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Magnetic vortex filament flows

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We exhibit a variational approach to study the magnetic flow associated with a Killing magnetic field in dimension 3. In this context, the solutions of the Lorentz force equation are viewed as Kirchhoff elastic rods and conversely. This provides an amazing connection between two apparently unrelated physical models and, in particular, it ties the classical elastic theory with the Hall effect. Then, these magnetic flows can be regarded as vortex filament flows within the localized induction approximation. The Hasimoto transformation can be used to see the magnetic trajectories as solutions of the cubic nonlinear Schrödinger equation showing the solitonic nature of those. © *2007 American Institute of Physics.*
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I. INTRODUCTION

The soliton equation theory, or the theory of integrable systems, has had (and has) an enormous impact in applied mathematics and in a wide variety of nonlinear phenomena in physics. This setting includes problems in nonlinear optics, field theories and sigma models, fluid dynamics, water wave theory, relativity, and so on. An important aspect of the study of soliton equations is that a single example can be of interest in many different contexts. This kind of universality may be strongly related to the fact that such equations frequently have an underlying geometric meaning. For example, the sine-Gordon equation, which first appeared in differential geometry, is used as a model for dislocation of crystals, and also in field theories and in nonlinear optics (self-induced transparency).

The localized induction equation (LIE), also called the filament equation or the Betchov-Da Rios equation,[6,26] is an idealized model of the evolution of the centerline of a thin vortex tube in a three-dimensional (3D) inviscid incompressible fluid. The connection of LIE with the theory of solitons was discovered by Hasimoto.[9] He showed that the solutions of LIE are related to solutions of the cubic nonlinear Schrödinger equation (NLSE) which is well known to be an equation with soliton solutions. That relationship is defined via the so-called Hasimoto transformation that gives the complex curvature of solitons in terms of the usual curvatures (see the Appendix for some details). Therefore, associated with LIE, there is an infinite sequence of conserved Hamiltonians that constitute the LIE hierarchy.

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In Ref. 10, Hasimoto proved that elastic curves may be regarded as solitons for LIE. The general version of this discovery was treated in Ref. 14. These results were complemented in Refs. 11 and 18, where the authors showed that the centerlines of the Kirchhoff elastic rods in equilibrium are soliton solutions of LIE. This constitutes a nice connection between two apparently unrelated physical models and ties the classical elasticity theory with current topics of solitons in hydrodynamics. The Kirchhoff elastic rods in 3D real-space forms with arbitrary constant curvature were studied in Ref. 12.

This paper deals with the following problem, which is related to the old idea of characterizing the magnetic flow lines from a global variational principle: To obtain the magnetic trajectories of (M,g,F) as solutions of a variational problem that neither involves any local potential nor constraints the topology of M given a magnetic field F on a Riemannian manifold (M,g).

In Ref. 4, the authors solved this problem for a certain class of magnetic fields in dimension 2, the so-called Gaussian magnetic fields. The associated Lorentz force equation of these magnetic fields corresponds with the field equations of a variational problem that describe the massive relativistic bosons, this is briefly recalled in Sec. III.

In this paper, we solve the problem for some magnetic fields in 3D, simply connected, complete, space forms $\mathbf{M}^3$ $(=\mathbb{R}^3, \mathbb{S}^3$, or $\mathbb{H}^3)$. First of all, we consider Killing magnetic fields. They are amply studied in Secs. V–IX. In particular, we have the following: The magnetic flow lines of a Killing magnetic field on $\mathbf{M}^3$ are equilibria (i.e., critical points) for a linear combination of the first three conserved Hamiltonians in the LIE hierarchy: the length, the total torsion, and the total squared curvature.

This result allows one to get a correspondence between Killing magnetic flow lines and Kirchhoff elastic rods where a flow line plays the role of centerline of the rod. Consequently, Killing magnetic flow lines provide initial conditions for soliton solutions of LIE and so they evolve according to the equation

$$\gamma_t = \gamma_s \wedge \gamma_{ss},$$

the Betchov–Da Rios filament equation or LIE.

To exhibit the solitonic nature of a Killing magnetic flow line, we consider the Hasimoto transformation. It associates to each space curve, $s \mapsto \gamma(s,t)$, with curvature $\kappa(s,t)$ and torsion $\tau(s,t)$, its complex wave function defined by

$$\psi(s,t) = \kappa(s,t)\exp\left(i\int_0^s \tau(u,t)\mathrm{d}u\right).$$

Now, the complex wave function of a Killing magnetic flow line is a solution of the cubic NLSE

$$i\psi_t + \psi_{ss} + \frac{1}{2}|\psi|^2\psi = 0,$$

In Sec. X, we pay attention to an important class of non-Killing magnetic fields. The Lorentz equation, associated with these magnetic fields, is completely integrated by proving that magnetic flow lines are Lancret helices. This constitutes a new link in the large chain of characterizations for these class of curves,[3] and allows one to solve the above stated problem in this class of magnetic fields.[1] In the next section, a general variational approach to solve the problem is discussed. Finally, in the last section, the existence of periodic magnetic flow lines is investigated. This belongs to a class of nontrivial problems known classically as *closed curve problem*. The solution is given in terms of data which are encoded in the underlying geometry governing the model.

II. PRELIMINARIES AND GENERALITIES

Let (M^n,g) be an $n(\geqslant 2)$-dimensional oriented Riemannian manifold with volume n-form Ω_n. Using the Hodge star operator $\star$ of (M^n,g), each p-form $\alpha \in \Lambda^p(M^n)$ on M^n can be equivalently described as its dual $(n-p)$-form $\star\alpha \in \Lambda^{n-p}(M^n)$. A closed 2-form $F \in \Lambda^2(M^n)$ is said to be a *magnetic field*. Let us mention a couple of situations where magnetic fields naturally appear:

(1) The rotational magnetic field. For an arbitrary vector field, $V \in \mathfrak{X}(M)$, and using the Levi-Civita covariant derivative ∇ of (M^n, g), one can define its associated *rotational* 2-form $\mathrm{rot}(V)$ (see, Ref. 24, for instance) by

$$\mathrm{rot}(V)(X,Y) = g(\nabla_X V, Y) - g(\nabla_Y V, X),$$

for all $X, Y \in \mathfrak{X}(M)$, which is known to be an exact 2-form. In fact, if $V^\flat$ denotes the g-equivalent 1-form of V, then

$$\mathrm{rot}(V) = dV^\flat.$$

(2) The magnetic field associated with a potential gauge. Magnetic fields are also related to connections on principal circle bundles. Indeed, let $P(M^n, \mathbb{S}^1)$ be a principal circle bundle on M^n and ω a potential gauge (i.e., a connection 1-form) on P. The strength or curvature 2-form is defined, via the structure equation, by

$$\mathrm{d}\omega = \mathbf{\Omega}.$$

For any local section $\sigma_A : A \subset M^n \to P$, we define $F_A := -i\sigma_A^*(\mathbf{\Omega}) \in \Lambda^2(A)$, which clearly satisfies $\mathrm{d}F_A = 0$. However, $F_A = F_B$ on the overlapping open subsets $A, B \subset M^n$, and consequently we get a closed 2-form F on the whole M^n. In other words, one has associated a magnetic field F on M^n to each potential over any circle bundle on M^n. Moreover, if ω' stands for another potential on the same circle bundle and F' is the associated magnetic field, then $F' - F = \mathrm{d}\alpha$, where $\alpha \in \Lambda^1(M^n)$. Hence, each principal circle bundle $P(M^n, \mathbb{S}^1)$ defines a De Rham cohomology class, $[F] \in \mathbf{H}^2(M^n)$. In particular, if M^n is simply connected, then the correspondence between principal circle bundles on M^n and $\mathbf{H}^2(M^n)$ is bijective.[15]

The Lorentz force of a magnetic field F on (M^n, g) is defined to be the skew-symmetric operator ϕ given by

$$g(\phi(X), Y) = F(X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$. The associated magnetic trajectories are curves γ in M^n that satisfy the Lorentz equation

$$\nabla_{\gamma'} \gamma' = \phi(\gamma'). \tag{1}$$

The Lorentz force is g-skew-symmetric and trajectories have constant speed, i.e., $\|\gamma'(t)\|$ is a constant.

A very special class of magnetic fields on a Riemannian manifold is that made up of parallel 2-forms, $\nabla F = 0$. These are called *uniform magnetic fields* and they play an important role in the classical Landau-Hall problem.[4] Notice that they correspond with parallel Lorentz forces, $\nabla \phi = 0$.

The existence and uniqueness of geodesics remain true when one considers magnetic curves associated with an arbitrary magnetic field. Thus, for each $p \in M^n$ and $v \in T_p M^n$, there exists exactly one inextensible (i.e., maximal) magnetic curve, $\gamma : (-\epsilon, \epsilon) \to M^n$, of (M^n, g, F), with $\gamma(0) = p$ and $\gamma'(0) = v$ (see Ref. 27, for instance). Nevertheless, the well known homogeneity result for geodesics works quite differently in nontrivial magnetic fields. Therefore, if γ is the inextensible magnetic curve of (M, g, F) determined from the initial data (p, v), the curve β, defined by $\beta(t) = \gamma(\lambda t), \lambda \in \mathbb{R} \setminus \{0\}$, is a magnetic trajectory of $(M, g, \lambda F)$ and also, when $\lambda > 0$, of $(M, (1/\lambda) g, F)$, in both cases determined from initial data $(p, \lambda v)$. Furthermore, the whole families of magnetic curves of (M, g, F) and $(M, \lambda g, \lambda F)$ coincide for any constant $\lambda > 0$. Consequently, we have the following:[4] Let F be a nontrivial magnetic field on a Riemannian manifold, (M^n, g). Then, there exists no affine connection on M^n whose geodesics are the magnetic curves of (M^n, g, F).

A magnetic background (M^n, g, F) with Lorentz force ϕ provides a unique vector field Q_ϕ on the tangent bundle TM^n. It is defined to have integral curves being the lifts to TM^n of the magnetic curves, that is, the curves $t \mapsto (\gamma(t), \gamma'(t))$, where γ is a magnetic curve of (M^n, g, F) (compare with Ref. 20). This vector field coincides with the geodesic flow of the metric g for trivial magnetic fields $F=0$. On the other hand, the fact that any integral curve of Q_ϕ is the velocity of its projection on M^n allows us to regard Q_ϕ as a nice example of the classically called *second order differential equation* on M^n. Notice that Q_ϕ is not a spray in general.

A dynamical system with complete trajectories is often thought in physics to be persisting eternally. However, in many circumstances one has to deal with incompleteness. So, because of its importance, we next give criteria to assert when it holds true. An important tool to study the completeness of the inextensible magnetic trajectories, i.e., under what assumptions all the inextensible magnetic curves are defined on all $\mathbb{R}$, is the vector field Q_ϕ. Using Lemma 1.56, in Ref. 24 it was shown in Ref. 4 that a magnetic curve $\gamma:(a,b) \to M^n$, $a, b \in \mathbb{R}$, $a<b$, of (M^n, g, F) can be extended to some bigger open interval I, $(a,b) \subset I$, if and only if $\gamma((a,b))$ is contained in a compact subset of M^n. Therefore we get in Ref. 4 the following result: Let γ be an inextensible magnetic curve of (M^n, g, F) and such that $\gamma((a,b))$ lies in a compact subset of M^n for any finite interval (a,b) in its domain. Then, γ must be complete. In particular, if M^n is compact, then each inextensible magnetic curve is complete.

Even more, we do not need the compactness of the space to guarantee the result. In fact, we have shown in Ref. 4 that the above result also holds if the Riemannian manifold (M^n, g) is geodesically complete.

The motion of a physical system, in particular, a charged particle in a Riemannian manifold (M^n, g), is defined by a principle of the *least action* (also called Maupertuis principle) and can be found by minimizing a certain Lagrangian (the action functional). Special cases of magnetic fields are those defined by exact 2-forms F. In these, F has a globally defined potential, i.e., there exists $\omega \in \Lambda^1(M^n)$ such that $F = \mathrm{d}\omega$. This happens with any magnetic field if the second De Rham cohomology group vanishes, $\mathbf{H}^2(M)=0$. Notice also that this potential always exists, at least locally. Let U be an open subset of M where $F=\mathrm{d}\omega$ for some potential 1-form ω. Denote by Γ the space of smooth curves that connect two fixed points of U. Now, we choose the action $\mathcal{L}: \Gamma \to \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) = \frac{1}{2}\int_\gamma g(\gamma', \gamma')\mathrm{d}t + \int_\gamma \omega(\gamma')\mathrm{d}t. \tag{2}$$

The tangent space of Γ in γ is made up of the smooth vector fields Z along γ that vanish at the end points. To compute the extremals of Eq. (2), we first observe that

$$Z(\omega(\gamma')) = \gamma'(\omega(Z)) - g(\phi(\gamma'), Z).$$

Now, a standard computation involving integration by parts allows one to compute the first variation of this action to be

$$(\delta\mathcal{L})(\gamma)[V] = \int_\gamma g(\nabla_{\gamma'}\gamma' - \phi(\gamma'), V)\mathrm{d}t + [\omega(Z)]_{\partial\gamma}.$$

As a consequence, since $[\omega(Z)]_{\partial\gamma}=0$ because Z vanishes at the end points, we get that

$$(\delta\mathcal{L})(\gamma)[V] = 0 \quad \text{for any } V \in T_\gamma\Gamma$$

if and only if γ is a solution of the Lorentz equation [Eq. (1)]. Therefore, the Lorentz equation is indeed the Euler-Lagrange equation associated with the functional $\mathcal{L}$.

However, it seems natural to characterize magnetic curves from a global variational principle. Thus, we propose the following general, interesting open problem: To obtain the magnetic trajectories of (M^n, g, F) as solutions of a variational problem that neither involves any local potential nor constraints the topology of M^n.

III. MAGNETIC FIELDS IN DIMENSION 2 AND THE CLASSICAL LANDAU-HALL PROBLEM

A magnetic field F on an oriented Riemannian surface (M^2,g) is completely determined by a smooth function, that expressing F as a multiple of the area 2-form, i.e., $F=f\Omega_2$, $f\in C^\infty(M^2)$. Uniform magnetic fields in this dimension are then determined by constant functions. Classically, the Landau-Hall problem consists of the motion study of a charged particle in the presence of a uniform magnetic field.[16] In other words, that problem consists of the study of solutions of the Lorentz equation associated with a uniform magnetic field. If one uses the Frenet equations for curves in (M^2,g), it is not difficult to see that the normalized (i.e., unit speed) magnetic curves, associated with a uniform magnetic field $F=c\Omega_2$, are curves with constant curvature, $\kappa=c$. Therefore, the classical Landau-Hall problem is equivalent to the old geometric problem of determining, on a given surface, those curves with constant curvature.

If M^2 is an oriented surface immersed in the Euclidean three-space $\mathbb{R}^3$, the metric g is the induced one. Then, besides the area 2-form Ω_2, one has a second natural 2-form on M^2. In fact, denote by N: $M^2\to\mathbb{S}^2$ the Gauss map and put $\mathrm{d}\sigma^2$ to name the area element on the unit round sphere $\mathbb{S}^2$. The natural 2-form N^* $(\mathrm{d}\sigma^2)$ on M^2 allows one to study magnetic fields of the type

$$F=\frac{1}{m}N^*(\mathrm{d}\sigma^2),$$

where m is a nonzero constant. It is well known that N^* $(\mathrm{d}\sigma^2)=G\Omega_2$, G denoting the Gaussian curvature of (M^2,g), and this, in particular, implies that we can consider these kinds of magnetic fields with no mention of the surrounding space. Namely, magnetic fields of the type

$$F=\frac{G}{m}\Omega_2. \tag{3}$$

These magnetic fields were considered in Ref. 4, under the name of *Gaussian magnetic fields*, where the previous problem was completely solved for them in the following terms: The magnetic curves of $(M^2,g,F=(G/m)\Omega_2)$ are just the critical points of the action

$$\mathcal{F}_m(\gamma)=\int_\gamma(\kappa+m)\mathrm{d}s,$$

where κ denotes the curvature of the curve γ acting on the space of curves that connect a pair of points of M^2 and satisfy a first order boundary condition.

It should be noticed that these kinds of actions where the Lagrangian density involves the proper acceleration of particles κ have been used to model relativistic particles with rigidity in the sense of Plyushchay.[25] Therefore, this result provides a nice relationship between the magnetic curves of Gaussian magnetic fields and the dynamics of the spinning relativistic particles with rigidity of order 1, which are two apparently unrelated physical models. Now, the next result gives a quantization principle for Gaussian magnetic fields and can be regarded as a topological characterization of them: A 2-form F on an oriented compact surface M^2 is a Gaussian magnetic field if and only if there exists $m\in\mathbb{R}$, $m\neq0$, satisfying

$$\int_{M^2}F=\frac{2\pi}{m}\chi(M^2),$$

where $\chi(M^2)$ is the Euler number of M^2.

In fact, the necessary condition follows from the classical Gauss-Bonnet theorem while the converse one is obtained using the well known techniques for prescribing the curvature form of a surface.[13,29] Even more, using elliptic theory, one can show that given a 2-form F in M^2, a nonzero real number m, and a Riemannian metric g_0 on M^2, there exists a Riemannian metric g on M^2, pointwise conformally related to g_0, such that F is a Gaussian magnetic field in (M^2,g), namely, $F=(G/m)\Omega_2$.

IV. MAGNETIC FIELDS IN DIMENSION 3

We are dealing here with time-independent magnetic fields in dimension 3, so that in physical terminology our approach belongs to the classical magnetostatic theory.[28] Magnetic fields in dimension 3 are quite special. In fact, there are several important facts which make their treatment very elegant. We will organize the main significant peculiarities in the following points:

(*a*) *2-forms and vector fields are the same thing*. Consider a 3D oriented Riemannian manifold (M^3,g). Given a 2-form $F\in\Lambda^2(M^3)$, one has its star 1-form $\star F\in\Lambda^1(M^3)$ and so the g-equivalent vector field $(\star F)^\sharp\in\mathfrak{X}(M^3)$. This defines a one-to-one map between 2-forms and vector fields. The converse trip is described as follows. Start with a vector field $V\in\mathfrak{X}(M^3)$, then consider its g-equivalent 1-form $V^\flat$ and then take its star, $\star V^\flat$. One so obtains a 2-form which can be also written, using the interior contraction, as $\star V^\flat=i_V\Omega_3$.

(*b*) *Magnetic fields mean divergence-free vector fields*. It is known that the Lie derivative of the volume form satisfies $\mathcal{L}_V\Omega_3=\mathrm{d}(i_V\Omega_3)=\mathrm{div}(V)\Omega_3$ and so, the 2-form $\star V^\flat=i_V\Omega_3$ is closed if and only if $\mathrm{div}(V)=0$, i.e., the volume element is invariant by the local flows of V. This will allow us to regard the magnetic fields in dimension 3 as divergence-free vector fields.

(*c*) *Uniform magnetic fields correspond to parallel vector fields*. If V is a vector field in M^3 which is parallel, i.e., $\nabla V=0$, then it obviously has divergence zero so $F=i_V\Omega_3$ is a closed 2-form defining a magnetic field on (M^3,g). Furthermore, it is clear that $\nabla F=0$. Conversely, suppose that F is a uniform magnetic field on (M^3,g) and take $V\in\mathfrak{X}(M^3)$ as its corresponding vector field, i.e., that defined by $F=i_V\Omega_3$. A direct computation shows that $\Omega_3(\nabla_X V,Y,Z)=(\nabla_X F)(Y,Z)=0$ for any X, Y, $Z\in\mathfrak{X}(M^3)$ which proves that $\nabla V=0$.

(*d*) *The Lorentz force is viewed in terms of the cross product*. One can define the cross product $X\wedge Y$ of any two vector fields X, $Y\in\mathfrak{X}(M^3)$ in a 3D oriented Riemannian manifold as follows:

$$g(X\wedge Y,Z)=\Omega_3(X,Y,Z).$$

Consequently, the Lorentz force ϕ associated with a magnetic field $F=i_V\Omega_3$ can be computed by

$$g(\phi(X),Y)=F(X,Y)=(i_V\Omega_3)(X,Y)=\Omega_3(V,X,Y)=g(V\wedge X,Y);$$

therefore, we have

$$\phi(X)=V\wedge X, \tag{4}$$

and consequently, the Lorentz force equation [Eq. (1)], which provides the magnetic flow, can be written by

$$\nabla_{\gamma'}\gamma'=\phi(\gamma')=V\wedge\gamma'. \tag{5}$$

Notice, in particular, that an integral curve of a magnetic field is a magnetic trajectory if and only if it is a geodesic.

Formula (5) allows one to talk about the spin Hall effect in a doubly extended direction, on one hand, for any magnetic field even if it is not uniform, and on the other hand, in spaces with nontrivial gravity, i.e., with nonzero curvature. This is an important approach and it is closer to reality than the classical approach is. Also, this formula and the Hall effect that it carries have important applications in many branches including those related to the mass spectrometer (analytical chemistry, biochemistry, atmospheric science, geochemistry, etc.) cyclotron, proton, cancer therapy, and velocity selector.

(*e*) *The length of a magnetic field*. The cross product on a 3D oriented Riemannian manifold satisfies the following two identities:

$$X \wedge (Y \wedge Z) = g(X,Z)Y - g(X,Y)Z,$$

$$g(X \wedge Y, X \wedge Z) = g(X,X)g(Y,Z) - g(X,Y)g(X,Z),$$

which hold true for any X, Y, $Z \in \mathfrak{X}(M^3)$. As a consequence, we have

$$\phi^2(X) = -g(V,V)X + g(V,X)V$$

for all $X \in \mathfrak{X}(M^3)$, and so

$$g(V,V) = -\frac{1}{2}\mathrm{trace}(\phi^2) \quad \text{and} \quad 2\|V\|^2 = \|\phi\|^2 := \|F\|^2.$$

(*f*) *Magnetic fields are rectifying sections of each magnetic curve*. Given a unit magnetic curve, let $\{T,N,B\}$ be its Frenet frame, and κ and τ the curvature and torsion functions, respectively. The Frenet equations are

$$\nabla_T T = \kappa N, \tag{6}$$

$$\nabla_T N = -\kappa T + \tau B, \tag{7}$$

$$\nabla_T B = -\tau N. \tag{8}$$

Now, the Lorentz force in the Frenet frame is written as

$$\phi(T) = \kappa N, \tag{9}$$

$$\phi(N) = -\kappa T + \omega B, \tag{10}$$

$$\phi(B) = -\omega N, \tag{11}$$

where ω is a certain function which is a kind of slope of the magnetic curves with respect to the magnetic field. To see this and more, set $V(s)=a_1T+a_2N+a_3B$, where a_i, $1 \leqslant i \leqslant 3$, are certain functions along a trajectory γ of V and assume V does not vanish on γ. Now, from $\phi(V)=0$ one obtains $a_2=0$; otherwise $\omega(s_0)=\kappa(s_0)=0$ for some s_0 and so $\phi=0$ at $\gamma(s_0)$, which implies $V(s_0)=0$. On the other hand, one uses Eqs. (5)–(7) to get $a_1=\omega$ and $a_3=\kappa$. This can be summarized in the following result: A unit speed curve γ is a magnetic trajectory of a magnetic field V if and only if V can be written along γ as

$$V(s) = \omega(s)T(s) + \kappa(s)B(s). \tag{12}$$

In particular, V lies in the field of rectifying planes along each trajectory.

From now on, the function $\omega(s)$ associated with each magnetic curve will be called its *quasislope* measured, obviously, with respect to the magnetic field V.

V. KILLING MAGNETIC FIELDS

Killing vector fields on a Riemannian manifold (M^n,g) are those generating local flows of isometries. It is known that $V \in \mathfrak{X}(M^n)$ is Killing if and only if $\mathcal{L}_V g=0$ or, equivalently, $\nabla V(p)$ is a skew-symmetric operator in T_pM^n, at each point $p \in M^n$. It is clear that any Killing vector field on (M^n,g) is divergence-free. In particular, if $n=3$, then every Killing vector field defines a magnetic field which will be called a *Killing magnetic field*. In particular, uniform magnetic fields, $\nabla V=0$, are obviously Killing. Therefore, the class of Killing magnetic fields constitutes an important family of magnetic fields whose study is one of the main purposes of this paper.

Besides the conservation law which asserts that the speed of any magnetic trajectory is constant, it deserves to be pointed out that the magnetic trajectories of Killing magnetic fields in dimension 3 have another additional conservation law. Even more, a magnetic field in a 3D Riemannian manifold is Killing if and only if for any magnetic curve γ the product $g(K,\gamma')$ is a constant along γ.

In fact, if K is Killing and γ is a magnetic trajectory of K, then

$$\frac{\mathrm{d}}{\mathrm{d}t}g(K,\gamma') = g(\nabla_{\gamma'}K,\gamma') + g(K,\nabla_{\gamma'}\gamma') = 0.$$

Conversely, for $p \in M$ and $v \in T_pM$, let γ be a magnetic trajectory of a magnetic field K such that $\gamma(0)=p$, $\gamma'(0)=v$. We have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}g(K,\gamma') = g(\nabla_{\gamma'}K,\gamma') + g(K,K \wedge \gamma') = g(\nabla_{\gamma'}K,\gamma').$$

Therefore, $g(\nabla_v K,v)=0$, which means that K is Killing.

Moreover, Killing magnetic fields exhibit a certain symmetry property between its magnetic trajectories. Concretely, Let $\{\varphi_t\}$ denote a (local) flow of a Killing vector field K. We have the following: (a) If γ is any magnetic trajectory of K, then $\varphi_t \circ \gamma$ is another magnetic trajectory of K. (b) For any magnetic trajectory γ of K, the constants associated with the two conservation laws for γ and $\varphi_t \circ \gamma$ are equal.

In fact, from Eq. (5) we have

$$\nabla_{(\varphi_t \circ \gamma)'}(\varphi_t \circ \gamma)' = \mathrm{d}\varphi_t(\nabla_{\gamma'}\gamma') = \mathrm{d}\varphi_t(K \wedge \gamma') = K \wedge (\varphi_t \circ \gamma)'.$$

On the other hand, assertion (b) follows easily because φ_t is an isometry.

Before going on, this notion should be distinguished from the classical one of Killing 2-form which appears in the literature. In fact, Yano[30] introduced the notion of a Killing 2-form F as that which satisfies $(\nabla_X F)(Y,Z)+(\nabla_Y F)(X,Z)=0$ for all $X,Y,Z \in \mathfrak{X}(M^n)$. Notice that a Killing magnetic field is not a Killing 2-form in general. In fact, the Killing vector field $x(\partial/\partial z)-z(\partial/\partial x)$ on $\mathbb{R}^3$ has $F=x\mathrm{d}x\wedge \mathrm{d}y - z\mathrm{d}y \wedge \mathrm{d}z$ as its associated Killing magnetic field F, but it is not a Killing 2-form in the sense of Yano. On the other hand, it can be also seen that the 2-form $F'=z\mathrm{d}x\wedge\mathrm{d}y + x\mathrm{d}y\wedge\mathrm{d}z + y\mathrm{d}z\wedge\mathrm{d}x$ is a Killing 2-form, but it is not a Killing magnetic field.

In this and in the next sections, we deal with the corresponding Landau problem for Killing magnetic fields in spaces with dimension 3 and constant curvature. To better understand the behavior of Killing magnetic curves, we start recalling some ideas relative to variations of curves in a certain space of curves in a Riemannian manifold.

Let $\beta: I \subset \mathbb{R} \to M^3$ be a curve in a Riemannian manifold with dimension 3, (M^3,g), and V a vector field along that curve. One can take a variation of β in the direction of V, say, a map $\Gamma: I \times (-\epsilon,\epsilon) \to M^3$ which satisfies

$$\Gamma(s,0) = \beta(s), \quad \left(\frac{\partial \Gamma}{\partial s}(s,t)\right)_{t=0} = V(s).$$

In this setting, we have three functions associated with the longitudinal curves $\beta_t(s)=\Gamma(s,t)$, namely,

1. the *speed* function $v(s,t)=\|(\partial\Gamma/\partial s)(s,t)\|$,
2. the *curvature* function $\kappa(s,t)$ of $\beta_t(s)$, and
3 the *torsion* function $\tau(s,t)$ of $\beta_t(s)$.

It should be noticed that, in spaces with constant curvature, those functions determine the curves and so the variation up to congruences in background.

A tedious computation allows one to compute formulas giving the variations of those functions at $t=0$,

$$V(v) = \left(\frac{\partial v}{\partial t}(s,t)\right)_{t=0} = g(\nabla_T V, T)v, \tag{13}$$

$$V(\kappa) = \left(\frac{\partial \kappa}{\partial t}(s,t)\right)_{t=0} = g(\nabla_T^2 V, N) - 2\kappa g(\nabla_T V, T) + g(R(V,T)T, N), \tag{14}$$

$$V(\tau) = \left(\frac{\partial \tau}{\partial t}(s,t)\right)_{t=0} = \left[\frac{1}{\kappa} g(\nabla_T^2 V + R(V,T)T, B)\right]_s + g(R(V,T)N, B) + \tau g(\nabla_T V, T) + \kappa g(\nabla_T V, B), \tag{15}$$

where R stands for the curvature tensor of M^3.

Proposition 1: Let $V(s)$ be the restriction to $\beta(s)$ of a Killing vector field, say, V, of M^3; then

$$V(v) = V(\kappa) = V(\tau) = 0.$$

Proof. Any local flow $\{\phi_t\}$ generated by V is made up of (local) isometries of M^3. Since the above formulas do not depend on the variation Γ but only on $V(s)$, we can variate $\beta(s)$ in the direction of $V(s)$ as follows:

$$\beta_t(s) = \Gamma(s,t) =: \phi_t(\beta(s)).$$

Now, the isometric nature of ϕ_t implies that the above considered three functions, $v(s,t)$, $\kappa(s,t)$, and $\tau(s,t)$, do not depend on t and so $V(v) = V(\kappa) = V(\tau) = 0$. ■

When M is a simply connected real-space form with constant curvature C, set as $M^3(C)$, then one has a converse of the stated Proposition 1.

Proposition 2: Let $\beta(s)$ be a curve in $M^3(C)$ and $V(s)$ a vector field along $\beta(s)$ that satisfies

$$V(v) = V(\kappa) = V(\tau) = 0;$$

then, $V(s)$ extends to a Killing vector field V of $M^3(C)$.

Proof: The equations $V(v) = V(\kappa) = V(\tau) = 0$ constitute a linear system in $V(s)$ whose solution space is six dimensional. Now, from Proposition 1, the restriction to $\beta(s)$ of any Killing field of $M^3(C)$ gives a solution of such a linear system. Since $M^3(C)$ is simply connected, the dimension of its isometry group is also six and then one obtains that the space of solutions is just made up of the restriction of Killing vector fields of $M^3(C)$. ■

The following result shows that each Killing magnetic flow line has constant quasislope. Furthermore it exhibits differential equations that characterize the Killing magnetic flow lines.

Theorem 1. (main result):

1. *Let V be a Killing vector field on a simply connected space form $(M^3(C), g)$. Then, the unit speed magnetic trajectories of $(M^3(C), g, V)$ are curves with curvature and torsion satisfying*

$$\kappa^2\left(\frac{1}{2}\omega - \tau\right) = A_1, \tag{16}$$

$$\kappa'' + \kappa\tau(\omega - \tau) + C\kappa + \frac{1}{2}\kappa^3 - A_2\kappa = 0, \tag{17}$$

where A_1 and A_2 are undetermined constants and ω is the constant quasislope defined to satisfy

$$V(s) = \omega T(s) + \kappa(s) B(s). \tag{18}$$

2. *Let γ be a unit speed curve in $M^3(C)$ whose curvature and torsion satisfy Eqs. (16) and (17)*

for certain constants A_1, A_2, and ω. Then, there exists a Killing vector field V on $M^3(C)$ such that γ is a magnetic trajectory of $(M^3(C),g,V)$ with quasislope ω. ■

Proof: Assume that V is a magnetic field in a Riemannian space M^3, not necessarily with constant curvature. Along any unit speed magnetic trajectory γ and according to Lemma 1, one has $V(s)=\omega(s)T(s)+\kappa(s)B(s)$. Now, if V is Killing, Proposition 1 implies that $V(v)=0$ and so, from Eq. (13), one obtains that $\nabla_T V$ has no tangential component, i.e., $\omega'(s)=0$. The constancy of the quasislope proves Eq. (18). Furthermore, one has

$$\nabla_T V = \kappa(\omega-\tau)N + \kappa' B.$$

Next, using once more the Frenet equations, one has

$$\nabla_T^2 V = -\kappa^2(\omega-\tau)T + [(\kappa(\omega-\tau))' - \kappa'\tau]N + [\kappa'' + \kappa\tau(\omega-\tau)]B.$$

This information is combined with $V(\kappa)=0$ in Eq. (14) to get

$$V(\kappa) = (\kappa(\omega-\tau))' - \kappa'\tau + g(R(V,T)T,N) = \omega\kappa' - 2\kappa'\tau - \kappa\tau' + g(R(V,T)T,N) = 0.$$

In the above equation, one multiplies by κ to have

$$\left(\kappa^2\left(\frac{1}{2}\omega-\tau\right)\right)' + \kappa g(R(V,T)T,N) = 0.$$

In particular, if M^3 has constant curvature C, then $g(R(V,T)T,N)=Cg(V,N)=0$ because of Eq. (18) and so

$$\left(\kappa^2\left(\frac{1}{2}\omega-\tau\right)\right)' = 0,$$

which shows Eq. (16).

One proceeds similarly with $V(\tau)=0$ and Eq. (15) to obtain

$$V(\tau) = \left[\frac{1}{\kappa}(\kappa'' + \kappa\tau(\omega-\tau) + g(R(V,T)T,B))\right]' + g(R(V,T)N,B) + \kappa\kappa' = 0.$$

Therefore, if M^3 has constant curvature C, then $g(R(V,T)T,B)=Cg(V,B)=C\kappa$ and $g(R(V,T)N,B)=0$; hence

$$\left[\frac{1}{\kappa}(\kappa'' + \kappa\tau(\omega-\tau) + C\kappa)\right]' + \left(\frac{1}{2}\kappa^2\right)' = 0.$$

This implies

$$\frac{1}{\kappa}(\kappa'' + \kappa\tau(\omega-\tau) + C\kappa) + \frac{1}{2}\kappa^2 = A_2$$

for a certain function A_2. This proves Eq. (17).

To prove the converse, let γ be a unit speed curve in $M^3(C)$ satisfying Eqs. (16) and (17) for arbitrary constants A_1, A_2, and ω. Define, along γ, the vector field $V(s)=\omega T(s)+\kappa(s)B(s)$, with the obvious meaning for ingredients. Using once more the Frenet equations, one has

$$\nabla_T V = \kappa(\omega-\tau)N + \kappa' B,$$

$$\nabla_T^2 V = -\kappa^2(\omega-\tau)T + [(\kappa(\omega-\tau))' - \kappa'\tau]N + [\kappa'' + \kappa\tau(\omega-\tau)]B.$$

All this information is carried out to Eqs. (13)–(15). Then, one uses that the background has constant curvature C and finally Eqs. (16) and (17) to obtain $V(v)=V(\kappa)=V(\tau)=0$.

Now, since $M^3(C)$ is simply connected, one uses Proposition 2 to extend $V(s)$ to a Killing vector field, say, V, on the whole space. Finally, point (f) in Sec. IV guarantees that γ is a magnetic trajectory of $(M^3(C),g,V)$. ■

This result has some interesting consequences. For example, it allows us to obtain a geometric meaning for that ω which we called quasislope of trajectories.

Corollary 1: Let V be a Killing vector field on a 3D Riemannian space (M^3,g). *Then each trajectory* γ *of* (M^3,g,V) *has constant quasislope, say,* ω. *Furthermore, if* $\theta(s)$ *denotes the angle that* $\gamma(s)$ *makes with* $V(s)$ *and* $\kappa(s)$ *is the curvature function of the trajectory, then the following conservation law holds:*

$$\kappa(s)\cot\,\theta(s)=\omega.$$

In particular, if $V(s)$ *has constant length, then the trajectory* γ *has constant curvature. Therefore the magnetic trajectories of a Killing field with constant length (infinitesimal translation) have both quasislope and curvature constant.*

Proof: Just notice that $\cos\,\theta(s)=\omega/\|V(s)\|$ and $\|V(s)\|=\sqrt{\omega^2+\kappa^2}$. Therefore,

$$\sin\,\theta(s)=\frac{\kappa(s)}{\sqrt{\omega^2+\kappa(s)^2}},$$

and so $\cot\,\theta(s)=\omega/\kappa(s)$. ■

VI. THE INTEGRATION OF THE KILLING MAGNETIC FLOW EQUATIONS

Once showed the constancy of the quasislope for magnetic trajectories of $(M^3(C),g,V)$; then the big consequence is that the corresponding equations [Eqs. (16) and (17)] can be nicely integrated by standard techniques in terms of elliptic functions (see Ref. 8 for details on elliptic integrals and elliptic functions). In particular, we completely solve the Landau-Hall problem for Killing magnetic fields in spaces with constant curvature. The way to do it works as follows. First, it should be noticed that a magnetic trajectory with constant curvature, which happens if and only if the length of the Killing vector field is constant along it, is automatically a helix [Eq. (16)]. Since in this case it makes no sense to integrate the equations, even if we will consider later the limit case when the Killing vector field is actually an infinitesimal translation, we will restrict ourselves to integrate the case of nontrivial trajectories or those with nonconstant curvature, which implies that $V(s)$ has no constant length.

Then, from Eq. (16) one can express the torsion of trajectories in terms of both the corresponding curvature and the constant quasislope by

$$\tau=\frac{1}{2}\omega-\frac{A_1}{\kappa^2}.$$

Now, this is combined with Eq. (17) to see that the curvature of magnetic trajectories must satisfy the following nonlinear second order differential equation:

$$\kappa''+\frac{1}{2}\kappa^3+\left(C-A_2+\frac{1}{4}\omega^2\right)\kappa-\frac{A_1^2}{\kappa^3}=0. \qquad (19)$$

This equation admits an obvious first integral. In fact, just multiply by $2\kappa'$ and integrate to get

$$(\kappa')^2+\frac{1}{4}\kappa^4+\left(C-A_2+\frac{1}{4}\omega^2\right)\kappa^2+\frac{A_1^2}{\kappa^2}=A_3,$$

where A_3 is an undetermined constant.

The last equation, in terms of the squared of the curvature, $h=\kappa^2$, can be written by

$$(h')^2 + h^3 + 4\left(C - A_2 + \frac{1}{4}\omega^2\right)h^2 - 4A_3 h + 4A_1^2 = 0. \qquad (20)$$

Since this equation is of the type $(h')^2 = P(h)$, where P is a polynomial of degree 3 in h, it can be solved using elliptic functions. To proceed, it is important to control the roots of P. To this respect, it is enough to observe the following properties of the cubic polynomial $P(h)$:

- $P(0) = -4A_1^2 \leqslant 0$.
- $\lim_{h\to+\infty} P(h) = -\infty$.
- $\lim_{h\to-\infty} P(h) = +\infty$.
- Finally, observe that if $h = \kappa^2$ is a nonconstant solution, then it must obviously take on values at which $P(h) < 0$.

This behavior allows one to assume that $P(h)$ has three real roots, say, a_1, a_2, and a_3 satisfying $a_1 \leqslant 0 \leqslant a_2 \leqslant a_3$ and, of course, treat the case where κ is constant (this case provides helices) separately. Now, the differential equation

$$(h')^2 + (h - a_1)(h - a_2)(h - a_3) = 0$$

is known to have the following general solution:

$$h(s) = a_3(1 - q^2 sn^2(rs, p)),$$

with parameters defined in terms of the roots as follows:

$$p^2 = \frac{a_3 - a_2}{a_3 - a_1}, \quad q^2 = \frac{a_3 - a_2}{a_3}, \quad r = \frac{\sqrt{a_3 q^2}}{\sqrt{4p^2}} = \frac{1}{2}\sqrt{a_3 - a_1}.$$

Obviously, the roots a_1, a_2, and a_3 are related to the coefficients of $P(h)$ by

$$4\left(C - A_2 + \frac{1}{4}\omega^2\right) = -a_1 - a_2 - a_3,$$

$$-4A_3 = a_1 a_2 + a_1 a_3 + a_2 a_3,$$

$$4A_1^2 = -a_1 a_2 a_3.$$

Therefore, given real numbers a_1, a_2, a_3, and ω with $a_1 \leqslant 0 \leqslant a_2 \leqslant a_3$, one constructs the parameters

$$A_1 = \pm\frac{1}{2}\sqrt{-a_1 a_2 a_3},$$

$$A_2 = C + \frac{1}{4}(\omega^2 + a_1 + a_2 + a_3),$$

$$A_3 = -\frac{1}{4}(a_1 a_2 + a_1 a_3 + a_2 a_3).$$

Now, with these values, one solves Eq. (20) to obtain

$$h(s) = a_3(1 - q^2 sn^2(rs, p)),$$

with the corresponding values of p, q, and r. Next, one considers the unit speed curve γ in $(M^3(C), g)$ with curvature and torsion given, respectively, by

$$\kappa(s)^2 = h(s), \quad \tau(s) = \frac{1}{2}\omega - \frac{A_1}{\kappa^2}.$$

Then, there exists a Killing vector field V on $M^3(C)$ such that γ is a magnetic trajectory of $(M^3(C), g, V)$. All the magnetic trajectories of a Killing vector field on $M^3(C)$ are constructed in this way, except the case of constant curvature. In this case the torsion is also a constant and then the trajectory is a helix along which the length of the Killing vector field is constant, too. This process can be summarized as follows: Let $a_1 \leq 0 \leq a_2 \leq a_3$ and ω be arbitrary real numbers. Then, the arclength parametrized curve $\gamma(s)$, with curvature and torsion given by

$$\kappa(s) = \sqrt{a_3(1 - q^2 sn^2(rs, p))}, \quad \tau(s) = \frac{1}{2}\omega - \frac{A_1}{\kappa^2},$$

with $A_1 = \pm \frac{1}{2}\sqrt{-a_1 a_2 a_3}$, is a magnetic trajectory with quasislope ω of a Killing magnetic field that extends the moment along the trajectory, $V(s) = \omega T(s) + \kappa(s) B(s)$. Furthermore, every Killing magnetic flow is made up to congruences of either helices or trajectories determined by curvature and torsion functions as above.

VII. THE CASE WHERE THE MAGNETIC FIELD IS AN INFINITESIMAL TRANSLATION

In this section, we study the case of a magnetic field being a Killing one with constant length. Then we have the following strong result which will be shown to be the best possible in all the directions.

Corollary 2: Let V be a (nonzero) Killing vector field, with constant length (an infinitesimal translation), in $(M^3(C), g)$; *then the magnetic trajectories of* $(M^3(C), g, V)$ *are helices with axis V.*

Proof: Since $\|V(s)\|^2 = \omega^2 + \kappa(s)^2$ is constant along any magnetic trajectory γ, one concludes that the magnetic trajectories have constant curvature. Next, one uses Eq. (16) and (17) of the main theorem to obtain the constancy of the torsion and so conclude the helicity nature of the magnetic trajectories. ■

Remark: The Killing vector fields with constant length are known as infinitesimal translations and they are characterized by the following property: *A Killing vector field V is an infinitesimal translation if and only if it has integral curves that are geodesics.* A well known obstruction to the existence of infinitesimal translations on a Riemannian manifold is obtained in terms of the Ricci curvature. In fact, for a Killing vector field of constant length, one has $0 = \triangle \frac{1}{2}\|V\|^2 = \|\nabla V\|^2 - \mathrm{Ric}(V, V)$. Therefore, a Riemannian manifold with negative definite Ricci tensor does not admit a nontrivial infinitesimal translation. Finally, obviously a nontrivial infinitesimal translation is not zero. Therefore, if it exists on a compact manifold then its Euler number must be zero.

Example 1: Take $(M = \mathbb{R}^3, g_{\mathbb{R}^3})$, the ordinary Euclidean space. Consider the magnetic field $V = \partial_z$ which is an infinitesimal translation. The trajectories of $(\mathbb{R}^3, g_{\mathbb{R}^3}, \partial_z)$ are, according to the above stated theorem, helices with axis ∂_z, that is,

$$\gamma(t) = (x_0 + a\cos t, y_0 + a \sin t, z_0 + bt),$$

where $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $a, b \in \mathbb{R}$.

Example 2: Let $\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ be the unit sphere endowed with its usual metric $g_{\mathbb{S}^3}$ induced from $\mathbb{R}^4 \equiv \mathbb{C}^2$. Consider the vector field V on $\mathbb{S}^3$ defined by $V(z) = iz = (iz_1, iz_2)$. Certainly, this is a Killing vector field with constant length 1 and so an infinitesimal translation on $\mathbb{S}^3$. In fact, the global flow generated by V is $\{\phi_t : \mathbb{S}^3 \to \mathbb{S}^3 : t \in \mathbb{R}\}$ with $\phi_t(z) = e^{it}z$, which are isometries of $\mathbb{S}^3$. Consequently, we use the above stated result to see that the trajectories of $(\mathbb{S}^3, g_{\mathbb{S}^3}, V)$ are helices in the 3-sphere. To understand helices in the 3-sphere better, one needs the usual Hopf mapping, $\pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$, where the basis is considered to have radius $1/2$ in order to be treated as a Riemannian submersion. In this context, one can describe completely helices, with axis V, in $\mathbb{S}^3$ as follows (see Ref. 3 for details):

First, observe that given a geodesic circle, say, $\beta = \beta(u)$, in $\mathbb{S}^2(1/2)$, its Hopf tube $\mathbf{T}_\beta = \pi^{-1}(\beta)$ is a flat torus with constant mean curvature $H = \frac{1}{2}\rho$ in $\mathbb{S}^3$, where ρ denotes the curvature of the geodesic circle β. That torus can be parametrized from the following Riemannian covering map:

$$\Phi : \mathbb{R}^2 \to \mathbf{T}_\beta \subset \mathbb{S}^3, \quad \Phi(u,t) = e^{it}\bar{\beta}(u),$$

where $\bar{\beta}$ stands for a horizontal lift of β. Therefore, the coordinate families of curves are the fibers (u=const) and the horizontal lifts of β (t=const). Now, each geodesic γ of $\mathbf{T}_\beta$ is determined from its slope $\sigma = b/a$ measured with respect to Φ, that is,

$$\gamma(s) = \Phi(as,bs), \quad \gamma'(s) = a\Phi_u + b\Phi_t.$$

An easy computation shows that γ is a helix of $\mathbb{S}^3$ with curvature and torsion given, respectively, by

$$\kappa = \frac{\rho + 2\sigma}{1+\sigma^2}, \quad \tau = \frac{1 - \sigma\rho - \sigma^2}{1+\sigma^2},$$

and making a constant angle with V, i.e., it is a helix in $\mathbb{S}^3$ with axis V.

However, the converse of the above fact also holds. Given any helix, say, γ, of $\mathbb{S}^3$ with axis V (curvature $\kappa \neq 0$ and torsion τ), one can consider the geodesic circle β of $\mathbb{S}^2(1/2)$ with curvature

$$\rho = \frac{\kappa^2 + \tau^2 - 1}{\kappa},$$

and then one chooses in its Hopf torus $\mathbf{T}_\beta$ the geodesic determined by the slope

$$\sigma = \frac{1-\tau}{\kappa}.$$

One obtains a curve with curvature κ and torsion τ which is congruent to γ in $\mathbb{S}^3$.

As a consequence the magnetic trajectories of $(\mathbb{S}^3, g_{\mathbb{S}^3}, V)$ are the geodesics of the Hopf tori in $\mathbb{S}^3$ constructed over geodesic circles in $\mathbb{S}^2(1/2)$.

VIII. THE HALL EFFECT DESCRIBED THROUGH ELASTIC CURVES

The Hall effect is a classical phenomenon for uniform magnetic fields in Euclidean space, and so in a free of gravity environment. It explains the dynamics of an electric current flow X in $\mathbb{R}^3$ when exposed to a perpendicular uniform magnetic field V. The basic physical principle underlying the Hall effect is the Lorentz force appearing in the Lorentz force equation. Therefore, X experiences a force, the Lorentz force, acting normal to both X and V, and it moves in response to this force and the force affected by its internal electric field.

Now, after Eq. (4), we notice that the Hall effect also appears in a more general context. For example, it applies to any magnetic field in $\mathbb{R}^3$, not necessarily uniform. Moreover, it also works in any Riemannian space (M^3, g) even with nontrivial gravity. Therefore, when an electric current flow X moves through a conductor (M^3, g) and perpendicular to an applied magnetic field V, it experiences a force, the Lorentz force given by Eq. (4), acting normal to both directions and it moves in response to this force and the force affected by its internal electric field.

In particular, if the conductor has constant curvature $(M^3(C), g)$ and the magnetic field V is Killing, then we are able to determine the above evolution using the theory of elasticae, and so to get at least two important consequences. First, we solve the associated Landau-Hall problem. Moreover, we obtain an amazing relationship between, on one hand, the Hall-Killing effect, and on the other hand, the elastic curves in $M^3(C)$. To better understand the latter fact, assume a Killing field V in $M^3(C)$. Now, suppose that γ is a unit speed curve in $M^3(C)$ which is orthogonal to V. Then, this curve is a magnetic trajectory of $(M^3(C),g,V)$, i.e., a trajectory of an electric current flow X orthogonally exposed to $(M^3(C),g,V)$ if and only if its curvature and torsion satisfy Eqs. (16) and (17) with $\omega=0$, that is,

$$\tau\kappa^2=-A_1,$$

$$2\kappa''+\kappa^3-2\kappa\tau^2+2C\kappa-2A_2\kappa=0.$$

Now, these two equations are just the Euler-Lagrange equations for elasticae in $M^3(C)$ with Lagrange multiplier $\lambda=2A_2$,[17,19] that is, the critical points of the action

$$\mathcal{E}(\gamma)=\int_\gamma(\kappa^2(s)+2A_2)\mathrm{d}s.$$

In particular, this shows that the Hall-Killing effect can be understood in a variational context and so be treated in a variational approach. Namely, electric current flows perpendicular to an applied Killing magnetic field in a space of constant curvature evolving through trajectories which are equilibria for a linear combination of the first two conserved Hamiltonians in the LIE hierarchy. Finally, notice that the zeros of the Killing magnetic field V correspond with a kind of sources in the sense that all possible orthogonal magnetic trajectories through these points must be elasticae of $M^3(C)$.

IX. MAGNETIC FLOWS, ELASTIC RODS, AND VORTICES

Roughly speaking, the general Hall effect describes what happens to current X flowing through a conducting material (M^3,g) if it is exposed to an arbitrary magnetic field V. In this sense, the surprise continues. Equation (4) also applies to electric current flows submitted to magnetic fields not necessarily perpendicular to the current. Moreover, as in the orthogonal case, for Killing magnetic fields in material with constant curvature, we can get all the solutions of the corresponding Lorentz equation as extremals of a variational problem associated with an action which is a linear combination of the first three conserved Hamiltonians of the LIE hierarchy. Recall that the elastic curves in $(M^3(C),g)$ are nothing but the critical points of the action

$$\mathcal{E}(\bar\gamma)=\int_{\bar\gamma}(\bar\kappa^2(\bar s)+\lambda)\mathrm{d}\bar s$$

on a suitable space of curves, where λ works as a Lagrange multiplier and overbars will used to distinguish elasticae from magnetic trajectories. The Euler-Lagrange equations of this variational problem can be computed to be

$$\bar\tau\bar\kappa^2=c, \tag{21}$$

$$\bar\kappa''+\frac{1}{2}\bar\kappa^3+\left(C-\frac{\lambda}{2}\right)\bar\kappa-\frac{c^2}{\bar\kappa^3}=0. \tag{22}$$

On the other hand, the magnetic trajectories of $(M^3(C),g,V)$, with V Killing, are solutions of Eqs. (16) and (19). Therefore, it seems natural to compare both pairs of equations. First, notice that Eqs. (22) and (19) coincide just when

$$\lambda=2A_2-\frac{1}{2}\omega^2,\quad A_1^2=c^2.$$

In this setting, let γ and $\bar\gamma$ be the unit speed curves that are solutions of Eqs. (16) and (19) and Eqs. (21) and (22), respectively. Then, they have the same curvature functions and the corresponding torsion functions coincide, up to the quasislope of γ, namely,

$$\kappa(s) = \bar{\kappa}(s), \quad \tau(s) = \bar{\tau}(s) + \frac{1}{2}\omega,$$

where one has chosen a suitable orientation. Therefore, one has showed the following:

Corollary 3: Let V be a Killing vector field on $(M^3(C),g)$ and γ a unit speed magnetic trajectory of $(M^3(C),g,V)$. Then, there exists an elastica $\bar{\gamma}$ in $M^3(C)$ which has the same curvature function as γ and the same torsion up to the quasislope of γ. ■

As an interesting consequence of the last result and Corollary 4.4 in Ref. 18, one has the following relationship between magnetic trajectories of $(M^3(C),g,V)$, V being any Killing vector field in the Euclidean space $\mathbb{R}^3$, and the Kirchhoff elastic rods (see also Refs. 7 and 12).

Corollary 4: For any Killing vector field V on $(M^3(C),g)$, the magnetic trajectories of $(M^3(C),g,V)$ are just the centerlines of elastic rods in $M^3(C)$. In other words, the magnetic trajectories of $(M^3(C),g,V)$ are the critical points of functionals of the type

$$\mathcal{F}(\gamma) = \int_\gamma \left[\frac{\lambda_3}{2}\kappa^2(s) + \lambda_2\tau(s) + \lambda_1\right] \mathrm{d}s$$

acting on suitable spaces of curves that satisfy corresponding second order boundary conditions. Furthermore, the quasislope of a magnetic trajectory is given by $\omega=\lambda_2/\lambda_3$. Conversely, the centerline of any elastic rod in $M^3(C)$ is a magnetic trajectory of a Killing magnetic field extending the moment along the rod. ■

Notice that when $\lambda_2=0$, then one has magnetic trajectories orthogonal to V and so elasticae in $M^3(C)$ (see the last section). Another interesting consequence of the main result is that Killing magnetic flows are constituted by filament vortices and so they have a solitonic nature.

Corollary 5: The Killing magnetic flows on a 3D simply connected Riemannian space of constant curvature are made up of vortex filament solutions of the LIE. More precisely, every Killing magnetic trajectory is an initial condition for a one soliton solution of the filament equation. ■

X. LANCRET CURVES AS MAGNETIC TRAJECTORIES

In this section, we consider a class of non-Killing magnetic fields in constant curvature backgrounds. The magnetic flow lines of these magnetic fields are Lancret curves in those spaces. Since these curves are trajectories of relativistic particles in models associated with Lagrangian densities that depend linearly on both the curvature and the torsion of the trajectories, we have, on one hand, that the magnetic flow lines of these magnetic fields are obtained as solutions of a variational problem. On the other hand, we obtain a connection between two apparently unrelated physical models that tie magnetic flow lines with current trajectories of relativistic particles with rigidity and torsion (see Ref. 1 and references therein).

Since in the hyperbolic space $\mathbb{H}^3$ Lancret curves are helices, and so this theory can be considered trivial in hyperbolic background,[3] we will restrict ourselves to $\mathbb{R}^3$ and $\mathbb{S}^3$ and will treat them separately.

A. Lancret curves in Euclidean space

On $M=\mathbb{R}^3$ consider the vector field $V(x,y,z)=f(x,y)\partial_z$, where f is a nonconstant function that does not depend on z. It is clear that for any vector field X on $\mathbb{R}^3$ one has $\nabla_X V = X(f)\partial_z$, because $\nabla\partial_z=0$. Consequently $\mathrm{div}(V)=\partial_z f=0$, and so it defines a magnetic field in $\mathbb{R}^3$. Moreover, it has no constant length $\|V(x,y,z)\|=|f(x,y)|$.

The trajectories, $\alpha(t)=(x(t),y(t),z(t))$, of the Killing magnetic field defined by V satisfy the following Lorentz equations:

$$x'' = -fy',$$

$$y'' = fx',$$

$$z'' = 0.$$

Clearly, the third equation gives $z(t)=at+b$ for certain constants a and b. This shows that the trajectories lie in right cylinders constructed on plane curves. More precisely, denote by $\gamma(t)=(x(t),y(t))$ the projection of a trajectory onto the plane $z=0$. The right cylinder on γ is

$$\mathbf{C}_\gamma = \{\bar{\mathbf{x}}(t,v) = \gamma(t) + v\partial_z : (t,v) \in \mathbb{R}^2\},$$

and so $\alpha(t)=\bar{\mathbf{x}}(t,at+b)$, which shows that the trajectory α is a geodesic of $\mathbf{C}_\gamma$, that is, a Lancret curve in $\mathbb{R}^3$ with axis ∂_z. The first two equations determine completely the projection of trajectories in the plane $z=0$. In fact, those two equations can be written by

$$\gamma''(t) = f(x(t),y(t))J(\gamma'(t)),$$

where J denotes the positive rotation of angle $\pi/2$ in the plane $z=0$. This allows one to compute the curvature of $\gamma(t)$ to be

$$\kappa_\gamma(t) = \frac{g(\gamma''(t),J(\gamma'(t)\ )}{\|\gamma'(t)\|^3} = \frac{f(x(t),y(t))}{\sqrt{r^2-a^2}},$$

where $r=\|\alpha'(t)\|$ is the constant speed of the trajectory. In particular, the curvature of the trajectories of $(\mathbb{R}^3, g_{\mathbb{R}^3}, V)$ is given by

$$\kappa_\alpha(t) = \frac{r^2-a^2}{r^2}\kappa_\gamma(t).$$

The converse of the above results also works. Let $\alpha(t)=(x(t),y(t),z(t))$ be a Lancret curve in $\mathbb{R}^3$ with axis ∂_z and assume that it is parametrized with constant speed $\|\alpha'(t)\|=r$. Then, it is a geodesic of a right cylinder over a plane curve, say, γ, in the plane $z=0$. Set $a=r\cos\varphi$, where φ is the slope of the Lancret curve, that is, the angle between the curve and its axis, then $z(t)=at+b$. The relationship between the curvature functions of α and γ is

$$\kappa_\alpha(t) = \kappa_\gamma(t)\sin^2\varphi.$$

On the other hand, since $\|\gamma'(t)\|=\sqrt{r^2-a^2}$ one has

$$\gamma''(t) = \|\gamma'(t)\|^2\kappa_\gamma(t)\frac{J(\gamma'(t))}{\|\gamma'(t)\|},$$

which implies that

$$x''(t) = -\|\gamma'(t)\|\kappa_\gamma(t)y'(t),$$

$$y''(t) = \|\gamma'(t)\|\kappa_\gamma(t)x'(t).$$

These equations, joint with $z(t)=at+b$, show that the Lancret curve α is a magnetic trajectory of $(\mathbb{R}^3, g_{\mathbb{R}^3}, V(x,y,z)=f(x,y)\partial_z)$ for any function f defined in the plane $z=0$ that satisfies

$$f(\gamma(t)) = \sqrt{r^2-a^2}\kappa_\gamma(t).$$

All the previous discussion can be summarized as follows: The Lancret curves in the Euclidean space are characterized from the fact that they are magnetic trajectories associated with magnetic fields parallel to their axis through a potential depending on both the curvature of the projection onto the plane orthogonal to the axis and the slope.

B. Lancret curves in the sphere

Perhaps, the more interesting setting to study Lancret curves is the 3D sphere. Here, the Hopf map provides a nice tool to understand the geometry of those curves.[3] Any smooth function, $f:\mathbb{S}^2(1/2)\to\mathbb{R}$, can be lifted to $\mathbb{S}^3$, via the Hopf map π, to get the potential $\bar{f}=f\circ\pi$, and then the vector field $\tilde{V}:=\bar{f}V$, where V is given in Example 2 of Sec. VII. Now, $\tilde{V}$ has divergence zero and so it defines a magnetic field in $\mathbb{S}^3$. To check this, let $\tilde{\nabla}$ be the Levi-Civita connection in the sphere, and denote, as usual, with overbars the horizontal lifts, via the Hopf map, of corresponding objects on $\mathbb{S}^2(1/2)$. Using the O'Neill formulas,[5,24] one has

$$g(\tilde{\nabla}_{\bar{X}}\tilde{V},\bar{X}) = \bar{X}(\bar{f})g(V,\bar{X}) + \bar{f}g(\tilde{\nabla}_{\bar{X}}V,\bar{X}) = 0,$$

because V is Killing, and

$$g(\tilde{\nabla}_{V}\tilde{V},V) = V(\bar{f})g(\tilde{V},V) + \bar{f}g(\tilde{\nabla}_{V}V,V) = 0.$$

The following statement gives a first approach to the relationship between the Lancret curves and the magnetic trajectories of these kinds of Killing fields: The magnetic trajectories of $(\mathbb{S}^3,g_{\mathbb{S}^3},\tilde{V}=\bar{f}V)$ are Lancret curves in $\mathbb{S}^3$ with axis V. Consequently, they are geodesics of Hopf tubes in $\mathbb{S}^3$ over curves in $\mathbb{S}^2(1/2)$.

To show it, recall that the magnetic trajectories α of $(\mathbb{S}^3,g_{\mathbb{S}^3},\tilde{V}=\bar{f}V)$ satisfy

$$\tilde{\nabla}_{\alpha'}\alpha' = \tilde{V}\wedge\alpha' \quad \text{and} \quad \|\alpha'(t)\| = r, \text{ const.}$$

If $\varphi(t)$ denotes the angle between $\alpha'(t)$ and $V(\alpha(t))$, then

$$\cos\varphi(t) = \frac{1}{r}g(\alpha'(t),V(\alpha(t))),$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\alpha'(t),V(\alpha(t))) = g(\tilde{V}(\alpha(t))\wedge\alpha'(t),V(\alpha(t))) = 0,$$

where we have used that V is Killing in $\mathbb{S}^3$. The last computation shows that the magnetic trajectories of $(\mathbb{S}^3,g_{\mathbb{S}^3},\tilde{V}=\bar{f}V)$ made a constant angle with the Killing vector field V with constant length. This proves that they are Lancret curves in $\mathbb{S}^3$ and so geodesics of Hopf tubes (see Ref. 3 for details).

Let $\gamma(t)=\pi(\alpha(t))$ be the projection in $\mathbb{S}^2(1/2)$ of a trajectory γ under the Hopf map π. Its Hopf tube is

$$\mathbf{T}_{\gamma} = \{\Phi(t,v) = e^{iv}\bar{\gamma}(t):(t,v)\in\mathbb{R}^2\},$$

where $\bar{\gamma}$ is a horizontal lift of γ. Since α is a geodesic in $\mathbf{T}_{\gamma}$, $\alpha(t)=\Phi(t,at+b)$ and so

$$\alpha'(t) = e^{i(at+b)}\bar{\gamma}'(t) + aV(\alpha(t)), \quad \|\alpha'(t)\|^2 = r^2 = a^2 + \|\gamma'(t)\|^2.$$

The curvature of trajectories can be computed in terms of the potential $\bar{f}$ as follows:

$$\tilde{\nabla}_{T_\alpha}T_\alpha = \frac{1}{r^2}\tilde{\nabla}_{\alpha'}\alpha' = \frac{1}{r^2}\tilde{V}\wedge\alpha',$$

where T_α denotes the unit tangent vector field of the trajectory. Therefore,

$$\tilde{\nabla}_{T_\alpha} T_\alpha = \frac{\bar{f}}{r^2} V \wedge (aV + e^{i(at+b)}\bar{\gamma}') = \frac{\bar{f}}{r^2} e^{i(at+b)} i\bar{\gamma}'.$$

Consequently,

$$\kappa_\alpha^2 = \|\bar{\nabla}_{T_\alpha} T_\alpha\|^2 = \frac{\bar{f}^2}{r^4}\|\bar{\gamma}'\|^2.$$

On the other hand, the curvature of γ is related to that of α by[3]

$$\kappa_\gamma(t) = \frac{r^2}{r^2 - a^2}\kappa_\alpha(t) + \frac{2a}{\sqrt{r^2 - a^2}},$$

and therefore,

$$\kappa_\gamma(t) = \frac{f(t) + 2a}{\sqrt{r^2 - a^2}}.$$

This can be used to show the converse of the stated fact. Let us assume now that α is a Lancret curve in $\mathbb{S}^3$ with $\|\alpha'(t)\| = r$. Then, it is a geodesic of a Hopf tube,[3] say $\mathbf{T}_\gamma$. Therefore, there exist $a, b \in \mathbb{R}$ such that

$$\alpha(t) = \Phi(t, at + b) = e^{i(at+b)}\bar{\gamma}(t).$$

A direct computation shows that α is a magnetic trajectory of $(\mathbb{S}^3, g_{\mathbb{S}^3}, \tilde{V} = \bar{f}V)$ for any function f in the 2-sphere $\mathbb{S}^2(1/2)$ that satisfies

$$f(\gamma(t)) = \sqrt{r^2 - a^2}\kappa_\gamma(t) - 2a.$$

XI. GENERAL VARIATIONAL APPROACH

In this section we deal with the following general variational problem. In $(M^3(C), g)$, we consider the space of clamped curves Λ that means curves in Λ are those smooth curves in $M^3(C)$ connecting a pair of points, say, $p, q \in M^3(C)$, and having the same Frenet frame at these points. Now, consider the action $\mathcal{F}: \Lambda \to \mathbb{R}$ defined by

$$\mathcal{F}(\gamma) = \int_\gamma F(\kappa, \tau)\mathrm{d}s,$$

where κ and τ are, as usual, the curvature and the torsion of the curve $\gamma \in \Lambda$ and F is a suitable function defining the Lagrangian density. Now the problem is to characterize the critical points (equilibria) of $(\Lambda, \mathcal{F})$. To do it, we first make a direct computation to see that the tangent space $T_\gamma\Lambda$ of Λ at $\gamma \in \Lambda$ is made up of the vector fields W along γ that satisfy the following boundary conditions:

1. W vanishes at the end points p and q.
2. $\nabla_T W$ lies in the tangent strait line of γ at the end points p and q.
3. $\nabla_T^2 W$ lies in the osculating plane of γ at the end points p and q.

Now, a standard argument which involves formulas (13)–(15) and some integrations by parts allows one to obtain

$$(\delta\mathcal{F})(\gamma): T_\gamma\Lambda \to \mathbb{R}$$

by

$$(\delta\mathcal{F})(\gamma)[W] = \int_\gamma g(\Omega(\gamma),W)\mathrm{d}s + \int_\gamma \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{B}(\gamma,W)\mathrm{d}s,$$

where $\Omega(\gamma)$ and $\mathcal{B}(\gamma,W)$ denote the Euler-Lagrange and the boundary operators which are, respectively, given by

$$\Omega(\gamma) = \nabla_T P + C\left(-F_\kappa N + \frac{1}{\kappa}\frac{\mathrm{d}F_\tau}{\mathrm{d}s}B\right), \tag{23}$$

where

$$P = \{F - (\kappa F_\kappa + \tau F_\tau)\}T - \left(\frac{\mathrm{d}F_\kappa}{\mathrm{d}s} + \frac{\tau}{\kappa}\frac{\mathrm{d}F_\tau}{\mathrm{d}s}\right)N + \left(-\tau F_\kappa + \kappa F_\tau + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\kappa}\frac{\mathrm{d}F_\tau}{\mathrm{d}s}\right)\right)B$$

and

$$\mathcal{B}(\gamma,W) = g\left(\nabla_T^2 W, \frac{F_\tau}{\kappa}B\right) + g\left(\nabla_T W, F_\kappa N - \frac{1}{\kappa}\frac{\mathrm{d}F_\tau}{\mathrm{d}s}B\right) + g(W,P). \tag{24}$$

The critical points are the curves $\gamma \in \Lambda$ that satisfy $\delta\mathcal{F}(\gamma)[W]=0$ for any $W \in T_\gamma\Lambda$. We can make successive choices for W in order to prove that the critical points satisfy the equation $\Omega(\gamma)=0$. However, to get a converse, and so a characterization of critical points, we need to use the boundary conditions that satisfy W. They automatically imply that the boundary terms appearing in the boundary operator [Eq. (24)] drop out, i.e.,

$$\int_\gamma \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{B}(\gamma,W)\mathrm{d}s = [\mathcal{B}(\gamma,W)]_0^L = 0.$$

Here $p=\gamma(0)$ and $q=\gamma(L)$.

As a consequence the critical points of the variational problem $(\Lambda,\mathcal{F})$ are those $\gamma \in \Lambda$ such that they satisfy the Euler-Lagrange equation $\Omega(\gamma)=0$.

It should be noticed that in a flat background $(C=0)$ the above stated Euler-Lagrange equation [Eq. (23)] is $\nabla_T P=0$. In particular, this means that P works as a Killing vector field along each equilibrium of $(\Lambda,\mathcal{F})$. However, this is not exclusive of a flat background. In fact, it also holds for any $M^3(C)$. One can see that $P(v)=P(\kappa)=P(\tau)=0$ along each critical point γ of $(\Lambda,\mathcal{F})$.

The equilibria of $(\Lambda,\mathcal{F})$ admit another Killing vector field. In fact, define along each equilibrium

$$J = F_\tau T + \frac{1}{\kappa}\frac{\mathrm{d}F_\tau}{\mathrm{d}s}N + F_\kappa B, \tag{25}$$

then it is not difficult to see that $J(v)=J(\kappa)=J(\tau)=0$ along each equilibrium. Hence, the following:

Proposition 3: There exist two Killing vector fields, P and J, *along each equilibrium* γ *of* $(\Lambda,\mathcal{F})$. *They are related by*

$$\nabla_T J = P \wedge T.$$

■

Furthermore, the Euler-Lagrange equation, $\Omega(\gamma)=0$, can be written as follows:

$$\nabla_T P = CJ \wedge T. \tag{26}$$

Consequently, it seems natural to state the following problem: Since P and J extend to Killing vector fields $\widetilde{P}$ and $\widetilde{J}$ on $(M^3(C),g)$, when are equilibria of $(\Lambda,\mathcal{F})$ magnetic flow lines of $(M^3(C),g,\lambda\widetilde{P})$ or $(M^3(C),g,\mu\widetilde{J})$ for some $\lambda,\mu \in \mathbb{R}$?

Proposition 4: The equilibria of $(\Lambda,\mathcal{F})$ *are magnetic flow lines of* $(M^3(C),g,\mu\widetilde{J})$ *if and only if the Lagrangian density corresponds with an elastic rod, i.e.*,

$$F(\kappa,\tau)=\lambda_1+\frac{\omega}{\mu}\tau+\frac{1}{2\mu}\kappa^2.$$

Proof: If equilibria of $(\Lambda,\mathcal{F})$ are magnetic flow lines of $(M^3(C),g,\mu\widetilde{J})$, then along each critical point, we have

$$\mu J=\omega T+\kappa B,$$

where ω is a constant, the quasislope. Now, we compare this expression with Eq. (25) to obtain

$$F_\kappa=\frac{1}{\mu}\kappa,\quad F_\tau=\frac{\omega}{\mu},$$

and so we can integrate to get

$$F(\kappa,\tau)=\lambda_1+\frac{\omega}{\mu}\tau+\frac{1}{2\mu}\kappa^2.$$

Conversely, on each equilibrium

$$\mu J=\mu F_\tau T+\mu\frac{1}{\kappa}\frac{\mathrm{d}F_\tau}{\mathrm{d}s}N+\mu F_\kappa B.$$

Thus, if the Lagrangian density has the above expression, then

$$\mu J=\omega T+\kappa B,$$

which proves that equilibria are magnetic flow lines of $(M^3(C),g,\mu\widetilde{J})$. ■

Proposition 5: The equilibria of $(\Lambda,\mathcal{F})$ *are magnetic flow lines of* $(M^3(C),g,\lambda\widetilde{P})$ *if and only if the following conditions hold:*

1. $C=0$ *and so* $\nabla_T P=0$.
2. *The Lagrangian density is linear in both the curvature and the torsion of trajectories which are helices with axis P.*

Proof: If equilibria of $(\Lambda,\mathcal{F})$ are magnetic flow lines of $(M^3(C),g,\lambda\widetilde{P})$, then

$$\lambda P=\omega T+\kappa B,$$

with ω constant. This formula is combined with Eqs. (25) and (26) to obtain

$$\lambda C F_\kappa=(\omega-\tau)\kappa,\quad -\frac{\lambda C}{\kappa}\frac{\mathrm{d}F_\tau}{\mathrm{d}s}=\frac{\mathrm{d}\kappa}{\mathrm{d}s}.$$

If $C\neq 0$, then we obtain

$$F_\kappa=\frac{\omega}{\lambda C}\kappa-\frac{1}{\lambda C}\kappa\tau,\quad F_\tau=-\frac{1}{2\lambda C}\kappa^2+b, \tag{27}$$

where b is a constant. These equations can be integrated to have

$$F(\kappa,\tau)=a+b\tau+\frac{\omega}{2\lambda C}\kappa^2-\frac{1}{2\lambda C}\kappa^2\tau.$$

On the other hand, the tangential component of P along each equilibrium is

$$\frac{\omega}{\lambda} = F - (\kappa F_\kappa + \tau F_\tau),$$

we put Eq. (27) in the above formula to get

$$F(\kappa,\tau) = \frac{\omega}{\lambda} + b\tau + \frac{\omega}{\lambda C}\kappa^2 - \frac{3}{2\lambda C}\kappa^2\tau.$$

Consequently $C=0$. Then, $\nabla_T P=0$ and so $\omega-\tau=\mathrm{d}\kappa/\mathrm{d}s=0$; the magnetic flow lines are helices with axis P. The converse is already known. ■

XII. THE EXISTENCE OF PERIODIC MAGNETIC FLOW LINES

The closed extremals of Eq. (2) correspond to periodic magnetic trajectories, i.e., periodic orbits in the motion of charged particles submitted to a magnetic field $F=\mathrm{d}\omega$ in a Riemannian manifold (M^n,g). The existence of these periodic trajectories is so important as it is difficult to prove its existence. Several methods have been used to this purpose, for example, some of Morse theory when the Lagrangian density is positive definite and so it defines a Finsler metric (see Ref. 2 and references therein). A systematic study of this problem was began by Novikov[21–23] who considered the case where $\mathcal{L}$ is a multivalued functional which happens if the magnetic field is a closed but not exact 2-form, i.e., the potential ω is not globally defined on the whole space M^n.

Our approach here allows to study this problem using a method that does not involve any local potential nor constraint the topology of M^n. In fact, in the cases that we have obtained the magnetic trajectories of (M^n,g,F) as solutions of a variational problem under the previous assumptions, we are able to solve the problem on the existence of periodic magnetic curves. Let us give some examples of magnetic fields which admit periodic orbits.

(1) *Periodic trajectories of a Hopf magnetic field*. On the unit 3-sphere $\mathbb{S}^3$ endowed with its canonical metric $g_{\mathbf{S}^3}$, we consider the Killing magnetic field $V(z)=iz$ (see Example 2 in Sec. VII). The magnetic curves of $(\mathbb{S}^3,g_{\mathbb{S}^3},V)$ are helices in $\mathbb{S}^3$ and so they are geodesics of Hopf tubes constructed over circles in the 2-sphere $\mathbb{S}^2(1/2)$. Those Hopf tubes are actually flat tori with constant mean curvature and the class includes the so-called Clifford torus which is obtained by lifting a geodesic of $\mathbb{S}^2(1/2)$. The isometry type of these tori can be determined (see Ref. 3) to be as follows: If β is a circle in $\mathbb{S}^2(1/2)$, with length L and enclosing an oriented area $A\in(-\pi/2,\pi/2)$, then its Hopf torus $\mathbf{T}_\beta$ is isometric to $\mathbb{R}^2/\Gamma$ where Γ is the lattice in the Euclidean plane $\mathbb{R}^2$ generated by $(L,2A)$ and $(0,2\pi)$. As we have seem in Sec. VII, each geodesic of $\mathbf{T}_\beta$ is completely determined by either (1) the curvature κ and the torsion τ in $\mathbb{S}^3$ or (2) the curvature ρ of β in $\mathbb{S}^2(1/2)$ and the slope σ in $\mathbf{T}_\beta$ measured with respect to the fibers.

It is not difficult to see that a geodesic γ closes if and only if its slope and the radius $r\in(0,1/2)$ satisfy the following quantization principle:

$$r\sigma + \frac{1}{2}\sqrt{1-4r^2} = q \quad \text{is a rational number.} \tag{28}$$

Now, we can compute the moduli space of periodic orbits of the Hopf magnetic field defined by V. To do it, we proceed as follows. Chose any point $z\in\mathbb{S}^3$ and a Hopf torus, $\mathbf{T}_\beta$ with $z\in\mathbf{T}_\beta$ where β is circle in $\mathbb{S}^2(1/2)$. Notice that this circle, and so its Hopf torus, is unique up to similarities. Now, for any unit vector $\vec{x}\in T_z(\mathbf{T}_\beta)$ with slope σ satisfying Eq. (28), we choose

- γ, the unique unit speed geodesic of $\mathbf{T}_\beta$ through z in the direction of $\vec{x}$, which we know is closed, and
- δ, the unique unit speed normalized magnetic curve of $(\mathbb{S}^3,g_{\mathbb{S}^3},V)$ through z in the direction of $\vec{x}$.

Since every magnetic curve of $(\mathbb{S}^3,g_{\mathbb{S}^3},V)$ is a helix and so a geodesic of a Hopf torus, both curves coincide. Therefore, we obtain the following: *The moduli space, up to similarities, of* *periodic magnetic curves of a Hopf magnetic field in the 3D sphere can be identified and so quantized in the set of rational numbers*.

(2) *Closed elastic curves as periodic magnetic trajectories*. The existence, up to similarities, of a countably infinite class of closed nonplanar elastic curves in $\mathbb{R}^3$ is known (see Ref. 17). These elasticae are free of self-intersections and lie in embedded tori of revolution. Let γ be a curve in the above class and denote, as previously, by κ and B the curvature function and the unit binormal of γ, respectively. Now, along this curve, define the vector field $V(s)=\kappa(s)B(s)$. A straightforward computation shows that $V(v)=V(\kappa)=V(\tau)=0$, and since $\mathbb{R}^3$ is simply connected, $V(s)$ extends to a Killing vector field, also named by V, on the whole $\mathbb{R}^3$ (see Proposition 2 of Sec. V). Moreover, γ is a magnetic curve of $(\mathbb{R}^3, g_{\mathbb{R}^3}, V)$ which is perpendicular to its own magnetic field. This process provides a map from the set $\mathbf{E}$ of closed elastic curves in $\mathbb{R}^3$ to the six-dimensional linear space $\mathcal{K}$ of Killing vector fields of $\mathbb{R}^3$. The image of this map is made up of those Killing magnetic fields in $\mathbb{R}^3$ that admits, at least, a periodic magnetic trajectory. The Hall-Killing effect provides periodic motion.

(3) *Closed Kirchhoff elastic rods and periodic magnetic trajectories*. The existence of closed Kirchhoff elastic rods has been studied in Refs. 11 and 12. In particular, the class of quasiperiodic elastic rod centerlines, up to similarities in $\mathbb{R}^3$, is parametrized by the closed unit disk. Here, a quasiperiodic centerline means either closed or densely wind around a torus. Now, choose a point x in the unit disk $\mathbb{D}$ and denote by γ_x the corresponding quasiperiodic elastic rod centerline. Then it is an extremal of an elastic energy action which is a linear combination of the first three conserved Hamiltonians of the LIE hierarchy. Denote, respectively, by λ_1, λ_2, and $\lambda_3/2$ the coefficients in that linear combination. Also denote by κ the curvature function of γ_x and by T and B its unit tangent and unit binormal vector fields, respectively. Along γ_x, we define the vector field

$$V(s) = \omega T(s) + \kappa(s)B(s) \quad \text{with } \omega = \frac{\lambda_2}{\lambda_3}.$$

We can use again here the argument in Sec. V to extend $V(s)$ to a Killing vector field, also denoted by V, on the whole Euclidean space. Moreover, γ_x is a quasiperiodic magnetic curve of $(\mathbb{R}^3, g_{\mathbb{R}^3}, V)$. This provides a map from the unit disk $\mathbb{D}$ to the six-dimensional linear space $\mathcal{K}$ of Killing vector fields of $\mathbb{R}^3$. The image of this map is constituted by those Killing fields which are viewed as magnetic ones and provide quasiperiodic motion for charged particles. Notice that this example contains as a particular case that viewed in the above item.

(4) *Closed Lancret curves as periodic magnetic trajectories*. The first considered framework can be generalized to certain magnetic fields in $\mathbb{S}^3$ which have, of course, divergence zero, but they are not Killing. Imagine the setting stated in the Sec. X B. The magnetic trajectories of $(\mathbb{S}^3, g_{\mathbb{S}^3}, \tilde{V}=\bar{f}V)$ are Lancret curves in $\mathbb{S}^3$ and so geodesics of Hopf tubes over curves in $\mathbb{S}^2(1/2)$. These tubes become tori when are shaped on closed curves in the 2-sphere. Since they are flat tori and we can determine their isometry type, we only need to determine closed geodesics in flat tori. However, we must be careful because the curvature of corresponding cross sections is controlled by the potential $\bar{f}$. Take an embedded, unit speed, closed curve γ in $\mathbb{S}^2(1/2)$, with length L and enclosing an oriented area $A \in (-\pi/2, \pi/2)$. Then the corresponding Hopf tube $\mathbf{T}_\gamma$ is a flat torus which is isometric to $\mathbb{R}^2/\Gamma$ and Γ the lattice in Euclidean plane generated by $(L, 2A)$ and $(0, 2\pi)$. On the other hand, let f be a smooth function in $\mathbb{S}^2(1/2)$ which restricted to the trace of γ is

$$f(\gamma(t)) = \kappa_\gamma(t) - 2\sigma,$$

where $\sigma \in \mathbb{R}$. Then, the geodesic of $\mathbf{T}_\gamma$ with slope σ (slope measured with respect to the fibers), say, α, is a magnetic curve of $(\mathbb{S}^3, g_{\mathbb{S}^3}, \tilde{V}=\bar{f}V)$. In particular, if we choose σ in such a way that

$$\frac{\sigma L - 2A}{\pi} = q \text{ is a rational number},$$

then α is closed and so a periodic magnetic trajectory of $(\mathbb{S}^3, g_{\mathbb{S}^3}, \tilde{V}=\bar{f}V)$.

XIII. FINAL REMARKS

For simplicity, we will consider $\mathbb{R}^3$ as surrounding space. Now, elastic curves in these backgrounds constitute a classical, old, and nice variational problem which can be considered as a special case of two equivalent variational approaches. On one hand, in a suitable space of curves Λ, one considers the action

$$\mathcal{F}(\gamma) = \int_\gamma \left[\frac{\lambda_3}{2}\kappa^2(s) + \lambda_2 \tau(s) + \lambda_1 \right] \mathrm{d}s, \quad \lambda_3 \neq 0.$$

The extremals of this action are called generalized elastic curves and elasticae appear in the particular case where $\lambda_2=0$.

On the other hand, to describe rod configurations, one uses adapted frame curves $\Upsilon = \{\gamma; T, U, W\}$, where γ is a unit speed curve, *the centerline of the rod*, and $\{T, U, W\}$ is a right-handed orthonormal frame along γ with $T(s)=\gamma'(s)$, which is called the *material frame of the rod*. Certainly, we have a redundant information in the above representation, so one can think of T as containing the bending information and U (or W) as containing the twisting information. The motion of the material frame may be described in terms of the Darboux (*angular velocity*) vector field,

$$\Theta = mT - m_2 U + m_1 W,$$

where m, m_1, and m_2 are the material stains by the equations

$$T' = \Theta \wedge T,$$

$$U' = \Theta \wedge U,$$

$$W' = \Theta \wedge W.$$

The *total elastic energy* of the adapted frame curve Υ is defined by

$$\mathcal{E}(\Upsilon) = \int_\gamma (a(m_1^2 + m_2^2) + bm^2)\mathrm{d}s = \int_\gamma (a\kappa^2 + bm^2)\mathrm{d}s.$$

The classical concept of Kirchhoff elastic rod corresponds with the critical points of this elastic energy. Again, one finds elasticae as a particular case of this variational problem.

These two stated variational problems are equivalent. In fact, the following result are already known:

- If γ is the centerline of a Kirchhoff elastic rod, i.e., a critical point of $\mathcal{E}$, then it is also a critical point of $\mathcal{F}$, i.e., a generalized elastica.[18]
- If γ is a generalized elastica or a critical point of $\mathcal{F}$, then it is the centerline of a Kirchhoff elastic rod.[11]

The relationship between the parameters involved in energies defining both variational problems is

$$\frac{bm}{a} = \frac{\lambda_2}{\lambda_3}.$$

In this paper, we have enlarged this equivalence chain by adding the Killing magnetism. In fact, when studying the magnetic curves associated with a Killing magnetic field V, i.e., the Hall-Killing effect, then we show that those curves correspond with these along which the Killing magnetic field can be written as $V(s)=\omega T(s)+\kappa(s)B(s)$, where ω is a constant that we called quasislope. Next, we used the Killing character of the magnetic field to prove that the geometric invariants of the magnetic curves must satisfy certain differential equations that, up to certain parameters, coincide with the Euler-Lagrange ones associated with $\mathcal{F}$. Namely, an electric current flow submitted to a Killing magnetic field experiences a force, the Lorentz force, and moves in response to this force through trajectories which are centerlines of Kirchhoff elastic rods associated with an elastic energy action constraint by the quasislope by

$$\frac{\lambda_2}{\lambda_3} = \omega, \quad \frac{\lambda_1}{\lambda_3} = A_2.$$

It allows one to see the Killing-Landau-Hall problem as a well stated variational one irrespective of the existence of a global potential for magnetic field or the topology of the background.

Consequently, the following theories are equivalents:

- generalized elasticae,
- Kirchhoff elastic rods, and
- the Killing-Landau-Hall problem.

In addition and as a consequence, we have proved the following: Each trajectory of the motion of an electric current flow submitted to a Killing magnetic field is a soliton for LIE or equivalently a solution of the cubic NLSE.

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APPENDIX: THE HASIMOTO TRANSFORMATION

The Hasimoto transformation identifies the space of connected smooth curves, free of inflections, up to rigid motions, with that of complex valued smooth functions free of zeros. It associates to each curve its complex wave function as follows:

$$\mathcal{H}(\gamma(s)) = \psi(s) \quad \text{with } \psi(s) = \kappa(s)\exp\left(i\int_0^s \tau(u)\mathrm{d}u\right),$$

where $\kappa > 0$ and τ denote the curvature and the torsion of the curve γ, respectively.[9,10] The curve can be recuperated from its complex wave function, up to rigid motions. In fact, the curvature κ and the torsion τ of γ are obtained in terms of ψ as follows:

$$\kappa(s) = |\psi(s)|, \quad \tau(s) = \frac{\mathrm{d}}{\mathrm{d}s}\arg \psi(s).$$

In this section we recall the main steps of the procedure employed by Hasimoto to connect the Betchov–Da Rios equation with the cubic Schrödinger one. Thus, we assume that the curve $\gamma(s)$ evolves in the space as a function of time t to produce a space curve $\gamma(s,t)$. The spatial variation of the Frenet frame, $\{T,N,B\}(s,t)$, is governed by the curvature and the torsion according to the Frenet equations [Eqs. (6)–(8)]. The latter two equations are conveniently combined into the complex form

$$\frac{\partial}{\partial s}(N+iB) + i\tau(N+iB) = -\kappa T.$$

This equation can be manipulated using as ingredients the unit complex vector field and the complex wave function given, respectively, by

$$\eta(s,t)=\exp\left(i\int_0^s \tau(u,t)\mathrm{d}u\right)(N+iB), \quad \psi(s,t)=\kappa(s,t)\exp\left(i\int_0^s \tau(u,t)\mathrm{d}u\right).$$

Using these data, the Frenet equations relative to the frame $\{T,\eta,\bar{\eta}\}(s,t)$, overbar indicating complex conjugate, are governed by the complex wave function ψ as follows:

$$\frac{\partial}{\partial s}\begin{pmatrix} T \\ \eta \\ \bar{\eta} \end{pmatrix}=\begin{pmatrix} 0 & \frac{1}{2}\bar{\psi} & \frac{1}{2}\psi \\ -\psi & 0 & 0 \\ -\bar{\psi} & 0 & 0 \end{pmatrix}\begin{pmatrix} T \\ \eta \\ \bar{\eta} \end{pmatrix}. \tag{A1}$$

On the other hand, a standard computation allows one to compute the temporal variation of $\{T,\eta,\bar{\eta}\}(s,t)$ which is now governed by a complex function φ and a real function R as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} T \\ \eta \\ \bar{\eta} \end{pmatrix}=\begin{pmatrix} 0 & -\frac{1}{2}\bar{\varphi} & -\frac{1}{2}\varphi \\ \varphi & iR & 0 \\ \bar{\varphi} & 0 & -iR \end{pmatrix}\begin{pmatrix} T \\ \eta \\ \bar{\eta} \end{pmatrix}. \tag{A2}$$

Equations (A1) and (A2) can be written, with the obvious meaning, by

$$\Omega_s = M\cdot\Omega, \quad \Omega_t = P\cdot\Omega,$$

and so the integrability condition, $\Omega_{st}=\Omega_{ts}$, is given by

$$M_t - P_s + [M,P] = 0.$$

This automatically implies

$$\psi_t + \varphi_s = -iR\psi \quad \text{with } R_s = \frac{1}{2}i(\psi\bar{\varphi}-\bar{\psi}\varphi). \tag{A3}$$

It should be noticed that in Eq. (A3) there are a pair of undetermined functions, φ and R. Now, the philosophy here is to assume some evolution of the original space curve $\gamma(s,t)$ in order to compute the corresponding evolution of its complex wave equation. Therefore, suppose that the evolution of $\gamma(s,t)$ is governed by the vortex filament (Betchov–Da Rios) equation

$$\gamma_t = \gamma_s \wedge \gamma_{ss} = \kappa B. \tag{A4}$$

Then, we can compute the temporal variation of $T=\gamma_s$ as follows:

$$T_t = \gamma_{st} = \gamma_{ts} = -\kappa\tau N + \kappa_s B.$$

On the other hand, it is not difficult to see that

$$\psi_s\bar{\eta} - \bar{\psi}_s\eta = 2i(\kappa\tau N - \kappa_s B).$$

This equation is now compared with Eq. (A2) to compute the value of the complex function φ

$$\varphi(s,t) = -i\psi_s(s,t). \tag{A5}$$

This is used in the second equation of Eq. (A2) to get the value of the real function R,

$$R_s = -\frac{1}{2}(\psi\bar{\psi}_s + \psi_s\bar{\psi}) = -\frac{1}{2}\frac{\partial}{\partial s}|\psi|^2 \Rightarrow R(s,t) = -\frac{1}{2}|\psi|^2 + A,$$

where A is a constant. With these values, the first equation of Eq. (A2) is just the following cubic Schrödinger equation:

$$i\psi_t + \psi_{ss} + \frac{1}{2}(|\psi|^2 + C)\psi = 0, \tag{A6}$$

for some constant C. Therefore, $\gamma(s,t)$ evolves according to the vortex filament equation if and only if its complex wave function evolves according to the above Schrödinger equation.

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