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Abstract

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On the Newton method for solving fuzzy optimization problems [4]

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Abstract

In this article we consider optimization problems where the objectives are fuzzy functions (fuzzy-valued functions). For this class of fuzzy optimization problems we discuss the Newton method to find a non-dominated solution. For this purpose, we use the generalized Hukuhara differentiability notion, which is the most general concept of existing differentiability for fuzzy functions. This work improves and correct the Newton Method previously proposed in the literature.

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1 Introduction

Fuzzy optimization problems have been studied by many researchers in several directions with a lot of applications. The collection of papers on fuzzy optimization edited by Delgado et al. [11], Lodwick and Kacprzyk [19], Inuiguchi and Ramík

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[15], Rommelfanger and Slowiński [26], Slowiński and Teghem [27], and the books by Lai and Hwang [16,17] provide reviews about this topic from a very broad point of view.

It is usually difficult to determine the coefficients of an objective function as a real number since, most often, these possess inherent uncertainty and/or inaccuracy. Given that this is the usual state, we consider fuzzy-valued objective function as one approach to tackle uncertainty and inaccuracies in the objective function coefficients of mathematical programming models (see, e.g., Lodwick [18]). Optimization problems with fuzzy-valued objective functions were studied by many researchers. For instance, see [5,8,23,20,30–37,39]. In particular, in [5,9,33,37] Karush-Kuhn-Tucker type optimality conditions for this class of fuzzy optimization problems were obtained. More recently, Pirzada and Pathak in [21] proposed a Newton method to find a non-dominated solution of a fuzzy optimization problem using the Hukuhara differentiability of fuzzy-valued functions.

The concept of Hukuhara differentibility (H-differentiability, for short) for fuzzy functions is very restrictive. For instance, $F(x) = C \cdot x$, where C is any fuzzy interval and x is a real number, is not H-differentiable being that F is a generalization of a linear function. In general, a fuzzy function defined by $F(x) = C \cdot g(x)$, where g is a differentiable real function and C is a fuzzy interval, is not always H-differentiable. However, it is always gH-differentiable. It is well-known that the concept of gH-differentiable fuzzy function (generalized Hukuhara differentiable fuzzy function) is a more general concept than level-wise differentiability [37,38], Hukuhara differentiability [14], and G-differentiability [1–3,6,7]. Thus, the more useful concept of differentiability for fuzzy functions is gH-differentiability.

The conditions imposed to implement the Newton method introduced by Pirzada and Pathak in [21] in Theorem 4.1 are very restrictive because they ask that in the neighborhood of a nondominated solution, all points must be comparable, but the order relation used is only partial. Moreover, it is also required that the nondominated solution is an ideal point of the endpoint functions of the objective fuzzy functions. The examples presented in the same paper do not obey the conditions required by Theorem 4.1. In addition, the objective functions of all examples they consider are not H-differentiable. Even so, they apply the Newton method to the examples. Not surprisingly, when they apply to the Example 4.1, they obtain a point that is not a non dominated solution, although the authors claim it is.

In this paper we formulate the Newton method to find a non-dominated solution of fuzzy optimization problems without the requirement that all feasible solutions in the neighborhood of a nondominated solution be comparable and use gH-differentibility instead of H-differentiability. Finally, we correct the examples considered in [21].

2 Notation and the space of fuzzy intervals

A fuzzy set on $\mathbb{R}^n$ is a mapping $u : \mathbb{R}^n \to [0, 1]$. For each fuzzy set u, we denote its α-level set as $[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$. The support of u is denoted by $supp(u)$, where $supp(u) = \{x \in \mathbb{R}^n \mid u(x) > 0\}$. The closure of $supp(u)$ defines the 0-level of u, .i.e. $[u]^0 = cl(supp(u))$, where $cl(M)$ means the closure of the subset $M \subset \mathbb{R}^n$.

Definition 1 *A fuzzy set u on $\mathbb{R}$ is said to be a **fuzzy interval** if:*

(1) u is normal, i.e. there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
(2) u is an upper semi-continuous function;
(3) $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$, $x, y \in \mathbb{R}$, $\lambda \in [0, 1]$;
(4) $[u]^0$ is compact.

Let $\mathcal{F}_C$ denote the family of all fuzzy intervals. So, for any $u \in \mathcal{F}_C$ we have that $[u]^\alpha \in \mathcal{K}_C$ for all $\alpha \in [0, 1]$, where $\mathcal{K}_C$ denotes the space of all compact intervals in $\mathbb{R}$, and thus the α-levels of a fuzzy interval are given by $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right]$, $\underline{u}_\alpha, \overline{u}_\alpha \in \mathbb{R}$ for all $\alpha \in [0, 1]$. If $[u]^1$ is a singleton then we say that u is a fuzzy number. Triangular fuzzy numbers are a special type of fuzzy numbers which are well determined by three real numbers $a \leq b \leq c$ and we write $u = (a, b, c)$ and

$$[u]^\alpha = [a + (b - a)\alpha, c - (c - b)\alpha],$$

for all $\alpha \in [0, 1]$.

For fuzzy intervals $u, v \in \mathcal{F}_C$ represented by $\left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ and $\left[\underline{v}_\alpha, \overline{v}_\alpha\right]$, respectively, and for any real number λ, we define the addition $u + v$ and scalar multiplication λu as follows:

$$(u + v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}$$

$$(\lambda u)(x) = \begin{cases} u\left(\frac{x}{\lambda}\right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

It is well known that, for every $\alpha \in [0, 1]$,

$$[u + v]^\alpha = \left[(\underline{u + v})_\alpha, (\overline{u + v})_\alpha\right] = \left[\underline{u}_\alpha + \underline{v}_\alpha\ ,\ \overline{u}_\alpha + \overline{v}_\alpha\right] \tag{1}$$

and

$$[\lambda u]^\alpha = \left[(\underline{\lambda u})_\alpha, (\overline{\lambda u})_\alpha\right] = \left[\min\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}, \max\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}\right]. \tag{2}$$

A crucial concept in obtaining a useful working definition of derivative for fuzzy functions is deriving a suitable difference between two fuzzy intervals. Toward this end we have the following definition.

Definition 2 *([29]) Given two fuzzy intervals u, v, the generalized Hukuhara difference (gH-difference for short) is the fuzzy interval w, if it exists, such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} \quad (i)\ u = v + w, \\ or\ \ (ii)\ v = u + (-1)w. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number. Note that the case (i) is coincident to Hukuhara difference (see [14]) and so the concept of gH-difference is more general than H-difference.

If $u \ominus_{gH} v$ exists then, in terms of α-levels, we have

$$[u \ominus_{gH} v]^{\alpha} = [u]^{\alpha} \ominus_{gH} [u]^{\alpha} = \left[\min\{\underline{u}_{\alpha} - \underline{v}_{\alpha}\}, \max\{\overline{u}_{\alpha} - \overline{v}_{\alpha}\}\right],$$

for all $\alpha \in [0, 1]$, where $[u]^{\alpha} \ominus_{gH} [u]^{\alpha}$ denotes the gH-difference between two intervals (see [28,29]).

Given $u, v \in \mathcal{F}_C$, we define the distance between u and v by

$$\begin{aligned} D(u, v) &= \sup_{\alpha \in [0,1]} H\left([u]^{\alpha}, [v]^{\alpha}\right) \\ &= \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}_{\alpha} - \underline{v}_{\alpha}\right|, \left|\overline{u}_{\alpha} - \overline{v}_{\alpha}\right|\right\}. \end{aligned}$$

So, $(\mathcal{F}_C, D)$ is a complete metric space.

3 Differentiable fuzzy functions

Henceforth, K denotes an open subset of $\mathbb{R}^n$. A function $F : K \to \mathcal{F}_C$ is said to be a fuzzy function. For each $\alpha \in [0, 1]$, we associate with F the family of interval-valued functions $F_{\alpha} : K \to \mathcal{K}_C$ given by $F_{\alpha}(x) = [F(x)]^{\alpha}$. For any $\alpha \in [0, 1]$, we denote

$$F_{\alpha}(x) = \left[\underline{f}_{\alpha}(x), \overline{f}_{\alpha}(x)\right].$$

Here, the endpoint functions $\underline{f}_{\alpha}, \overline{f}_{\alpha} : K \to \mathbb{R}$ are called upper and lower functions of F, respectively.

Next we present the concept of gH-differentiability of fuzzy functions in the one dimensional case.

Definition 3 *([3]) Let $K \subset \mathbb{R}$ with $F : K \to \mathcal{F}_C$ a fuzzy function and $x_0 \in K$ and h be such that $x_0 + h \in K$. Then the generalized Hukuhara derivative (gH-derivative,* *for short) of F at x_0 is defined as*

$$F'(x_0) = \lim_{h\to 0} \frac{F(x_0+h) \ominus_{gH} F(x_0)}{h}. \tag{3}$$

If $F'(x_0) \in \mathcal{F}_C$ satisfying (3) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at x_0.

The gH-derivative for an interval-valued function [28] is similar to Definition 3. More precisely, an interval-valued function $F : K \to \mathcal{K}_C$ is gH-differentiable at $x_0 \in K$, with gH-derivative $F'(x_0) \in \mathcal{K}_C$, if (3) exists with respect to the limit in the metric space $(\mathcal{K}_C, H)$, where the difference is given by the gH-difference between intervals (see [28]).

Theorem 1 *Let $F : K \to \mathcal{F}_C$ be a fuzzy function. If F is gH-differentiable then the interval-valued function $F_\alpha : K \to \mathcal{K}_C$ is gH-differentiable for each $\alpha \in [0,1]$. Moreover*

$$[F'(x)]^\alpha = F'_\alpha(x). \tag{4}$$

Proof. The proof is a consequence of the definition of gH-differentiability. ■

Example 1 *Consider the fuzzy mapping $F : \mathbb{R} \to \mathcal{F}_C$ defined by $F(x) = C \cdot x$, where C is a fuzzy interval and $[C]^\alpha = [\underline{C}_\alpha, \overline{C}_\alpha]$ with $\underline{C}_\alpha < \overline{C}_\alpha$. Note that F is a generalization of a linear function and for each $\alpha \in [0,1]$ we have*

$$F_\alpha(x) = \begin{cases} \left[\underline{C}_\alpha x, \overline{C}_\alpha x\right] & \text{if } x \geq 0; \\ \left[\overline{C}_\alpha x, \underline{C}_\alpha x\right] & \text{if } x < 0. \end{cases}$$

Thus the endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are not differentiable at $x = 0$. However F is gH-differentiable on $\mathbb{R}$ and $F'(x) = C$ for all $x \in \mathbb{R}$. In general, if $F(x) = C \cdot g(x)$, where $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $C \in \mathcal{F}_C$, then it follows relatively easy that the gH-derivative exists and it is $F'(x) = C \cdot g'(x)$, but the endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are not necessarily differentiable.

In general we have the following result which connects gH-differentiability of F and the differentiability of its endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$.

Theorem 2 *Let $F : K \to \mathcal{F}_C$ be a fuzzy function. If F is gH-differentiable at $x_0 \in K$ then, for each $\alpha \in [0,1]$, one of the following cases hold:*

(*a*) *$\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at x_0 and*

$$[F'(x_0)]^\alpha = \left[\min\left\{(\underline{f}_\alpha)'(x_0), (\overline{f}_\alpha)'(x_0)\right\}, \max\left\{(\underline{f}_\alpha)'(x_0), (\overline{f}_\alpha)'(x_0)\right\}\right];$$

(b) $(\underline{f}_\alpha)'_-(x_0)$, $(\underline{f}_\alpha)'_+(x_0)$, $(\overline{f}_\alpha)'_-(x_0)$ *and* $(\overline{f}_\alpha)'_+(x_0)$ *exist and satisfy* $(\underline{f}_\alpha)'_-(x_0) = (\overline{f}_\alpha)'_+(x_0)$ *and* $(\underline{f}_\alpha)'_+(x_0) = (\overline{f}_\alpha)'_-(x_0)$. *Moreover*

$$\begin{aligned}[F'(t_0)]^\alpha &= \left[\min\left\{(\underline{f}_\alpha)'_-(x_0), (\overline{f}_\alpha)'_-(x_0)\right\}, \max\left\{(\underline{f}_\alpha)'_-(x_0), (\overline{f}_\alpha)'_-(x_0)\right\}\right] \\ &= \left[\min\left\{(\underline{f}_\alpha)'_+(x_0), (\overline{f}_\alpha)'_+(x_0)\right\}, \max\left\{(\underline{f}_\alpha)'_+(x_0), (\overline{f}_\alpha)'_+(x_0)\right\}\right]\end{aligned}$$

Proof. The proof is a consequence of Theorem 9 in [6] and Theorem 1. ■

Remark 1 *Note that the gH-differentiability is coincident with the H-differentiability (differentiability in the sense of Hukuhara introduced by Puri and Ralescu [22] as a generalization of the Hukuhara derivative for set-valued functions [14]) only when* $\underline{f}_\alpha$ *and* $\overline{f}_\alpha$ *are differentiable and* $(\underline{f}_\alpha)'(x) \leq (\overline{f}_\alpha)'(x)$ *for all* $\alpha \in [0,1]$. *Thus, the gH-differentiability is a more general concept of differentiability for fuzzy functions than the H-differentiability. The gH-differentiability concept is also more general than G-differentiability, see [1].*

We are now going to define the partial derivative for a fuzzy function F defined on $K \subset \mathbb{R}^n$, i.e., $F(x) = F(x_1, ..., x_n) \in \mathcal{F}_C$ for each $x = (x_1, ..., x_n) \in K$. For this, given a fuzzy function $F : K \to \mathcal{F}_C$, we denote the fuzzy interval $F(x)$ by $F(x) = \left[\underline{f}(x), \overline{f}(x)\right]$ and, for each $\alpha \in [0,1]$,

$$F_\alpha(x) = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right].$$

Definition 4 *Let F be a fuzzy function defined on $K \subset \mathbb{R}^n$ and let* $x_0 = \left(x_1^{(0)}, ..., x_n^{(0)}\right)$ *be a fixed element of K. We consider the fuzzy function* $h_i(x_i) = F(x_1^{(0)}, ..., x_{i-1}^{(0)}, x_i, x_{i+1}^{(0)}, ..., x_n^{(0)})$. *If h_i is gH-differentiable at* $x_i^{(0)}$, *then we say that F has the ith partial gH-derivative at x_0 (denoted by* $(\partial F/\partial x_i)(x_0)$*) and* $(\partial F/\partial x_i)(x_0) = (h_i)'(x_i^{(0)})$.

Definition 5 *Let F be a fuzzy function defined on K and let* $x_0 = \left(x_1^{(0)}, ..., x_n^{(0)}\right) \in K$ *be fixed. We say that F is gH-differentiable at x_0 if all the partial gH-derivatives* $(\partial F/\partial x_1)(x_0)$,..., $(\partial F/\partial x_n)(x_0)$ *exist on some neighborhood of x_0 and are continuous at x_0.*

Note that if F is gH-differentiable at x_0, then $(\partial F/\partial x_i)(x_0)$ is a fuzzy interval. So, for each $\alpha \in [0,1]$, we denote

$$\left[\frac{\partial F}{\partial x_i}(x_0)\right]^\alpha = \frac{\partial F_\alpha}{\partial x_i}(x_0) = \left[\frac{\partial \underline{F_\alpha}}{\partial x_i}(x_0), \frac{\partial \overline{F_\alpha}}{\partial x_i}(x_0)\right].$$

We obtain $\left(\partial \underline{F_\alpha}/\partial x_i\right)(x_0)$ and $\left(\partial \overline{F_\alpha}/\partial x_i\right)(x_0)$ from Theorem 2.

Next we present an interesting proposition which will be used to obtain our main results.

Proposition 1 *Let $F : K \to \mathcal{F}_C$ be a fuzzy function. If F is gH-differentiable at $x_0 \in K$ then, for each $\alpha \in [0, 1]$, the real-valued function $\underline{f}_\alpha + \overline{f}_\alpha : K \to \mathbb{R}$ is differentiable at x_0. Moreover,*

$$\frac{\partial \underline{F_\alpha}}{\partial x_i}(x_0) + \frac{\partial \overline{F_\alpha}}{\partial x_i}(x_0) = \frac{\partial \left(\underline{f}_\alpha + \overline{f}_\alpha\right)}{\partial x_i}(x_0). \tag{5}$$

Proof. The proof is a consequence of Theorem 2. ∎

From previous definition we can define the gradient of a fuzzy function as follows.

Definition 6 *Given the fuzzy function $F : K \to \mathcal{F}_C$, the gradient of F at x_0, denoted by $\tilde{\nabla} F(x_0)$, is defined by*

$$\tilde{\nabla} F(x_0) = \left(\left(\frac{\partial F}{\partial x_1}\right)(x_0), ..., \left(\frac{\partial F}{\partial x_n}\right)(x_0)\right), \tag{6}$$

where $(\partial F/\partial x_j)(x_0)$ is the jth partial G-derivative of F at x_0.

Note that $\tilde{\nabla} F(x)$ is a n-dimensional fuzzy vector. For the gradient of a fuzzy function we use the symbol $\tilde{\nabla}$, whereas for the gradient of a real-valued function we use the symbol ∇.

Definition 7 *Let $F : K \subset \mathbb{R}^n \to \mathcal{F}_C$ be a fuzzy function, where $K \subset \mathbb{R}^n$ is an open set. Suppose now that there is $x^0 \in K$ such that gradient of F, $\tilde{\nabla} F$, is itself gH-differentiable at x^0, that is, for each i, the function $\frac{\partial F}{\partial x_i} : K \to \mathcal{F}_C$ is gH-differentiable at x^0. Denote the gH-partial derivative of $\frac{\partial F}{\partial x_i}$ by*

$$D^2_{ij} F(x^0) \quad or \quad \frac{\partial^2 F}{\partial x_i x_j}(x^0), \quad if\ i \neq j,$$

and

$$D^2_{ii} F(x^0) \quad or \quad \frac{\partial^2 F}{\partial x_i^2}(x^0), \quad if\ i = j.$$

If F is twice gH-differentiable at each x^0 in K, we say that F is twice gH-differentiable on K, and if for each $i, j = 1, 2, ..., n$, the cross-partial derivative $\frac{\partial^2 F}{\partial x_i x_j}$ is continuous function from K to $\mathcal{F}_C$, we say that F is twice continuously gH-differentiable on K.

We define a m-times continuously gH-differentiable fuzzy function in way similar to Definition 7, that is, $F : K \to \mathcal{F}_C$ is m-times continuously gH-differentiable on K if and only if all of the partial gH-derivatives of order $m \in \mathbb{N}$ exist and are continuous (in the sense of fuzzy function).

If F is gH-differentiable we have that the endpoint function $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are not necessarily differentiable. However, from Proposition 1, we have that $\underline{f}_\alpha + \overline{f}_\alpha$ is always differentiable for all $\alpha \in [0,1]$. This property holds for the case of m-times gH-differentiability of $\underline{f}_\alpha + \overline{f}_\alpha$.

Proposition 2 *Let $F : K \to \mathcal{F}_C$ be a fuzzy function. If F is m-times gH-differentiable at $x_0 \in K$ then, for each $\alpha \in [0,1]$, the real-valued function $\underline{f}_\alpha + \overline{f}_\alpha : K \to \mathbb{R}$ is m-times differentiable at x_0.*

Proof. The proof follows from Proposition 1. ∎

4 Fuzzy optimization

We consider the following order relations on the space $\mathcal{F}_C$. Let u and v be two fuzzy intervals, so $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ and $[v]^\alpha = \left[\underline{v}_\alpha, \overline{v}_\alpha\right]$ are two intervals for all $\alpha \in [0,1]$. We write:

$$u \preceq v, \quad \text{iff} \quad [u]^\alpha \preceq [v]^\alpha, \quad \text{for all } \alpha \in [0,1]; \tag{7}$$

which is equivalent to writing $\underline{u}_\alpha \le \underline{v}_\alpha$ and $\overline{u}_\alpha \le \overline{v}_\alpha$ for all $\alpha \in [0,1]$.

$$u \prec v, \quad \text{iff} \quad u \preceq v \quad \text{and } u \neq v; \tag{8}$$

which is equivalent to $[u]^\alpha \preceq [v]^\alpha$ for all $\alpha \in [0,1]$ and there exists $\alpha^* \in [0,1]$ such that $\underline{u}_{\alpha^*} < \underline{v}_{\alpha^*}$ or $\overline{u}_{\alpha^*} < \overline{v}_{\alpha^*}$.

Now we consider the following optimization problem with fuzzy-valued objective function

$$(\textbf{FO}) \quad \min F(x), \quad x \in X$$

where $X \subset \mathbb{R}^n$ and $F : X \to \mathcal{F}_C$ is a fuzzy function.

Since "$\preceq$" and "$\prec$" are partial orderings on $\mathcal{F}_C$, we may follow the similar solution concept used in multiobjective programming problems.

Definition 8 *Let $X \subset \mathbb{R}^n$ be an open set. We say that $x^* \in X$ is a locally non-dominated solution of problem (FO) if there exists no $x \in N_\epsilon(x^*) \cap X$ such that $F(x) \prec F(x^*)$, where $N_\epsilon(x^*)$ is a ϵ-neighborhood of x^*.*

The following result has been proved in [21] and it is essential to Newton method implementation which was also proposed in the same article.

Theorem 3 *Let $F : X \to \mathcal{F}_C$ be a fuzzy function, where $X \subset \mathbb{R}^n$ is an open set. If $x^* \in X$ is a locally non-dominated solution of (FO) and for any direction d and for any $\delta > 0$ there exists $\lambda \in (0,\delta)$ such that $F(x^* + \lambda \cdot d)$ and $F(x^*)$ are comparable, then x^* is a local minimizer of the real-valued functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$, for all $\alpha \in [0,1]$.*

In Theorem 3, the condition: $x^* \in X$ is a locally non-dominated solution of (FO) and for any direction d and for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $F(x^* + \lambda \cdot d)$ and $F(x^*)$ are comparable, is very restrictive.

In fact, the order relation $\preceq$ is a partial order in the space of fuzzy intervals. For example, if we consider $F(x) = u \cdot x$, where $u = (-1, 0, 1)$, we have that the previous condition is not satisfied, i.e,. for all x and y, $F(x)$ and $F(y)$ are not comparable, where 0 is a nondominated solution of F

On the other hand, to require that the nondominated solution x^* be a local minimizer of both real-valued functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$, for all $\alpha \in [0, 1]$, is very restrictive. If this point existed, it would be an ideal point, as it is known in the multiobjective literature. It is very difficult to find fuzzy functions that possess ideal points. In fact, the Examples 4.1, 4.2, 4.3 and 4.4 of the article [21] do not have ideal point for all $\alpha \in [0, 1]$. For instance, take the Example 4.2, in which it has been introduced the fuzzy function

$$F(x_1, x_2) = (-1, 1, 3) \cdot x_1^2 + (0, 1, 2) \cdot x_1 x_2 + (1, 2, 4) \cdot x_2^2,$$

with $x_1,\ x_2 \in \mathbb{R}$. In this case, $x^* = (0, 0)$ is a non-dominated solution of F, however $x^* = (0, 0)$ is not a local minimizer of the endpoint function $\underline{f}_0$ since $\underline{f}_0(\epsilon, 0) < 0 = \underline{f}_0(0, 0)$, for all $\epsilon > 0$ near zero.

Next we provide a sufficient condition for a nondominated solution of F based only on the sum of the objective endpoint functions. This result generalizes the previous theorem and is more suitable for application of the Newton method.

Theorem 4 *Let $F : X \to \mathcal{F}_C$ be a fuzzy function, $X \subset \mathbb{R}^n$ is an open set. If x^* is a local minimizer of the real-valued function $\underline{f}_\alpha + \overline{f}_\alpha$, for all $\alpha \in [0, 1]$ then x^* is a locally non-dominated solution of (FO)*

Proof. Suppose that x^* is not a locally non-dominated solution of (FO). Then, there exist $x \in N_\epsilon(x^*)$ such the $F(x) \prec F(x^*)$. So, there exist $\alpha^* \in [0, 1]$ such that

$$\underline{f}_{\alpha^*}(x) \leq \underline{f}_{\alpha^*}(x^*) \text{ and } \overline{f}_{\alpha^*}(x) \leq \overline{f}_{\alpha^*}(x^*),$$

where at least one of the inequality is strict. So

$$(\underline{f}_{\alpha^*} + \overline{f}_{\alpha^*})(x) < (\underline{f}_{\alpha^*} + \overline{f}_{\alpha^*})(x^*).$$

Therefore, x^* is not a local minimizer of the real-valued function $\underline{f}_{\alpha^*} + \overline{f}_{\alpha^*}$. This complete the proof. ■

5 Newton method

In this section we propose a Newton method to find a non-dominated solution of (FO). For this, we assume that at each measurement point $x^{(k)}$ we can calculate $F\left(x^{(k)}\right)$, $\tilde{\nabla} F\left(x^{(k)}\right)$ and $\tilde{\nabla}^2 F\left(x^{(k)}\right)$. Thus, taking into account Propositions 1 and 2, we can also calculate $\underline{f}_\alpha\left(x^{(k)}\right)$, $\overline{f}_\alpha\left(x^{(k)}\right)$, $\nabla\left(\underline{f}_\alpha + \overline{f}_\alpha\right)\left(x^{(k)}\right)$ and $\nabla^2\left(\underline{f}_\alpha + \overline{f}_\alpha\right)\left(x^{(k)}\right)$ for all $\alpha \in [0, 1]$. Hence, for each $\alpha \in [0, 1]$, we can approximate the real-valued function $\underline{f}_\alpha + \overline{f}_\alpha$ by a quadratic real-valued function h_α, with the help of Taylor's formula and obtain

$$\begin{aligned} h_\alpha(x) = &\left(\underline{f}_\alpha + \overline{f}_\alpha\right)\left(x^{(k)}\right) + \nabla\left(\underline{f}_\alpha + \overline{f}_\alpha\right)\left(x^{(k)}\right) \cdot \left(x - x^{(k)}\right) \\ &+ \left\{\frac{1}{2}\left(x - x^{(k)}\right)^T \cdot \nabla^2\left(\underline{f}_\alpha + \overline{f}_\alpha\right)\left(x^{(k)}\right) \cdot \left(x - x^{(k)}\right)\right\}, \end{aligned}$$

for all $\alpha \in [0, 1]$.

If x^* is a minimizer of $\underline{f}_\alpha + \overline{f}_\alpha$, for all $\alpha \in [0, 1]$, given $x^{(k)}$ we try to approximate a minimizer of $\underline{f}_\alpha + \overline{f}_\alpha$ by finding a minimizer of h_α, for all $\alpha \in [0, 1]$. From first-order necessary condition for h_α we have

$$\nabla h_\alpha(x^*) = 0,$$

for all $\alpha \in [0, 1]$. This implies that

$$\int_0^1 \nabla\left(\underline{f}_\alpha + \overline{f}_\alpha\right)\left(x^{(k)}\right) d\alpha + \int_0^1 \nabla^2\left(\underline{f}_\alpha + \overline{f}_\alpha\right)\left(x^{(k)}\right) d\alpha\left(x - x^{(k)}\right) = 0. \tag{9}$$

Define the real-function $H : \mathbb{R}^n \to \mathbb{R}$ by

$$H(x) = \int_0^1 \left(\underline{f}_\alpha + \overline{f}_\alpha\right)(x)\, d\alpha.$$

If the fuzzy function F is twice continuously gH-differentiable then, from Proposition 2, $\underline{f}_\alpha + \overline{f}_\alpha$ is twice continuously gH-differentiable for all $\alpha \in [0, 1]$. Thus H is also twice continuously gH-differentiable. Therefore, from (9) we have

$$\nabla H\left(x^{(k)}\right) + \nabla^2 H\left(x^{(k)}\right) \cdot \left(x - x^{(k)}\right) = 0, \tag{10}$$

By putting $x = x^{(k+1)}$ in (10), we arrive at

$$x^{(k+1)} = x^{(k)} - \nabla H(x^{(k)}) \cdot \left[\nabla^2 H\left(x^{(k)}\right)\right]^{-1}, \tag{11}$$

where $\left[\nabla^2 H\left(x^{(k)}\right)\right]^{-1}$ is the inverse of matrix $\nabla^2 H\left(x^{(k)}\right)$. Thus, starting with an initial approximation to a minimizer of F, we can generate a sequence of approximations to this minimizer of F by using formula (11). The procedure is terminated when $\|x^{(k+1)} - x^{(k)}\| < \epsilon$, where ϵ is a pre-specified termination scalar.

Note that by using formula (11) we are able to find stationary points of $\underline{f}_\alpha + \overline{f}_\alpha$ for all $\alpha \in [0,1]$. However, to determine if these points are minimizers of $\underline{f}_\alpha + \overline{f}_\alpha$, one needs second order sufficient conditions or some other type of conditions, such as convexity or generalized convexity of $\underline{f}_\alpha + \overline{f}_\alpha$, for all $\alpha \in [0,1]$.

5.1 Convergence of the Newton Method

In this subsection we will show convergence of the Newton method.

Theorem 5 *Suppose that F is a three times continuously gH-differentiable fuzzy function defined on $\mathbb{R}^n$ and $x^* \in \mathbb{R}^n$ is a point such that*

(*i*) $\nabla H(x^*) = 0$*;*
(*ii*) $\nabla^2 H(x^*)$ *is invertible;*

Then for all $x^{(0)}$ sufficiently close to x^*, the Newton method is well defined for all k, and converges to x^* with order of convergence at least 2.

Proof. Since F is three times continuously gH-differentiable fuzzy function then H is three times continuously differentiable function. So, the proof follows as in the classical proof of the newton method. ■

5.2 Numerical Examples

In this section we present some examples to justify the proposed method. In addition we correct the examples given in [21].

Example 2 *(Example 4.1, [21]) Consider the following nonlinear fuzzy optimization problem,*

$$\min F(x_1, x_2) = \tilde{1} \cdot x_1^3 \oplus \tilde{2} \cdot x_2^3 \oplus \tilde{1} \cdot x_1 \cdot x_2, \qquad x_1, x_2 \in \mathbb{R},$$

where $\tilde{1} = (-1, 1, 3)$ and $\tilde{2} = (1, 2, 3)$ are triangular fuzzy numbers. So, for each $\alpha \in [0,1]$, we have

$$
\begin{aligned}
&[F(x_1, x_2)]^\alpha \\
&= [\tilde{1}]^\alpha x_1^3 + [\tilde{2}]^\alpha x_2^3 + [\tilde{1}]^\alpha x_1 x_2 \\
&= [-1 + 2\alpha, 3 - 2\alpha] x_1^3 + [1 + \alpha, 3 - \alpha] x_2^3 + [-1 + 2\alpha, 3 - \alpha] x_1 x_2 \\
&= \begin{cases}
[(-1 + 2\alpha)x_1^3, (3 - 2\alpha)x_1^3] + [(1 + \alpha)x_2^3, (3 - \alpha)x_2^3] & \\
\quad +[(-1 + 2\alpha)x_1 x_2, (3 - 2\alpha)x_1 x_2], & \text{if } x_1 \geq 0,\ x_2 \geq 0; \\
[(-1 + 2\alpha)x_1^3, (3 - 2\alpha)x_1^3] + [(3 - \alpha)x_2^3, (1 + \alpha)x_2^3] & \\
\quad +[(3 - 2\alpha)x_1 x_2, (-1 + 2\alpha)x_1 x_2], & \text{if } x_1 \geq 0,\ x_2 < 0; \\
[(3 - 2\alpha)x_1^3, (-1 + 2\alpha)x_1^3] + [(3 - \alpha)x_2^3, (1 + \alpha)x_2^3] & \\
\quad +[(-1 + 2\alpha)x_1 x_2, (3 - 2\alpha)x_1 x_2], & \text{if } x_1 < 0,\ x_2 < 0; \\
[(3 - 2\alpha)x_1^3, (-1 + 2\alpha)x_1^3] + [(1 + \alpha)x_2^3, (3 - \alpha)x_2^3] & \\
\quad +[(3 - 2\alpha)x_1 x_2, (-1 + 2\alpha)x_1 x_2], & \text{if } x_1 < 0,\ x_2 \geq 0.
\end{cases}
\end{aligned}
$$

Therefore the endpoint functions $\underline{f}_\alpha$ *and* $\overline{f}_\alpha$ *are defined by*

$$
\underline{f}_\alpha(x_1, x_2) = \begin{cases}
(-1 + 2\alpha)x_1^3 + (1 + \alpha)x_2^3 + (-1 + 2\alpha)x_1 x_2, & \text{if } x_1 \geq 0,\ x_2 \geq 0; \\
(-1 + 2\alpha)x_1^3 + (3 - \alpha)x_2^3 + (3 - 2\alpha)x_1 x_2, & \text{if } x_1 \geq 0,\ x_2 < 0; \\
(3 - 2\alpha)x_1^3 + (3 - \alpha)x_2^3 + (-1 + 2\alpha)x_1 x_2, & \text{if } x_1 < 0,\ x_2 < 0; \\
(3 - 2\alpha)x_1^3 + (1 + \alpha)x_2^3 + (3 - 2\alpha)x_1 x_2, & \text{if } x_1 < 0,\ x_2 \geq 0.
\end{cases}
$$

and

$$
\overline{f}_\alpha(x_1, x_2) = \begin{cases}
(3 - 2\alpha)x_1^3 + (3 - \alpha)x_2^3 + (3 - 2\alpha)x_1 x_2, & \text{if } x_1 \geq 0,\ x_2 \geq 0; \\
(3 - 2\alpha)x_1^3 + (1 + \alpha)x_2^3 + (-1 + 2\alpha)x_1 x_2, & \text{if } x_1 \geq 0,\ x_2 < 0; \\
(-1 + 2\alpha)x_1^3 + (1 + \alpha)x_2^3 + (3 - 2\alpha)x_1 x_2, & \text{if } x_1 < 0,\ x_2 < 0; \\
(-1 + 2\alpha)x_1^3 + (3 - \alpha)x_2^3 + (-1 + 2\alpha)x_1 x_2, & \text{if } x_1 < 0,\ x_2 \geq 0.
\end{cases}
$$

In the Example 4.1 in [21], the authors have obtained other endpoint functions which are not the correct ones.

In our developments, we can clearly see that $\underline{f}_\alpha$ *and* $\overline{f}_\alpha$ *are not differentiable and thus F is not H-differentiable. Therefore we can not apply the procedure of the Newton method given in [21].*

Nonetheless, we note that F is three times gH-differentiable and from Proposition 2 we have that $\underline{f}_\alpha + \overline{f}_\alpha$ *is three times differentiable and*

$$
\left(\underline{f}_\alpha + \overline{f}_\alpha\right)(x_1, x_2) = 2x_1^3 + 4x_2^3 + 2x_1 x_2,
$$

for all $\alpha \in [0,1]$. *Therefore*

$$H(x) = \int_0^1 \left(\underline{f}_\alpha + \overline{f}_\alpha\right)(x_1, x_2)d\alpha = 2x_1^3 + 4x_2^3 + 2x_1x_2.$$

Then

$$\nabla H(x_1, x_2) = \begin{pmatrix} 6x_1^2 + 2x_2 \\ 12x_2^2 + 2x_1 \end{pmatrix}$$

and

$$\nabla^2 H(x_1, x_2) = \begin{pmatrix} 12x_1 & 2 \\ 2 & 24x_2 \end{pmatrix}.$$

Now we obtain a sequence $\{x^{(k)}\}$, $k = 1, 2, ...$ *by using the following equation*

$$x^{(k+1)} = x^{(k)} - \nabla H(x^{(k)}) \cdot \left[\nabla^2 H\left(x^{(k)}\right)\right]^{-1}, \tag{12}$$

and get that $x^* = (0,0)$ *is a stationary point of* $\underline{f}_\alpha + \overline{f}_\alpha$, *for all* $\alpha \in [0,1]$, *with accuracy* 10^{-3}. *Since* $\underline{f}_\alpha + \overline{f}_\alpha$ *is not invex then we can not ensure that* $x^* = (0,0)$ *is a non-dominated solution of* F. *In fact,* $x^* = (0,0)$ *is not a non-dominated solution since* $F(0, -\epsilon) \prec F(0,0)$. *This corrects Example 4.1 in [21], where the authors claim that* $x^* = (0,0)$ *is a non-dominated solution.*

Example 3 *Consider the following problem*

$$\min F(x_1, x_2) = (-1, 1, 3) \cdot x_1^3 \oplus (0, 1, 2) \cdot x_1 \cdot x_2 \oplus (1, 2, 4) \cdot x_2^2, \quad x_1, x_2 \in \mathbb{R}.$$

In this case F *is not H-differentiable but it is three times gH-differentiable. Also*

$$\left(\underline{f_\alpha} + \overline{f_\alpha}\right)(x_1, x_2) = 2x_1^2 = 2x_1x_2 + (5 - \alpha)x_2^2.$$

We search for the stationary point of $\underline{f_\alpha} + \overline{f_\alpha}$, *for all* $\alpha \in [0,1]$, *using the Newton method previously proposed. For this, we consider the initial point* $x_0 = (2, -2)$ *and calculate the sequence* $\{x^{(k)}\}$, $k = 1, 2, ...$ *by making use of equation (11). We obtain* $x^* = (0,0)$ *as a stationary point of* $\underline{f_\alpha} + \overline{f_\alpha}$, *for all* α, *with accuracy* 10^{-3}. *Since* $\underline{f_\alpha} + \overline{f_\alpha}$ *is convex,* x^* *is minimizer of* $\underline{f_\alpha} + \overline{f_\alpha}$. *So, from Theorem 4, we have* x^* *is a nondomited point of* F.

The Newton method proposed in [21] could not be applied for this example, because F is not H-differentiable and does not satisfy the required conditions for the implementation of this method.

The fuzzy functions of the other examples considered by Pirzada and Pathak in [21] are also not H-differentiable and do not possess the required conditions for the implementation of the Newton method they proposed. Nonetheless, these examples do satisfy the conditions to the Newton method proposed here and, as in the previous examples, we may obtain nondominated solutions.

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