# On minimax-regret Huff location models*

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**Abstract**

We address the following single-facility location problem: a firm is entering into a market by locating one facility in a region of the plane. The demand captured from each user by the facility will be proportional to the users buying power and inversely proportional to a function of the user-facility distance. Uncertainty exists on the buying power (weight) of the users. This is modeled by assuming that a set of scenarios exists, each scenario corresponding to a weight realization. The objective is to locate the facility following the Savage criterion, i.e., the minimax-regret location is sought. The problem is formulated as a global optimization problem with objective written as difference of two convex monotonic functions. The numerical results obtained show that a branch and bound using this new method for obtaining bounds clearly outperforms benchmark procedures.

**Keywords:** continuous location, Huff model, DC functions, DCM functions, global optimization, minimax regret.

## 1 The model

The Huff location model [2, 10, 13, 17, 18] in the plane can be described as follows: Let $A \subset \mathbb{R}^2$ be a set of users, asking for a certain service. Each user $a \in A$ has demand $\omega_a$. Such demand is being patronized by the different existing facilities, located at points $x_1, \ldots, x_r$, so that the demand captured by facility at $x_i$ from user $a$ is inversely proportional to a positive non-decreasing function of the distance $\|a - x_i\|$ from the user at $a$ to the facility at $x_i$. In other words, the demand captured by the facility at $x_i$ from the user at $a$ is given by

$$\omega_a \frac{1/\varphi_{ai}(\|a - x_i\|)}{\sum_{j=1}^{r} 1/\varphi_{aj}(\|a - x_j\|)}. \tag{1}$$

The norm $\|\cdot\|$ is typically the Euclidean norm, and the usual choice for each $\varphi_{aj}$ has the form $\varphi_{aj}(d) = d^\alpha$. When $\alpha = 2$, we have the so-called gravitational model.

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A new firm is entering the market, by locating one facility at some $x \in S$. This perturbs market share, since the new facility at $x$ will capture a demand from $a \in A$ equal to

$$\omega_a \frac{1/\varphi_a(\|a-x\|)}{1/\varphi_a(\|a-x\|) + \sum_{j=1}^{r} 1/\varphi_{aj}(\|a-x_j\|)}. \tag{2}$$

Here $\varphi_a$ is assumed to be non-negative, non-decreasing and continuous in $\mathbb{R}_+$. The goal of the entering firm is the maximization of its market share. This is written as the following optimization problem:

$$\max_{x \in S} \sum_{a \in A} \omega_a \frac{1/\varphi_a(\|a-x\|)}{1/\varphi_a(\|a-x\|) + \sum_{j=1}^{r} 1/\varphi_{aj}(\|a-x_j\|)}. \tag{3}$$

Defining for each $a \in A$ the positive constant $\beta_a$,

$$\beta_a = \sum_{j=1}^{r} \frac{1}{\varphi_{aj}(\|a-x_j\|)}, \tag{4}$$

it follows that Problem (3) can be rewritten as

$$\max_{x \in S} \sum_{a \in A} \omega_a \frac{1}{1 + \beta_a \varphi_a(\|a-x\|)}. \tag{5}$$

Problem (5) is a multimodal problem, solved heuristically in [7], and via Global-Optimization methods, among others, in [2, 10].

An important limitation in practice of (5) is the assumption that weights are known. Since weights are affected, among other things, by the demographic growth of each $a \in A$, a more realistic model would accommodate some type of uncertainty in these parameters. In a recent paper by Tammy Drezner, [8], uncertainty is modeled by assuming the existence of a (finite) set $E$ of different scenarios, where, for each scenario $e \in E$, a vector $(\omega_a^e)_{a \in A}$ of weights is given. Under scenario $e \in E$, the ideal market share for the entering firm is $z^e$,

$$z^e = \max_{x \in S} \sum_{a \in A} \omega_a^e \frac{1}{1 + \beta_a \varphi_a(\|a-x\|)}. \tag{6}$$

The aim in [8], also pursued here, is to find the location for the facility achieving a market share closest to the ideal, i.e, by solving

$$\min_{x \in S} F(x) := \left\| \left( z^e - \sum_{a \in A} \omega_a^e \frac{1}{1 + \beta_a \varphi_a(\|a-x\|)} \right)_{e \in E} \right\|_p, \tag{7}$$

where $\|\cdot\|_p$ is the $\ell_p$ norm, and $1 \leq p \leq \infty$. See [6, 20, 21] for further details on minimax regret, and also [9] for alternative approaches to modeling uncertainty in continuous location with competition.

In [8] the following procedure is suggested to obtain a global optimum of (7): a branch-and-bound method proposed in [12] is used. Lower bounds for $F$ are obtained in two steps:

1. Upper bounds $U^e$ for each term $\sum_{a \in A} \omega_a^e \frac{1}{1+\beta_a \varphi_a(\|a-x\|)}$ are constructed following [10].

2. A lower bound for $F$ is given by $\| \left(z^e - U^e\right)_{e \in E} \|_p$.

The purpose of this paper is to show how a new bounding strategy, proposed by the authors in [2], can notably reduce the computing times and storage requirements, enabling one to solve in reasonable times more complex problems, with many more data points or scenarios. In our procedure, applicable also to more general settings than those described in [8], the objective function $F$ is expressed as $F(x) = G(D(x))$, with $D(x) = (\|a - x\|)_{a \in A}$ and $G : \mathbb{R}^{|A|} \to \mathbb{R}$ defined as

$$G((d_a)_{a \in A}) = \left\| \left( z^e - \sum_{a \in A} \omega_a^e \frac{1}{1 + \beta_a \varphi_a(d_a)} \right)_{e \in E} \right\|_p, \tag{8}$$

and then the properties of the function $G$ are successfully exploited. As shown in Section 2, $G$ is *DCM*, i.e., it can be written as a difference of two convex and monotonic functions [2]. As a consequence, the bounding strategy for DCM functions described in [2] can be used to solve (7).

Section 3 presents numerical results, showing that our strategy outperforms the method suggested in [8].

# 2 Properties

We discuss some general properties of the objective function in (7), which will be used in Section 2.2 to construct bounds for $F$.

## 2.1 $G$ is DCM

We recall the reader that, given a convex set $\Omega \subset \mathbb{R}^n$, a function is said to be *DC* on $\Omega$ if it can be written as a difference of two convex functions on $\Omega$. DC functions constitute a wide class of functions (note that convex and concave functions are dc), that can be found in many applications fields, Locational Analysis being one of them, [1, 2, 4, 5, 11, 15, 19]. In [2] a proper subset of dc functions has been introduced, namely the set of functions that can be expressed as the difference of two convex and monotonic (DCM) functions. When the objective function can be written as the composition of a DCM function with a convex function, sharp bounds can be obtained, enabling one to design more efficient branch and bound procedures.

In what follows we assume that each $\varphi_a$ in (7) is such that the function $1/(1 + \beta_a \varphi_a(d))$ is DCM in $\mathbb{R}_+$ and there exists a DCM decomposition where both functions are non-decreasing or non-increasing simultaneously. A sufficient condition for this to happen is given in the following result.

**Proposition 1** *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a convex and non-decreasing function and let $\beta \geq 0$. Then $\Phi(d) = 1/(1 + \beta\varphi(d))$ is DCM in $\mathbb{R}_+$ and a DCM decomposition with non-decreasing components is given by:*

$$\Phi(d) = (\Phi(d) + (1 + \beta\varphi(d))) - (1 + \beta\varphi(d)). \tag{9}$$

**Proof.** The function $h(t) = 1/t + t$ is convex and non-decreasing in $[1, +\infty]$, since $h'(t) = -1/t^2 + 1 \geq 0 \; \forall \; t \geq 1$ and $h''(t) = 2/t^3 \geq 0 \; \forall \; t \geq 1$. Taking into account that $1 + \beta\varphi(d)$ is convex and non-decreasing, it follows that the function $h(1 + \beta\varphi(d))$ has the same properties and the result holds. $\square$

An alternative DCM decomposition can be easily obtained when $\varphi(d) = d^\alpha$ with $\alpha \geq 1$. In such a case, the function $\Phi$ in Proposition 1 is concave in the interval $[0, d_0)$ and convex in $[d_0, +\infty)$, where

$$d_0 = \left( \frac{\alpha - 1}{(\alpha + 1)\beta} \right)^{\frac{1}{\alpha}}.$$

Hence, $\Phi^1 - \Phi^2$ is a DC decomposition of $\Phi$ where

$$\Phi^1(d) = \begin{cases} \Phi(d_0) + \Phi'(d_0)(d - d_0) & \text{if } d \leq d_0 \\ \Phi(d) & \text{if } d > d_0 \end{cases} \qquad \Phi^2(d) = \begin{cases} \Phi(d_0) + \Phi'(d_0)(d - d_0) - \Phi(d) & \text{if } d \leq d_0 \\ 0 & \text{if } d > d_0 \end{cases} \tag{10}$$

Moreover, $\Phi^1 - \Phi^2$ is a DCM decomposition of $\Phi$. Indeed, $\Phi^1$ is clearly non-increasing, as $\Phi$ has the same property. On the other hand, $\Phi^2$ is also a non-increasing function since its derivative, $\Phi'(d_0) - \Phi'(d)$, is non-positive in $[0, d_0)$ due to the concavity of $\Phi$ in such interval.

The concept of DCM function was originally introduced in [2] for a function $\varphi : K \to \mathbb{R}$ with $K \subset \mathbb{R}$. Now we extend that definition to a real function defined over a subset of $\mathbb{R}^n$.

**Definition 2** *Given $K \subset \mathbb{R}^n$, a function $\varphi : K \to \mathbb{R}$ is said to be monotonic in $K$ if $\varphi$ is a non-decreasing componentwise function or a non-increasing componentwise function in $K$, i.e. either one has for all $x, y \in K$ with $x_i \leq y_i$ , $1 \leq i \leq n$, that $\varphi(x) \leq \varphi(y)$ or one has that $\varphi(x) \geq \varphi(y)$.*

**Definition 3** *Given $K \subset \mathbb{R}^n$, a function $\varphi : K \to \mathbb{R}$ is said to be difference of convex monotonic (DCM) in $K$ if there exist $\varphi^1, \varphi^2 : K \to \mathbb{R}$, convex and monotonic in $K$ such that $\varphi = \varphi^1 - \varphi^2$.*

We claim that, under mild assumptions, the function $G$ defined in (8) is DCM, and a decomposition as a difference of two convex and monotonic functions is then given.

**Proposition 4** *For each $a \in A$, let $1/(1 + \beta_a\varphi_a(d)) = f_a(d) - g_a(d)$, with $f_a, g_a$ convex and monotonic in $\mathbb{R}_+$, and assume that the functions $\{f_a, g_a\}_{a\in A}$ are all of them non-decreasing or non-increasing simultaneously.*

*Define for each $a \in A$ the scalar $\Omega_a = \|(\omega_a^e)_{e\in E}\|_p$. Then, $G$ as defined in (8) is DCM, and can be written as*

$$G((d_a)_{a\in A}) = \left( G((d_a)_{a\in A}) + \sum_{a\in A} \Omega_a \left(f_a(d_a) + g_a(d_a)\right) \right) - \sum_{a\in A} \Omega_a \left(f_a(d_a) + g_a(d_a)\right), \tag{11}$$

*where $G((d_a)_{a\in A}) + \sum_{a\in A} \Omega_a \left(f_a(d_a) + g_a(d_a)\right)$, $\sum_{a\in A} \Omega_a \left(f_a(d_a) + g_a(d_a)\right)$ are both convex and monotonic in $\mathbb{R}_+^{|A|}$.*

**Proof.** The idea of the proof is similar to the one given in [3]. Let $\|\cdot\|_q$ be the dual norm to $\|\cdot\|_p$ in $\mathbb{R}^{|E|}$. By definition of dual norm, we have that

$$\begin{array}{rcl} G((d_a)_{a\in A}) & = & \max_{u\in\mathbb{R}^{|E|}:\,\|u\|_q\le 1}\left\{\sum_{e\in E} u_e\left(z^e - \sum_{a\in A}\omega_a^e\left(f_a(d_a)-g_a(d_a)\right)\right)\right\} \\ & = & \max_{u\in\mathbb{R}^{|E|}:\,\|u\|_q\le 1}\left\{\sum_{e\in E} u_e z^e + \sum_{a\in A}\left(\Omega_a + \sum_{e\in E} u_e\omega_a^e\right) g_a(d_a) + \right. \\ & & \left. + \sum_{a\in A}\left(\Omega_a - \sum_{e\in E} u_e\omega_a^e\right) f_a(d_a)\right\} - \sum_{a\in A}\Omega_a\left(f_a(d_a)+g_a(d_a)\right). \end{array} \tag{12}$$

By Hölder's inequality, it follows for all $u\in\mathbb{R}^{|E|}$, $\|u\|_q\le 1$, that

$$\pm\sum_{e\in E} u_e\omega_a^e \le \Omega_a \qquad \forall a\in A, \tag{13}$$

thus

$$\Omega_a \pm \sum_{e\in E} u_e\omega_a^e \ge 0 \qquad \forall a\in A. \tag{14}$$

This implies that, for each $u$, $\|u\|_q\le 1$, the following functions are convex:

$$\begin{array}{l} \sum_{a\in A}\left(\Omega_a + \sum_{e\in E} u_e\omega_a^e\right) g_a(d_a) \\ \sum_{a\in A}\left(\Omega_a - \sum_{e\in E} u_e\omega_a^e\right) f_a(d_a) \\ \sum_{e\in E} u_e z^e + \sum_{a\in A}\left(\Omega_a + \sum_{e\in E} u_e\omega_a^e\right) g_a(d_a) + \sum_{a\in A}\left(\Omega_a - \sum_{e\in E} u_e\omega_a^e\right) f_a(d_a). \end{array}$$

Since the maximum of convex functions is convex, we have shown convexity of the function $\max_{u\in\mathbb{R}^{|E|}:\,\|u\|_q\le 1}\left\{\sum_{e\in E} u_e z^e + \sum_{a\in A}\left(\Omega_a + \sum_{e\in E} u_e\omega_a^e\right) g_a(d_a) + \sum_{a\in A}\left(\Omega_a - \sum_{e\in E} u_e\omega_a^e\right) f_a(d_a)\right\}$, which, by (12), equals $G((d_a)_{a\in A}) + \sum_{a\in A}\Omega_a\left(f_a(d_a)+g_a(d_a)\right)$. Moreover, since each $\Omega_a\ge 0$, the function $\sum_{a\in A}\Omega_a\left(f_a(d_a)+g_a(d_a)\right)$ is also convex. Hence, (11) gives a DC decomposition of $G$.

Finally, since all the functions $f_a$ and $g_a$ are non-decreasing or non-increasing at the same time and the coefficients $\Omega_a$, $\Omega_a \pm \sum_{e\in E} u_e\omega_a^e$ are, by (14), non-negative, it follows that the two functions involved in (11) yield also a DCM decomposition of $G$, as asserted. □

By applying Proposition 4, we are in position to derive a bounding procedure for the objective function $F$ of Problem (7).

### 2.2 Constructing lower bounds for $F$

The bounding procedure proposed here is an extension of that obtained in [2] for a DCM function defined over $\mathbb{R}$. Let $S\subset\mathbb{R}^2$ be a polytope where a lower bound of $F$ is computed, and let $G^1 - G^2$ be a DCM decomposition of $G$, for instance, following Proposition 4.

A lower bound of $F(x)$ over $S$ is obtained as follows:

1. Construct a concave minorant $L(x)$ of $F^1(x) = G^1(D(x))$ over $S$.

2. Construct a convex majorant $U(x)$ of $F^2(x) = G^2(D(x))$ over $S$.

3. Compute a lower bound of $F$ on $S$ from $L$ and $U$.

The three steps are now detailed.

### 2.2.1 Constructing a concave minorant of $F^1$

1. If $G^1$ is componentwise non-decreasing , then $F^1(x) = G^1(D(x))$ is the composition of a convex and non-decreasing componentwise function with a convex function. Hence, it is also convex, [16], and it can be bounded below by an affine function. Indeed, if $x^0 \in S \setminus A$ and $\xi$ is a subgradient at $x^0$ of the convex function $F^1(x) = G^1(D(x))$, one has the following:
$$G^1(D(x)) \geq G^1(D(x^0)) + \xi^T(x - x^0) \qquad \forall\, x \in S$$
by definition of subgradient.

2. If $G^1$ is componentwise non-increasing , then given $x^0 \in S \setminus A$, for any $\xi$, subgradient at $d^0 = D(x^0)$ of $G^1$, by definition of subgradient, one has
$$G^1(d) \geq G^1(d^0) + \xi^T(d - d^0) \qquad \forall d$$
and then,
$$G^1(D(x)) \geq G^1((D(x^0)) + \xi^T(D(x) - D(x^0))$$
Since $G^1$ is assumed to be componentwise non-increasing, one has that $\xi_a \leq 0\ \forall\ a \in A$, and hence the minorant found is concave.

### 2.2.2 Constructing a convex majorant of $F^2$

1. If $G^2$ is componentwise non-decreasing, then $F^2(x) = G^2(D(x))$ is convex, since it is the composition of a convex and non-decreasing componentwise function with a convex function, [16]. Hence, we can take $U(x) = F^2(x)$.

2. If $G^2$ is componentwise non-increasing, then given $x^0 \in S \setminus A$, for each $a \in A$ let $\xi_a$ be a subgradient of $\|a - \cdot\|$ at $x^0$. Then, by definition of subgradient, one has
$$\|a - x\| \geq \|a - x^0\| + \xi_a^T(x - x^0) \qquad \forall\ a \in A$$
and, since $G^2$ is componentwise non-increasing,
$$G^2(D(x)) \leq G^2((\|a - x^0\| + \xi_a^T(x - x^0))_{a \in A})$$
we obtain a convex majorant, since it is the composition of a convex function and a componentwise affine function.

### 2.2.3 Computing a lower bound of $F$

Once a concave minorant $L$ of $F^1$ and a convex majorant $U$ of $F^2$ have been obtained, one has that $L(x) - U(x)$ is a concave function bounding $F$. Hence, denoting by $V = \{v_i\ :\ i \in I\}$ the set of extreme points of $S$, one has
$$F(x) = G(D(x)) = F^1(x) - F^2(x) \geq L(x) - U(x) \geq \min_{i \in I}(L(v_i) - U(v_i)) \qquad \forall\ x \in S.$$

In summary, the previous procedure yields the following lower bound $\underline{F}$ on $S$ for $F$:
$$F(x) \geq \underline{F} := \min_{i \in I}(L(v_i) - U(v_i)) \qquad \forall\ x \in S.$$

#### 2.2.4 Bounding procedure: An example

In order to illustrate the bounding procedure proposed in this paper, we address here the problem of locating a competitive facility in the square $S = [0, 10] \times [0, 10]$, with a set $A$ of four users located at positions $(2, 1)$, $(9, 4)$, $(6, 5)$ and $(3, 9)$, and two existing facilities, at $(7, 2)$ and $(3, 5)$. Three scenarios for the demand have been considered, with weights given in the following table:

| | User | | | |
|---|---|---|---|---|
| Scenario | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| $e_1$ | 4 | 1 | 1 | 2 |
| $e_2$ | 1 | 1 | 1 | 1 |
| $e_3$ | 1 | 6 | 3 | 2 |

All functions $\varphi_{aj}$ in (5) are assumed to be $\varphi_{aj}(d) = d^2$, and the norm there considered is the Euclidean. As regards the $\ell_p$ norm in (7), we have taken $p = \infty$, as in [8]. The constants $\beta_a$ in (5) have been computed according to (4), giving the following values:

$$\beta_1 = 0.097285 \qquad \beta_2 = 0.152027 \qquad \beta_3 = 0.211111 \qquad \beta_4 = 0.077885.$$

First, the values $z^e$ in (6) have been computed. To do this, for each scenario $e$, a branch-and-bound algorithm has been used. Bounds were calculated using the fact that each term in (4) is a DCM function of $\|a - .\|$ (see [2] for details). This yields:

$$z^1 = 4.570515 \qquad z^2 = 1.982744 \qquad z^3 = 7.546552$$

As previously stated in the paper, the objective function $F(x)$ of Problem (7) can be written as $F(x) = G(D(x))$, where $D(x) = (\|a - x\|)_{a \in A}$ and $G$ is a DCM function. Moreover, a DCM decomposition $G^1 - G^2$ of $G$ is obtained by using Proposition 4, provided that a DCM decomposition of the function $1/(1 + \beta_a \varphi_a(d))$ is known. The constant values $\Omega_a$ in (11) turned out to be:

$$\Omega_1 = \|(4, 1, 1)\|_\infty = 4 \quad \Omega_2 = \|(1, 1, 6)\|_\infty = 6 \quad \Omega_3 = \|(1, 1, 3)\|_\infty = 3 \quad \Omega_4 = \|(2, 1, 2)\|_\infty = 2$$

We have computed two bounds, DCM1 and DCM2, for the objective function $F$ in (7) over the initial square $S$. Both are based on the DCM approach described in the paper, and differ from each other in the DCM decomposition used: DCM1 has been obtained from (9), whereas (10) has been used in DCM2.

Firstly we analyze DCM1. Take the DCM decomposition of $1/(1+\beta_a \varphi_a(d))$ given by (9) and consider the initial square $S$, whose extreme points are $v_1 = (0, 0)^\top$, $v_2 = (10, 0)^\top$, $v_3 = (10, 10)^\top$ and $v_4 = (0, 10)^\top$. Since the components of the DCM decomposition (9) are non-decreasing, we have that $G^1$ is non-decreasing too. Hence, a concave minorant of $F^1(x) = G^1(D(x))$ is given by:

$$L(x) = G^1(D(x^0)) + \xi^\top (x - x^0)$$

where $x^0 \in S \setminus A$ and $\xi$ is a subgradient of $F^1$ at $x^0$. If we choose $x^0 = (5, 5)^\top$, then we have $D(x^0) = (5, \sqrt{17}, 1, 2\sqrt{5})^\top$ and $G^1(D(x^0)) = 96.388609$. On the other hand, using the algebra of subgradients it turns out that $\xi = (-11.360750, 7.184373)^\top$ is a subgradient of $F^1$ at $x^0$. We evaluate $L$ at each extreme point of $S$, with the following results:

$$\begin{array}{rcl} L(v_1) &=& 96.388609 + (-11.360750, 7.184373)(-5,-5)^\top &=& 117.270495 \\ L(v_2) &=& 96.388609 + (-11.360750, 7.184373)(5,-5)^\top &=& 3.662993 \\ L(v_3) &=& 96.388609 + (-11.360750, 7.184373)(5,5)^\top &=& 75.506723 \\ L(v_4) &=& 96.388609 + (-11.360750, 7.184373)(-5,5)^\top &=& 189.114224 \end{array}$$

Regarding $F^2(x) = G^2(D(x))$, we have that $G^2$ is non-decreasing, so that an upper convex majorant is given directly by:

$$U(x) = F^2(x)$$

Evaluating $U$ at each extreme point of $S$, we obtain:

$$\begin{array}{rcl} U(v_1) &= G^2((\sqrt{5}, \sqrt{97}, \sqrt{61}, 3\sqrt{10})^\top) &=& 319.693917 \\ U(v_2) &= G^2((\sqrt{65}, \sqrt{17}, \sqrt{41}, \sqrt{130})^\top) &=& 206.745637 \\ U(v_3) &= G^2((\sqrt{145}, \sqrt{37}, \sqrt{41}, 5\sqrt{2})^\top) &=& 279.750731 \\ U(v_4) &= G^2((\sqrt{85}, 3\sqrt{13}, \sqrt{61}, \sqrt{10})^\top) &=& 392.073243 \end{array}$$

Finally, a lower bound $\underline{F}$ for $F$ on $S$ is computed by evaluating its lower concave minorant $L - U$ at the extreme points of $S$,

$$\begin{array}{rcl} L(v_1) - U(v_1) &=& -202.423422 \\ L(v_2) - U(v_2) &=& -203.082643 \\ L(v_3) - U(v_3) &=& -204.244009 \\ L(v_4) - U(v_4) &=& -202.959019 \end{array}$$

and taking the minimum, yielding $\underline{F} = -204.244009$.

Let us assume now that the DCM decomposition of $1/(1 + \beta_a \varphi_a(d))$ is given by (10) and consider again the initial square $S$. This time $G^1$ is a non-increasing function, since the components of the DCM decomposition (10) have the same property. Hence, a concave minorant of $F^1(x) = G^1(D(x))$ is given by:

$$L(x) = G^1(D(x^0)) + \xi^\top (D(x) - D(x^0))$$

where $x^0 \in S \backslash A$ and $\xi$ is a subgradient of $G^1$ at $d^0 = D(x^0)$. Choosing $x^0 = (5,5)^\top$, we have $d^0 = (5, \sqrt{17}, 1, 2\sqrt{5})^\top$ and $G^1(D(x^0)) = 8.425917$. On the other hand, a subgradient of $G^1$ at $d^0$ can be calculated by using the algebra of subgradients, yielding $\xi = (-0.247765, 0, -0.063473, 0)^\top$. Evaluating $L$ at each extreme point of $S$, one obtains the following:

$$\begin{array}{rclcrcl} D(v_1) &=& (\sqrt{5}, \sqrt{97}, \sqrt{61}, 3\sqrt{10})^\top & \Rightarrow & L(v_1) &=& 8.678456333 \\ D(v_2) &=& (\sqrt{65}, \sqrt{17}, \sqrt{41}, \sqrt{130})^\top & \Rightarrow & L(v_2) &=& 7.324242191 \\ D(v_3) &=& (\sqrt{145}, \sqrt{37}, \sqrt{41}, 5\sqrt{2})^\top & \Rightarrow & L(v_3) &=& 6.338299846 \\ D(v_4) &=& (\sqrt{85}, 3\sqrt{13}, \sqrt{61}, \sqrt{10})^\top & \Rightarrow & L(v_4) &=& 6.948191851 \end{array}$$

As for $F^2(x) = G^2(D(x))$, it turns out that $G^2$ is non-increasing and, as a consequence, an upper convex majorant of $F^2$ is given by:

$$U(x) = G^2((\|a - x^0\| + \xi_a^T (x - x^0))_{a \in A})$$

where $\xi_a$ is a subgradient of $\|a - \cdot\|$ at $x^0$. Such subgradients can be easily computed, yielding:

$$\xi_1 = (3/5, 4/5)^\top \qquad \xi_2 = (-4/\sqrt{17}, 1/\sqrt{17})^\top \qquad \xi_3 = (-1, 0)^\top \qquad \xi_4 = (1/\sqrt{5}, -2/\sqrt{5})^\top$$

Evaluating $U$ at each extreme point of $S$, we obtain:

$$\begin{array}{rclcl} U(v_1) & = & G^2((-2, 32/\sqrt{17}, 6, 3\sqrt{5})^\top) & = & 10.74544876 \\ U(v_2) & = & G^2((4, -8/\sqrt{17}, -4, 5\sqrt{5})^\top) & = & 6.276804878 \\ U(v_3) & = & G^2((12, 2/\sqrt{17}, -4, \sqrt{5})^\top) & = & 21.16614784 \\ U(v_4) & = & G^2((6, 42/\sqrt{17}, 6, -\sqrt{5})^\top) & = & 6.276804878 \end{array}$$

Finally, a lower bound $\underline{F}$ for $F$ on $S$ is obtained by computing the minimum of its lower concave minorant $L - U$ on the set of extreme points of $S$. Since

$$\begin{array}{rcr} L(v_1) - U(v_1) & = & -2.066992427 \\ L(v_2) - U(v_2) & = & -23.23436763 \\ L(v_3) - U(v_3) & = & -14.82784798 \\ L(v_4) - U(v_4) & = & 0.671386973 \end{array}$$

it follows that the lower bound is $\underline{F} = -23.23436763$.

For the sake of completeness, we have also computed a lower bound for $F$ on $S$ making use of the method proposed by T. Drezner in [8]. Once the values $z^e$ are computed, we find an upper bound $U^e$ for each term $\sum_{a \in A} \omega_a^e \frac{1}{1+\beta_a \varphi_a(\|a-x\|)}$ on $S$, using the bounding procedure described in [10], as mentioned in Section 1. This yields:

$$U^1 = 999.0303230 \qquad U^2 = 569.683472 \qquad U^3 = 2640.205088$$

Afterwards, the $\ell_\infty$-norm of the vector $(z^e - U^e)_{e \in E}$ provides a lower bound for $F$ on $S$:

$$\|(z^e - U^e)_{e \in E}\|_\infty = \|(-994.459808, -567.700728, -2632.658536)\|_\infty = -567.700728$$

The incidence of the bounding procedure is evident in this case: The three upper bounds obtained in this particular instance via DCM1, DCM2 and the procedure of [8] are rather different in value, namely -204.244009, -23.23436763, -567.700728. As we will show in the following section, the bounding approach based on DCM2 (the one yielding the sharpest bound in this particular instance) turns out to be the most suitable in terms of overall running times and memory requirements.

# 3 Computational experience

In order to solve Problem (7), the branch and bound method BSSS, [14], has been implemented using two alternative bounding techniques, namely, the DCM bounding scheme proposed by Blanquero and Carrizosa (2008) in [2], as well as the resolution procedure suggested by Drezner in [8]. It is worth recalling that this bounding strategy for DCM functions was recently compared favorably with other proposals such as [11] for constructing bounds.
Since a DCM decomposition is not unique, different DCM decompositions may yield different bounds, and thus different running times. In the numerical results in Section 3.2, two different DCM decompositions have been used and compared. The results obtained by using the best of them are then compared with those provided by the method proposed in [8].

| Type | Number of scenarios | Number of facilities |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 5 | 5 |
| 3 | 10 | 10 |

Table 1: Problem Types

### 3.1 Description of the experiments

The problems addressed in the numerical experiments are described in Table 1. Three types of problems were considered, with different number of scenarios and existing facilities. All functions $\varphi_{aj}$ are assumed to be $\varphi_{aj}(d) = d^2$, and the $\ell_p$ norm is chosen with $p = \infty$, as in [8]. Proposition 4 is first used to express $G$ as

$$G((d_a)_{a\in A}) = G^1((d_a)_{a\in A}) - G^2((d_a)_{a\in A})$$

where:

$$\begin{aligned} G^1((d_a)_{a\in A}) &= G((d_a)_{a\in A}) + \sum_{a\in A} \Omega_a(f_a(d_a) + g_a(d_a)) \\ G^2((d_a)_{a\in A}) &= \sum_{a\in A} \Omega_a(f_a(d_a) + g_a(d_a)) \\ \Omega_a &= \|(\omega_a^e)_{e\in E}\|_\infty \end{aligned}$$

The two DCM decompositions, DCM1, DCM2, differ the way functions $f_a$, $g_a$ are chosen. Following (9) , for DCM1,

$$\begin{aligned} f_a(d_a) &= \frac{1}{1+\beta_a d_a^2} + (1+\beta_a d_a^2) \\ g_a(d_a) &= 1+\beta_a d_a^2 \end{aligned}$$

The second DCM decomposition, DCM2, is given by setting $f_a$, $g_a$ as in (10):

$$f_a(d) = \begin{cases} \Phi_a(d_0) + \Phi_a'(d_0)(d-d_0) & \text{if } d \leq d_0 \\ \Phi_a(d) & \text{if } d > d_0 \end{cases} \qquad g_a(d) = f_a(d) - \Phi_a(d)$$

where:

$$\Phi_a(d) = \frac{1}{1+\beta_a d^2} \quad \text{y} \quad d_0 = \left(\frac{1}{3\beta_a}\right)^{\frac{1}{2}}$$

### 3.2 Numerical results

The three types of problems described in Table 1 were considered. For each problem type, problems of different number $N$ of demand points, ranging from very small ($N = 10$) to large ($N = 10.000$) were constructed, by generating $N$ random points in the feasible region $S$, always the unit square $[0, 1] \times [0, 1]$.

For each problem type $T = 1, 2, 3$ and number $N$ of users, 10 instances were generated and solved, using the BSSS method with the abovementioned DCM decompositions DCM1, DCM2 and the bounding strategy suggested in [8], hereafter referred as DRZ. The program code was written in Fortran, compiled by Intel Fortran 10.1, and run on a 2.4 Ghz computer under Windows XP. The solutions were found to a relative accuracy of $10^{-5}$.

Table 2 reports, for DCM1, DCM2 and DRZ, some statistics (minimum, maximum and average) for three indicators of the algorithm performance on Problem Type 1: number of iterations, maximum number of squares in the branch and bound list and running time (see also Figure 1 for a plot of running times for the different values of $N$). The first two indicators come from the resolution of Problem (7), whereas the running time also includes the resolution of Problem (6). DCM2 clearly outperforms DCM1. This also happens with the remaining types of problems, so that in the sequel we only present the results for problems of type 2-3 for DCM2 and DRZ, as reported in Tables 3-4. Comparing both strategies, we note that DCM2 drastically reduces the computational burden needed to solve the problems. Finally, Figures 2-3 show, for DCM2, how running times and number of iterations are affected by the number $N$ of demand points when the number of scenarios and existing facilities vary. The number of iterations is not affected by $N$ whereas the running times increase linearly in all problems types addressed.

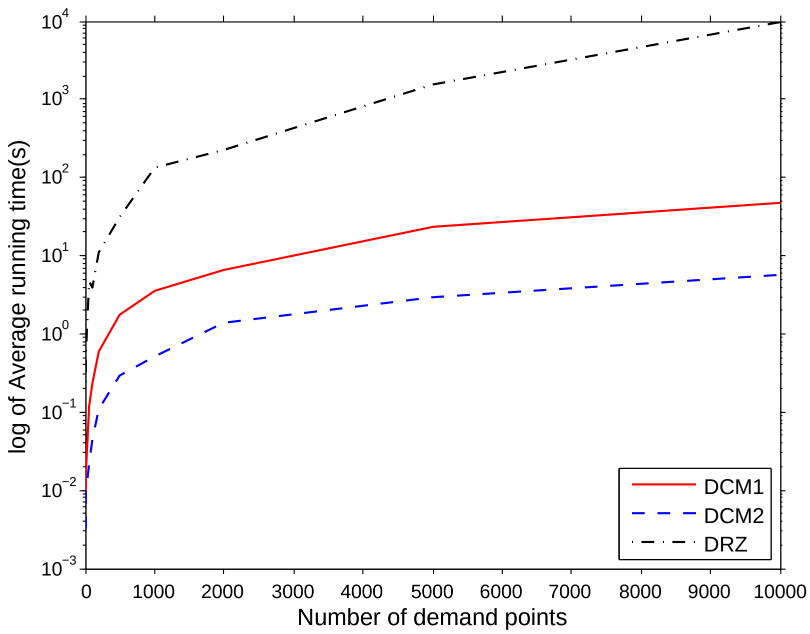


Figure 1: Comparing average running times of DCM1, DCM2 and DRZ for Problem Type 1

### 3.3 Concluding remarks

A competitive location problem recently addressed in [8], namely, a minimax regret Huff location problem in the plane, is shown to have an objective function which can be written as a difference of two convex monotonic functions.

| DCM1 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $N$ | Iterations | | | Max squares | | | Time (s) | | |
| | Min | Max | Ave | Min | Max | Ave | Min | Max | Ave |
| 10 | 139 | 673 | 408 | 36 | 183 | 106 | 0.00 | 0.02 | 0.01 |
| 20 | 351 | 1429 | 670 | 77 | 432 | 185 | 0.00 | 0.05 | 0.02 |
| 50 | 568 | 4057 | 1807 | 168 | 1418 | 589 | 0.05 | 0.27 | 0.12 |
| 100 | 555 | 2831 | 1675 | 117 | 1037 | 546 | 0.06 | 0.39 | 0.23 |
| 200 | 1129 | 3429 | 2096 | 385 | 1146 | 636 | 0.30 | 0.95 | 0.58 |
| 500 | 1357 | 8660 | 2713 | 351 | 2670 | 766 | 0.84 | 5.59 | 1.74 |
| 1000 | 1620 | 7486 | 2899 | 426 | 1631 | 756 | 2.11 | 8.53 | 3.60 |
| 2000 | 1472 | 4482 | 2677 | 346 | 1259 | 713 | 3.36 | 10.91 | 6.57 |
| 5000 | 2226 | 6773 | 4060 | 490 | 1716 | 1015 | 11.97 | 41.56 | 23.77 |
| 10000 | 2910 | 7139 | 4352 | 643 | 1676 | 1024 | 31.53 | 78.00 | 48.20 |
| DCM2 | | | | | | | | | |
| 10 | 80 | 357 | 149 | 16 | 73 | 31 | 0.00 | 0.02 | 0.00 |
| 20 | 137 | 307 | 221 | 25 | 53 | 41 | 0.00 | 0.02 | 0.01 |
| 50 | 122 | 401 | 217 | 23 | 100 | 48 | 0.02 | 0.03 | 0.02 |
| 100 | 122 | 461 | 259 | 24 | 113 | 56 | 0.03 | 0.06 | 0.05 |
| 200 | 117 | 668 | 315 | 23 | 109 | 59 | 0.05 | 0.19 | 0.11 |
| 500 | 255 | 856 | 364 | 45 | 219 | 81 | 0.22 | 0.61 | 0.30 |
| 1000 | 143 | 544 | 299 | 26 | 146 | 57 | 0.34 | 0.84 | 0.52 |
| 2000 | 246 | 661 | 440 | 38 | 152 | 94 | 0.89 | 1.98 | 1.40 |
| 5000 | 228 | 478 | 360 | 27 | 110 | 72 | 2.11 | 3.63 | 3.00 |
| 10000 | 214 | 581 | 342 | 31 | 128 | 64 | 3.92 | 8.67 | 5.77 |
| DRZ | | | | | | | | | |
| 10 | 369 | 82677 | 15065 | 71 | 50563 | 8260 | 0.01 | 0.64 | 0.24 |
| 20 | 1366 | 229046 | 25726 | 338 | 136799 | 14646 | 0.07 | 3.00 | 0.57 |
| 50 | 2694 | 1418445 | 147465 | 924 | 864362 | 88640 | 0.25 | 41.26 | 4.79 |
| 100 | 2304 | 56830 | 22513 | 476 | 23423 | 8919 | 0.68 | 7.25 | 3.99 |
| 200 | 3153 | 110483 | 46851 | 916 | 45088 | 18490 | 0.79 | 23.43 | 11.07 |
| 500 | 6887 | 138294 | 73889 | 2448 | 55739 | 29250 | 3.65 | 55.45 | 30.49 |
| 1000 | 36185 | 710544 | 203260 | 14195 | 325963 | 85808 | 28.95 | 398.53 | 134.72 |
| 2000 | 9857 | 387122 | 171598 | 3541 | 157596 | 69810 | 16.67 | 522.21 | 221.44 |
| 5000 | 12761 | 1362762 | 549849 | 3926 | 567341 | 225325 | 38.35 | 3965.93 | 1577.28 |
| 10000 | 700949 | 3042246 | 1753495 | 279254 | 1246240 | 342237 | 3979.95 | 16688.20 | 9718.02 |

Table 2: Computational results for Problem Type 1

| $N$ | Iterations | | | Max squares | | | Time (s) | | |
|---|---|---|---|---|---|---|---|---|---|
| | Min | Max | Ave | Min | Max | Ave | Min | Max | Ave |
| DCM2 | | | | | | | | | |
| 10 | 76 | 484 | 215 | 19 | 112 | 47 | 0.00 | 0.02 | 0.01 |
| 20 | 128 | 465 | 260 | 27 | 117 | 53 | 0.02 | 0.02 | 0.02 |
| 50 | 172 | 453 | 316 | 31 | 82 | 60 | 0.03 | 0.06 | 0.05 |
| 100 | 302 | 771 | 498 | 42 | 186 | 95 | 0.08 | 0.16 | 0.12 |
| 200 | 102 | 663 | 346 | 25 | 181 | 60 | 0.13 | 0.30 | 0.20 |
| 500 | 101 | 1066 | 384 | 27 | 244 | 88 | 0.31 | 1.00 | 0.53 |
| 1000 | 122 | 470 | 237 | 30 | 87 | 46 | 0.63 | 1.17 | 0.84 |
| 2000 | 127 | 1025 | 437 | 34 | 221 | 89 | 1.44 | 4.19 | 2.49 |
| 5000 | 144 | 887 | 415 | 44 | 218 | 92 | 3.97 | 8.47 | 5.70 |
| 10000 | 146 | 1129 | 503 | 35 | 274 | 110 | 6.77 | 21.47 | 13.74 |
| DRZ | | | | | | | | | |
| 10 | 282 | 78374 | 18201 | 77 | 46309 | 10637 | 0.17 | 1.21 | 0.50 |
| 20 | 2056 | 94877 | 28094 | 690 | 55370 | 15838 | 0.39 | 3.75 | 1.37 |
| 50 | 2309 | 484289 | 93686 | 586 | 284326 | 55172 | 1.14 | 34.96 | 8.36 |
| 100 | 3027 | 12256 | 7731 | 883 | 4312 | 2687 | 2.42 | 10.51 | 5.55 |
| 200 | 837 | 2283002 | 235552 | 148 | 1365311 | 139150 | 5.73 | 636.01 | 74.32 |
| 500 | 974 | 30781 | 9359 | 181 | 10956 | 3122 | 10.60 | 54.06 | 33.39 |
| 1000 | 2573 | 47123 | 16986 | 428 | 17798 | 5522 | 60.79 | 147.68 | 91.73 |
| 2000 | 1205 | 155069 | 69125 | 273 | 61728 | 26696 | 17.10 | 882.75 | 381.33 |
| 5000 | 5197 | 594461 | 164540 | 848 | 241285 | 64433 | 235.93 | 4643.84 | 1581.68 |
| 10000 | 8130 | 1430585 | 292907 | 1858 | 601821 | 115503 | 252.67 | 20326.59 | 4623.17 |

Table 3: Computational results for Problem Type 2

| $N$ | Iterations | | | Max squares | | | Time (s) | | |
|---|---|---|---|---|---|---|---|---|---|
| | Min | Max | Ave | Min | Max | Ave | Min | Max | Ave |
| DCM2 | | | | | | | | | |
| 10 | 63 | 574 | 311 | 17 | 254 | 81 | 0.02 | 0.03 | 0.02 |
| 20 | 125 | 419 | 234 | 29 | 70 | 44 | 0.02 | 0.03 | 0.03 |
| 50 | 119 | 614 | 270 | 28 | 106 | 51 | 0.05 | 0.09 | 0.07 |
| 100 | 151 | 346 | 249 | 34 | 73 | 51 | 0.13 | 0.16 | 0.14 |
| 200 | 108 | 666 | 366 | 24 | 204 | 88 | 0.20 | 0.47 | 0.33 |
| 500 | 150 | 1233 | 571 | 33 | 301 | 125 | 0.63 | 1.69 | 1.13 |
| 1000 | 177 | 998 | 552 | 61 | 221 | 122 | 1.70 | 2.83 | 2.24 |
| 2000 | 170 | 1145 | 470 | 51 | 377 | 124 | 3.38 | 6.36 | 4.51 |
| 5000 | 211 | 1228 | 562 | 57 | 301 | 124 | 8.92 | 20.42 | 13.15 |
| 10000 | 186 | 1198 | 513 | 55 | 263 | 121 | 14.22 | 36.11 | 23.02 |
| DRZ | | | | | | | | | |
| 10 | 258 | 17749 | 3860 | 76 | 10489 | 1975 | 0.17 | 0.90 | 0.48 |
| 20 | 701 | 42783 | 12007 | 136 | 23350 | 6449 | 0.84 | 3.12 | 1.43 |
| 50 | 660 | 43985 | 11894 | 157 | 24127 | 5260 | 1.15 | 10.98 | 4.51 |
| 100 | 1099 | 226590 | 47230 | 189 | 122814 | 26060 | 3.92 | 71.87 | 19.27 |
| 200 | 649 | 1487426 | 306025 | 143 | 912071 | 185354 | 11.23 | 837.90 | 182.25 |
| 500 | 1212 | 77159 | 18260 | 205 | 41040 | 7991 | 16.04 | 211.93 | 78.19 |
| 1000 | 820 | 1230194 | 146594 | 258 | 739272 | 83364 | 19.15 | 3396.71 | 511.92 |
| 2000 | 831 | 243705 | 45081 | 230 | 103900 | 18113 | 51.25 | 1529.32 | 422.41 |
| 5000 | 2555 | 208155 | 40999 | 363 | 83246 | 14903 | 319.95 | 4997.31 | 1371.84 |
| 10000 | 1235 | 345119 | 85420 | 301 | 138347 | 32797 | 147.28 | 12093.65 | 3674.73 |

Table 4: Computational results for Problem Type 3

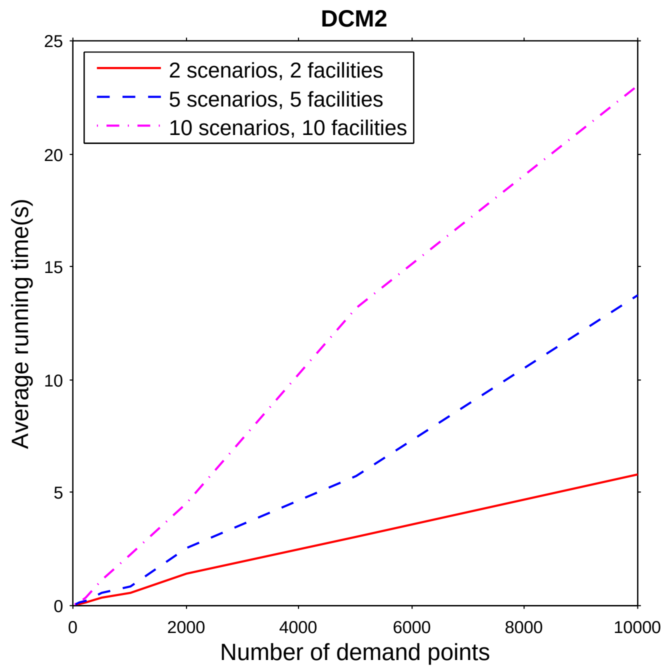


Figure 2: Comparing running times in the three Problem Types

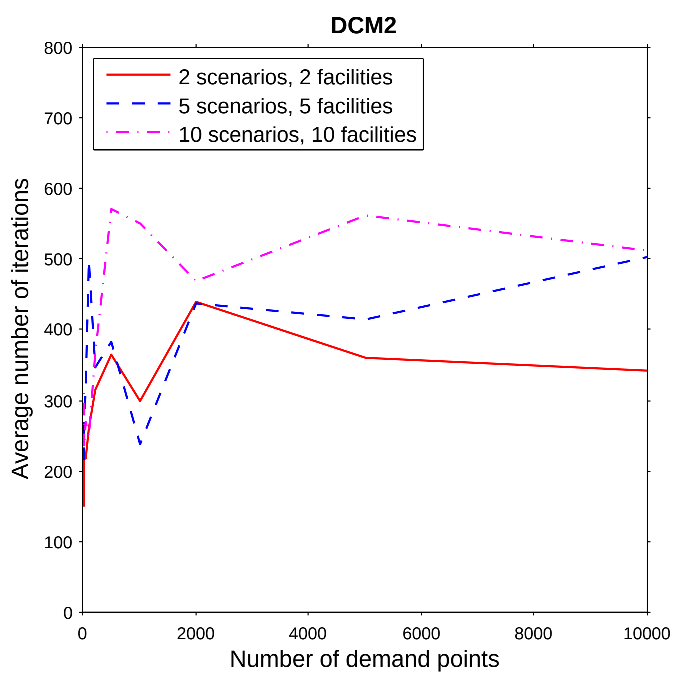


Figure 3: Comparing the number of iterations in the three Problem Types

The DCM bounding scheme proposed by the authors in [2] is analyzed for this problem and compared with the one suggested in [8].

Two different DCM decompositions are used and tested in a series of numerical experiments with to up to 10000 demand points. As the computational results clearly show, the bounds provided by both DCM decompositions outperform those obtained by using the procedure suggested in [8]. Moreover, we observe that using the decomposition DCM2 proposed in this paper reduces considerably the running times and allows one to solve in very reasonable time problems with a large number of points.

For simplicity, in the model proposed in this paper, only the weights are affected by uncertainty, and thus modeled by means of a set $E$ of scenarios. However, the analysis can be directly extended to the case in which we also have uncertainty on the locations of the competing firms. Indeed, for each scenario, let $I(e)$ denote a finite indexset, representing the set of competing facilities, so that each facility $i \in I(e)$ will be located at $x_i^e$, and will attract a market inversely proportional to $\varphi_{ai}^e(\|a - x_i^e\|)$. In this case, Problem (7) would become

$$\min_{x \in S} \left\| \left( z^e - \sum_{a \in A} \omega_a^e \frac{1}{1 + \beta_a^e \varphi_a(\|a - x\|)} \right)_{e \in E} \right\|_p , \tag{15}$$

where, for each $e \in E$, $\beta_a^e$ is given by

$$\beta_a^e = \sum_{j \in I(e)} \frac{1}{\varphi_{aj}^e(\|a - x_j^e\|)}. \tag{16}$$

Moreover, distances to the ideal point are measured by $\| \cdot \|_p$, an $\ell_p$ norm. Extensions to arbitrary absolute symmetric norms, as discussed in [6], are straightforward, since Proposition 4 can be easily extended to this more general case.

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