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REMARKS ON MULTIVALUED NONEXPANSIVE MAPPINGS

BY

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Abstract. Convergence of fixed point sets of multivalued nonexpansive mappings is studied under both the Mosco and Hausdorff senses. A characterization for *-nonexpansive multivalued mappings is given. Also a counterexample is constructed to show a negative answer to a question raised by A. Canbtti, G. Marino and P. Pibtramala.

Let H be a Hilbert space, C a bounded closed convex subset of H and T: $C \to C$ is a (single-valued) nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$). Then for each fixed $x_0 \in C$ and $\lambda \in [0,1)$, the mapping $T_\lambda : C \to C$ defined by

$$T_\lambda x = (1-\lambda)x_0 + \lambda Tx, \quad x \in C \tag{1}$$

is a contraction on C. Hence, Banach's Contraction Principle yields a unique $x_\lambda \in C$ such that $T_\lambda x_\lambda = x_\lambda$; namely,

$$x_\lambda = (1-\lambda)x_0 + \lambda Tx_\lambda. \tag{2}$$

An elegant result in the fixed point theory of (single-valued) nonexpansive mappings is Browder's theorem [1] which states that the approximating curve x_λ defined by (2) converges strongly as $\lambda \to 1$ to a fixed point of T. This result

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was extended by Reich [11] to a framework of a uniformly smooth Banach space. For recent progress along the line, the reader is referred to [12], [7], [14].

Now we turn to the multivalued case. For a metric space (X, d), we use $CB(X)$ to denote the family of all nonempty closed bounded subsets of X, $K(X)$ the family of all nonempty compact subsets of X, and H the Hausdorff metric on $CB(X)$ induced by the metric d of X; that is, for A, $B \in CB(X)$,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,E) = \inf\{d(x,y) : y \in E\}$ is the distance from a point $x \in X$ to a subset $E \subset X$. Now recall that a multivalued mapping $T : X \to CB(X)$ is said to be *nonexpansive* if

$$H(Tx,Ty) \leq d(x,y), \quad x,y \in X.$$

Recall also that a sequence $\{A_n\}$ in $CB(X)$ is said to converge to an element $A \in CB(X)$ under the Mosco sense if

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A,$$

where $\limsup_{n\to\infty} A_n = \{x \in X$: there are subsequences $\{n_k\}$ and $\{x_{n_k}\}$ with $x_{n_k} \in A_{n_k}$ such that $x_{n_k} \to x\}$ and $\liminf_{n\to\infty} A_n = \{x \in X$: there exists $x_n \in A_n$ for each n such that $x_n \to x\}$. It is not hard to see that if $H(A_n, A) \to 0$ $(A_n, A \in CB(X))$, then $A_n \to A$ under the sense of Mosco. Assume now H and C are as above and $T : C \to K(C)$ is nonexpansive. For each fixed $x_0 \in C$ and $\lambda \in [0,1)$, we define the mapping $T_\lambda : C \to K(C)$ by the same formula (1) above. Then T_λ is a multivalued contraction and hence has a (nonunique, in general) fixed point $x_\lambda \in C$ (see[8]); i.e.,

$$x_\lambda \in (1-\lambda)x_0 + \lambda T x_\lambda. \tag{3}$$

Let $y_\lambda \in Tx_\lambda$ be such that

$$x_\lambda = (1-\lambda)x_0 + \lambda y_\lambda. \tag{4}$$

A natrual question now gives rise to whether Browder's theorem can be extended to the multivalued case. The following simple example presents a negative answer.

Example 1.[10] Let $C = [0,1] \times [0,1]$ be the square in the real plane and $T : C \to K(C)$ be defined by

$$T(a,b) = \text{the triangle with vertices } (0,0), (a,0), (0,b), \quad (a,b) \in C.$$

Then it is easy to see that for any $(a_i, b_i) \in C$, $i = 1, 2$,

$$H(T(a_1,b_1), T(a_2,b_2)) = \max\{|a_1 - a_2|, |b_1 - b_2|\} \leq \|(a_1,b_1) - (a_2,b_2)\|,$$

showing that T is nonexpansive. It is also easy to see that the fixed point set of T is $F(T) = \{(a,0) : 0 \leq a \leq 1\} \cup \{(0,b) : 0 \leq b \leq 1\}$. Let $x_0 = (1,0)$. Then the map T_λ defined by (1) has the fixed point set

$$F(T_\lambda) = \{(a,0) : 1 - \lambda \leq a \leq 1\}.$$

Let

$$x_\lambda = \begin{cases} (\frac{1}{n}, 0), & \text{if } \lambda = 1 - \frac{1}{n}; \\ (1,0), & \text{otherwise.} \end{cases}$$

Then $\{x_\lambda\}$ satisfies (3) but is not convergent.

The same example also shows that the net $\{F(T_\lambda)\}$ of fixed point sets of the T_λ's does not converge as $\lambda \to 1$ to the fixed point set $F(T)$ of T under either the Hausdorff metric or the Mosco sense. However, this will be so if we put some restrictions on the fixed point set $F(T)$ of T. First recall that a Banach space X is said to satisfy *Opial's property* [9] if for any sequence $\{x_n\}$ in X, the condition that $\{x_n\}$ converges weakly to x implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \quad \forall y \in X, \quad y \neq x.$$

Spaces saisfying this property include all Hilbert spaces and ℓ^p for $1 < p < \infty$. Also it is known [3] that any separable Banach space can be equivalently renormed so that it possesses Opial's property.

Theorem 1. *Let C be a nonempty closed bounded convex subset of a Banach space X satisfying Opial's property and $T : C \to K(C)$ be a nonexpansive mapping such that $F(T) = \{z\}$. Then for any $x_0 \in C$, the net $\{F(T_\lambda)\}$ of fixed point sets of the T_λ's weakly converges as $\lambda \to 1$ to the fixed point set $F(T)$ of T under the Mosco sense, i.e.,*

$$w - \limsup_{\lambda\to 1} F(T_\lambda) = w - \liminf_{\lambda\to 1} F(T_\lambda) = F(T).$$

Proof. It is sufficient to show that

(i) $F(T) \supseteq w - \limsup_{\lambda\to 1} F(T_\lambda)$ and

(ii) $w - \liminf_{\lambda\to 1} F(T_\lambda) \supseteq F(T)$.

To show (i), we assume that $x \in w - \limsup_{\lambda\to 1} F(T_\lambda)$, which means that there exist a sequence $\lambda_n \in [0,1)$ converging to 1 and a sequence $\{x_n\}$ such that $x_n \in F(T_{\lambda_n})$ and $x_n \to x$ weakly. Let $y_n \in Tx_n$ be such that $x_n = (1-\lambda_n)x_0 + \lambda_n y_n$. Choose $z_n \in Tx$ satisfying

$$\|y_n - z_n\| \le H(Tx_n, Tx) \le \|x_n - x\|. \tag{5}$$

Since Tx is compact, we may assume that $z_n \to z_\infty \in Tx$ strongly. Noting that $\|x_n - y_n\| \to 0$, we obtain by (5) that

$$\limsup \|x_n - z_\infty\| \le \limsup \|x_n - x\|. \tag{6}$$

Since $x_n \to x$ weakly, it follows from (6) and Opial's property that $x = z_\infty$ and $x \in Tx$. This concludes the proof of (i). Next we show (ii). For each $\lambda \in [0,1)$, choose any $x_\lambda \in F(T_\lambda)$ and $y_\lambda \in Tx_\lambda$ satisfying (4). Then by the same proof as above, we see that every weak cluster point of $\{x_\lambda\}$ is a fixed point of T. But, by assumption, $F(T) = \{z\}$. Hence $\{x_\lambda\}$ converges weakly as $\lambda \to 1$ to z.

If the unique fixed point z of T is such that $Tz = \{z\}$, then we have the following strong convergence result.

Theorem 2. *Let C be a nonempty closed convex subset of a Hilbert space H and T: $C \to K(C)$ be a nonexpansive mapping with a unique fixed point z. Suppose in addition that $Tz = \{z\}$. Then $H(F(T_\lambda), F(T)) \to 0$ as $\lambda \to 1$.*

Proof. First we observe that $\{F(T_\lambda)\}$ is uniformly bounded. In fact, given any $x_\lambda \in F(T_\lambda)$, we have some $y_\lambda \in Tx_\lambda$ such that $x_\lambda = (1-\lambda)x_0 + \lambda y_\lambda$. However,

$$\|y_\lambda - z\| = d(y_\lambda, Tz) \le H(Tx_\lambda, Tz) \le \|x_\lambda - z\|.$$

Hence

$$\|x_\lambda - z\| \le \lambda\|y_\lambda - z\| + (1-\lambda)\|x_0 - z\| \le \lambda\|x_\lambda - z\| + (1-\lambda)\|x_0 - z\|.$$

This implies that $\|x_\lambda - z\| \le \|x_0 - z\|$ and $\{x_\lambda\}$ is uniformly bounded. Now choose $x_\lambda \in F(T_\lambda)$ such that

$$H(F(T_\lambda), F(T)) = \sup_{x \in F(T_\lambda)} \|x - z\| < \|x_\lambda - z\| + 1 - \lambda.$$

We shall show that $\|x_\lambda - z\| \to 0$ as $\lambda \to 1$. Indeed, we have $y_\lambda \in Tx_\lambda$ satisfying (4). Since $\|y_\lambda - z\| = d(y_\lambda, Tz) \le H(Tx_\lambda, Tz) \le \|x_\lambda - z\|$, we obtain

$$\left\|\frac{x_\lambda - (1-\lambda)x_0}{\lambda} - z\right\| \le \|x_\lambda - z\|; \quad \text{that is,}$$

$$\left\|\frac{x_\lambda - x_0}{\lambda} + (x_0 - z)\right\|^2 \le \|(x_\lambda - x_0) + (x_0 - z)\|^2,$$

which leads to

$$\|\lambda - x_0\|^2 \le \frac{2\lambda}{1+\lambda}\langle x_\lambda - x_0, z - x_0\rangle \le \|x_\lambda - x_0\|\|z - x_0\|.$$

Therefore,

$$\|x_\lambda - x_0\| \le \|z - x_0\|. \tag{7}$$

From the proof of theorem 1, we know that $x_\lambda \to z$ weakly as $\lambda \to 1$. It then easily follows from (7) that $\limsup_{\lambda\to 1}\|x_\lambda\| \le \|z\|$. On the other hand, due to the lower weak continutiy of the norm of H, we have $\liminf_{\lambda\to 1}\|x_\lambda\| \ge \|z\|$. Therefore, we have $\lim_{\lambda\to 1}\|x_\lambda\| = \|z\|$ and

$$\|x_\lambda - z\|^2 = \|x_\lambda\|^2 - 2\langle x_\lambda, z\rangle + \|z\|^2 \to 0 \quad \text{as} \quad \lambda \to 1.$$

This completes the proof of the theorem.

Corollary 1. [10] *Let the assumptions of theorem* 2 *be satisfied. Then*

$$w - \limsup_{\lambda - 1} F(T_\lambda) = \|\cdot\| - \liminf_{\lambda \to 1} F(T_\lambda) = F(T).$$

Remark 1. The example above shows that the conclusions of theorems 1 and 2 are not valid if the fixed point set $F(T)$ of T is not a singleton. However, it is an open question whether the restriction $Tz = \{z\}$ in Theorem 2 can be removed. We also do not know if Theorem 2 is valid outside a Hilbert space.

Next we let (X, d) be a metric space. A multivalued map $f : X \to K(X)$ is said to be *-nonexpansive [4] if for all $x, y \in X$ and $u_x \in f(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \in f(x)\}$, there exists $u_y \in f(y)$ with $d(y, u_y) = \inf\{d(y, w) : w \in f(y)\}$ such that

$$d(u_x, u_y) \leq d(x, y).$$

It is obvious that this notion is identical with the notion of nonexpansiveness for singlevalued mappings. But they are different for multivalued mappings (see [13]). We now give a characterization of multivalued *-nonexpansive mappings. Denote by P_f the map $x \mapsto \{u_x \in f(x) : d(x, u_x) = \inf\{d(u, x) : u \in f(x)\}\}$. Note that $P_f(x)$ is nonempty for $f(x)$ is compact.

Theorem 3. *A multivalued map* $f : X \to K(X)$ *is *-nonexpansive if and only if the associated map* $P_f : X \to K(X)$ *is nonexpansive.*

Proof. First assume that f is *-nonexpansive. Given any $x, y \in X$ and $u_x \in P_f(x)$. By definiition, there is $u_x \in f(y)$ such that $d(u_x, u_y) \leq d(x, y)$. It follows that

$$\sup_{u_x \to P_f(x)} d(u_x, P_f(y)) \leq \sup_{u_x \to P_f(x)} d(u_x, u_y) \leq d(x, y).$$

The same argument shows that

$$\sup_{u_y \to P_f(y)} d(u_y, P_f(x)) \leq d(x, y).$$

Hence

$$H(P_f(x), P_f(y)) \leq d(x, y)$$

and P_f is nonexpansive. Conversely, we assume that P_f is nonexpansive. Then given any $x, y \in X$ and $u_x \in f(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \to f(x)\}$ (i.e., $u_x \in P_f(x)$). By compactness, we can choose $u_y \in P_f(y)$ such that $d(u_x, u_y) = d(u_x, P_f(y))$. Hence

$$d(u_x, u_y) \leq H(P_f(x), P_f(y)) \leq d(x, y)$$

and f is *-nonexpansive.

Remark 2. Theorem 3 indicates that the fixed point theory of multivalued nonexpansive mappings applies to multivalued *-nonexpansive mappings; in particular, we have the following results whose nonexpansive counterparts were proved in [5] and [6], respectively.

Corollary 2 *Let X be a Banach space satisfying Opial's property, C a nonempty weakly compact convex subset of X , and $T : C \to K(C)$ a *-nonexpansive mapping. Then T has a fixed point.*

Corollary 3. *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X, and $T : C \to K(C)$ a *-nonexpansive mapping. Then T has a fixed point.*

Corollaries 2 and 3 improve upon the corresponding results of [4] and [13].

We conclude the paper with a counterexample that presents a negative answer to a question raised by A. Canbtti, G. Marino and P. Pibtramala [2].

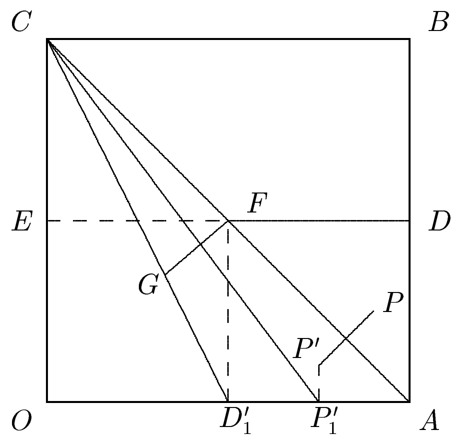
Suppose that H is a Hilbert space and K is a nonempty closed convex subset of H. We denote by $\mathcal{KC}(K)$ the family of all nonempty compact convex subsets of K, and by $d(A, B)$ the distance between two subsets $A, B \subset H$, i.e., $d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$. With each mapping $T : K \to \mathcal{KC}(H)$ one can associate a multivalued mapping $\hat{T} : K \to \mathcal{KC}(H)$ defined as follows:

$$\hat{T}x := \{y \in Tx : d(y, K) = d(Tx, K)\}.$$

The question raised by A. Canbtti, G. Marino and P. Pibtramala (see [2, Remark 1, p. 207]) is whether the nonexpansiveness of T implies that $\hat{T}$ is nonexpansive. The following example shows that the answer is negative.

Example 2. Let OABC be the unit square $[0,1]\times[0,1]$ in the plane H. Let D and E be the midpoints of the segments $\overline{AB}$ and $\overline{OC}$, respectively. Let K be the triangle $\triangle ADF$. To each point $P\in K$, let P' be the symmetric point of P with respect to the diagonal segment $\overline{AC}$. Let P_1' be the projection of P' onto the segment $\overline{OA}$. Now we define a map $T: K\to \mathcal{KC}(H)$ by setting (see the figure below)

$$T(P) := \text{The segment } \overline{CP_1'}.$$



It is then easy to see that T is a nonexpansive mappipng with the unique fixed point A. We also have the following facts:

(i) $T(A)=\overline{AC}$ and hence $d(T(A),K)=0$;

(ii) $\hat{T}(A)$ is the segment $\overline{AF}$;

(iii) $T(D)=\overline{CD_1'}$, where D_1' is the midpoint of $\overline{OA}$;

(iv) $\hat{T}(D)=\{G\}$, where G is the nearest point projection of F onto the segment $\overline{CD_1'}$. Hence

$$\begin{aligned}H(\hat{T}(A),\hat{T}(D)) &= \sup\{d(G,M): M\in\overline{AF}\}\\ &= \text{The length of the segment } \overline{GA}\\ &> \text{The length of the segment } \overline{AD}\\ &= d(A,D),\end{aligned}$$

showing that $\hat{T}$ is not nonexpansive.

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References

[1] F. E. Browder, *Convergenceofapproximantstofixedpointsofnonexpansivenonlinearmappings inBanachspaces*, Arch. Rational Mech. Anal., 24(1967), 82-90.

[2] A. Canbtti, G. Marino and P. Pibtramala, *Fanφsbestapproximationandfixedpointtheorems formultivaluednonexpansivemappingsinHilbertspaces*, Indian J. Math., 33(1991), 203-220.

[3] D. van Dulst, *Equivalentnormsandthefixedpointpropertyfornonexpansivemappings*, J. London Math. Soc., 25(1982), 139-144.

[4] T. Husain and A. Latif, *Fixedpointsofmultivaluednonexpansivemaps*, Math. Japonica, 33(1988), 385-391.

[5] E. Lami Dozo, *MultivaluednonexpansivemappingsandOpialφscondition*, Proc. Amer. Math. Soc., 38(1973),286-292.

[6] T. C. Lim, *Afixedpointtheoremformultivaluednonexpansivemappingsinauniformlyconvex Ba› nachspace*, Bull. Amer. Math. Soc., 80(1974), 1123-1126.

[7] G. Marino and G. Trometta, *Onaℓproximationfixedpointsfornonexpansivemaps*, Indian J. math., 34(1992), 91-98.

[8] S. Nadler, *Multivaluedcontractionmappings*, Pacific J. Math., 30(1969), 475-488.

[9] Z. Opial, *Weakconvegenceofthesequenceofsuccessiveapproximationsfornonexpansivemappings*, Bull. Amer. Math. Soc., 73(1973), 591-597.

[10] P. Pietramala, *Convergenceofapproximatingfixedpointssetsformultivaluednonexpansivemap› pings*, Comment. Math. Univ. Carolinae, 32(1991), 697-701.

[11] S. Reich, *StrongconvergencetheoremsforresolventsofaccretiveoperatorsinBanachspaces*, J. Math. Anal. Appl., 75(1980), 287-292.

[12] S. P. Singh and B. Watson, *OnapproximatingfixedpointsProc▷SymposiaPureMath▷*, 45, pt. 2(1986), 393-395.

[13] H. K. Xu, *Onweaklynonexpansiveand⇀›nonexpansivemultivaluedmappings*, Math. Japonica, 39(1991), 441-445.

[14] H. K. Xu and X. M. Yin, *Strongconvergencetheoremsfornonexpansivenon›selfmappings‹Non› linearAnalysis*, to appear.

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