\partial^l F_k / \partial z^l \to 0$ $(k \to \infty)$ uniformly on compacta in $G \times \mathbb{C}$ for each $l \in \mathbb{N}_0$. Finally, this combined with (5) shows that, for every compact set $L \subset G$, one has $S(\varphi, \zeta, p_k) \to \varphi$ $(k \to \infty)$ in $A^\infty(G)$ uniformly in $\zeta \in L$. This concludes the proof of (A).

5. It remains to prove (B) and that $\varphi \in H_e(G)$. With this aim, fix any $K \in \mathcal{M}(\overline{G})$ and any $f \in A(K)$. By Mergelyan's theorem (see [12]) there exists a sequence of polynomials $\{\Pi_{m_j}^\star\}_{j \geq 1}$ with

$$\Pi_{m_j}^\star \to f \quad (k \to \infty) \quad \text{uniformly on} \quad K. \tag{6}$$

The set K is contained in some $K_\nu^\star$ and by our construction there exists a sequence $\{k_j\}_{j \geq 1} \subset \mathbb{N}$ with $K_\nu^\star = K_{k_j}$, $\Pi_{m_j}^\star = \Pi_{k_j}$ $(j \in \mathbb{N})$.
From (3) we obtain

$$\max_{z \in K} \left| \sum_{\nu=1}^{k_j} P_\nu(z) - \Pi_{k_j}(z) \right| < \frac{1}{k_j},$$

and together with (6) we get

$$S(\varphi, 0, p_{k_j})(z) = \sum_{\nu=0}^{p_{k_j}} a_\nu z^\nu = \sum_{\nu=1}^{k_j} P_\nu(z) \to f(z) \quad (k \to \infty) \quad \textbf{uniformly on} \quad K.$$

6. The power series (4) has Ostrowski gaps (p_k, q_k) $(k \in \mathbb{N})$ with $q_k/p_k \to \infty$. By the universal properties established in step (5), the sequence $\{S(\varphi, 0, p_{k_j})\}_{j \geq 1}$ cannot converge at any point of $\overline{G}^c$. It therefore follows from Ostrowski's theorem on overconvergence (see for instance [page 314][15]) that $\varphi \in H_e(G)$.

7. Finally, fix again a set $K \in \mathcal{M}(\overline{G})$ and a function $f \in A(K)$. Fix also a compactly $\overline{G}$-externally controlled operator $T : \mathcal{E} \to \mathcal{E}$ with dense range. Let $\varepsilon > 0$. Then there exists $g \in \mathcal{E}$ such that

$$\sup_{z \in K} |(Tg)(z) - f(z)| < \frac{\varepsilon}{2}. \tag{7}$$

By the control property, we can find a number $\delta > 0$ and a set $M \in \mathcal{M}(\overline{G})$ such that

$$\left[h \in \mathcal{E} \quad \textbf{and} \quad \sup_{z \in M} |h(z)| < \delta \right] \qquad \textbf{implies} \qquad \sup_{z \in K} |(Th)(z)| < \frac{\varepsilon}{2}. \tag{8}$$

By step 5 (with K, f replaced by M, g, respectively) we can get a number $k = k(\varepsilon) \in \mathbb{N}$ satisfying

$$\sup_{z \in M} |S(\varphi, 0, p_k)(z) - g(z)| < \delta.$$

Hence (8) tells us that

$$\sup_{z \in K} |(TS(\varphi, 0, p_k))(z) - (Tg)(z)| < \frac{\varepsilon}{2}, \tag{9}$$

where we have used the linearity of T. Therefore, (7), (9) and the triangle inequality yield

$$\sup_{z \in K} |(TS(\varphi, 0, p_k))(z) - f(z)| < \varepsilon.$$

By choosing $\varepsilon = 1/j$ $(j \in \mathbb{N})$, it is evident that there is a sequence $\{k(1) < k(2) < \cdots\} \subset \mathbb{N}$ for which

$$\sup_{z \in K} |(TS(\varphi, 0, p_{k(j)}))(z) - f(z)| \to 0 \qquad (j \to \infty). \tag{10}$$

In order to prove (B) it is enough –thanks to (10) and the linearity of T– to select a sequence $\{j(\nu)\}_{\nu \geq 1} \subset \mathbb{N}$ such that

$$\sup_{\zeta \in L} \sup_{z \in K} |(T(S(\varphi, \zeta, p_{k(j(\nu))}) - S(\varphi, 0, p_{k(j(\nu))})))(z)| \to 0 \qquad (j \to \infty) \tag{11}$$

for all compact sets $L \subset G$. Finally, we would re-label $p_{k(j)} \equiv p_{k(j(\nu))}$ and this would conclude the proof. With this aim, the control property of T comes anew to our help.

Fix an increasing sequence of comact sets $L_\nu \subset G$ ($\nu \in \mathbb{N}$) with the property that every compact set $L \subset G$ is included in some L_ν (see for instance [30, Chapter 13]). By Lemma 2.2, we have for all compact sets $L \subset G$, $M \subset \mathbb{C}$ that

$$\sup_{\zeta \in L} \sup_{z \in M} |S(\varphi, \zeta, p_{k(j)})(z) - S(\varphi, 0, p_{k(j)})(z)| \to 0 \qquad (j \to \infty). \qquad (12)$$

Given $\nu \in \mathbb{N}$, there exist $\delta_\nu > 0$ and $M_\nu \in \mathcal{M}(\overline{G})$ such that $\sup_{z \in K} |(Th)(z)| < 1/\nu$ for every $h \in \mathcal{E}$ with $\sup_{z \in M_\nu} |h(z)| < \delta_\nu$. From (12), there is $j(\nu) \in \mathbb{N}$ (by induction, it can be obtained $j(1) < j(2) < \cdots$) with

$$\sup_{z \in M_\nu} |S(\varphi, \zeta, p_{k(j(\nu))})(z) - S(\varphi, 0, p_{k(j(\nu))})(z)| < \delta_\nu \qquad (\zeta \in L_\nu). \qquad (13)$$

Let us prescribe a compact set $L \subset G$. Then there is $\nu_0 \in \mathbb{N}$ such that $L \subset L_\nu$ for all $\nu \geq \nu_0$. Consequently, (13) and the control property give us for all $\nu \geq \nu_0$ that

$$\sup_{\zeta \in L} \sup_{z \in K} |(T(S(\varphi, \zeta, p_{k(j(\nu))}) - S(\varphi, 0, p_{k(j(\nu))})))(z)|$$
$$\leq \sup_{\zeta \in L_\nu} \sup_{z \in K} |(T(S(\varphi, \zeta, p_{k(j(\nu))}) - S(\varphi, 0, p_{k(j(\nu))})))(z)| \leq \frac{1}{\nu} \to 0 \qquad (\nu \to \infty).$$

Thus, (11) is derived, as required, and the proof is complete.

We conclude the paper by gathering a number of comments concerning Theorem 3.1 and its proof.

Remarks 3.2. 1. Observe that the proof of the last theorem is rather constructive, in the sense that it is not based on Baire-category arguments.

2. A closer look at the proof reveals that one can weaken slightly the hypothesis of denseness of the range of T. In fact, it is enough to assume that $T(\mathcal{E})$ is dense for the topology on $\mathcal{E}$ defined by the uniform convergence on all sets in $\mathcal{M}(\overline{G})$. For instance, if $\alpha \in \mathcal{E}$ and $\emptyset \neq \alpha^{-1}(\{0\}) \subset \overline{G}$, then the multiplication operator M_α has not dense range but it still satisfies the conclusion of Theorem 3.1.

3. In the case that $d(S) = 1$ our theorem remains valid if the Jordan domain G is replaced by, more generally, a bounded domain G with $G = \overline{G}^0$ and $\overline{G}^c$ connected. Indeed, in step 1 we still can find Jordan domains G_n with $\overline{G} \subset G_n$ and $\overline{G_n} \cap K_n = \emptyset$ ($n \in \mathbb{N}$); then one would take $\delta_n := \min\{1, \text{dist}(\overline{G}, \partial G_n)\}$ to make the adequate estimations. Finally, in step 6, from the application of Ostrowski's theorem it follows that the largest domain contained in $\overline{G}$ –that is, $\overline{G}^0$– is the domain of holomorphy of φ. But $G = \overline{G}^0$, so $\varphi \in H_e(G)$. The remaining steps of the proof may stay unchanged. Consequently, we have obtained the one-dimensional case of

Siciak's theorem mentioned in Section 1, but enriched with lacunarity and universality properties.

4. Concerning (A), even in the familiar case $G = \mathbb{D}$, $\zeta = 0$ one may well have for a function $\varphi \in A^\infty(\mathbb{D})$ that $S(\varphi, 0, n) \not\to \varphi$ $(n \to \infty)$ in $A^\infty(\mathbb{D})$. In fact, there are functions $\varphi \in A(\overline{\mathbb{D}})$ such that its sequence of Taylor polynomials at the origin does not converge to φ in $A(\overline{\mathbb{D}})$ (that is, uniformly on $\overline{\mathbb{D}}$). Only it is true that $\varphi(rz) \to \varphi(z)$ $(r \to 1^-)$ in $A(\overline{\mathbb{D}})$ for all $\varphi \in A(\overline{\mathbb{D}})$, see [8, p. 286]).

5. Concerning (B), we may wonder whether the compact sets K might be allowed to satisfy merely $K \cap G = \emptyset$ instead of the stronger condition $K \cap \overline{G} = \emptyset$. The answer is negative. In fact, we cannot even construct a universal function $\varphi \in A(\overline{G})$, see [25, Proposition 5.6].

Резюме. В работе рассматривается голоморфная функция φ в жордановой области G комплексной плоскости, все производные которой непрерывно продолжены до границы ∂G, являющейся естественной границей функции φ. Далее, определяется действие некоторого класса операторов на функцию φ и исследуются некоторые универсальные свойства явления сверхсходимости.

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