Programa de Doctorado "Matemáticas"

PhD Dissertation

Comportamiento asintótico en tiempo de ecuaciones en derivadas parciales no locales

Long-time behaviour of nonlocal partial differential equations

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A mis padres, por su apoyo incondicional

A Antonio Jesús, mi amigo, mi confidente,
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Contents

Notation

$\lvert\cdot\rvert$	norm in $\mathbb{R}^N$ and the Lebesgue measure of a subset of $\mathbb{R}^N$ for $N \geq 1$
$\lvert\cdot\rvert_r$	norm in $L^r(\Omega)$ with $r \geq 1$
$\lVert\cdot\rVert_r$	norm in $W_0^{1,r}(\Omega)$ with $r \geq 2$, given by de $L^r(\Omega)$ norm of the grandient
$\lVert\cdot\rVert_{L^r(\tau,T;X)}$	norm in $L^r(\tau, T; X)$ where $r \geq 1$ and X is a separable Banach space
$\lVert\cdot\rVert_*$	norm in the dual space of $W_0^{1,r}(\Omega)$ with $r \geq 2$
$(\cdot,\cdot)$	inner product in $L^2(\Omega)$
$((\cdot,\cdot))$	inner product in $H_0^1(\Omega)$ given by the product in $(L^2(\Omega))^N$ of the gradients
$\langle\cdot,\cdot\rangle$	duality product between the dual space of $W_0^{1,r}(\Omega)$ and $W_0^{1,r}(\Omega)$ for $r \geq 2$
$\overline{\{\dots\}}^X$	closure in X of a subset of X
a.e.	almost everywhere (in the sense of the Lebesgue measure)
$B_X(x,r)$	ball in the metric space X of center x and radius r
$\overline{B}_X(x,r)$	closed ball in X of center x and radius r
$\mathcal{C}_+(L^2(\Omega))$	positive cone of $L^2(\Omega)$, i.e. the set $\{g \in L^2(\Omega) : g \geq 0 \quad \text{a.e. } \Omega\}$
d_X	distance defined on a (metric) space X
$\text{dist}_X(\mathcal{O}_1, \mathcal{O}_2)$	Hausdorff semi-distance in the metric space (X, d_X), i.e. $\sup_{x\in\mathcal{O}_1} \inf_{y\in\mathcal{O}_2} d_X(x,y)$ for $\mathcal{O}_1, \mathcal{O}_2 \subset X$
$\mathcal{D}(\Omega)$	$C^\infty(\Omega) \cap C_c^0(\Omega)$
$\mathcal{D}'(\Omega)$	$\mathcal{L}(\mathcal{D}(\Omega))$
f^+	positive part of f, i.e. $f^+(x) = \max\{0, f(x)\}$
$\mathcal{L}(X,\mathbb{R})$	the space of continuous linear forms from X into $\mathbb{R}$
$\mathcal{P}(X)$	the family of all nonempty subsets of X
$\mathbb{R}_d^2$	the set $\{(t,\tau) \in \mathbb{R}^2 : \tau \leq t\}$

After identifying $L^2(\Omega)$ with its dual, it is also denoted by:

$(\cdot,\cdot)$	the duality product between $L^p(\Omega)$ and $L^q(\Omega)$, where p is the conjugate exponent of q
$\langle\cdot,\cdot\rangle$	the duality product between $H^{-1}(\Omega) + L^q(\Omega)$ and $H_0^1(\Omega) \cap L^p(\Omega)$, where p and q are conjugate exponents.

Introduction

In recent decades, nonlocal problems have arisen in modeling with great interest by its usefulness in real applications (e.g. cf. [59, 19, 67, 109, 12]). For instance, in Biology, the evolution of some species might be better represented by a nonlocal equation than within the corresponding local simplification. Of course, the disadvantage is that sometimes it is very complicated to deal with the nonlocal operators and terms since they are more involved.

In 1989, within of this nonlocal framework, Furter & Grindfeld published [60], a paper in which models of populations with nonlocal effects are analysed. They stated that in the ecological context, there did not exist any reason why the interactions in single-species population dynamics should be local. A few years later, in [46], Chipot & Rodrigues studied the behaviour of a population of bacterias within a container, which was modelled by the nonlocal elliptic problem

$$\begin{cases} -a\left(\displaystyle\int_\Omega u\right)\Delta u + \lambda u = f \quad \text{in } \Omega, \\ \partial_n u + \gamma\left(\displaystyle\int_{\Omega'} u\right) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, the boundary $\partial\Omega$ is Lipschitz, $\Omega' \subset \Omega$, $\lambda > 0$, the functions a and γ belong to $C(\mathbb{R};\mathbb{R}_+)$, $f \in L^2(\Omega)$ and $\partial_n u$ is the normal derivative of u.

In the following papers by Chipot and his collaborators, instead of considering the nonlocal term $a\left(\int_\Omega u\right)$, the authors use a more general nonlocal operator $a(l(u))$ where $l \in \mathcal{L}(L^2(\Omega),\mathbb{R})$, i.e. for some $g \in L^2(\Omega)$

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx.$$

Namely, a lot of attention has been paid to the nonlocal parabolic equation

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f, \tag{1}$$

where the function $a \in C(\mathbb{R};\mathbb{R}_+)$ and there exist positive constants m, $M > 0$ such that

$$0 < m \le a(s) \le M \quad \forall s \in \mathbb{R}.$$

These are natural conditions of non-degeneracy of a in order to avoid the extinction and the existence of the solutions only in finite time intervals. Under these conditions, the equation possesses a parabolic nature and therefore, classical results such as the Maximum Principle and the sub-supersolution method can be applied (for more details, see [87]).

It is worth highlighting that equation (1) is not a trivial perturbation of the heat equation and great difficulties arise in different contexts. For instance, some common manipulations such as multiplying by u_t do not give any additional information in the a priori estimates, unlike what happens in the local case (see [5, Chapter 2, p. 32]), since it is not possible to consider $a(l(u))\frac{d}{dt}|\nabla u|^2$ as the temporal derivative of a function due to the fact that the nonlocal term $a(l(u))$ depends on time. Regarding the existence of a Lyapunov function, it is not guaranteed in a general framework. Additional requirements (see for more detail [45]) or more specific nonlocal operators, which are strongly related to the diffusion terms (see [49, 47]), are needed to build this function. Furthermore, some obstacles arise when the nonlocal equation is set in unbounded domains. It seems that due to the presence of the nonlocal operator in the diffusion term, Rosa's method, which is detailed in [101], cannot be applied to problem (2). As far as we know, the analysis in unbounded domains is still an open problem.

From a biological point of view, the function u might represent the density of a population. Additional assumptions could be imposed on the function a to better reflect the behaviour of the community. For instance, to model species with a tendency to leave crowded zones, a natural assumption would be to assume that a is an increasing function of its argument. On the other hand, if we are dealing with species attracted by growing population, one would assume a to decrease. In addition, nonlocal models have been used in epidemic theory and from a physical point of view, to study the heat propagation (cf. [120, 18, 38, 119]).

To analyse a complete model, it considers the nonlocal problem studied by Chipot & Lovat in [44]

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f(t) \quad \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \; \dfrac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{2}$$

where Ω is a smooth open subset of $\mathbb{R}^N$, whose boundary is split into two parts, Γ_0 and Γ_1. Furthermore, $f \in L^2(0,T;V')$ where V' is the dual space of $V = \{v \in H^1(\Omega)/v = 0 \text{ on } \Gamma_0\}$ and $u_0 \in L^2(\Omega)$. The existence of weak solution is shown making use of the Galerkin approximations and compactness arguments. To deal with the limit of the sequence of Galerkin approximations associated to the nonlocal term $-a(l(u))\Delta u$, the Aubin-Lions Lemma and [85, Lemme 1.3, p. 12] are applied. In addition, to prove the uniqueness of solution, due to the nonlinearity generated in the diffusion term by the nonlocal operator, the function a is assumed to be globally Lipschitz. This condition can be weakened assuming only that the function a is locally Lipschitz.

The stationary study is the most interesting analysis of the cited paper. The existence of stationary solution to (2) are related (if and only if) to the solution of the scalar equation

$$a(\mu)\mu = l(\varphi), \tag{3}$$

where φ is the weak solution to the problem

$$\begin{cases} -\Delta\varphi = f & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma_0, \end{cases}$$

(see [44, Theorem 3.2] for more details). Observe that the number of stationary solutions to (2) is characterised by the number of intersection points of the curve $y = a(s)$ and the hyperbola $y = l(\varphi)/s$, since the intersection points fulfil $a(s)s = l(\varphi)$, which is the expression (3). Therefore, the uniqueness of stationary solution will be guaranteed, for instance, if we assume that a is a non-decreasing function, since a cuts just once the hyperbola. In this framework, it makes sense to study the exponential decay of the solution of the evolution problem (2) towards the unique stationary solution. Namely, this result can be proved as long as it satisfies the condition

$$\frac{(L_a)^2\|l\|^2_{\mathcal{L}(L^2(\Omega),\mathbb{R})}\|\varphi\|_2^2(\mu^*)^2}{(l(\varphi))^2} < m^2\lambda_1,$$

where L_a is the globally Lipschitz constant of the function a, λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions and μ^* is the unique solution to (3). This result does not appear in [44], but it is a contribution to the outstanding study made by Chipot & Lovat, which continues as follows.

Assume that

$$l > 0, \quad \text{i.e. } l(u) > 0 \quad \forall u \geq 0 \;/\; u \not\equiv 0 \text{ a.e. } \Omega,$$

and $f \in V'$ fulfils

$$f \not\equiv 0 \quad \text{and} \quad \langle f, v\rangle \geq 0 \quad \forall v \in V \;/\; v \geq 0 \text{ a.e. } \Omega,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between V and V'.

Suppose that there exist two stationary solutions to (2). Therefore, the equation (3) possesses two solutions called μ_1 and μ_2. Namely, we consider (see Figure 1)

$$l(\varphi) \leq a(s)s \quad \forall s \in [\mu_1, \mu_2],$$

and

$$a(\mu_2) \leq a(s) \leq a(\mu_1) \quad \forall s \in [\mu_1, \mu_2].$$

For $i = 1, 2$, we denote by u_i the solution to

$$\begin{cases} -a(\mu_i)\Delta u_i = f & \text{in } \Omega \\ u_i = 0 & \text{on } \Gamma_0. \end{cases}$$

Then, when the initial datum u_0 fulfils

$$u_1 \leq u_0 \leq u_2 \quad \text{a.e. } \Omega, \quad u_0 \not\equiv u_2,$$

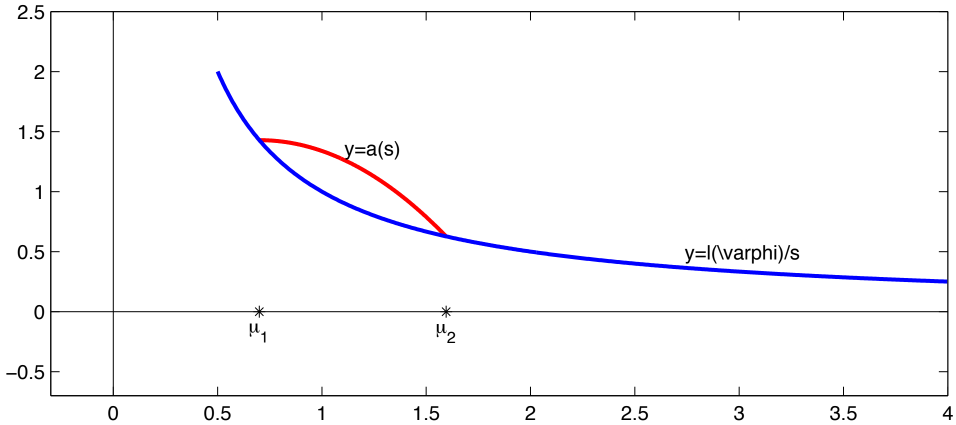

Figure 1: $a(s)$ and $\dfrac{l(\varphi)}{s}$ intersect at two points.

it holds (cf. [44, Theorem 4.1])

$$u(t) \to u_1 \quad \text{strongly in } L^2(\Omega) \text{ as } t \to \infty.$$

Observe that when the function f appearing in problem (2) depends on the unknown u, the difficulty of the problem increases considerably, making the previous result not be available in this more complex framework. As far as we know, there is only a comparison result between the solution of the evolution problem and two (assumed to exist) stationary solutions (cf. Theorem 3.10).

Later, Chipot & Molinet in [45] generalise the results obtained in [44], dealing with a continuum of steady states using dynamical systems. In the same lines of [44, 45], in [48] Chipot & Siegwart consider a more general elliptic operator than the $-\Delta$ and study the asymptotic behaviour of the solutions to problems with nonlocal diffusion and mixed boundary conditions. In [35], Chang & Chipot are also interested in the asymptotic behaviour of the solutions to nonlocal problems, but in this case they deal with two nonlocal operators. In particular, they prove results which establish relationships between the solution to the evolution problem and stationary solutions. These results are similar to those given in the simpler framework of paper [44] which have been detailed previously. Considering also zero Dirichlet boundary conditions, in [50], Chipot & Zheng analyse the convergence of the solution of the evolution problem to one of the equilibria without assuming uniqueness of stationary solutions.

Observe that not only have authors been interested in analysing problems in which the nonlocal operator is defined by $a(l(u))$, but they have also studied other variants, such as $a(|\nabla u|^2)$ and $a(\|\nabla u\|_p^p)$. The first one

$$\frac{\partial u}{\partial t} - a(|\nabla u|^2)\Delta u = f$$

was analysed in [49] by Chipot et al. The main advantage of considering this new variation is that it allows to study the long-time behaviour of weak solutions making use of global minimizers. In [47], Chipot & Savitska consider the p-Laplacian in the diffusion term instead of the $-\Delta$ together with the nonlocal operator $a(\|\nabla u\|_p^p)$, which involve several difficulties, since although the p-Laplacian is a monotone operator, its lack of linearity makes it more complicated to deal with the nonlocal diffusion as we will show later (see Chapter 7).

Another variant, which has been analysed when f is still independent of u, deals with local nonlocal operators, i.e. the nonlocal operator is not acting globally in the whole domain but in a part of it contained in a ball centered on each position point:

$$\begin{array}{rcl} l_r(\cdot)(x): \quad L^2(\Omega) & \rightarrow & \mathbb{R} \\ u & \mapsto & l_r(u)(x) = \displaystyle\int_{B_{\mathbb{R}^N}(x,r)\cap\Omega} u(y)g(y)dy \quad [g \in L^2(\Omega)]. \end{array}$$

Problems with this kind of operators have been analised by Andami Ovono & Rougirel in [2, 3]. The authors study radial solutions, bifucation, the existence of branch of solutions and stability. Namely, in [2], Andami Ovono proves the existence of local branches of solutions in a radial setting by bifurcation analysis. Furthermore, in the cited paper, the existence of the compact global attractor in $L^2(\Omega)$ is analysed.

For f depending on the unknown u, the situation is more involved. In [70], Hilhorst & Rodrigues analyse the parabolic equation

$$\frac{\partial u}{\partial t} = a\left(\frac{1}{|\Omega|}\int_\Omega u(x)dx\right)\Delta u + f\left(u, \frac{1}{|\Omega|}\int_\Omega u(x)dx\right),$$

studying the existence of weak solutions and obtaining a rigorous derivation of a class of diffusion equations that have been used to model the threshold phenomena in porous media combustion. Later, in [97], Menezes analyses the equation

$$u_t - a(l(u))\Delta u + f(u) = h(t) \quad \text{in } \Omega \times (0, T),$$

where f is a Lipschitz function, h belongs to $L^2(0, T; H^{-1}(\Omega))$ and Ω has a smooth boundary. Using fixed point techniques, the existence and uniqueness of weak solutions are analysed. Moreover, making use of the Galerkin approximations, the existence of periodic solutions is also studied.

In this more complex framework of f depending on u, Corrêa considers in [53] the nonlocal elliptic problem

$$\begin{cases} -a\left(\displaystyle\int_\Omega |u|^q\right)\Delta u = H(x)f(u) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and the existence of positive solutions through fixed point theorems is analysed. An analogous result is proved by Corrêa et al. in [54], analysing the existence of positive solutions to

$$\begin{cases} -a\left(\displaystyle\int_\Omega u\right)\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Recently, Figuereido de Sousa et al. have also made interesting contributions in this framework (see [58]), showing the existence of positive solutions to the non-local logistic equation

$$\begin{cases} -a\left(\int_\Omega u\right)\Delta u = \lambda u - b(x)u^2 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

As we can appreciate by the large number of references cited along this introduction, there have been many advances related to equation (1) and its variants, with contributions on existence and uniqueness of weak solutions, radial solutions, periodic solutions or convergence of the solution of the evolution problem towards a stationary solution, amongst others.

Concerning the long-time behaviour of solutions, except for some concrete problems, it is an intractable task to study the existence of stationary solutions and their stability or Lyapunov functions, amongst others. That is why it is worth considering the information that can be obtained by the theory of attractors to study the asymptotic behaviour of the solutions (see [2], also some previous results on this direction by Lovat [87]).

In the context of attractors, the compact global attractor is a useful tool that has been developed to study autonomous dynamical systems in the last few decades. Namely, this object has been deeply analysed by Hale [69], Ladyzhenskaya [83], Babin & Vishik [16], Vishik [113], Ball [17], Temam [111], Robinson [100], Sell & You [105] or Babin [15]. A global attractor is characterised for being a compact set in a given metric space which is maximal and invariant for the corresponding semiflow and attracts through the semiflow to all fixed nonempty bounded subsets of the metric space. In [69, 16, 100], the authors provided conditions that guarantee the existence of global attractors as well as examples.

However, after including time-dependent terms, which allow to model more complex situations, studying the asymptotic behaviour of the solutions through the compact global attractor may not make much sense. For instance, consider the Cauchy problem

$$\begin{cases} x'(t) = -\alpha x(t) + t \\ x(\tau) = x_\tau, \end{cases} \tag{4}$$

where $\alpha > 0$. It is easy to check that the solution to (4) is defined by

$$x(t;\tau,x_\tau) = \left(x_\tau + \frac{1}{\alpha^2} - \frac{\tau}{\alpha}\right)e^{-\alpha(t-\tau)} + \frac{t}{\alpha} - \frac{1}{\alpha^2}.$$

Taking limit when $t \to \infty$, we do not obtain any information about the behaviour of the solutions.

What happens is that the trajectories are attracted by a time-dependent family defined by $\mathcal{A} = \{\mathcal{A}(t) = t/\alpha - 1/\alpha^2\}$. For a fixed $t \in \mathbb{R}$, the section $\mathcal{A}(t)$ is a compact set which attracts the solutions in a pullback sense, i.e. when the initial time $\tau \to -\infty$. This approach allows us to establish not only the asymptotic behaviour of the dynamical system but also what the current attractions sections are when the initial data come from long time ago in the past. Furthermore, observe that in the autonomous framework, both concepts, forward attraction and pullback attraction, coincide.

In addition to this approach, there are other different ones from the point of view of non-autonomous dynamical systems, like skew-product flows (see Sell [104]) or uniform attractors and their kernel sections, which seem to be the natural generalisation of the compact global attractor, studying the asymptotic behaviour of the solutions when the time goes to infinity (see Chepizhov & Vishik [40]). All of them are valid to analyse different features of the evolution of a non-autonomous dynamical system.

In this thesis, we choose the approach of pullback attractors (see Kloeden & Schmalfuß [81, 82] and Kloeden [76], also related to random dynamical systems [55]), since it allows us to minimize the assumptions on the forcing terms and the resultant objects are invariant (in a suitable "non-autonomous-dynamical-system sense"), unlike what happens with the uniform attractors which do not fulfil any property of invariance in general. However, this new object, the pullback attractor, may not be unique like the compact global attractor. For instance, see [93, Example 11]. In the cited example, Marín-Rubio & Real consider a continuous function $f = f(t, x) : \mathbb{R}^2 \mapsto \mathbb{R}$, which is globally Lipschitz w.r.t. x and fulfils that $f(t, x) = -x$ if $|x| \leq e^{-t}$. Then, they define a process U on $\mathbb{R}$ given by

$$U(t, \tau)x_\tau = x(t; \tau, x_\tau) \quad \forall x_\tau \in \mathbb{R} \quad \forall t \geq \tau,$$

where $x(t; \tau, x_\tau)$ is the unique solution to the Cauchy problem

$$\begin{cases} x'(t) = f(t, x) \\ x(\tau) = x_\tau. \end{cases}$$

To prove that there exist more than one pullback attractor consider $\mathcal{A}_1 = \{0\}$ and $\mathcal{A}_2 = \{\mathcal{A}_2(t) = [-e^{-t}, e^{-t}] : t \in \mathbb{R}\}$. Both of them are families of compact and nonempty sets, invariant for the process U in a non-autonomous-dynamical-system sense, and attract fixed nonempty bounded subsets of $\mathbb{R}$ (for more detail about pullback attractors see Chapter 1). To solve this problem, a minimality condition is imposed (cf. Definition 1.11). This way the uniqueness is guaranteed, being $\mathcal{A}_1$ the minimal pullback attractor.

In this approach of pullback attractors, many new results have appeared over the last years. Some authors have been interested in studying the pullback attractor in the classical sense, i.e. the pullback attractor of solutions starting in fixed bounded sets. Others, though, have employed the concept of attraction related to a class of families, called universe $\mathcal{D}$, which is made up of sets which are allowed to move in time and are usually defined in terms of a tempered condition (e.g. cf. [51, 31, 32]). In [93], Marín-Rubio & Real analyse these two different concepts of pullback attractor with detail, highlighting some difficulties which appear in the framework of the universe of fixed nonempty bounded sets and which can be solved making use of tempered tools. Finally, the authors establish relationships between these two notions of attractors proving that in fact both families coincide under a suitable assumption (see [93, Proposition 23]).

This PhD project is split into seven chapters. In Chapter 1, abstract results on pullback attractors within the framework of universes are analysed. Chapters 2, 3 and 4 are devoted to studying the existence and uniqueness of solutions as well as the existence of minimal pullback attractors in the phase spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ for non-autonomous nonlocal parabolic equations. Next, in Chapter 5, the theory of pullback attractors for multi-valued non-autonomous dynamical systems will be analysed to be applied later in the following two chapters, Chapters 6 and 7. In Chapter 6, a non-autonomous nonlocal reaction-diffusion equation with a small perturbation in the nonlocal diffusion term and the non-autonomous force is analysed in a multi-valued framework. The existence of weak solutions and pullback attractors in $L^2(\Omega)$ is proved. In addition, the upper semicontinuous behaviour of attractors will be analysed when the perturbation goes to zero. Finally, in Chapter 7, the existence of solutions for an autonomous nonlocal p-Laplacian equation is shown. Furthermore, the asymptotic behaviour of the solutions is studied proving the existence of the compact global attractor in $L^2(\Omega)$. The study of problem $(\widetilde{P})$ in the non-autonomous framework is also possible, but for the sake of simplicity, we have focused on a problem without non-autonomous terms to make the proof clearer.

Chapter 1 is split in three sections. In Section 1.1, we consider a universe $\mathcal{D}$ composed of families parameterised in time. We analyse some basic concepts as well as some abstract results which will be crucial to prove the existence of pullback attractors under minimal assumptions.

In Section 1.2, we study the main result of this chapter. Namely, Theorem 1.13 guarantees the existence of the minimal pullback $\mathcal{D}$-attractor. Making use of this result, in Corollary 1.15, we establish relationships between the attractor of the universe of fixed nonempty bounded sets and the attractor associated to the universe $\mathcal{D}$ (for more details see [93, Proposition 23]). In addition, in Theorem 1.16, we also establish relationships between attractors related to general universes associated to different phase spaces (see [62, Theorem 3.15]).

To end this chapter, in Section 1.3, the flattening property is analysed. The notion provided in this thesis is a slight modification of the well-known "Condition (C)" introduced by Ma, Wang and Zhong in [88], coined by Kloeden & Langa as the flattening property in [77]. It is a useful tool that allows to prove one of the main ingredients in order to guarantee the existence of the minimal pullback attractor, the pullback asymptotic compactness (cf. Proposition 1.18).

In Chapter 2, the non-autonomous nonlocal parabolic problem

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases}$$

is analysed, where Ω is an open bounded subset of $\mathbb{R}^N$, $\tau \in \mathbb{R}$ and the function $a \in C(\mathbb{R};\mathbb{R}_+)$ is locally Lipschitz and there exists a positive constant m, such that

$$0 < m \le a(s) \quad \forall s \in \mathbb{R}. \tag{5}$$

In addition, $l \in \mathcal{L}(L^2(\Omega),\mathbb{R})$, $f \in C(\mathbb{R})$ and there exist two constants $\eta > 0$ and $C_f \ge 0$ such that

$$\begin{gathered} |f(s)| \le C_f(1+|s|) \quad \forall s \in \mathbb{R}, \\ (f(s)-f(r))(s-r) \le \eta(s-r)^2 \quad \forall s,r \in \mathbb{R}. \end{gathered} \tag{6}$$

To conclude the setting of (P), we assume that the initial datum $u_\tau \in L^2(\Omega)$ and the non-autonomous term $h \in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$.

We divide Chapter 2 into four sections. Section 2.1 is devoted to existence and uniqueness results. First of all, the existence of local solution to (P) is proved making use of [52, Theorem 1.1, p.43], which is a generalisation of the Peano Theorem. To guarantee the uniqueness, the function a also needs to be locally Lipschitz. Then, in Theorem 2.4, the existence and uniqueness of weak (global) solutions are analysed making use of the Galerkin approximations and compactness arguments. The main difficulty of this result consists in dealing with the limit of the Galerkin approximations of the nonlinear terms $-a(l(u))\Delta u$ and $f(u)$. To do this, in addition to applying the Aubin-Lions lemma, we use [85, Lemme 1.3, p. 12], which allows us to work with the nonlinearities. Furthermore, in this section, namely in Theorem 2.5, the regularising effect of the equation is stated as well as the existence of strong solution in a more regular framework.

In Section 2.2, we study the existence and uniqueness of stationary solutions and their stability. For the existence result, we make use of a corollary of the Brouwer fixed point theorem (see [85, Lemme 4.3, p. 53]). Furthermore, the uniqueness of stationary solution is shown under additional requirements. The exponential decay of the solution of the evolution problem (P) towards the unique stationary solution is also analysed.

Section 2.3 is devoted to proving the existence of minimal pullback attractors in the phase space $L^2(\Omega)$ as well as some relationships between these families (cf. Theorem 1.15). In this case, to build a suitable tempered universe for our purposes, we need to make this additional assumption on the function f

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < m\lambda_1, \tag{7}$$

where λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions. Observe that to prove the most difficult result of this section, the pullback asymptotic compactness (cf. Proposition 2.15), which leads immediately to the existence of attractors, we use an energy method which relies on the continuity of the solutions (cf. for more details [73, 92, 94, 62]).

Finally, in Section 2.4, in a more regular framework we show the existence of minimal pullback attractors in the phase space $H^1_0(\Omega)$ and establish some relationships between these objects and the attractors analysed in Section 2.3. In this case, the pullback asymptotic compactness is also proved using the energy method that has been applied in Section 2.3, but making use of a more regular energy equality associated to strong solutions (cf. Proposition 2.22). All the results of this chapter can be found in [21].

Chapter 3 is devoted to studying problem (P) when the function $f \in C(\mathbb{R})$ satisfies (6) and there exist positive constants α_1, α_2, κ and $p > 2$ such that

$$-\kappa - \alpha_1|s|^p \leq f(s)s \leq \kappa - \alpha_2|s|^p \quad \forall s \in \mathbb{R}. \tag{8}$$

Although we relax the assumptions on f, since now we are not dealing with semilinear reaction terms, we need to impose some strong smoothness condition on the domain Ω. Nevertheless, we do not impose any restriction on the dimension of the domain $\Omega \subset \mathbb{R}^N$, which allows to deal with problems which have a strong dependence not only on the spatial variables, but also on others.

The structure of Chapter 3 is as follows. In Section 3.1, the existence and uniqueness of weak solutions to (P) is analysed in Theorem 3.3. To do this, we use the Galerkin approximations and compactness arguments, together with some projection operators which are well-defined thanks to the regularity imposed to the domain Ω. Next, the existence and uniqueness of strong solutions to (P) and the regularising effect of the equation are studied in Theorem 3.4. Whereas in local reaction-diffusion equations, strong solutions belong to $L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C([\tau, T]; H_0^1(\Omega))$ for all $T > \tau$ (cf. [100, 5]), in nonlocal problems like problem (P) we are not able to obtain the regularity $C([\tau, T]; H_0^1(\Omega))$ for the solution u, due to the fact that in general, $u' \in L^q(\tau, T; L^q(\Omega))$ $(1 \leq q < 2)$, instead of belonging to $L^2(\tau, T; L^2(\Omega))$. Therefore, making use of the fact that $u \in L^\infty(\tau, T; H_0^1(\Omega))$ together with the cited regularity of u', we can only obtain that $u \in C_w([\tau, T]; H_0^1(\Omega))$ for all $T \geq \tau$ (cf. [108, Theorem 2.1, p. 544] or [111, Lemma 3.3, p. 74]). Furthermore, a Maximum Principle is provided for problem (P), which is an essential tool to study biological models, because in order to analyse population dynamics it is crucial to guarantee that the solution is positive (for more details see Remark 3.25).

Section 3.2 is in a certain sense complementary to our main results on attractors. We analyse some results concerning the stationary solutions. Observe that due to the presence of the nonlinear terms $-a(l(u))\Delta u$ and $f(u)$, this problem is far from being trivial. Therefore we can only provide some partial results. Namely, we establish the existence of nontrivial solutions for a special choice of f making use of a result by Chipot & Corrêa (cf. [42, Theorem 2.1]). In a more general framework, we provide a conditional result in the same line as [44, Lemma 4.1]. Namely, we obtain a comparison result between the solution to problem (P) and two (assumed to exist) stationary solutions.

Section 3.3 focuses its study on the existence of minimal pullback attractors in the phase space $L^2(\Omega)$ and some relations between them are obtained. Regarding f, while in the sublinear framework (see Chapter 2) we also needed to impose the usual assumption (7) to study the asymptotic behaviour of the solutions through the theory of attractors, here condition (8) will be enough.

To conclude this chapter, Section 3.4 is devoted to analysing the existence of minimal pullback attractors in the phase space $H_0^1(\Omega)$ in the strong-solutions framework. Due to the fact that in general $f(u)$ belongs to $L^q(\tau,T;L^q(\Omega))$, it does not make any sense to multiply the equation of problem (P) by $-\Delta u \in L^2(\tau,T;L^2(\Omega))$. Therefore, in the general case in which $f(u)$ does not belong to $L^2(\tau,T;L^2(\Omega))$, we cannot use an energy equality in a strong sense. However, to prove the existence of pullback attractors in $H_0^1(\Omega)$, we need to deal with this kind of equalities. That is the reason why in this last section we impose the additional assumption

$$f(u) \in L^2(\tau,T;L^2(\Omega)) \quad \forall u \in L^2(\tau,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap L^\infty(\tau,T;H_0^1(\Omega)),$$

which will be replaced along this section by

$$\|f(u)\|^2_{L^2(\tau,T;L^2(\Omega))} \leq C_f \|u\|^{2\tilde{b}}_{L^\infty(\tau,T;H_0^1(\Omega))} \|u\|^{2\hat{b}}_{L^2(\tau,T;H^2(\Omega)\cap H_0^1(\Omega))},$$

for some $\hat{b}$, $\tilde{b}$, $C_f > 0$. This assumption has been obtained using interpolation results (cf. [116, Lemma II.4.1, p. 72]) and the regularity of the domain Ω. Then, taking this into account, the existence of attractors is guaranteed, where the pullback asymptotic compactness has been proved using the same kind of energy method as the one applied in Chapter 2 for the same purpose. All the results of these chapter have been analysed in [23].

In Chapter 4, we continue analysing problem (P) in the reaction-diffusion framework, i.e. when the function f fulfils the assumptions (6) and (8), like in Chapter 3. In this chapter 4, we get rid of the strong assumptions made on the domain Ω in Chapter 3, which allows to model real phenomena with more accuracy since they tend to be posed in nonsmooth domains (see [68] for more details). Nevertheless, in this case we need to impose some restrictions on either the dimension N (cf. Theorem 4.8) or the reaction term (see Theorem 4.10) or even both of them (cf. Corollary 4.11) when we are analysing the existence of strong solutions and the regularising effect of the equation, unlike what happened in Chapter 3 that it was only necessary smoothness assumptions on the domain Ω. To prove the existence of pullback attractors in $H_0^1(\Omega)$ in this chapter 4, it is not enough with assuming only the restrictions on the dimension of the domain Ω that were made in Theorem 4.8, we need to study the existence of these families in the settings of Theorem 4.10, imposing requirements on the reaction term, or Corollary 4.11, making assumptions on the dimension of the domain and the reaction term. Observe that thanks to the restrictions made on the dimension of the domain in Corollary 4.11, the reaction term can be taken more general in this corollary than in Theorem 4.10.

We split this chapter into three sections. In Section 4.1, we study the existence of weak and strong solutions to the nonlocal reaction-diffusion problem (P). In the first part, Section 4.1.1, we briefly recall the monotonicity method for solving nonlinear PDEs (cf. [85, Chapitre 2]). Next, in Section 4.1.2, the existence and uniqueness of weak solutions to (P) are shown making use of an iterative method together with monotonicity and compactness arguments. Namely, we consider the sequence of problems

$$
(\mathrm{P}_n)\begin{cases} u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega)) \quad \forall T \geq \tau, \\ \dfrac{d}{dt}(u(t), v) + a(l(u^{n-1}(t)))((u(t), v)) = (f(u(t)), v) + \langle h(t), v \rangle, \\ u(\tau) = u_\tau, \end{cases}
$$

where $u^0 \equiv 0$ and u^n is the solution to (P_n) if $n \geq 1$.

Observe that the existence and uniqueness of weak solutions to (P_n) are guaranteed by the monotonicity method. Then, applying compactness arguments to the sequence $\{u^n\}$, we can prove the existence of weak solutions to (P). Observe that this result is an improvement compared to the existence result of Chapter 3 in the weak-solutions framework (cf. Theorem 3.3), since in this case we are able to prove the existence of weak solutions to (P) without making any smoothness assumptions on the domain Ω. The uniqueness holds immediately when we also assume that the function a is locally Lipschitz. Next, in Section 4.1.3, the existence and uniqueness of strong solutions as well as the regularising effect of the equation are proved without assuming any smoothness conditions on the domain Ω as in Chapter 3. In return, we need to impose some restrictions on either the dimension N (cf. 4.11) like in Theorem 4.8 or the reaction term. Namely in Theorem 4.10, to prove the existence and uniqueness of strong solutions and the regularising effect of the equation, we assume that

$$
|f(s)| \leq C(1 + |s|^{\gamma+1}) \quad \forall s \in \mathbb{R}, \tag{9}
$$

with $\gamma = 2/N$ if $N \geq 3$, where this estimate has been obtained applying interpolation results (cf. [116, Lemma II.4.1, p. 72]) to the Sobolev spaces $L^\infty(\tau, T; L^2(\Omega))$ and $L^2(\tau, T; H_0^1(\Omega))$. Observe that when $N = 1, 2$, γ can be any positive value (cf. Remark 4.9 (i)). In fact, the assumption (9) can be improved if we impose some requirements on the dimension of the domain Ω. Namely, $\gamma = 2/(N-2)$ when $3 \leq N \leq 2p/(p-2)$ (see Corollary 4.11 for more details).

Section 4.2 is devoted to studying the existence of pullback attractors in the phase space $L^2(\Omega)$ in the more general setting of Section 4.1.2. Although this result is not new in this PhD project, since the existence of these families has been proved in Chapter 3 (see Theorem 3.17 for more details), the method applied to prove the pullback asymptotic compactness is. Namely, we argue similarly as it was done in [101].

For the sake of completeness, in Section 4.3, we analyse the existence of pullback attractors in $H_0^1(\Omega)$ in the framework of universes and establish some relationships amongst these families of attractors and the ones analysed in Section 4.2. Observe that the existence of pullback attractor in H^1-norm has been analysed in Chapter 3 (cf. Theorem 3.23). Nevertheless, in this chapter to prove the pullback asymptotic compactness we use the pullback flattening property, which is a tool that has not been used before in this thesis. In addition, it is worth highlighting that to prove these results, we need to work in the setting of Theorem 4.10 or Corollary 4.11, since assuming only restrictions on the dimension of the domain Ω is not enough to guarantee the existence of stronger energy equalities as well as the continuity of the strong solutions in $H_0^1(\Omega)$. The results of this chapter can be found in [25].

From Chapter 5 forward, all the results provided in this PhD project are set in a multi-valued framework. Many authors have been interested in analysing problems in which the uniqueness of solution is not guaranteed. Amongst them, it is worth highlighting 3D incompressible Navier-Stokes equations (cf. [17]), differential inclusions (cf. [96]), reaction-diffusion equations (cf. [6]) and delay differential equations (cf.[89]).

In Chapter 5, we describe abstract results on multi-valued non-autonomous dynamical systems in the framework of universes. This chapter is divided into two sections. In Section 5.1, we analyse the basic concepts studied in Chapter 1 in this new setting. In addition, the notion of upper semicontinuous process (cf. Definition 5.2) is shown. Later, in Section 5.2, the existence of pullback attractors and some relationships between them are established in this multi-valued framework (cf. Theorem 5.11, Corollary 5.13 and Theorem 5.14).

Chapter 6 is devoted to the study of the perturbed non-autonomous nonlocal reaction-diffusion problem

$$(P_\varepsilon)\begin{cases} \dfrac{\partial u}{\partial t} - (1-\varepsilon)a(l(u))\Delta u = f(u) + \varepsilon h(t) \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases}$$

where $\varepsilon \in [0,1)$, $\tau \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded open set, the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (5), $f \in C(\mathbb{R})$ satisfies (8), and $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$. In addition, the initial datum $u_\tau \in L^2(\Omega)$ and the non-autonomous term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$.

This chapter is split into three sections. In Section 6.1, the existence of weak solutions to (P_ε) is analysed in Theorem 6.2 making use of the Galerkin approximations and compactness arguments. Observe that although we are analysing a reaction-diffusion equation similar to the one studied in (P), it is not necessary to assume any smoothness condition on the domain Ω. The reason is that along the proof of this result, we do not need to obtain a uniform estimate of the Galerkin approximations associated to u', the temporal derivative of a solution u to (P_ε), since instead of applying the Aubin-Lions lemma, we make use of the compactness by translation (cf. [110, Theorem 13.2, p. 97] and [110, Remark 13.1, p. 100]) in the compactness arguments.

In Section 6.2, namely in Theorem 6.12, the existence of minimal pullback attractors in the phase space $L^2(\Omega)$ is stated and some relationships between them are established. In this case, to prove that the multi-valued process U is asymptotically compact, we make use of the same kind of energy method applied in Chapters 2 and 3 for the same purpose.

In Section 6.3, we study the upper semicontinuous behaviour of attractors. In Theorem 6.15, we prove that the family of pullback attractors indexed by ε converges to the global attractor associated to (P_0) when the parameter ε goes to zero.

Finally, in Section 6.4, we analyse some regularity results. Namely, in Theorem 6.17 the existence of strong solutions and the regularising effect of the equation are studied. Observe that to prove this result, since the uniqueness of solution is not guaranteed, we make use of an argument of a posteriori regularity. Then, the existence of pullback attractors in $H_0^1(\Omega)$ as well as the upper semicontinuos behaviour of attractors in $H_0^1(\Omega)$ are proved in Theorems 6.23 and 6.26 respectively. All these results have been studied in [22, 26].

In Chapter 7 we generalise the diffusion term, analysing a nonlocal problem for the p-Laplacian. This operator appears in a wide range of areas in Physics. For instance, in Fluid Dynamics, where $p = 2$ if the fluid is Newtonian, $p < 2$ when it is pseudoplastic and $p > 2$ when the fluid is dilatant. In addition, this operator is also essential in the study of flow through porous media ($p = 3/2$), Nonlinear Elasticity ($p \geq 2$), Glaciology ($1 < p \leq 4/3$) and Image Restoration (for more detail cf. [118, 121, 99, 107]). As it has been mentioned before, in [47], Chipot & Savistka analyse a nonlocal problem for the p-Laplacian, $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$. Namely,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f \quad \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $1 < p < \infty$, the function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (5), $f \in W^{-1,q}(\Omega)$, where q is the conjugate exponent of p, and the initial datum $u_0 \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. The existence and uniqueness of weak solutions are proved rescaling the time as follows

$$\alpha(t) = \int_0^t a(l(u(s)))ds, \tag{10}$$

and making use of the Galerkin approximations and compactness arguments.

Although this change of variable has already been used by Chipot et al. in [49] in order to prove the uniqueness of solution for a nonlocal problem, as far as we know, it is the first time that it is used as a tool to prove the existence of solutions for nonlocal diffusion problems. The main reason is that in the previous papers (cf. [43, 44, 45, 35, 36, 48, 49, 37, 50, 97, 2]), the diffusion term contained the Laplacian, which is linear. Then, although the nonlocal term generated a nonlinear diffusion, making use of [85, Lemme 1.3, p. 12], it was not difficult to ensure the existence of solution. However, for the p-Laplacian, it does not seem to be possible to argue in the same way, even using monotonicity arguments.

In this chapter, we consider the nonlocal problem for the p-Laplacian

$$(\widetilde{P}) \begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta_p u = f \quad \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

where $p \geq 2$, the function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (5), $f \in W^{-1,q}(\Omega)$ (where q is the conjugate exponent of p), $l \in \mathcal{L}(L^2(\Omega),\mathbb{R})$ and $u_0 \in L^2(\Omega)$.

We split Chapter 7 into two sections. In Section 7.1, the existence of weak solutions to $(\widetilde{P})$ is shown in Theorem 7.2 using the Galerkin approximations, the change of variable (10) and compactness arguments. However, under the assumptions made on the function a, the uniqueness of solution to $(\widetilde{P})$ is not guaranteed. In addition, in this result we also show a regularising effect of the equation. Next, in Section 7.2, we prove the main result of this chapter, Theorem 5.11, in which we study the asymptotic behaviour of the solutions to $(\widetilde{P})$ through the theory of attractors. To do this, we prove the existence of the compact global attractor for a multi-valued semiflow in the phase space $L^2(\Omega)$, since $(\widetilde{P})$ is an autonomous problem. To establish the existence of this object, the main difficulty lies in proving the asymptotic compactness. To that end, we build an absorbing set in $W_0^{1,p}(\Omega)$ (cf. Proposition 7.8) and make use of the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$. Observe that the analysis of problem $(\widetilde{P})$ in the non-autonomous framework is also possible. However, for the sake of simplicity the study has been made in the autonomous setting, since the proof of Theorem 7.2 involves a change of variable that makes it quite technical.

To conclude this PhD project, we also provide a non-exhaustive list of problems that we would like to study in the near future like nonlocal equations in unbounded domains, the Kneser property or nonlocal delays problems, amongst others. In addition, we describe some of the works in progress and highlight the difficulties that have arisen.

Spanish Summary

En las últimas décadas, muchos autores han estado interesados en analizar problemas no locales por su utilidad en aplicaciones reales (e.g. cf. [59, 19, 67, 109, 12]). Concretamente, se ha prestado especial atención a ecuaciones parabólicas no locales del tipo

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f, \tag{11}$$

(cf. [87, 43, 44, 35, 36, 48, 49, 37, 50, 97, 2], para análisis con el p-Laplaciano ver [103]), donde la función a es continua y está acotada inferior y superiormente por constantes positivas, es decir

$$0 < m \leq a(s) \leq M \quad \forall s \in \mathbb{R}. \tag{12}$$

Obsérvese que en muchas ocasiones la constante M puede ser calculada localmente. A lo largo de este proyecto de tesis, la hipótesis (12) sólo será impuesta para el análisis de problemas elípticos (cf. Secciones 2.2 y 3.2).

Desde un punto de vista biológico, la función u que aparece en la ecuación (11) representa la densidad de una determinada población. Las características de la misma pueden reflejarse en la ecuación imponiendo ciertas condiciones a la función a. Por ejemplo, si se pretende modelar el comportamiento de una especie que tiene tendencia a alejarse de zonas donde la densidad de población es alta, esta actitud se traduce imponiendo que la función a sea creciente. De este modo la difusión será mayor.

Con respecto a las condiciones impuestas a la función a, la continuidad y la acotación inferior por una constante estrictamente positiva parecen las hipótesis mínimas necesarias para evitar que la especie exista sólo en intervalos finitos de tiempo y además, permiten que la ecuación conserve su carácter parabólico. Por tanto, resultados clásicos como el Principio del Máximo y métodos de sub-supersoluciones pueden aplicarse (ver [87] para más detalles).

En este proyecto de tesis se estudia el comportamiento asíntotico de las soluciones de problemas no locales (variantes de (11) con condiciones de contorno Dirichlet homogéneas) haciendo uso de la teoría de atractores. En este marco hay varias tendencias a seguir dependiendo de si el problema es autónomo o no.

En la teoría autónoma, el principal objeto de estudio es el atractor global, analizado con detalle por Hale [69], Temam [111], Ladyzhenskaya [83], Babin & Vishik [16], Vishik [113], Robinson [100] o Sell & You [105].

Sin embargo, para modelar situaciones más complejas que conlleven la aparición de términos dependientes del tiempo es necesario recurrir a una teoría más general, los sistemas dinámicos no autónomos. Dentro de este marco, existen varias tendencias que permiten analizar el comportamiento asintótico de las soluciones. Por ejemplo, el estudio de los atractores uniformes (cf. Chepizhov & Vishik [40]) es una generalización natural del atractor global, ya que se analiza el comportamiento de las soluciones cuando el tiempo tiende a infinito. Sin embargo, los atractores no satisfacen en general la propiedad de invarianza y los términos no autónomos necesitan satisfacer determinadas restricciones. Otra opción posible es considerar la teoría de atractores pullback[1], la cual ha sido ampliamente desarrollada en la última década (cf. Kloeden & Schmalfuß [81, 82]; Kloeden [76]; Caraballo, Łukaszewicz y Real [31, 32]); Kloeden & Rasmussen [80]; y Carvalho, Langa y Robinson [33]). Dentro de esta tendencia, algunos autores están interesados en estudiar atractores pullback en el sentido clásico, i.e. atractores pullback de soluciones comenzando en conjuntos acotados fijos. Otros, en cambio, emplean el concepto de atracción asociado a una clase de familia, llamada universo $\mathcal{D}$, constituida por conjuntos dependientes del tiempo y definidos a partir de una condición temperada (e.g. cf. [51, 31, 32, 62]). Este último concepto de atracción será el que se aplique en este proyecto de tesis. Además se establecerán relaciones entre estas familias de atractores y aquellos dados en el sentido clásico.

Este trabajo está dividido en siete capítulos. En los Capítulos 1 y 5, analizamos resultados abstractos de sistemas dinámicos no autónomos univaluados y multivaluados dentro del marco de los procesos. Estos resultados son utilizados en los Capítulos 2, 3, 4 y 6 para analizar el comportamiento asintótico de las soluciones de problemas parabólicos no locales con términos no autónomos. Finalmente, en el Capítulo 7, las variantes autónomas de estos resultados abstractos se aplican a una ecuación no local para el p-Laplaciano sin unicidad de solución. El estudio en el marco no autónomo también es posible, pero por simplicidad y claridad en las pruebas, hemos decidido analizar el problema sin términos dependientes del tiempo.

El primer capítulo está dedicado a realizar una descripción de la teoría abstracta de atractores pullback en el marco de los universos. Está dividido en tres secciones. En la Sección 1.1, consideramos un universo $\mathcal{D}$ formado por familias dependientes del tiempo y analizamos algunos conceptos básicos como la definición de familia pullback $\mathcal{D}$-absorbente y la compacidad asintótica pullback con respecto al universo $\mathcal{D}$. Además, desarrollamos algunos resultados abstractos que serán esenciales para demostrar la existencia de los atractores pullback asumiendo las hipótesis más débiles.

A continuación, en la Sección 1.2, demostramos el principal resultado de este capítulo, el Teorema 1.13, el cual garantiza la existencia del $\mathcal{D}$-atractor pullback minimal. Haciendo uso de este resultado, en el Corolario 1.15 establecemos relaciones entre el atractor del universo de los acotados fijos y el asociado al universo $\mathcal{D}$. Para concluir esta sección, demostramos el Teorema 1.16, el cual nos permite comparar

[1]No existe una traducción literal en castellano de esta noción, la cual trata el comportamiento asintótico de las soluciones cuando el dato inicial viene desde menos infinito.

atractores para universos más generales (no sólo el universo de los acotados fijos) asociados a diferentes espacios de fases.

Finalmente, en la Sección 1.3, analizamos la propiedad flattening[2], una herramienta muy útil que nos permite probar uno de los ingredientes claves para determinar la existencia del atractor pullback, la compacidad asintótica, de forma inmediata (cf. Proposición 1.18).

En el Capítulo 2 consideramos una ecuación parabólica no local con términos sublineales y no autónomos. Este capítulo está dividido en cuatro secciones. En la Sección 2.1, estudiamos la existencia y unicidad de soluciones débiles y fuertes aplicando las aproximaciones de Galerkin y argumentos de compacidad. En primer lugar, probamos la existencia de solución local empleando una generalización del Teorema de Peano (cf. [52, Theorem 1.1, p. 43]). A continuación, demostramos la unicidad de solución local imponiendo al operador no local el carácter localmente lipschitziano. Posteriormente, en el Teorema 2.4, probamos la existencia y unicidad de soluciones (globales) débiles utilizando [85, Lemme 1.3, p. 12], ya que necesitamos pasar al límite en las aproximaciones de Galerkin asociadas a los términos no lineales $-a(l(u))\Delta u$ y $f(u)$. Además, en un marco más regular demostramos el efecto regularizante de la ecuación así como la existencia de soluciones fuertes.

En la Sección 2.2 estudiamos la existencia de soluciones estacionarias aplicando un corolario del teorema del punto fijo de Brouwer (ver [85, Lemme 4.3, p.53]). Mostramos la unicidad de solución de forma análoga a como hicimos anteriormente en el caso parabólico, pero en un marco más restrictivo. Finalmente, bajo las hipótesis que garantizan la unicidad de solución estacionaria, obtenemos el decaimiento exponencial de la solución del problema evolutivo hacia la estacionaria.

En la Sección 2.3, concretamente en el Teorema 2.16, probamos la existencia de atractores pullback minimales en $L^2(\Omega)$ en el marco de los universos mediante un método de energía que utiliza la continuidad de las soluciones débiles (véase [73, 92, 94, 62]). Además, establecemos relaciones entre estas familias de atractores.

Para finalizar este capítulo, en la Sección 2.4, demostramos la existencia de atractores pullback en $H_0^1(\Omega)$ usando un método de energía del mismo tipo que el usado en la Sección 2.3 y establecemos relaciones entre estas nuevas familias y los atractores obtenidos en el Teorema 2.16. Todos los resultados de este capítulo han sido tratados en la publicación [21].

En el Capítulo 3 analizamos una ecuación de reacción-difusión no local en presencia de términos no autónomos. A lo largo de este capítulo imponemos que el dominio Ω sea regular, pero no imponemos ninguna restricción a la dimensión del mismo, lo que nos permite tratar problemas que tienen fuertes dependencias de otras variables no sólo la espacial. Este capítulo está dividido en cuatro secciones. En la Sección 3.1 estudiamos la existencia y unicidad de soluciones débiles empleando las aproximaciones de Galerkin, argumentos de compacidad y la regularidad del

[2]Esta propiedad consiste en un aplanamiento del sistema dinámico usando sólo un número finito de nodos. En español se usa esta palabra ya que su traducción literal al castellano, aplanamiento, no ha tenido mucha trascendencia.

dominio (véase el Teorema 3.3). Además analizamos la existencia y unicidad de solución fuerte así como el efecto regularizante de la ecuación. Obsérvese que mientras que en las ecuaciones de reacción-difusión locales, la solución fuerte u pertenece a $L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C([\tau, T]; H_0^1(\Omega))$ para todo $T > \tau$ (cf. [100, 5]), en las variantes no locales analizadas en este capítulo no podemos alcanzar en general la regularidad $C([\tau, T]; H_0^1(\Omega))$, debido a que $u' \in L^q(\tau, T; L^q(\Omega))$ (donde $1 \leq q < 2$), en lugar de pertenecer a $L^2(\tau, T; L^2(\Omega))$, que junto con $u \in L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$, garantizarían la citada continuidad en $H_0^1(\Omega)$ de la solución.

A continuación, en la Sección 3.2, realizamos algunas aportaciones en el marco estacionario. Concretamente, en los Teoremas 3.8 and 3.10 estudiamos respectivamente la existencia de soluciones no triviales y un resultado de comparación entre la solución del problema evolutivo y dos soluciones estacionarias.

Finalmente, en las dos últimas secciones analizamos el comportamiento asintótico de las soluciones a través de la teoría de atractores pullback.

En la Sección 3.3 demostramos la existencia de atractores pullback en $L^2(\Omega)$ en el marco de los universos y establecemos relaciones entre estas familias (véase el Teorema 3.17).

Para concluir este capítulo, en la Sección 3.4, estudiamos la existencia de atractores pullback en $H_0^1(\Omega)$. Para ello, necesitamos imponer hipótesis adicionales sobre la función f que garanticen que $f(u)$ pertenece a $L^2(\tau, T; L^2(\Omega))$. Estas hipótesis son construidas usando resultados de interpolación y la regularidad del dominio Ω. De esta forma, $u' \in L^2(\tau, T; L^2(\Omega))$ y por tanto, las manipulaciones con $-\Delta u$ tienen sentido así como es posible obtener estimaciones más regulares usando la igualdad de energía asociada a las soluciones fuertes (cf. (3.55)). Estos resultados han sido analizados en el trabajo [23].

En el Capítulo 4 continuamos analizando la ecuación de reacción-difusión no local con términos no autónomos estudiada en el Capítulo 3 bajo otras condiciones. A lo largo de este capítulo no imponemos ninguna regularidad al dominio Ω. Esto nos permite modelar problemas reales con más precisión ya que muchos de ellos están planteados en dominios no regulares (véase [68] para más detalles). Este capítulo está dividido en tres secciones. En la Sección 4.1 analizamos la existencia de soluciones débiles y fuertes. A diferencia de lo que ocurría en el Capítulo 3, en el Teorema 4.6 se demuestra la existencia y unicidad de solución débil sin imponer ninguna regularidad al dominio, utilizando un método iterativo y argumentos de compacidad y de monotonía. Este resultado supone una mejoría con respecto al resultado de existencia de solución débil del Capítulo 3 (cf. Teorema 3.3), en el que la regularidad del dominio Ω era imprescindible. La existencia y unicidad de solución fuerte así como el efecto regularizante de la ecuación se prueba haciendo uso de las aproximaciones de Galerkin y argumentos de compacidad. Como no asumimos que el dominio Ω es regular, como se hizo en el Capítulo 3, a cambio necesitamos imponer ciertas restricciones a la dimensión del dominio N (cf. Teorema 4.8), al término de reacción (cf. Teorema 4.10) o a ambos (cf. Corolario 4.11).

A continuación, en la Sección 4.2, estudiamos la existencia de atractores pullback en $L^2(\Omega)$ en el marco de los universos. Aunque este resultado no es nuevo en esta tesis, la existencia de estas familias ha sido probada en el Capítulo 3 (cf. Teorema 3.17), el método usado para probar la compacidad asintótitca pullback sí lo es. Concretamente, se emplea el método usado por Rosa en [101] adaptado al marco no local.

Finalmente, en la Sección 4.3, analizaremos la existencia de atractores pullback en $H_0^1(\Omega)$. Mientras que en el Capítulo 3, para garantizar que $f(u) \in L^2(\tau, T; L^2(\Omega))$ en el marco de las soluciones fuertes usábamos que el dominio Ω fuese regular, en este capítulo para obtener que $f(u) \in L^2(\tau, T; L^2(\Omega))$, como no estamos asumiendo ninguna condición de regularidad al dominio Ω, tenemos que imponer algunas restricciones al término de reacción (cf. Teorema 4.10). Obsérvese que estos requerimientos pueden ser debilitados imponiendo ciertas restricciones a la dimensión del dominio Ω (cf. Corolario 4.11).

Los problemas analizados en los restantes capítulos de esta tesis, concretamente en los Capítulos 6 and 7, están planteados en un marco multivaluado, ya que no podemos garantizar la unicidad de solución bajo las hipótesis impuestas. Por ello, en el Capítulo 5, estudiamos algunos resultados abstractos sobre sistemas dinámicos multivaluados para procesos. Este Capítulo 5 está constituido por dos secciones. En la Sección 5.1 definimos algunos conceptos básicos y estudiamos varios resultados abstractos que nos permitirán demostrar el teorema principal de existencia de atractores pullback en la Sección 5.2. Concretamente, dicho resultado se corresponde con el Teorema 5.11. A continuación, en el Corolario 5.13 se establecen relaciones entre el atractor de los acotados fijos y el atractor asociado a un universo $\mathcal{D}$ constituido por familias parametrizadas en tiempo. Para concluir el capítulo, en el Teorema 5.14, estudiamos más relaciones que se pueden establecer entre atractores asociados a universos más generales.

En el Capítulo 6, estudiamos una ecuación de reacción-difusión no local sin unicidad de solución con una pequeña perturbación ε en el término de difusión y en la fuerza no autónoma. Este capítulo está dividido en tres secciones. En la Sección 6.1 demostramos la existencia de soluciones débiles usando las aproximaciones de Galerkin y argumentos de compacidad. Obsérvese que a diferencia de lo que ocurría en el Capítulo 3, en este caso no es necesario imponer ninguna regularidad al dominio Ω. Esto es debido a que a lo largo de la prueba del Teorema 6.2 no realizamos ninguna estimación uniforme de la aproximación de Galerkin asociada a la derivada temporal de una solución, ya que en los argumentos de compacidad, en lugar de usar el lemma de Aubin-Lions, hacemos uso de la compacidad por traslación (cf. [110, Theorem 13.2, p. 97] y [110, Remark 13.1, p. 100]).

A continuación, en la Sección 6.2, demostramos la existencia de atractores pullback en $L^2(\Omega)$. Para ello, en la Proposición 6.11 se analiza la compacidad asintótica pullback aplicando el mismo tipo de método de energía que el usado en los Capítulos 2 y 3.

En la Sección 6.3, estudiamos la propiedad de semicontinuidad superior de atractores. Concretamente, en el Teorema 6.15 se prueba que la familia de atractores pullback dependiente del parámetro ε, cuya existencia ha sido demostrada en el

Teorema 6.12, converge, cuando el parámetro tiende a cero, al atractor global del semiflujo multivaluado asociado al problema autónomo inicial con $\varepsilon = 0$.

Para finalizar este capítulo, en la Sección 6.4, estudiamos algunos resultados de regularidad. Concretamente, estudiamos la existencia de soluciones fuerte para (P_ε) así como el efecto regularizante de la ecuación. A continuación, demostramos la existencia de atractores pullback en $H_0^1(\Omega)$ así como generalizamos el resultado de semicontinuidad superior de atractores estudiado en la sección anterior, demostrando la convergencia en la H^1-norma. Para estudiar el comportamiento de las soluciones en este marco más regular, como la unicidad de solución no está garantizada, usamos un razonamiento de regularidad a posteriori (véase el Teorema 6.17). Los resultados de este capítulo han sido analizados en los trabajos [22, 26].

En el Capítulo 7, analizamos un problema autónomo en el que el término de difusión está constituido por un operador no local y el p-Laplaciano, generalizando así la difusión con respecto a los capítulos anteriores, en los que todos los análisis han sido hechos para el Laplaciano. Este capítulo está dividido en dos secciones. En la Sección 7.1 probamos la existencia de soluciones débiles. A través de un cambio de variable temporal transformamos un problema con difusión no local en un problema con difusión local, análogamente a como fue hecho por Chipot & Savitska en [47]. Sin embargo, la unicidad de solución no está garantizada debido a la generalidad del operador no local. Además de la existencia de soluciones débiles, en el Teorema 7.2, se demuestra una propiedad de regularización del problema analizado.

A continuación, en la Sección 7.2, analizamos el comportamiento asintótico de las soluciones demostrando la existencia del atractor global en $L^2(\Omega)$ ya que el problema estudiado no posee términos no autónomos. Para ello, en la Proposición 7.8 construimos un conjunto absorbente en $W_0^{1,p}(\Omega)$ empleando la propiedad regularizante citada anteriormente. Finalmente, teniendo esto en cuenta junto con la compacidad de la inyección $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, concluimos este trabajo demostrando la existencia del atractor global en el Teorema 7.9. Obsérvese que también es posible analizar el problema $(\widetilde{P})$ en el marco no autónomo. Sin embargo, por simplicidad el estudio se ha hecho sin términos dependientes del tiempo, ya que la prueba del Teorema 7.2 emplea un cambio de variable que la hace muy técnica. Estos resultados han sido analizados en [24].

Para concluir este proyecto de tesis, proporcionamos una lista no exhaustiva de problemas que nos gustaría estudiar en el futuro como por ejemplo, ecuaciones no locales en dominios no acotados, la propiedad Kneser o problemas no locales con retardo, entre otros. Además, describimos algunos de los trabajos en curso y resaltamos las dificultades que nos han aparecido.

Chapter 1

Abstract results on the theory of pullback attractors. Pullback flattening property

The modelisation of real phenomena in different scientific fields like Physics, Biology or Chemistry, makes the equations more and more complex when they try to reproduce the reality with accuracy. As a consequence, the study of the existence of points of equilibrium and their stability or Lyapunov functions associated to partial differential equations proves to be an intractable task in many occasions. On the other hand, as a natural generalisation of the behaviour of the solutions around points of equilibrium and thanks to the presence of chaos and turbulence phenomena in the reality, the dynamical systems field, which involves the theory of attractors, inertial manifolds or fractal dimension analysis in diverse senses, amongst others, has been developing in the last few decades.

In the context of attractors there are several choices to study the asymptotic behaviour of the solutions of evolution problems. One can prove the existence of the global compact attractor in the autonomous framework (cf. [100]). However, when the equation possesses time-dependent terms, several approaches from non-autonomous dynamical systems can be used. Namely, one can do attempts with uniform attractors (cf. [40]), skew-product flows (cf. [105]) and pullback attractors (see [76, 31, 32, 93, 62] for more details; also related to random dynamical systems, cf. [55]).

In this chapter, we analyse abstract results of the theory of pullback attractors, which allow us to study not only the future of the dynamical system but also what the current attracting sections are when the initial data come from $-\infty$. In addition, making use of this approach, we can analyse the existence of attractors for equations with general forcing terms and the resultant objects are invariant in a "suitable non-autonomous-dynamical-system sense", unlike what happens with the uniform attractors. The theory of pullback attractors has been used for a wide range of problems such as non-autonomous difference equations (cf. [76]), non-autonomous and stochastic multi-valued dynamical systems (cf. [29]), non-autonomous difference inclusions (cf. [79]), non-autonomous 2D-Navier-Stokes equations in some unbounded domains (cf. [32, 84]), non-autonomous differential equations (cf. [14]), non-autonomous reaction-diffusion equations in unbounded domains (cf. [114, 7, 6, 115]), 2D or 3D-Navier-Stokes equations with delay (cf. [95, 90, 63, 64, 66]). Within this framework, some authors are interested in studying the pullback attractor in the classical sense, i.e. the pullback attractor of solutions starting in "fixed" bounded sets. Others, though, employ the concept of attraction related to a class of families, called universe $\mathcal{D}$, made up by sets which are allowed to move in time and usually defined in terms of a tempered condition (cf. [31, 32, 61]). This approach will be the one analysed along this chapter.

The results of this chapter can be found in [93, 62, 33, 65].

1.1 Basic concepts

Consider given a metric space (X, d_X).

Definition 1.1.

(a) A process on X (also called a two-parameter semigroup) is a mapping $\mathbb{R}^2_d \times X \ni (t, \tau, x) \mapsto U(t, \tau)x \in X$ such that $U(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t, s)(U(s, r)x) = U(t, r)x$ for any $r \leq s \leq t$ and all $x \in X$.

(b) A process U on X is said to be

1. *continuous if for any pair $(t, \tau) \in \mathbb{R}^2_d$, the mapping $U(t, \tau) : X \to X$ is continuous;*
2. *strong-weak (also known as norm-to-weak) continuous if for any pair $(t, \tau) \in \mathbb{R}^2_d$, the map $U(t, \tau)$ is continuous from X with the strong topology into X with the weak topology;*
3. *closed if for any pair $(t, \tau) \in \mathbb{R}^2_d$ and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $U(t, \tau)x_n \to y \in X$, then $U(t, \tau)x = y$.*

Remark 1.2. *It is clear that every continuous process is strong-weak continuous and every strong-weak continuous process is closed.*

Let $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ be a family of nonempty sets. Observe that we do not require any additional condition on these sets such as compactness or boundedness.

Definition 1.3. *A process U on X is said to be pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$, and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.*

Denote the omega-limit set of $\widehat{D}_0$ by

$$\Lambda(\widehat{D}_0, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau) D_0(\tau)}^X \quad \forall t \in \mathbb{R}. \tag{1.1}$$

Proposition 1.4 (Sequential characterisation of the omega-limit set). *It holds that $y \in \Lambda(\widehat{D}_0, t)$ if and only if there exist sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$, with $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, such that $U(t, \tau_n)x_n \to y$.*

Then, we have the following result.

Proposition 1.5. *If the process U on X is pullback $\widehat{D}_0$-asymptotically compact, then for any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}_0, t)$ given by (1.1) is a nonempty compact subset of X and*

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t, \tau)D_0(\tau), \Lambda(\widehat{D}_0, t)) = 0. \tag{1.2}$$

In addition, the family $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t, \tau)D_0(\tau), C(t)) = 0,$$

then $\Lambda(\widehat{D}_0, t) \subset C(t)$.

Proof. Consider fixed $t \in \mathbb{R}$, and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ such that $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n. Since the process U is pullback $\widehat{D}_0$-asymptotically compact, there exist subsequences $\{\tau_n\}$ and $\{x_n\}$ (relabeled the same) and $y \in X$ such that $U(t, \tau_n)x_n \to y$ in X. Then, from the sequential characterization of $\Lambda(\widehat{D}_0, t)$, it holds that $y \in \Lambda(\widehat{D}_0, t)$. Therefore, $\Lambda(\widehat{D}_0, t)$ is nonempty.

Now we will show that the set $\Lambda(\widehat{D}_0, t)$ is compact. To do this, since $\Lambda(\widehat{D}_0, t)$ is closed (cf. (1.1)), we only need to prove that this set is relatively compact in X. To that end, consider $\{y_n\} \subset \Lambda(\widehat{D}_0, t)$. Since $y_n \in \Lambda(\widehat{D}_0, t)$ for all n, from the sequential characterization of $\Lambda(\widehat{D}_0, t)$, we deduce that there exist $\tau_n \leq t - n$ and $x_n \in D_0(\tau_n)$ such that

$$d_X(y_n, U(t, \tau_n)x_n) \leq \frac{1}{n}. \tag{1.3}$$

Now, taking into account that the process U is pullback $\widehat{D}_0$-asymptotically compact, it holds that there exists a convergent subsequence of $\{U(t, \tau_n)x_n\}$. Then, from (1.3), there exists a convergent subsequence of $\{y_n\}$. Thus, $\Lambda(\widehat{D}_0, t)$ is a compact set.

Thereupon, to prove (1.2), we argue by contradiction. Assume that there exists $t \in \mathbb{R}$ such that (1.2) does not hold. Then, there exist $\varepsilon > 0$ and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$, such that

$$d_X(U(t, \tau_n)x_n, \Lambda(\widehat{D}_0, t)) > \varepsilon \quad \forall n \geq 1. \tag{1.4}$$

On the other hand, since the process U is pullback $\widehat{D}_0$-asymptotically compact, there exists a subsequence of $\{U(t, \tau_n)x_n\}$ which converges to an element of $\Lambda(\widehat{D}_0, t)$, which is a contradiction with (1.4).

Finally, we will prove that given a family of closed sets $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ that fulfils

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t, \tau)D_0(\tau), C(t)) = 0, \tag{1.5}$$

the relationship $\Lambda(\widehat{D}_0, t) \subset C(t)$ holds.

Consider fixed $t \in \mathbb{R}$, $\varepsilon > 0$ and $x \in \Lambda(\widehat{D}_0, t)$. Let us see that $x \in C(t)$. To do this, we will prove that

$$B_X(x, \varepsilon) \cap C(t) \neq \emptyset. \tag{1.6}$$

Since $x \in \Lambda(\widehat{D}_0, t)$, from the sequential characterization of $\Lambda(\widehat{D}_0, t)$, it holds that there exist sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that

$$\lim_{n\to\infty} d_X(x, U(t, \tau_n)x_n) = 0.$$

Therefore, there exists $n_0(\varepsilon) \geq 1$ such that

$$d_X(x, U(t, \tau_n)x_n) < \frac{\varepsilon}{2} \quad \forall n \geq n_0(\varepsilon). \tag{1.7}$$

On the other hand, from (1.5) we deduce that there exists $n_1(\varepsilon) \geq 1$ such that

$$d_X(U(t, \tau_n)x_n, C(t)) < \frac{\varepsilon}{2} \quad \forall n \geq n_1(\varepsilon). \tag{1.8}$$

Then, taking into account (1.7) and (1.8), (1.6) holds. □

Now, if we assume that the process U is closed, the invariance of the family of sets $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$ is fulfilled.

Proposition 1.6. *If the process U on X is pullback $\widehat{D}_0$-asymptotically compact and closed, then the family of sets $\{\Lambda(\widehat{D}_0, t); t \in \mathbb{R}\}$ is invariant for U, that is*

$$\Lambda(\widehat{D}_0, t) = U(t, \tau)\Lambda(\widehat{D}_0, \tau) \quad \forall \tau \leq t.$$

Proof. Consider fixed $\tau < t$ and $y \in \Lambda(\widehat{D}_0, \tau)$. Then, from the sequential characterization of $\Lambda(\widehat{D}_0, \tau)$, there exist sequences $\{\tau_n\} \subset (-\infty, \tau]$ and $\{x_n\} \subset X$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that $U(\tau, \tau_n)x_n \to y$. From this, taking into account that $U(t, \tau_n) = U(t, \tau)U(\tau, \tau_n)$ for all n, and the process U is closed and pullback $\widehat{D}_0$-asymptotically compact, it holds that $U(t, \tau)y \in \Lambda(\widehat{D}_0, t)$. Thus, $U(t, \tau)\Lambda(\widehat{D}_0, \tau) \subset \Lambda(\widehat{D}_0, t)$.

Thereupon, we will prove that $\Lambda(\widehat{D}_0, t) \subset U(t, \tau)\Lambda(\widehat{D}_0, \tau)$. To do this, consider $y \in \Lambda(\widehat{D}_0, t)$ fixed. Then, there exists sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, such that

$$\lim_{n\to\infty} d_X(U(t, \tau_n)x_n, y) = 0. \tag{1.9}$$

Since $\tau_n \to -\infty$, there exists $n(\tau) \geq 1$ such that $\tau_n \leq \tau$ for all $n \geq n(\tau)$. Therefore, we have

$$U(t, \tau_n)x_n = U(t, \tau)U(\tau, \tau_n)x_n \quad \forall n \geq n(\tau). \tag{1.10}$$

Now, since the process U on X is pullback $\widehat{D}_0$-asymptotically compact, there exists subsequences $\{\tau_n\}_{n\geq n(\tau)}$ and $\{x_n\}_{n\geq n(\tau)}$ (relabeled the same), such that $U(\tau, \tau_n)x_n \to z \in \Lambda(\widehat{D}_0, \tau)$. From this, taking into account that the process U is closed, (1.9) and (1.10), it satisfies that $U(t, \tau)z = y$. Therefore, $\Lambda(\widehat{D}_0, t) \subset U(t, \tau)\Lambda(\widehat{D}_0, \tau)$. □

In what follows, consider a nonempty class $\mathcal{D}$ of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ is called a universe in $\mathcal{P}(X)$.

Then, we have the following definition.

Definition 1.7. *The family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists $\tau_0(\widehat{D}, t) < t$ such that $U(t,\tau)D(\tau) \subset D_0(t)$ for all $\tau \le \tau_0(\widehat{D}, t)$.*

Observe that in the above definition, $\widehat{D}_0$ does not necessarily belong to the class $\mathcal{D}$.

Proposition 1.8. *If the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the process U on X, then $\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t)$ for all $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$. Moreover, if $\widehat{D}_0 \in \mathcal{D}$, then $\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.*

Proof. Consider fixed $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$. If $\Lambda(\widehat{D}, t)$ is nonempty, then given $y \in \Lambda(\widehat{D}, t)$, from the sequential characterization of $\Lambda(\widehat{D}, t)$, there exist sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ with $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n, such that $U(t,\tau_n)x_n \to y$. On the other hand, since the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the process U, there exists $\tau_0(\widehat{D}, t) \le t$ such that $U(t,\tau)D(\tau) \subset D_0(t)$ for all $\tau \le \tau_0(\widehat{D}, t)$. Then, we deduce that there exists $n(\tau_0) \ge 1$ such that $U(t,\tau_n)x_n \in D_0(t)$ for all $n \ge n(\tau_0)$. Now, consider a subsequence of $\{\tau_n\}$ which satisfies that $\tau_{n_k} \le t-k$ and $y_{n_k} := U(t-k, \tau_{n_k})x_{n_k} \in D_0(t-k)$ for all $k \ge 1$. From this, taking into account that $U(t,t-k)y_{n_k} = U(t,\tau_{n_k})x_{n_k}$ for all $k \ge 1$, it fulfils that $y \in \Lambda(\widehat{D}_0, t)$. Therefore, $\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t)$ for all $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$.

Finally, if $\widehat{D}_0 \in \mathcal{D}$, we will prove that $\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

Consider $t \in \mathbb{R}$ fixed. If $\Lambda(\widehat{D}_0, t)$ is nonempty, given $y \in \Lambda(\widehat{D}_0, t)$, there exist $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$, such that $U(t,\tau_n)x_n \to y$. Now, taking into account that the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing, there exists $n_0 \in \mathbb{N}$ such that $U(t,\tau_n)D_0(\tau_n) \subset D_0(t)$ for all $n \ge n_0$. Therefore, $y \in \overline{D_0(t)}^X$. □

Now, we have the following definition.

Definition 1.9. *A process U on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$ (cf. Definition 1.3).*

Proposition 1.10. *If $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a pullback $\mathcal{D}$-absorbing family for the process U on X and the process U is pullback $\widehat{D}_0$-asymptotically compact, then the process U is also pullback $\mathcal{D}$-asymptotically compact.*

Proof. Consider fixed $t \in \mathbb{R}$, $\widehat{D} \in \mathcal{D}$, and the sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ such that $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$ for all n. Our aim is to prove that the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Since the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the process U on X, for any $k \ge 1$, there exist $\tau_{n_k} \in \{\tau_n\}$ such that $\tau_{n_k} \le t-k$ and $y_{n_k} := U(t-k, \tau_{n_k})x_{n_k} \in$

$D_0(t-k)$. Now, taking into account that U is pullback $\widehat{D}_0$-asymptotically compact, it satisfies that the sequence $\{U(t,t-k)y_{n_k}\}$ is relatively compact. Finally, since $U(t,t-k)y_{n_k} = U(t,\tau_{n_k})x_{n_k}$, then we have proved that the sequence $\{U(t,\tau_{n_k})x_{n_k}\}$ is relatively compact in X. $\square$

1.2 Existence and relationships between pullback attractors

Definition 1.11. *Consider a family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. Then, it is called the minimal pullback $\mathcal{D}$-attractor for the process U if the following properties are satisfied:*

(a) the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X for any $t \in \mathbb{R}$,

(b) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e.

$$\lim_{\tau\to-\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \quad \forall t \in \mathbb{R} \quad \forall \widehat{D} \in \mathcal{D},$$

(c) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \le t$,

(d) $\mathcal{A}_{\mathcal{D}}$ is minimal, i.e. if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets which is pullback $\mathcal{D}$-attracting, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

The uniqueness of the minimal pullback $\mathcal{D}$-attractor comes from its own definition (cf. (d)). See also Remark 1.14 (i).

Now, to prove the main result of this section, we need the following proposition, which is a consequence of Propositions 1.5 and 1.6.

Proposition 1.12. *Suppose that the process U on X is pullback $\mathcal{D}$-asymptotically compact and closed. Then, for any $\widehat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$, the set $\Lambda(\widehat{D},t)$ is a nonempty compact subset of X, which is invariant for U and attracts $\widehat{D}$ in the pullback sense. In addition, for each $\widehat{D} \in \mathcal{D}$, the family $\{\Lambda(\widehat{D},t) : t \in \mathbb{R}\}$ is minimal amongst all the pullback attracting families of closed sets.*

The following theorem guarantees the existence of the minimal pullback attractor.

Theorem 1.13. *Assume that $U : \mathbb{R}^2_d \times X \to X$ is a closed process, $\mathcal{D}$ is a universe in $\mathcal{P}(X)$ and $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a pullback $\mathcal{D}$-absorbing family for U. Moreover, suppose that U is pullback $\widehat{D}_0$-asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$, which is given by*

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D},t)}^X \quad \forall t \in \mathbb{R}, \tag{1.11}$$

is the minimal pullback $\mathcal{D}$-attractor for the process U. In addition, if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$.

Proof. Firstly, we will prove that $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact set and fulfils $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t)$ for all $t \in \mathbb{R}$. Consider $t \in \mathbb{R}$ fixed. Making use of Propositions 1.10 and 1.12, and taking into account (1.11), $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty closed set. In fact, thanks to Propositions 1.5 and 1.8, we have that $\mathcal{A}_{\mathcal{D}}(t)$ is compact and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t)$.

Moreover, the family $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting. Namely, from (1.11) we deduce

$$\mathrm{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) \leq \mathrm{dist}_X(U(t,\tau)D(\tau), \Lambda(\widehat{D}, t)),$$

for all $\widehat{D} \in \mathcal{D}$ and $(t,\tau) \in \mathbb{R}^2_d$. From this, taking into account Proposition 1.12, it holds that the family $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting.

Thereupon, the invariance of $\mathcal{A}_{\mathcal{D}}$ will be proved. Firstly, we will show that $\mathcal{A}_{\mathcal{D}}(t) \subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$ for all $(t,\tau) \in \mathbb{R}^2_d$. Consider fixed $(t,\tau) \in \mathbb{R}^2_d$ and $y \in \mathcal{A}_{\mathcal{D}}(t)$. From (1.11), we deduce that there exist two sequences $\{\widehat{D}_n\} \subset \mathcal{D}$ and $\{y_n\} \subset X$ such that $y_n \in \Lambda(\widehat{D}_n, t)$ for all n and $y_n \to y$. Since $\Lambda(\widehat{D}, t)$ is invariant (cf. Proposition 1.6), there exists a sequence $\{x_n\} \subset X$ with $x_n \in \Lambda(\widehat{D}_n, \tau) \subset \mathcal{A}_{\mathcal{D}}(\tau)$ such that $y_n = U(t,\tau)x_n$ for all n. Since $\mathcal{A}_{\mathcal{D}}(\tau)$ is a compact set and the process U is closed, $y \in U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$. Now, to prove that $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) \subset \mathcal{A}_{\mathcal{D}}(t)$, we fix $(t,\tau) \in \mathbb{R}^2_d$ and $y \in \mathcal{A}_{\mathcal{D}}(\tau)$. We will check that $U(t,\tau)y \in \mathcal{A}_{\mathcal{D}}(t)$. Since $y \in \mathcal{A}_{\mathcal{D}}(\tau)$, there exist two sequences $\{\widehat{D}_n\} \subset D$ and $\{y_n\} \subset X$ such that $y_n \in \Lambda(\widehat{D}_n, \tau)$ and $y_n \to y$. From this, taking into account that $\Lambda(\widehat{D}_n, t) = U(t,\tau)\Lambda(\widehat{D}_n, \tau)$ and $\mathcal{A}_{\mathcal{D}}(t)$ is compact, it holds that the sequence $\{x_n\} \subset X$, given by $x_n := U(t,\tau)y_n \in \mathcal{A}_{\mathcal{D}}(t)$ for all n, is relatively compact in X. Finally, bearing in mind that the process U is closed, we conclude that $U(t,\tau)y \in \mathcal{A}_{\mathcal{D}}(t)$.

The minimality of $\mathcal{A}_{\mathcal{D}}$ is due to Proposition 1.12 and (1.11). Namely, assume that $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), C(t)) = 0.$$

Consider fixed $t \in \mathbb{R}$ and $y \in \mathcal{A}_{\mathcal{D}}(t)$. From (1.11), we deduce that there exists two sequences $\{\widehat{D}_n\} \subset \mathcal{D}$ and $\{y_n\} \subset X$ such that $y_n \in \Lambda(\widehat{D}_n, t)$ for all n and $y_n \to y$. In fact, $\{y_n\} \subset C(t)$ (cf. Proposition 1.12). Then, $y \in C(t)$ and therefore, $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Finally, when $\widehat{D}_0 \in \mathcal{D}$, it holds that $\mathcal{A}_{\mathcal{D}}(t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$, thanks to (1.11) and Proposition 1.8. □

Remark 1.14. *(i) If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b) and (c) in Definition 1.11. A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$ and the universe $\mathcal{D}$ is inclusion-closed, that means that if $\widehat{D} \in \mathcal{D}$ and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ satisfies that $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, then $\widehat{D}' \in \mathcal{D}$.*

(ii) The universe of fixed nonempty bounded subsets of X is denoted by $\mathcal{D}^X_F$. Then, the corresponding minimal pullback $\mathcal{D}^X_F$-attractor for the process U is the attractor defined by Crauel, Debussche and Flandoli (cf. [55, Theorem 1.1]).

Now it is not difficult to conclude the following result (see [93, Proposition 23] for more details).

Corollary 1.15. *Under the assumptions of Theorem 1.13, if* $\mathcal{D}_F^X \subset \mathcal{D}$*, then the minimal pullback attractors* $\mathcal{A}_{\mathcal{D}_F^X}$ *and* $\mathcal{A}_{\mathcal{D}}$ *exist and* $\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ *for all* $t \in \mathbb{R}$*.*

Besides, if for some $T \in \mathbb{R}$*, the set* $\bigcup_{t\le T} D_0(t)$ *is a bounded subset of* X*, then* $\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t)$ *for all* $t \le T$*.*

Thanks to the following result, we can compare two attractors for a process (see [62, Theorem 3.15]).

Theorem 1.16. *Suppose that* $\{(X_i, d_{X_i})\}_{i=1,2}$ *are two metric spaces such that* $X_1 \subset X_2$ *with continuous injection,* $\mathcal{D}_i$ *is a universe in* $\mathcal{P}(X_i)$ *for* $i = 1, 2$*, and* $\mathcal{D}_1 \subset \mathcal{D}_2$*. Assume that* U *is a map that acts as a process in both cases, i.e.* $U : \mathbb{R}_d^2 \times X_i \to X_i$ *for* $i = 1, 2$ *is a process. For each* $t \in \mathbb{R}$*,*

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i} \quad i = 1, 2,$$

where the subscript i *in the symbol of the omega-limit set* Λ_i *is used to denote the dependence on the respective topology. Then,* $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ *for all* $t \in \mathbb{R}$*.*

If moreover

(i) $\mathcal{A}_1(t)$ *is a compact subset of* X_1 *for all* $t \in \mathbb{R}$*,*

(ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ *and* $t \in \mathbb{R}$*, there exist a family* $\widehat{D}_1 \in \mathcal{D}_1$ *and a* $t^*_{\widehat{D}_1}$ *such that* U *is pullback* $\widehat{D}_1$*-asymptotically compact, and for any* $s \le t^*_{\widehat{D}_1}$ *there exists a* $\tau_s < s$ *such that*

$$U(s,\tau)D_2(\tau) \subset D_1(s) \quad \forall \tau \le \tau_s,$$

then $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ *for all* $t \in \mathbb{R}$*.*

Proof. Consider $t \in \mathbb{R}$ fixed. From the sequential characterization of the omega-limit set, taking into account that $X_1 \subset X_2$ with continuous injection and $\mathcal{D}_1 \subset \mathcal{D}_2$, we deduce

$$\overline{\bigcup_{\widehat{D}_1 \in \mathcal{D}_1} \Lambda_1(\widehat{D}_1, t)}^{X_1} \subset \overline{\bigcup_{\widehat{D}_2 \in \mathcal{D}_2} \Lambda_2(\widehat{D}_2, t)}^{X_2}.$$

Therefore, taking into account (1.11), $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$.

To prove the opposite inclusion, fix $\widehat{D}_2 \in \mathcal{D}_2$ and $t \in \mathbb{R}$. Given $x \in \Lambda_2(\widehat{D}_2, t)$, from the sequential characterization of $\Lambda_2(\widehat{D}_2, t)$, we deduce that there exist two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X_2$, with $\tau_n \to -\infty$ and $x_n \in D_2(\tau_n)$ for all n, such that

$$\lim_{n\to\infty} \mathrm{dist}_{X_2}(U(t,\tau_n)x_n, x) = 0.$$

By the assumption (ii), there exist a $\widehat{D}_1 \in \mathcal{D}_1$ and an integer $k_{\widehat{D}_1} \ge 1$ such that U is pullback $\widehat{D}_1$-asymptotically compact and for any $k \ge k_{\widehat{D}_1}$ and there exist

$x_{n_k} \in \{x_n\}$ and $\tau_{n_k} \leq t-k$ such that $y_{n_k} = U(t-k,\tau_{n_k})x_{n_k} \in D_1(t-k)$. Then, since the process U is pullback $\widehat{D}_1$-asymptotically compact, there exist a subsequence of $\{y_{n_k}\}$ (relabeled the same) and $z \in \Lambda_1(\widehat{D}_1, t)$ such that

$$\lim_{n\to\infty} \operatorname{dist}_{X_1}(U(t,t-k)y_{n_k}, z) = 0.$$

Now, taking into account that $U(t,t-k)y_{n_k} = U(t,\tau_{n_k})x_{n_k}$ and $X_1 \subset X_2$ with continuous injection, $z = x$ holds. Therefore,

$$\bigcup_{\widehat{D}_2\in\mathcal{D}_2} \Lambda_2(\widehat{D}_2, t) \subset \bigcup_{\widehat{D}_1\in\mathcal{D}_1} \Lambda_1(\widehat{D}_1, t) \subset \mathcal{A}_1(t).$$

Finally, bearing in mind that thanks to the continuous injection $X_1 \subset X_2$, $\mathcal{A}_1(t)$ is not only compact in X_1, but also in X_2, $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ holds. □

1.3 Pullback flattening property

Thereupon, we recall some results about the pullback $\widehat{D}$-flattening property, a useful tool that will be very helpful to prove the pullback $\mathcal{D}$-asymptotic compactness. This notion, introduced by Ma, Wang and Zhong in [88], was called "Condition (C)". In [77], it was re-christened by Kloeden & Langa like flattening property. In both papers, it was necessary to assume the existence of a projection operator P_ε in order to prove this property. Later, in [33, Definition 2.24] and [65, Definition 8], the assumptions on P_ε were weakened and it does not need to be a projection operator anymore.

Definition 1.17. *Consider a Banach space X with norm $\|\cdot\|_X$ and a family $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. If for any $t \in \mathbb{R}$ and $\varepsilon > 0$, there exist $\tau_\varepsilon < t$, a finite-dimensional subspace X_ε of X and a map $P_\varepsilon : X \to X_\varepsilon$, all depending on $\widehat{D}$, t and ε, such that $\{P_\varepsilon U(t,\tau)u_\tau : \tau \leq \tau_\varepsilon, u_\tau \in D(\tau)\}$ is bounded in X and*

$$\|(I - P_\varepsilon)U(t,\tau)u_\tau\|_X < \varepsilon \quad \forall \tau \leq \tau_\varepsilon \quad \forall u_\tau \in D(\tau),$$

then the process U on X is said to satisfy the pullback $\widehat{D}$-flattening property.

The following result establishes a relationship between the pullback $\widehat{D}$-flattening property and the pullback $\widehat{D}$-asymptotic compactness. We show the proof for the sake of completeness (cf. [88, 77, 33, 65]).

Proposition 1.18. *If X is a Banach space and $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family such that the process U on X fulfils the pullback $\widehat{D}$-flattening property, then the process U is pullback $\widehat{D}$-asymptotically compact.*

Proof. Consider fixed $t \in \mathbb{R}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ such that $\tau_n \to -\infty$ and a sequence $\{x_n\}$ such that $x_n \in D(\tau_n)$ for all n. Our aim is to prove that the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

For each $k \geq 1$, making use of the pullback $\widehat{D}$-flattening property, there exists $N_k \geq 1$, a finite dimensional space of X denoted by X_k and an application $P_k : X \to X_k$ such that $\{P_k U(t, \tau_n)x_n : n \geq N_k\}$ is a bounded set of X_k. Therefore, it is a relatively compact subset of X. In addition, we also have

$$\|(I - P_k)U(t, \tau_n)x_n\|_X \leq \frac{1}{2k} \quad \forall n \geq N_k. \tag{1.12}$$

Then, the set $\{U(t, \tau_n)x_n : n \geq 1\}$ can be covered by a finite number of balls in X with radius $1/k$. Let us prove it.

Case 1: $n \geq N_k$**.**

Since the set $\{P_k U(t, \tau_n)x_n : n \geq N_k\}$ is relatively compact in X, we have

$$\{P_k U(t, \tau_n)x_n : n \geq N_k\} \subset \bigcup_{x \in I_k \subset X} B_X\left(x, \frac{1}{2k}\right), \tag{1.13}$$

where I_k is a finite dimensional subset of X.

Consider $n \geq N_k$ fixed. From (1.13), we deduce that there exists $x \in I_k$ such that

$$\|P_k U(t, \tau_n)x_n - x\|_X \leq \frac{1}{2k}.$$

Taking this into account together with (1.12), we deduce

$$\|U(t, \tau_n)x_n - x\|_X \leq \frac{1}{k}.$$

Therefore, $U(t, \tau_n)x_n \subset B_X\left(x, \frac{1}{k}\right)$ with $x \in I_k$.

Case 2: $n < N_k$**.**

This steps is immediate since

$$\{U(t, \tau_n)x_n : n < N_k\} \subset \bigcup_{n < N_k} B_X\left(U(t, \tau_n)x_n, \frac{1}{k}\right).$$

Now, bearing in mind that k is arbitrary and making use of a diagonal procedure, it is not difficult to check that the sequence $\{U(t, \tau_n)x_n : n \geq 1\}$ has a Cauchy subsequence in X. Then, since X is a Banach space, we have that the sequence $\{U(t, \tau_n)x_n : n \geq 1\}$ is relatively compact in X. □

Chapter 2

Non-autonomous nonlocal parabolic equation with sublinear terms

Over the last few decades the study of nonlocal problems has taken a keen interest (e.g., cf. [59, 19, 67, 109, 12] amongst many others), especially those of diffusion type (see e.g. [46, 87, 43, 41, 11]). Namely, many authors have analysed this kind of nonlocal parabolic equations

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f. \tag{2.1}$$

where $a \in C(\mathbb{R}; \mathbb{R}_+)$ and $l(u) = \int_\Omega g(x)u(x)dx$ with $g \in L^2(\Omega)$.

Prof. Chipot and his collaborators have studied the asymptotic behaviour of the solution of nonlocal evolution problems with uniqueness of solution similar to (2.1) considering mixed boundary conditions (cf. [35, 48]), different nonlocal terms (cf. [43, 45, 49]) and even they have analysed other types of nonlocal evolution equations like the nonlocal p-Laplacian equation (cf. [47]). To that end, different techniques have been applied such as dynamical systems (cf. [87, 45, 50]), energy functionals, global minimizers (cf. [47]) and Lyapunov functions (cf. [49]), which do not always exist (see [50] for more details). In addition, some results that establish order relationships among two stationary solutions and the long-time behaviour of the solution of the evolution problem have also been studied (cf. [44, 45, 35, 36, 37]).

In this chapter, we will analyse a non-autonomous nonlocal parabolic equation of the same type as (2.1). However, in this case f, which has sublinear growth, depends on the unknown u. First, we will show the existence and uniqueness of weak and strong solutions using the Galerkin approximations and compactness arguments. Later, the existence and uniqueness of stationary solution are analysed as well as its global exponential stability. Then, the existence of several pullback attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$ is shown. The proof of the asymptotic compactness, which is an essential ingredient to prove the existence of these families, is based on an energy method which relies on the continuity of solutions (e.g. cf. [62, 94, 92, 73]). In addition, we establish some relationships between the attractors.

The results of this chapter can be found in [21].

2.1 Statement of the problem. Existence results

Consider the following problem for a non-autonomous nonlocal parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases} \tag{2.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$, the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and there exists a positive constant m, such that

$$0 < m \leq a(s) \quad \forall s \in \mathbb{R}. \tag{2.3}$$

This condition of non-degeneracy of a is essential to guarantee the existence of solution not only in finite-time intervals (see [87] for more details).

In addition, we assume that $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, $f \in C(\mathbb{R})$ and there exist constants $\eta > 0$ and $C_f \geq 0$, such that

$$|f(s)| \leq C_f(1 + |s|) \quad \forall s \in \mathbb{R}, \tag{2.4}$$

$$(f(s) - f(r))(s - r) \leq \eta(s - r)^2 \quad \forall s, r \in \mathbb{R}. \tag{2.5}$$

Finally, $u_\tau \in L^2(\Omega)$ and the non-autonomous term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$.

From now on, we identify $L^2(\Omega)$ with its dual. Therefore, we have the usual chain of dense and compact embeddings $H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. Observe that thanks to the previous identification, $l(u)$ is in fact (l, u). However, we keep the usual notation in the existing previous literature $l(u)$ instead of (l, u) for the operator l acting on u.

Now we will show the existence and uniqueness of solutions.

Definition 2.1. *A weak solution to (2.2) is a function $u \in L^2(\tau, T; H^1_0(\Omega)) \cap L^\infty(\tau, T; L^2(\Omega))$ for all $T > \tau$, with $u(\tau) = u_\tau$, such that*

$$\frac{d}{dt}(u(t), v) + a(l(u(t)))((u(t), v)) = (f(u(t)), v) + \langle h(t), v\rangle \quad \forall v \in H^1_0(\Omega), \tag{2.6}$$

where the previous equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

Observe that if u is a weak solution to (2.2), the continuity the a, $l \in L^2(\Omega)$, (2.4) and (2.6) imply that $u' \in L^2(\tau, T; H^{-1}(\Omega))$ for any $T > \tau$. Therefore, $u \in C([\tau, \infty); L^2(\Omega))$ and the initial datum in (2.2) makes sense. Moreover, the following energy equality holds

$$|u(t)|_2^2 + 2\int_s^t a(l(u(r)))\|u(r)\|_2^2 dr = |u(s)|_2^2 + 2\int_s^t (f(u(r)), u(r))dr + 2\int_s^t \langle h(r), u(r)\rangle dr \tag{2.7}$$

for all $\tau \leq s \leq t$ (cf. [56, Théorème 2, p. 575] or [111, Lemma 3.2, p. 71]).

A notion of more regular solution is also suitable for the problem.

Definition 2.2. *A strong solution to (2.2) is a weak solution u which also satisfies that $u \in L^2(\tau, T; D(-\Delta)) \cap L^\infty(\tau, T; H_0^1(\Omega))$ for all $T > \tau$.*

When $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, as long as u is a strong solution to (2.2), we have that $u' \in L^2(\tau, T; L^2(\Omega))$ for all $T > \tau$. As a consequence, $u \in C([\tau, \infty); H_0^1(\Omega))$ and the following energy equality holds

$$\|u(t)\|_2^2 + 2\int_s^t a(l(u(r)))|-\Delta u(r)|_2^2 dr = \|u(s)\|_2^2 + 2\int_s^t (f(u(r)) + h(r), -\Delta u(r))dr \tag{2.8}$$

for all $\tau \le s \le t$.

To prove the existence of weak solution and strong solution to (2.2), we will use the Faedo-Galerkin approximations and pass to the limit by using compactness arguments. Using spectral theory, it holds that there exists a sequence $\{w_i\}_{i\ge 1}$, which is a Hilbert basis of $L^2(\Omega)$ composed by the eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$.

Firstly, we consider the function $u_n(t;\tau,u_\tau) = \sum_{j=1}^n \varphi_{nj}(t)w_j$ (for short denoted $u_n(t)$) for all $n \ge 1$, the unique local solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) + a(l(u_n(t)))((u_n(t), w_j)) = (f(u_n(t)), w_j) + \langle h(t), w_j\rangle, & t \in (\tau, \infty), \\ (u_n(\tau), w_j) = (u_\tau, w_j), & j = 1, \dots, n. \end{cases} \tag{2.9}$$

Observe that (2.9) is a Cauchy problem for the following ordinary differential system in $\mathbb{R}^n$

$$\varphi_{nj}'(t) + \lambda_j a(l(u_n(t)))\varphi_{nj}(t) = (f(u_n(t)), w_j) + \langle h(t), w_j\rangle, \quad j = 1, \dots, n, \tag{2.10}$$

where $t \ge \tau$, λ_j is the eigenvalue associated to the eigenfunction w_j and the vector $(\varphi_{n1}, \dots, \varphi_{nn})$ is the unknown.

Proposition 2.3. *Suppose $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (2.3), $f \in C(\mathbb{R})$ verifies (2.4), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, for each initial datum $u_\tau \in L^2(\Omega)$, there exists $(\varphi_{n1}, \dots, \varphi_{nn})$ local solution of the ordinary differential system (2.10) defined on some interval (τ, t_n). Furthermore, if the function a is locally Lipschitz and f satisfies (2.5), the uniqueness of local solution is guaranteed.*

Proof. We split the proof into two steps.

Step 1. Existence of local solution. To do this, we are going to use [52, Theorem 1.1, p. 43], which is a generalization of Peano's Theorem. We define

$$\begin{array}{rccl} g: & \mathcal{R} & \to & \mathbb{R}^n \\ & (t,x) & \mapsto & (-\lambda_1 z(x)x_1 + (f(\sum_{i=1}^n x_i w_i), w_1) + \langle h(t), w_1\rangle, \dots, \\ & & & -\lambda_n z(x)x_n + (f(\sum_{i=1}^n x_i w_i), w_n) + \langle h(t), w_n\rangle), \end{array}$$

where

$$\mathcal{R} = \{(t,x) \in [\tau, T] \times \mathbb{R}^n : \tau \le t \le T, \ |x - ((u_\tau, w_1), \dots, (u_\tau, w_n))| \le b\},$$

for any fixed $b \in \mathbb{R}^+$ and

$$x = (x_1, \ldots, x_n) \mapsto z(x) = a(l(\sum_{i=1}^{n} x_i w_i)). \tag{2.11}$$

In what follows, for simplicity we denote $\xi = ((u_\tau, w_1), \ldots, (u_\tau, w_n))$.

Firstly, we are going to prove that g is a Caratheodory function.
Consider x fixed. The function $g(\cdot, x)$ is measurable because

$$g_j(\cdot, x) = -\lambda_j z(x) x_j + (f(\sum_{i=1}^{n} x_i w_i), w_j) + \langle h(\cdot), w_j \rangle$$

is measurable function as a consequence of Fubini's Theorem.
Secondly, we need to check that the function $g(t, \cdot)$ is continuous a.e. $t \in [\tau, T]$. Indeed

$$g_j(t, x) = -\lambda_j z(x) x_j + (f(\sum_{i=1}^{n} x_i w_i), w_j) + \langle h(t), w_j \rangle$$

is a continuous function with respect to the second variable, because the functions z and $x \in \mathbb{R}^n \mapsto (f(\sum_{i=1}^{n} x_i w_i), w_j)$ are continuous.

Now we are going to prove that there exists a function $m \in L^1(\tau, T)$ such that

$$|g(t, x)| \le m(t) \quad \forall (t, x) \in \mathcal{R}.$$

From the definition of $\mathcal{R}$, we deduce

$$|x| \le b + |\xi| =: C_{\mathcal{R}},$$
$$|(f(\sum_{i=1}^{n} x_i w_i), w_j)| \le 2^{1/2} C_f |\Omega|^{1/2} |w_j|_2 + 2^{1/2} C_f C_{\mathcal{R}} (\sum_{i=1}^{n} |w_i|_2) |w_j|_2.$$

Observe that taking into account (2.11) and making use of the continuity of the function a in the compact interval $I' := [-|l|_2 C_{\mathcal{R}} \sum_{i=1}^{n} |w_i|_2, |l|_2 C_{\mathcal{R}} \sum_{i=1}^{n} |w_i|_2]$, there exists $M > 0$ such that

$$z(x) \le M \quad \forall x \in \mathbb{R}^n : |x| \le C_{\mathcal{R}}.$$

Then, bearing this in mind, we deduce

$$\begin{aligned} &|g_j(t, x)| \\ &\le \lambda_j z(x) |x_j| + |(f(\sum_{i=1}^{n} x_i w_i), w_j)| + |\langle h(t), w_j \rangle| \\ &\le \lambda_j M C_{\mathcal{R}} + 2^{1/2} C_f |\Omega|^{1/2} |w_j|_2 + 2^{1/2} C_f C_{\mathcal{R}} (\sum_{i=1}^{n} |w_i|_2) |w_j|_2 + |\langle h(t), w_j \rangle|. \end{aligned}$$

Therefore,

$$
\begin{aligned}
m = & \sum_{j=1}^{n} \lambda_j M C_{\mathcal{R}} + \sum_{j=1}^{n} 2^{1/2} C_f [|\Omega|^{1/2} + C_{\mathcal{R}}(\sum_{i=1}^{n} |w_i|_2)] |w_j|_2 \\
& + \sum_{j=1}^{n} |\langle h(\cdot), w_j \rangle| \in L^1(\tau, T).
\end{aligned}
$$

In conclusion, there exists a local solution to (2.10).

Step 2. Uniqueness of local solution. Since the function a is locally Lipschitz, for any bounded interval $[-R, R]$ of $\mathbb{R}$, there exists a positive constant $L_a(R)$ such that

$$
|a(x) - a(y)| \leq L_a(R)|x - y| \quad \forall x, y \in [-R, R].
$$

Assume that there exist two solutions φ_n^1, φ_n^2 of the ordinary differential system (2.10) in (τ, t_1) and (τ, t_2) respectively. Then, it holds

$$
\begin{cases}
(\varphi_{nj}^1(t) - \varphi_{nj}^2(t))' = g_j(t, \varphi_n^1(t)) - g_j(t, \varphi_n^2(t)), \ t \in (\tau, \min\{t_1, t_2\}), \\
\quad (\varphi_{nj}^1 - \varphi_{nj}^2)(\tau) = 0, \qquad\qquad\qquad\qquad\qquad j = 1, \ldots, n,
\end{cases}
\tag{2.12}
$$

where $g_j(t, \varphi_n^i(t)) = -\lambda_j a(l(u_n^i(t)))\varphi_{n_j}^i(t) + (f(u_n^i(t)), w_j) + \langle h(t), w_j \rangle$ for $i = 1, 2$.

Then, multiplying (2.12) by $\varphi_{nj}^1 - \varphi_{nj}^2$, summing from $j = 1$ to n and making use of (2.5), we obtain

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|u_n^1(t) - u_n^2(t)|_2^2 + a(l(u_n^1(t)))\|u_n^1(t) - u_n^2(t)\|_2^2 \\
\leq & \ |a(l(u_n^1(t))) - a(l(u_n^2(t)))||((u_n^2(t), u_n^1(t) - u_n^2(t)))| + \eta |u_n^1(t) - u_n^2(t)|_2^2.
\end{aligned}
$$

Since $u_n^1, u_n^2 \in C([\tau, \min\{t_1, t_2\}]; L^2(\Omega))$, it fulfils that $u_n^1(t), u_n^2(t) \in S$ for all $t \in [\tau, \min\{t_1, t_2\}]$, where S is a bounded set of $L^2(\Omega)$. In addition, taking into account that $l \in L^2(\Omega)$, it satisfies that $\{l(u_n^i(t))\}_{t \in [\tau, \min\{t_1, t_2\}]} \in [-R, R]$ for $i = 1, 2$, for some $R > 0$. Hence, using (2.3), (2.5) and the fact that the function a is locally Lipschitz (with Liptschitz constant $L_a(\cdot)$), we deduce

$$
\frac{d}{dt}|u_n^1(t) - u_n^2(t)|_2^2 \leq C|u_n^1(t) - u_n^2(t)|_2^2,
$$

where

$$
C = \frac{(L_a(R))^2 |l|_2^2 \lambda_n C_{\mathcal{R}}^2 + 4m\eta}{2m}.
$$

Then, using the Gronwall Lemma, we have

$$
|u_n^1(t) - u_n^2(t)|_2^2 \leq |u_n^1(\tau) - u_n^2(\tau)|_2^2 e^{C(t-\tau)}.
$$

□

Now, we will show the existence and uniqueness of weak solutions and the continuity of the solution in $L^2(\Omega)$ with respect to the initial data.

Theorem 2.4. *Suppose that the function a is locally Lipschitz and fulfils (2.3), $f \in C(\mathbb{R})$ satisfies (2.4) and (2.5), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, for each initial datum $u_\tau \in L^2(\Omega)$, there exists a weak solution to the problem (2.2). In addition, this solution behaves continuously in $L^2(\Omega)$ w.r.t. the initial data.*

Proof. We split the proof into two steps.

Step 1. Existence of weak solution. Multiplying by $\varphi_{nj}(t)$ in (2.9), summing from $j = 1$ to n and using (2.3), we obtain

$$\frac{d}{dt}|u_n(t)|_2^2 + 2m\|u_n(t)\|_2^2 \leq 2(f(u_n(t)), u_n(t)) + 2\langle h(t), u_n(t)\rangle \quad \text{a.e. } t \in (\tau, t_n),$$

where (τ, t_n) is the interval of existence of maximal solution. By the Cauchy inequality (cf. [57, Appendix B, p. 622]) and (2.4),

$$\frac{d}{dt}|u_n(t)|_2^2 + m\|u_n(t)\|_2^2 \leq \frac{4C_f^2|\Omega|}{\lambda_1 m} + \frac{4C_f^2}{\lambda_1 m}|u_n(t)|_2^2 + \frac{2}{m}\|h(t)\|_*^2 \quad \text{a.e. } t \in (\tau, t_n),$$

where λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions.

Integrating between τ and t with $\tau < t < t_n$, we obtain

$$\begin{aligned}
&|u_n(t)|_2^2 + m\int_\tau^t \|u_n(s)\|_2^2 ds \\
&\leq |u_\tau|_2^2 + \frac{4C_f^2|\Omega|(T-\tau)}{\lambda_1 m} + \frac{4C_f^2}{\lambda_1 m}\int_\tau^T |u_n(s)|_2^2 ds + \frac{2}{m}\int_\tau^T \|h(s)\|_*^2 ds.
\end{aligned}$$

Therefore, the Gronwall lemma implies that $\{u_n\}$ is well defined for all time $t \geq \tau$, and actually bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega))$ for all $T > \tau$. Thus, taking this into account together with the fact that each $u_n \in C([\tau, T]; L^2(\Omega))$, we deduce that there exists a positive constant C_∞ such that

$$|u_n(t)|_2 \leq C_\infty \quad \forall t \in [\tau, T] \quad \forall n \geq 1.$$

From this, bearing in mind that $a \in C(\mathbb{R}; \mathbb{R}_+)$ and $l \in L^2(\Omega)$, it satisfies that there exists a positive constant $M_{C_\infty} > 0$ such that

$$a(l(u_n(t))) \leq M_{C_\infty} \quad \forall t \in [\tau, T] \quad \forall n \geq 1.$$

Now, we have

$$\int_\tau^T |a(l(u_n(t)))|^2 \| -\Delta u_n(t)\|_*^2 dt \leq (M_{C_\infty})^2 \int_\tau^T \|u_n(t)\|_2^2 dt. \tag{2.13}$$

Taking into account that $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$, we deduce that the sequence $\{-a(l(u_n))\Delta u_n\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$.

On the other hand, using (2.4), we have

$$\int_\tau^T \int_\Omega |f(u_n(x,t))|^2 dxdt \leq \int_\tau^T \int_\Omega 2C_f^2(1+|u_n(x,t)|^2)dxdt$$
$$\leq 2C_f^2|\Omega|(T-\tau) + 2C_f^2 \int_\tau^T |u_n(t)|_2^2 dt. \tag{2.14}$$

Now, using that $\{u_n\}$ is bounded in $L^\infty(\tau,T;L^2(\Omega))$, we have that $\{f(u_n)\}$ is bounded in $L^2(\tau,T;L^2(\Omega))$.

To prove that the sequence $\{u_n'\}$ is bounded in $L^2(\tau,T;H^{-1}(\Omega))$, we need first to define the following projectors:

$$\begin{array}{rccl} \widetilde{P}_n: & H^{-1}(\Omega) & \longrightarrow & H^{-1}(\Omega) \\ & f & \longmapsto & [\phi \in H_0^1(\Omega) \mapsto \langle \widetilde{P}_n f, \phi\rangle := \langle f, P_n\phi\rangle], \end{array}$$

where

$$\begin{array}{rccl} P_n: & L^2(\Omega) & \longrightarrow & V_n := \text{span}[w_1,\ldots,w_n] \\ & \phi & \longmapsto & \sum_{j=1}^n (\phi, w_j)w_j. \end{array}$$

Observe that $\widetilde{P}_n$ is the continuous extension in $H^{-1}(\Omega)$ of P_n. Then, in what follows, we will make an abuse of notation and denote this projection by P_n.

Bearing in mind (2.13), (2.14) and the definitions of the above projectors, we have that the sequence $\{u_n'\}$ is bounded in $L^2(\tau,T;H^{-1}(\Omega))$, since

$$\begin{aligned} &\int_\tau^T \|u_n'(t)\|_*^2 dt \\ =& \int_\tau^T \|a(l(u_n(t)))\Delta u_n(t) + P_n f(u_n(t)) + P_n h(t)\|_*^2 dt \\ \leq& \, 3(M_{C_\infty})^2 \int_\tau^T \|u_n(t)\|_2^2 dt + \frac{3}{\lambda_1}\int_\tau^T |P_n f(u_n(t))|_2^2 dt + 3\int_\tau^T \|P_n h(t)\|_*^2 dt \\ \leq& \, (3(M_{C_\infty})^2 + 6C_f^2\lambda_1^{-1}) \int_\tau^T \|u_n(t)\|_2^2 dt + 6C_f^2\lambda_1^{-1}|\Omega|(T-\tau) + 3\int_\tau^T \|h(t)\|_*^2 dt. \end{aligned}$$

Therefore, making use of compactness arguments and the Aubin-Lions lemma, there exist a subsequence of $\{u_n\}$ (relabeled the same) and $u \in L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega))$ with $u' \in L^2(\tau,T;H^{-1}(\Omega))$, such that

$$\left\{ \begin{array}{rl} u_n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)), \\ u_n' \rightharpoonup u' & \text{weakly in } L^2(\tau,T;H^{-1}(\Omega)), \\ u_n \to u & \text{strongly in } L^2(\tau,T;L^2(\Omega)), \\ u_n(x,t) \to u(x,t) & \text{a.e. } (x,t) \in \Omega\times(\tau,T), \\ u_n(t) \to u(t) & \text{strongly in } L^2(\Omega) \text{ a.e. } t\in(\tau,T), \\ f(u_n) \rightharpoonup \xi_1 & \text{weakly in } L^2(\tau,T;L^2(\Omega)), \\ a(l(u_n))u_n \rightharpoonup \xi_2 & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)), \end{array}\right. \tag{2.15}$$

for all $T > \tau$.

Now, we need to check that $\xi_1 = f(u)$ and $\xi_2 = a(l(u))u$. Since u_n converges to u strongly in $L^2(\Omega)$, we deduce that

$$u_n(x,t) \to u(x,t) \quad \forall (x,t) \in \Omega \times (\tau, T)\backslash N_1, \tag{2.16}$$

$$u_n(t) \to u(t) \quad \text{strongly in } L^2(\Omega) \quad \forall t \in (\tau, T)\backslash N_2, \tag{2.17}$$

where N_1 is a null set in $\mathbb{R}^{N+1}$ and N_2 is a null set in $\mathbb{R}$.

From this, we can deduce that $\xi_1 = f(u)$ applying [85, Lemme 1.3, p. 12], since $f \in C(\mathbb{R})$, $\{f(u_n)\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$ and converges pointwisely to $f(u)$ a.e. $\Omega \times (\tau, T)$ making use of (2.16).

Finally, we will prove that $\xi_2 = a(l(u))u$. As $a \in C(\mathbb{R}; \mathbb{R}_+)$, $l \in L^2(\Omega)$ and (2.17) holds, it satisfies

$$a(l(u_n(t))) \to a(l(u(t))) \quad \forall t \in (\tau, T)\backslash N_2.$$

Therefore,

$$a(l(u_n(t)))u_n(x,t) \to a(l(u(t)))u(x,t) \quad \forall (x,t) \in \Omega \times (\tau, T)\backslash (N_1 \cup (\Omega \times N_2)),$$

where $N_1 \cup (\Omega \times N_2)$ is a null set in $\mathbb{R}^{N+1}$. In addition, $\{a(l(u_n))u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$. Then, applying again [85, Lemme 1.3, p. 12], $\xi_2 = a(l(u))u$ follows. Now, passing to the limit in (2.9), taking into account (2.15) together with the fact that $\cup_{n\in\mathbb{N}} V_n$ is dense in $H_0^1(\Omega)$, (2.6) holds for all $v \in H_0^1(\Omega)$. Therefore, to prove that u is a weak solution to (2.2), we only need to check that $u(\tau) = u_\tau$, which makes complete sense since $u \in C([\tau, T]; L^2(\Omega))$.

On the one hand, consider fixed n, $\varphi \in H^1(\tau, T)$ with $\varphi(T) = 0$ and $\varphi(\tau) \neq 0$, and $w \in V_n$. Multiplying by φ in (2.9), integrating between τ and T, we obtain for all $\mu > n$

$$\begin{aligned} &-(u_\tau, w)\varphi(\tau) - \int_\tau^T \varphi'(t)(u_\mu(t), w)dt + \int_\tau^T a(l(u_\mu(t)))\langle -\Delta u_\mu(t), w\rangle \varphi(t)dt \\ &= \int_\tau^T (f(u_\mu(t)), w)\varphi(t)dt + \int_\tau^T \langle h(t), w\rangle \varphi(t)dt. \end{aligned} \tag{2.18}$$

Taking limit when $\mu \to \infty$ and using (2.15), we deduce

$$\begin{aligned} &-(u_\tau, w)\varphi(\tau) - \int_\tau^T \varphi'(t)(u(t), w)dt + \int_\tau^T a(l(u(t)))\langle -\Delta u(t), w\rangle \varphi(t)dt \\ &= \int_\tau^T (f(u(t)), w)\varphi(t)dt + \int_\tau^T \langle h(t), w\rangle \varphi(t)dt. \end{aligned} \tag{2.19}$$

On the other hand, multiplying by φ in (2.6) and integrating between τ and T, we obtain

$$\begin{aligned} &-(u(\tau), w)\varphi(\tau) - \int_\tau^T \varphi'(t)(u(t), w)dt + \int_\tau^T a(l(u(t)))\langle -\Delta u(t), w\rangle \varphi(t)dt \\ &= \int_\tau^T (f(u(t)), w)\varphi(t)dt + \int_\tau^T \langle h(t), w\rangle \varphi(t)dt. \end{aligned}$$

Comparing (2.19) with the above expression, we have $(u(\tau), w)\varphi(\tau) = (u_\tau, w)\varphi(\tau)$. As $\varphi(\tau) \neq 0$ and $\{w_j\}$ is a Hilbert basis of $L^2(\Omega)$, we conclude $u(\tau) = u_\tau$.

Step 2. Uniqueness of solution and continuity w.r.t. initial data. Assume that there exist two weak solutions, $u_1(\cdot;\tau,u_\tau^1)$ and $u_2(\cdot;\tau,u_\tau^2)$, to (2.2). For short, we will denote $u_i(\cdot) = u_i(\cdot;\tau,u_\tau^i)$ for $i = 1,2$. From the energy equality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_1(t)-u_2(t)|_2^2+a(l(u_1(t)))\|u_1(t)-u_2(t)\|_2^2\\ &=[a(l(u_2(t)))-a(l(u_1(t)))]((u_2(t),u_1(t)-u_2(t)))+(f(u_1(t))-f(u_2(t)),u_1(t)-u_2(t))\end{aligned}$$

a.e. $t \in [\tau, T]$.

Since $u_1, u_2 \in C([\tau,T];L^2(\Omega))$, there exists a bounded set $S \subset L^2(\Omega)$ such that $\{u_i(t)\}_{t\in[\tau,T]} \subset S$ for $i = 1,2$. Besides, taking into account that $l \in L^2(\Omega)$, there exists a constant $R > 0$ such that $\{l(u_i(t))\}_{t\in[\tau,T]} \subset [-R,R]$ for $i = 1,2$. Then, making use of (2.3), (2.5) and the locally Lipschitz continuity of the function a, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_1(t)-u_2(t)|_2^2 + m\|u_1(t)-u_2(t)\|_2^2\\ &\leq L_a(R)|l|_2|u_1(t)-u_2(t)|_2\|u_2(t)\|_2\|u_1(t)-u_2(t)\|_2 + \eta|u_1(t)-u_2(t)|_2^2,\end{aligned}$$

where $L_a(R)$ denotes the Lipschitz constant of the function a in $[-R,R]$.

Now, applying the Cauchy inequality to the above expression, we have

$$\frac{d}{dt}|u_1(t)-u_2(t)|_2^2 \leq C(t)|u_1(t)-u_2(t)|_2^2 \quad \text{a.e. } t \in (\tau,T)$$

where

$$C(t) = \frac{(L_a(R))^2|l|_2^2\|u_2(t)\|_2^2 + 2m\eta}{m}.$$

Thus, we deduce

$$|u_1(t)-u_2(t)|_2^2 \leq |u_\tau^1 - u_\tau^2|_2^2 e^{\int_\tau^t C(s)ds} \quad \forall t \in [\tau,T].$$

Both results, the uniqueness of solution to (2.2) and the continuity w.r.t. the initial data, follow immediately. □

In the following result, we will study the regularising effect of the equation. In addition, taking a more regular initial datum, the existence of a strong solution will be analysed.

Theorem 2.5. *Under the assumptions of Theorem 2.4, if $h \in L^2_{loc}(\mathbb{R};L^2(\Omega))$, for every $\varepsilon > 0$ and $T > \tau + \varepsilon$, the weak solution u belongs to $C((\tau,T],H_0^1(\Omega))\cap L^2(\tau+\varepsilon,T;D(-\Delta))$. In fact, if the initial condition $u_\tau \in H_0^1(\Omega)$, then the function $u \in C([\tau,T],H_0^1(\Omega)) \cap L^2(\tau,T;D(-\Delta))$ for every $T > \tau$.*

Proof. We split the proof into two steps.

Step 1. Regularising effect. Multiplying by $\lambda_j \varphi_{nj}(t)$ in (2.9), summing from $j=1$ to n, and using (2.3), (2.4) and the Cauchy and Poincaré inequalities, it yields

$$\frac{d}{dt}\|u_n(t)\|_2^2 + m|-\Delta u_n(t)|_2^2 \le \frac{4C_f^2|\Omega|}{m} + \frac{4C_f^2}{\lambda_1 m}\|u_n(t)\|_2^2 + \frac{2}{m}|h(t)|_2^2 \quad \text{a.e. } t \ge \tau.$$

Integrating between s and t with $\tau < s \le t \le T$, we obtain

$$\begin{aligned} &\|u_n(t)\|_2^2 + m\int_s^t |-\Delta u_n(r)|_2^2 dr \\ &\le \frac{4C_f^2|\Omega|(T-\tau)}{m} + \frac{4C_f^2}{\lambda_1 m}\int_\tau^T \|u_n(r)\|_2^2 dr + \frac{2}{m}\int_\tau^T |h(r)|_2^2 dr + \|u_n(s)\|_2^2. \end{aligned} \tag{2.20}$$

Now, integrating in s between τ and t, we deduce

$$\begin{aligned} (t-\tau)\|u_n(t)\|_2^2 \le \frac{4C_f^2|\Omega|(T-\tau)^2}{m} &+ \left(\frac{4C_f^2(T-\tau)}{\lambda_1 m} + 1\right)\int_\tau^T \|u_n(r)\|_2^2 dr \\ &+ \frac{2(T-\tau)}{m}\int_\tau^T |h(r)|_2^2 dr. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_n(t)\|_2^2 \le \frac{4C_f^2|\Omega|(T-\tau)^2}{\varepsilon m} &+ \left(\frac{4C_f^2(T-\tau) + \lambda_1 m}{\varepsilon \lambda_1 m}\right)\int_\tau^T \|u_n(r)\|_2^2 dr \\ &+ \frac{2(T-\tau)}{\varepsilon m}\int_\tau^T |h(r)|_2^2 dr, \end{aligned}$$

for all $t \in [\varepsilon+\tau, T]$ with $\varepsilon \in (0, T-\tau)$.

From this and taking into account the boundedness of $\{u_n\}$ in $L^2(\tau,T;H_0^1(\Omega))$ (cf. Theorem 2.4), we deduce that $\{u_n\}$ is bounded in $L^\infty(\varepsilon+\tau,T;H_0^1(\Omega))$. As a byproduct, the boundedness of $\{u_n\}$ in $L^2(\tau+\varepsilon,T;D(-\Delta))$ is immediate just taking $s=\varepsilon$ and $t=T$ in (2.20). In addition, making use of this more regular boundedness, we deduce that the sequence $\{u_n'\}$ is bounded in $L^2(\tau+\varepsilon,T;L^2(\Omega))$. Thanks to the uniqueness of the weak solution, u_n converge to u weakly in $L^2(\tau+\varepsilon,T;D(-\Delta))$ and u_n' converge to u' weakly in $L^2(\tau+\varepsilon,T;L^2(\Omega))$. As a consequence, $u \in L^2(\tau+\varepsilon,T;D(-\Delta)) \cap C((\tau,T];H_0^1(\Omega))$.

Step 2. Strong solution. In this step if $u_\tau \in H_0^1(\Omega)$, we will show that $u \in L^2(\tau,T;D(-\Delta)) \cap C([\tau,T];H_0^1(\Omega))$ for all $T > \tau$. To that end, we multiply by $\lambda_j\varphi_{nj}$ in (2.9), sum from $j=1$ to n and use (2.3), obtaining

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2 + m|-\Delta u_n(t)|_2^2 \le (f(u_n(t)), -\Delta u_n(t)) + (h(t), -\Delta u_n(t)) \tag{2.21}$$

a.e. $t \in (\tau,T)$.

Now, from (2.4) and the Cauchy inequality, we deduce

$$\begin{aligned}
(f(u_n(t)), -\Delta u_n(t)) &\leq \frac{1}{m}|f(u_n(t))|_2^2 + \frac{m}{4}|-\Delta u_n(t)|_2^2 \\
&\leq \frac{2C_f^2|\Omega|}{m} + \frac{2C_f^2}{m}|u_n(t)|_2^2 + \frac{m}{4}|-\Delta u_n(t)|_2^2, \\
(h(t), -\Delta u_n(t)) &\leq \frac{1}{m}|h(t)|_2^2 + \frac{m}{4}|-\Delta u_n(t)|_2^2.
\end{aligned}$$

Taking this into account, from (2.21), we deduce that

$$\frac{d}{dt}\|u_n(t)\|_2^2 + m|-\Delta u_n(t)|_2^2 \leq \frac{4C_f^2}{\lambda_1 m}\|u_n(t)\|_2^2 + \frac{4C_f^2|\Omega|}{m} + \frac{2}{m}|h(t)|_2^2$$

a.e. $t > \tau$.

Integrating between τ and $t \in [\tau, T]$, we have

$$\begin{aligned}
&\|u_n(t)\|_2^2 + m\int_\tau^t |-\Delta u_n(s)|_2^2 ds \\
&\leq \|u_\tau\|_2^2 + \frac{4C_f^2}{\lambda_1 m}\int_\tau^T \|u_n(s)\|_2^2 ds + \frac{4C_f^2|\Omega|(T-\tau)}{m} + \frac{2}{m}\int_\tau^T |h(s)|_2^2 ds.
\end{aligned}$$

Taking into account that $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$ (cf. Theorem 2.4), we deduce that $\{u_n\}$ is bounded in $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$. As a result, the sequence $\{u_n'\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$. Then, thanks to the uniqueness of a weak solution, it holds that u_n converge to u weakly-star in $L^\infty(\tau, T; H_0^1(\Omega))$ and weakly in $L^2(\tau, T; D(-\Delta))$, and u_n' converge to u' weakly in $L^2(\tau, T; L^2(\Omega))$. Therefore, since $u \in L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$ and $u' \in L^2(\tau, T; L^2(\Omega))$, we obtain that $u \in C([\tau, T]; H_0^1(\Omega))$. □

2.2 Analysis of the stationary problem

In this section we study the elliptic problem

$$\begin{cases} -a(l(u))\Delta u = f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

where the functions a and f are globally Lipschitz, with respective Lipschitz constants L_a, $L_f \geq 0$ and there exists a positive constant $M > 0$ such that

$$0 < m \leq a(s) \leq M \quad \forall s \in \mathbb{R}. \tag{2.23}$$

In addition, h is time-independent, i.e. $h \in H^{-1}(\Omega)$.

We analyse the existence of solutions to (2.22) making use of a corollary of the Brouwer fixed point theorem. The uniqueness as well as the global exponential stability are also studied under suitable assumptions.

Definition 2.6. *A solution to (2.22) is a function $u^* \in H_0^1(\Omega)$ such that*

$$a(l(u^*))((u^*, v)) = (f(u^*), v) + \langle h, v\rangle \quad \forall v \in H_0^1(\Omega).$$

In the following result we analyse the existence, uniqueness and regularity of the stationary solutions of the problem (2.2) (the idea of the proof is close to that in [91]).

Theorem 2.7. *Assume that a and f are globally Lipschitz functions, with Lipschitz constants L_a and L_f respectively, (2.23) is satisfied, $h \in H^{-1}(\Omega)$, $l \in L^2(\Omega)$ and $m > \lambda_1^{-1} L_f$. Then:*

1. *There exists at least one solution to (2.22). In addition, any solution u^* to (2.22) fulfils*
$$\|u^*\|_2 \leq \Upsilon := \frac{\lambda_1^{-1/2}|\Omega|^{1/2}|f(0)| + \|h\|_*}{m - \lambda_1^{-1}L_f}. \tag{2.24}$$
Further, if $h \in L^2(\Omega)$, then the solutions given above belong in fact to $D(-\Delta)$.

2. *Under the additional assumption*
$$\lambda_1^{-1/2}|l|_2 L_a \Upsilon < m - \lambda_1^{-1}L_f, \tag{2.25}$$
problem (2.22) possesses a unique solution.

Proof. We split the proof into four steps.

Step 1. Existence. Let us consider the orthonormal Hilbert basis $\{w_j : j \geq 1\}$ of $L^2(\Omega)$ consisting of the eigenvectors associated with eigenvalues $\{\lambda_j : j \geq 1\}$ of the operator $-\Delta$ with zero Dirichlet boundary condition in Ω. For each $n \geq 1$, let us denote $V_n = \text{span}[w_1, \ldots, w_n]$, with the inner product $((\cdot,\cdot))$ and norm $\|\cdot\|_2$.

Now, the operators $R_n : V_n \to V_n$ for all $n \geq 1$ are defined as follows

$$((R_n u, v)) = \langle -a(l(u))\Delta u, v\rangle - (f(u), v) - \langle h, v\rangle \quad \forall u, v \in V_n.$$

Observe that each $R_n u \in V_n$ is well defined thanks to the Riesz Theorem, since the right hand side is a continuous linear map from V_n to $\mathbb{R}$. In addition, R_n is continuous. Namely, making use of (2.23), the Poincaré inequality and the Lipschitz continuity of the functions a and f, we deduce

$$\begin{aligned}
&((R_n u - R_n\tilde{u}, v)) \\
&= \langle -a(l(u))\Delta u + a(l(\tilde{u}))\Delta\tilde{u} - f(u) + f(\tilde{u}), v\rangle \\
&= \langle a(l(u))(-\Delta(u-\tilde{u}) + (a(l(\tilde{u})) - a(l(u)))\Delta\tilde{u}, v\rangle + (f(\tilde{u}) - f(u), v) \\
&\quad \leq (M + L_a|l|_2\lambda_1^{-1/2}\|\tilde{u}\|_2 + L_f\lambda_1^{-1})\|\tilde{u} - u\|_2\|v\|_2,
\end{aligned}$$

for all $u, \tilde{u}, v \in V_n$. Therefore,

$$\|R_n u - R_n\tilde{u}\|_2 \leq (M + L_a|l|_2\lambda_1^{-1/2}\|\tilde{u}\|_2 + L_f\lambda_1^{-1})\|\tilde{u} - u\|_2.$$

for all $u, \tilde{u} \in V_n$.

On the other hand, making use again of (2.23), the Poincaré inequality and the global Liptchiz continuity of f, we have

$$\begin{aligned}((R_n u, u)) &= \langle -a(l(u))\Delta u, u\rangle - (f(u), u) - \langle h, u\rangle \\ &= \langle -a(l(u))\Delta u, u\rangle - (f(u) - f(0), u) - (f(0), u) - \langle h, u\rangle \\ &\geq m\|u\|_2^2 - L_f \lambda_1^{-1}\|u\|_2^2 - |f(0)||\Omega|^{1/2}\lambda_1^{-1/2}\|u\|_2 - \|h\|_*\|u\|_2,\end{aligned}$$

for all $u \in V_n$.

Therefore, taking

$$\Upsilon := \frac{\lambda_1^{-1/2}|\Omega|^{1/2}|f(0)| + \|h\|_*}{m - \lambda_1^{-1}L_f},$$

we obtain

$$((R_n u, u)) \geq 0 \quad \forall u \in V_n / \|u\|_2 = \Upsilon.$$

Now, making use of a corollary of the Brouwer fixed point theorem (see [85, Lemme 4.3, p.53]), we deduce that for each $n \geq 1$ there exists $u_n \in V_n$ such that $R_n(u_n) = 0$, with

$$\|u_n\|_2 \leq \Upsilon. \tag{2.26}$$

Therefore, it verifies

$$\langle -a(l(u_n))\Delta u_n, v\rangle = (f(u_n), v) + \langle h, v\rangle \quad \forall v \in V_n. \tag{2.27}$$

Now, using the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$ by Υ and the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we can extract a subsequence $\{u_{n_k}\} \subset \{u_n\}$ which fulfils

$$\begin{cases} u_{n_k} \rightharpoonup u^* & \text{weakly in } H_0^1(\Omega), \\ u_{n_k} \to u^* & \text{strongly in } L^2(\Omega), \end{cases}$$

where $u^* \in H_0^1(\Omega)$ is a solution to (2.22). To check that, just take limit in (2.27) and make use of the assumptions made on a, l and f. In addition, observe that u^* fulfils (2.26).

Step 2. The a priori estimate (2.24). So far, we only have proven that there exists at least a solution to problem (2.22) and u^* verifies (2.24). But this does not imply that any u_* solution to problem (2.22) fulfils (2.24) since the uniqueness of solution is not guaranteed. Therefore, let us prove that any u_* solution to problem (2.22) verifies (2.24).

Consider fixed u_* a solution to (2.22). It holds

$$m\|u_*\|_2^2 \leq |f(u_*) - f(0)|_2|u_*|_2 + |f(0)||\Omega|^{1/2}|u_*|_2 + \|h\|_*\|u_*\|_2.$$

Using that the function f is globally Lipschitz and the Poincaré inequality, we obtain

$$m\|u_*\|_2^2 \leq L_f\lambda_1^{-1}\|u_*\|_2^2 + |f(0)||\Omega|^{1/2}\lambda_1^{-1/2}\|u_*\|_2 + \|h\|_*\|u_*\|_2.$$

Therefore, u_* satisfies (2.24).

Step 3. Regularity. Now, we will check that if $h \in L^2(\Omega)$, any solution u_* to (2.22) belong to $D(-\Delta)$. In what follows, we represent $u_n^* = P_n u_* := \sum_{i=1}^n (u_*, w_i) w_i$.

Since u_* is a solution to (2.22), taking $v = -\Delta u_n^*$ in Definition 2.6, we deduce

$$a(l(u_*))|-\Delta u_n^*|_2^2 = (f(u_*), -\Delta u_n^*) + (h, -\Delta u_n^*).$$

Using the Cauchy-Schwartz and Cauchy inequalities, the fact that f is globally Lipschitz and (2.24), we deduce

$$\begin{aligned}(f(u_*), -\Delta u_n^*) &\leq \frac{1}{m}|f(u_*)|_2^2 + \frac{m}{4}|-\Delta u_n^*|_2^2 \\ &\leq \frac{1}{m}(2\lambda_1^{-1}L_f^2\|u_*\|_2^2 + 2|f(0)|^2|\Omega|) + \frac{m}{4}|-\Delta u_n^*|_2^2 \\ &\leq \frac{2\lambda_1^{-1}L_f^2\Upsilon^2 + 2|f(0)|^2|\Omega|}{m} + \frac{m}{4}|-\Delta u_n^*|_2^2, \\ (h, -\Delta u_n^*) &\leq \frac{1}{m}|h|_2^2 + \frac{m}{4}|-\Delta u_n^*|_2^2.\end{aligned}$$

Thus, from above we obtain

$$|-\Delta u_n^*|_2^2 \leq \frac{2}{m^2}(2\lambda_1^{-1}L_f^2\Upsilon^2 + 2|f(0)|^2|\Omega| + |h|_2^2).$$

Then, as the sequence $\{P_n u_*\}$ is bounded in $D(-\Delta)$ and $P_n u_*$ converge to u_* strongly in $L^2(\Omega)$, it holds that $u_* \in D(-\Delta)$.

Step 4. Uniqueness. Let u_1 and u_2 be two solutions to (2.22). Then,

$$\langle -a(l(u_1))\Delta u_1 + a(l(u_2))\Delta u_2, v\rangle = (f(u_1) - f(u_2), v) \quad \forall v \in H_0^1(\Omega).$$

Adding $\pm a(l(u_1))\Delta u_2$ and taking $v = u_1 - u_2$, we obtain

$$m\|u_1 - u_2\|_2^2 \leq (\lambda_1^{-1/2}|l|_2 L_a\|u_2\|_2 + \lambda_1^{-1}L_f)\|u_1 - u_2\|_2^2.$$

Now we argue by contradiction. Assume that $u_1 \neq u_2$. Then, we can simplify the above expression, dropping the factor $\|u_1 - u_2\|_2^2$. However, using the a priori estimate (2.24) for u_2, we would arrive at the opposite inequality to that one in (2.25), what is a contradiction. Therefore, $u_1 = u_2$ holds. □

To conclude this section, we will show that the unique stationary solution to (2.2) is globally asymptotically exponentially stable.

Theorem 2.8. *Under the assumptions of Theorem 2.7, if (2.25) also holds, the difference between any solution to (2.2) and the unique solution u^* to (2.22) fulfils*

$$|u(t;\tau,u_\tau) - u^*|_2^2 \leq e^{-\lambda(t-\tau)}|u_\tau - u^*|_2^2 \quad \forall t \geq \tau,$$

where $\lambda = 2\lambda_1(m - \lambda_1^{-1/2}|l|_2 L_a\Upsilon - \lambda_1^{-1}L_f) > 0$.

Proof. For short, denote by $u(\cdot)$ the weak solution to the problem (2.2). Then, from the energy equality,

$$\frac{1}{2}\frac{d}{dt}|u(t)-u^*|_2^2 = \langle a(l(u(t)))\Delta u(t) - a(l(u^*))\Delta u^* + f(u(t)) - f(u^*), u(t)-u^*\rangle,$$

a.e. $t \in (\tau, T)$.

Adding $\pm a(l(u))\Delta u^*$ and using (2.23), the Poincaré inequality and the global Lipschitz continuity of the functions a and f, we have

$$\frac{1}{2}\frac{d}{dt}|u(t)-u^*|_2^2 \leq (-m + \lambda_1^{-1/2}|l|_2 L_a\|u^*\|_2 + L_f\lambda_1^{-1})\|u(t)-u^*\|_2^2.$$

Finally, making use of (2.24), (2.25) and the Poincaré inequality, we deduce

$$\frac{d}{dt}|u(t)-u^*|_2^2 \leq -\lambda|u(t)-u^*|_2^2 \quad \text{a.e. } t > \tau,$$

where λ is given in the statement. □

Remark 2.9. *(i) The upper bound M in (2.23) can be removed to obtain Theorems 2.7 and 2.8. Indeed, consider the function a substituted by*

$$\begin{cases} a(\widetilde{M}) & if \quad s \geq \widetilde{M} \\ a(s) & if \quad |s| \leq \widetilde{M} \\ a(-\widetilde{M}) & if \quad s \leq -\widetilde{M}, \end{cases}$$

with $\widetilde{M} = \lambda_1^{-1/2}|l|_2\Upsilon$, thanks to the a priori estimate (2.24).
(ii) The same argument allows to remove the global Liptchitz character of the function a in Theorem 2.7. However, for Theorem 2.8, it seems to be necessary to keep the function a globally Lipschitz since $|u(t)|_2$ can take arbitrary large values.

2.3 Minimal pullback attractors in L^2-norm

Now, under the initial setting of Section 2.1, fulfilled with some more general assumptions, we are going to analyse the long-time behaviour of the solutions to (2.2) in $L^2(\Omega)$ making use of the results on pullback attractors shown in Chapter 1.

First of all, thanks to Theorem 2.4, the map $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$ defined as

$$U(t,\tau)u_\tau = u(t;\tau,u_\tau) \quad \forall u_\tau \in L^2(\Omega) \quad \forall \tau \leq t, \tag{2.28}$$

where $u(t;\tau,u_\tau)$ is the weak solution to (2.2), is a process on $L^2(\Omega)$. In addition, as a consequence of Theorem 2.4, we have the following result.

Proposition 2.10. *Suppose that the function a is locally Lipschitz and fulfils (2.3), $f \in C(\mathbb{R})$ satisfies (2.4) and (2.5), $h \in L^2_{loc}(\mathbb{R}, H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, for any pair $(t,\tau) \in \mathbb{R}_d^2$, the map U is continuous from $L^2(\Omega)$ into itself.*

Proof. Consider $(t,\tau) \in \mathbb{R}^2_d$ fixed and let u_1 and u_2 be two solutions to (2.2) corresponding to the initial condition $u^1_\tau \in L^2(\Omega)$ and $u^2_\tau \in L^2(\Omega)$ respectively, and with the same non-autonomous term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$.

From the proof of Theorem 2.4, namely Step 2, we deduce

$$|u_1(t) - u_2(t)|_2^2 \leq |u^1_\tau - u^2_\tau|_2^2 e^{\int_\tau^t C(s)ds}, \tag{2.29}$$

where $C(s)$ is given by

$$C(s) = \frac{(L_a(R))^2 |l|_2^2 \|u_2(s)\|_2^2 + 2m\eta}{m},$$

with $L_a(R)$ the Lipschitz constant of the function a in $[-R,R] \supset \{l(u_i(t))\}_{t\in[\tau,T]}$ for $i = 1,2$.

Therefore, using (2.28), we can rewrite (2.29) as follows

$$|U(t,\tau)u^1_\tau - U(t,\tau)u^2_\tau|_2 \leq |u^1_\tau - u^2_\tau|_2 e^{\frac{1}{2}\int_\tau^t C(s)ds}.$$

□

From now on, we assume that the function f also fulfils

$$f(s)s \leq \alpha|s|^2 + \beta \quad \forall s \in \mathbb{R}, \tag{2.30}$$

where $\alpha \in [0, \lambda_1 m)$ and $\beta \geq 0$. Observe that if the constant C_f appearing in the assumption (2.4) belongs to $[0, \lambda_1 m)$, this new assumption would be redundant.

Now, we have the following estimate.

Lemma 2.11. *Suppose that the function a is locally Lipschitz and satisfies (2.3), $f \in C(\mathbb{R})$ fulfils (2.4), (2.5) and (2.30), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$, $l \in L^2(\Omega)$ and $u_\tau \in L^2(\Omega)$. Then, the solution u to (2.2) fulfils*

$$|u(t)|_2^2 \leq \frac{2\beta|\Omega|}{\mu} + e^{-\mu(t-\tau)}|u_\tau|_2^2 + \frac{e^{-\mu t}}{2(m - \alpha\lambda_1^{-1}) - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu s}\|h(s)\|_*^2 ds \quad \forall t \geq \tau \tag{2.31}$$

for any $\mu \in (0, 2(\lambda_1 m - \alpha))$.

Proof. From the energy equality, the Cauchy-Schwartz inequality, (2.3) and (2.30), we deduce

$$\frac{d}{dt}|u(t)|_2^2 + \mu|u(t)|_2^2 + 2m\|u(t)\|_2^2 \leq (2\alpha + \mu)|u(t)|_2^2 + 2\beta|\Omega| + 2\|h(t)\|_*\|u(t)\|_2.$$

Applying the Poincaré and Cauchy inequalities in the above expression, we obtain

$$\frac{d}{dt}|u(t)|_2^2 + \mu|u(t)|_2^2 \leq 2\beta|\Omega| + \frac{1}{2m - (2\alpha+\mu)\lambda_1^{-1}}\|h(t)\|_*^2.$$

Finally, multiplying by $e^{\mu t}$ and integrating between τ and t, (2.31) holds. □

Thanks to the previous estimate, now we can define a suitable tempered universe in $\mathcal{P}(L^2(\Omega))$.

Definition 2.12. *The class of all families of nonempty subsets* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ *such that*

$$\lim_{\tau \to -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|_2^2 \right) = 0$$

is denoted by $\mathcal{D}_\mu^{L^2}$ *for all* $\mu > 0$.

Now, if we assume that h satisfies a suitable growth condition, using the above estimates, we can prove the existence of a $\mathcal{D}_\mu^{L^2}$-absorbing family for the process U.

Proposition 2.13. *Assume that the function a is locally Lipschitz and satisfies (2.3), $f \in C(\mathbb{R})$ fulfils (2.4), (2.5) and (2.30), $l \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ fulfils that there exists some $\mu \in (0, 2(\lambda_1 m - \alpha))$ such that*

$$\int_{-\infty}^{0} e^{\mu s} \|h(s)\|_*^2 ds < \infty. \tag{2.32}$$

Then, the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t))$, where

$$R_{L^2}(t) = 1 + \frac{2\beta|\Omega|}{\mu} + \frac{e^{-\mu t}}{2(m - \alpha\lambda_1^{-1}) - \mu\lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s} \|h(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_\mu^{L^2}$-absorbing for the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$. Moreover, $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$.

Proof. Let us fix $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$. Using Lemma 2.11, (2.28) and (2.32), it holds

$$|U(t,\tau)u_\tau|_2^2 \leq \frac{2\beta|\Omega|}{\mu} + e^{-\mu(t-\tau)}|u_\tau|_2^2 + \frac{e^{-\mu t}}{2(m - \alpha\lambda_1^{-1}) - \mu\lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s} \|h(s)\|_*^2 ds \tag{2.33}$$

for all $u_\tau \in D(\tau) \in \widehat{D}$ and $\tau \leq t$.

Since $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_0(\widehat{D}, t) < t$ such that

$$e^{-\mu(t-\tau)}|u_\tau|_2^2 \leq 1 \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_0(\widehat{D}, t). \tag{2.34}$$

Now, simply replacing the estimation (2.34) in (2.33), we obtain

$$U(t,\tau)u_\tau \in D_0(t) \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_0(\widehat{D}, t).$$

Finally, thanks to (2.32) it is not difficult to prove that $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$. □

To prove the existence of minimal pullback attractors in $L^2(\Omega)$, we only need to check that the process U is pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact. To that end, we need first to establish some estimates.

Lemma 2.14. *Under the assumptions of Proposition 2.13, for any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, *there exists* $\tau_1(\widehat{D}, t) < t-2$ *such that for any* $\tau \leq \tau_1(\widehat{D}, t)$ *and any* $u_\tau \in D(\tau)$, *it fulfils*

$$\begin{cases} |u(r;\tau,u_\tau)|_2^2 \leq \rho_1(t) \quad \forall r \in [t-2,t], \\ \displaystyle\int_{r-1}^{r} \|u(s;\tau,u_\tau)\|_2^2 ds \leq \rho_2(t) \quad \forall r \in [t-1,t], \\ \displaystyle\int_{r-1}^{r} \|u'(s;\tau,u_\tau)\|_*^2 ds \leq \rho_3(t) \quad \forall r \in [t-1,t], \end{cases} \tag{2.35}$$

where

$$\rho_1(t) = 1 + \frac{2\beta|\Omega|}{\mu} + \frac{e^{-\mu(t-2)}}{2(m-\alpha\lambda_1^{-1}) - \mu\lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s}\|h(s)\|_*^2 ds,$$

$$\rho_2(t) = \frac{1}{m-\alpha\lambda_1^{-1}}\left(2\beta|\Omega| + \rho_1(t) + \frac{1}{m-\alpha\lambda_1^{-1}} \max_{r\in[t-1,t]} \int_{r-1}^{r} \|h(s)\|_*^2 ds\right),$$

$$\rho_3(t) = 3\left((M_{(\rho_1(t),l)})^2\rho_2(t) + 2C_f^2\lambda_1^{-1}(|\Omega| + \rho_1(t)) + \max_{r\in[t-1,t]} \int_{r-1}^{r} \|h(s)\|_*^2 ds\right),$$

where $M_{(\rho_1(t),l)}$ *is a positive constant.*

Proof. Let $\tau_1(\widehat{D}, t) < t-2$ be such that

$$e^{-\mu(t-2)}e^{\mu\tau}|u_\tau|_2^2 \leq 1 \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_1(\widehat{D}, t).$$

Consider fixed $\tau \leq \tau_1(\widehat{D}, t)$ and $u_\tau \in D(\tau)$.

The first estimate in (2.35) follows directly from (2.31), using the increasing character of the exponential.

Now, we will prove the other two inequalities in (2.35) for the Galerkin approximations and later, making use of compactness arguments, we will obtain the same ones for the solution. Observe that the first estimate in (2.35) also holds for the Galerkin approximations.

From the energy equality for the Galerkin approximation, making use of (2.3), we have

$$\frac{d}{dt}|u_n(t)|_2^2 + 2m\|u_n(t)\|_2^2 \leq 2(f(u_n(t)), u_n(t)) + 2\langle h(t), u_n(t)\rangle \quad \text{a.e. } t > \tau.$$

Applying (2.30) and the Poincaré inequality, we deduce

$$\frac{d}{dt}|u_n(t)|_2^2 + 2(m-\alpha\lambda_1^{-1})\|u_n(t)\|_2^2 \leq 2\beta|\Omega| + 2\langle h(t), u_n(t)\rangle \quad \text{a.e. } t > \tau.$$

Now, using the Cauchy inequality, we obtain

$$\frac{d}{dt}|u_n(t)|_2^2 + (m-\alpha\lambda_1^{-1})\|u_n(t)\|_2^2 \leq 2\beta|\Omega| + \frac{1}{m-\alpha\lambda_1^{-1}}\|h(t)\|_*^2 \quad \text{a.e. } t > \tau.$$

Integrating between $r-1$ and r when $r \in [t-1, t]$, we deduce for all $n \in \mathbb{N}$

$$\begin{aligned}\int_{r-1}^{r} \|u_n(s)\|_2^2 ds \leq & \frac{1}{m-\alpha\lambda_1^{-1}} \left(|u_n(r-1)|_2^2 + 2\beta|\Omega| + \frac{1}{m-\alpha\lambda_1^{-1}} \int_{r-1}^{r} \|h(s)\|_*^2 ds \right) \\ \leq & \rho_2(t), \end{aligned} \tag{2.36}$$

where $\rho_2(t)$ is given in the statement, thanks to the first inequality in (2.35) for u_n. Taking inferior limit in (2.36) and using the well-known fact that u_n converge to $u(\cdot;\tau, u_\tau)$ weakly in $L^2(r-1, r; H_0^1(\Omega))$ for all $r \in [t-1, t]$ (cf. Theorem 2.4), the second inequality in (2.35) holds.

Finally,

$$\|u_n'(t)\|_*^2 \leq 3a(l(u_n(t)))^2 \| - \Delta u_n(t)\|_*^2 + \frac{3}{\lambda_1}|f(u_n(t))|_2^2 + 3\|h(t)\|_*^2$$

a.e. $t > \tau$.

Observe that making use of the continuity of the function a and the fact that $l \in L^2(\Omega)$ and $|u_n(r)|_2^2 \leq \rho_1(t)$ for all $r \in [t-2, t]$ and $n \in \mathbb{N}$, there exists a positive constant $M_{(\rho_1(t),l)}$ such that

$$a(l(u_n(r))) \leq M_{(\rho_1(t),l)} \quad \forall r \in [t-2, t] \quad \forall n \in \mathbb{N}.$$

Taking this into account together with the fact that the function f satisfies (2.4), $-\Delta$ is an isometric isomorphism from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$, and the already proved first two estimates of (2.35) for u_n, we deduce

$$\int_{r-1}^{r} \|u_n'(s)\|_*^2 ds \leq \rho_3(t) \quad \forall r \in [t-1, t] \quad \forall n \in \mathbb{N}, \tag{2.37}$$

where $\rho_3(t)$ is the expression given in the statement. Now, taking inferior limit in (2.37) and bearing in mind that u_n' converge to $u'(\cdot;\tau, u_\tau)$ weakly in $L^2(r-1, r; H^{-1}(\Omega))$ for all $r \in [t-1, t]$ (cf. Theorem 2.4), the third estimate in (2.35) holds. □

Now we are ready to prove that the process U is pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact. To that end, we apply an energy method with continuous functions (e.g. cf. [73, 92, 94, 62]).

Proposition 2.15. *Under the assumptions of Proposition 2.13, the process* $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$ *is pullback* $\mathcal{D}_\mu^{L^2}$*-asymptotically compact.*

Proof. Consider $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, a sequence $\{\tau_n\} \subset (-\infty, t-2]$ with $\tau_n \to -\infty$ and $u_{\tau_n} \in D(\tau_n)$ for all n. Our aim is to prove that the sequence $\{u(t;\tau_n, u_{\tau_n})\}$ is relatively compact in $L^2(\Omega)$. For short we will denote $u^n(\cdot) = u(\cdot;\tau_n, u_{\tau_n})$.

Making use of Lemma 2.14, the continuity of the function a, $l \in L^2(\Omega)$ and (2.4), we know that there exists $\tau_1(\widehat{D}, t) < t-2$ satisfying that, if $n_1 \geq 1$ is such that $\tau_n \leq \tau_1(\widehat{D}, t)$ for all $n \geq n_1$, $\{u^n\}_{n \geq n_1}$ is bounded in $L^\infty(t-2, t; L^2(\Omega)) \cap L^2(t-2, t; H_0^1(\Omega))$, $\{f(u^n)\}_{n \geq n_1}$ is bounded in $L^2(t-2, t; L^2(\Omega))$, and the sequences

$\{-a(l(u^n))\Delta u^n\}_{n\geq n_1}$ and $\{(u^n)'\}_{n\geq n_1}$ are bounded in $L^2(t-2,t;H^{-1}(\Omega))$. Then, using the Aubin-Lions lemma, there exists $u \in L^\infty(t-2,t;L^2(\Omega))\cap L^2(t-2,t;H_0^1(\Omega))$ with $u' \in L^2(t-2,t;H^{-1}(\Omega))$, such that for a subsequence (relabeled the same) it holds

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(t-2,t;L^2(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^2(t-2,t;H_0^1(\Omega)), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(t-2,t;H^{-1}(\Omega)), \\ u^n \to u \quad \text{strongly in } L^2(t-2,t;L^2(\Omega)), \\ u^n(s) \to u(s) \quad \text{strongly in } L^2(\Omega) \quad \text{a.e. } s \in (t-2,t), \\ f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^2(t-2,t;L^2(\Omega)), \\ -a(l(u^n))\Delta u^n \rightharpoonup -a(l(u))\Delta u \quad \text{weakly in } L^2(t-2,t;H^{-1}(\Omega)). \end{cases} \tag{2.38}$$

Observe that the last two convergences in (2.38) have been obtained arguing in the same way as in the proof of Theorem 2.4, making use of [85, Lemme 1.3, p. 12].

Furthermore, $u \in C([t-2,t];L^2(\Omega))$ and using (2.38), it is not difficult to prove that u fulfils (2.6) in the interval $(t-2,t)$.

Since $\{(u^n)'\}_{n\geq n_1}$ is bounded in $L^2(t-2,t;H^{-1}(\Omega))$, we have that $\{u^n\}_{n\geq n_1}$ is equicontinuous in $H^{-1}(\Omega)$ on $[t-2,t]$. Namely, fixed $\varepsilon > 0$, consider $s_1, s_2 \in [t-2,t]$ with $|s_2 - s_1| < 1$, then

$$\begin{aligned} \|u^n(s_2) - u^n(s_1)\|_*^2 &\leq \left(\sup_{v\in H_0^1(\Omega)/\|v\|_2=1} \left| \langle \int_{s_1}^{s_2} (u^n(r))' dr, v\rangle \right| \right)^2 \\ &\leq \left(\int_{s_1}^{s_2} \|(u^n(r))'\|_* dr \right)^2 \\ &\leq \rho_3(t)|s_2 - s_1|. \end{aligned}$$

Then, it just simply takes $\delta_\varepsilon = \min\{\varepsilon^2/\rho_3(t), 1\}$. In addition, as $\{u^n\}_{n\geq n_1}$ is bounded in $C([t-2,t];L^2(\Omega))$ and the embedding $L^2(\Omega) \subset H^{-1}(\Omega)$ is compact, by the Arzela-Ascoli theorem, we obtain (for another subsequence, relabeled again the same)

$$u^n \to u \quad \text{strongly in } C([t-2,t];H^{-1}(\Omega)). \tag{2.39}$$

Now, consider a sequence $\{s_n\} \subset [t-2,t]$ which converges to s_*. Since $\{u^n\}_{n\geq n_1}$ is bounded in $C([t-2,t];L^2(\Omega))$, there exist a subsequence of $\{u^n(s_n)\}_{n\geq n_1}$ (relabeled the same) and $v \in L^2(\Omega)$ such that

$$u^n(s_n) \rightharpoonup v \quad \text{weakly in } L^2(\Omega). \tag{2.40}$$

Let us prove that $v = u(s_*)$. Fixed $\varepsilon > 0$, from (2.39) we deduce that there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|u^n(s) - u(s)\|_* \leq \frac{\varepsilon}{2} \quad \forall n \geq n_\epsilon \quad \forall s \in [t-2,t].$$

From this and using that the function $u \in C([t-2,t];H^{-1}(\Omega))$, we deduce

$$u^n(s_n) \to u(s_*) \quad \text{strongly in } H^{-1}(\Omega). \tag{2.41}$$

Then, from (2.40) and (2.41), by the uniqueness of the limit we obtain

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } L^2(\Omega). \tag{2.42}$$

Observe that if we prove

$$u^n \to u \quad \text{strongly in } C([t-1,t];L^2(\Omega)), \tag{2.43}$$

in particular the sequence $\{u(t;\tau_n,u_{\tau_n})\}$ is relatively compact in $L^2(\Omega)$.

We establish (2.43) by contradiction. We suppose that there exist $\varepsilon > 0$, a sequence $\{t_n\} \subset [t-1,t]$, without loss of generality converging to some t_*, with

$$|u^n(t_n) - u(t_*)|_2 \geq \varepsilon \quad \forall n \geq 1. \tag{2.44}$$

On the other hand, using the energy equality (2.7), the Cauchy inequality, (2.3) and (2.30), the estimate

$$|z(s)|_2^2 \leq |z(r)|_2^2 + 2\beta|\Omega|(s-r) + \frac{1}{2(m-\alpha\lambda_1^{-1})}\int_r^s \|h(\xi)\|_*^2 d\xi \quad \forall t-2 \leq r \leq s \leq t$$

holds with z replaced by u or any u^n.

Now we define the functions

$$J_n(s) = |u^n(s)|_2^2 - 2\beta|\Omega|s - \frac{1}{2(m-\alpha\lambda_1^{-1})}\int_{t-2}^s \|h(r)\|_*^2 dr,$$
$$J(s) = |u(s)|_2^2 - 2\beta|\Omega|s - \frac{1}{2(m-\alpha\lambda_1^{-1})}\int_{t-2}^s \|h(r)\|_*^2 dr.$$

From the regularity of u and all u^n, and the above inequality it holds that these functions J and J_n are continuous and non-increasing on $[t-2,t]$. In addition, observe that using (2.38), we have

$$J_n(s) \to J(s) \quad \text{a.e. } s \in (t-2,t).$$

Hence, there exists a sequence $\{\tilde{t}_k\} \subset (t-2,t_*)$ such that $\tilde{t}_k \to t_*$ when $k \to \infty$ and

$$\lim_{n\to\infty} J_n(\tilde{t}_k) = J(\tilde{t}_k) \quad \forall k \geq 1.$$

Fix an arbitrary value $\epsilon > 0$. From the continuity of J on $[t-2,t]$, there exists $k(\epsilon) \geq 1$ such that

$$|J(\tilde{t}_k) - J(t_*)| \leq \epsilon/2 \quad \forall k \geq k(\epsilon).$$

Now consider $n(\epsilon) \geq 1$ such that

$$t_n \geq \tilde{t}_{k(\epsilon)} \quad \text{and} \quad |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| \leq \epsilon/2 \quad \forall n \geq n(\epsilon).$$

Then, since all J_n are non-increasing, we deduce

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k(\epsilon)}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| + |J(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &\leq \epsilon \quad \forall n \geq n(\epsilon). \end{aligned}$$

As $\epsilon > 0$ is arbitrary, from above we deduce

$$\limsup_{n\to\infty} J_n(t_n) \le J(t_*).$$

Thus,

$$\limsup_{n\to\infty} |u^n(t_n)|_2 \le |u(t_*)|_2.$$

From this, (2.42) applied to the sequence $\{t_n\}$, it satisfies that the sequence $\{u^n(t_n)\}$ converges to $u(t_*)$ strongly in $L^2(\Omega)$, which is contradictory with (2.44). Therefore, (2.43) holds. □

As a consequence of the previous results, we obtain the existence of the minimal pullback attractors for the process U on $L^2(\Omega)$.

Theorem 2.16. *Suppose that the function a is locally Lipschitz and satisfies (2.3), $f \in C(\mathbb{R})$ fulfils (2.4), (2.5) and (2.30), $l \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ satisfies condition (2.32) for some $\mu \in (0, 2(\lambda_1 m - \alpha))$. Then, there exist the minimal pullback $\mathcal{D}_F^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and the minimal pullback $\mathcal{D}_\mu^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ for the process $U : \mathbb{R}^2_d \times L^2(\Omega) \to L^2(\Omega)$. The family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$ and the following relationship holds*

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \subset \overline{B}_{L^2}(0, R^{1/2}_{L^2}(t)) \quad \forall t \in \mathbb{R}. \tag{2.45}$$

Finally, if moreover h satisfies

$$\sup_{s\le 0}\left(e^{-\mu s}\int_{-\infty}^{s} e^{\mu r}\|h(r)\|_*^2 dr\right) < \infty, \tag{2.46}$$

then

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \quad \forall t \in \mathbb{R}.$$

Proof. The process U is continuous on $L^2(\Omega)$ and pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact (cf. Propositions 2.10 and 2.15 respectively). In addition, there exists a pullback $\mathcal{D}_\mu^{L^2}$-absorbing family (cf. Proposition 2.13) and $\mathcal{D}_F^{L^2} \subset \mathcal{D}_\mu^{L^2}$. Then, from Corollary 1.15, we deduce the existence of $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ and $\mathcal{A}_{\mathcal{D}_F^{L^2}}$, as well as the first relationship appearing in (2.45). The second relation in (2.45) is straightforward making use of Theorem 1.13 and the fact that $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$.

In addition, using that $\mathcal{D}_\mu^{L^2}$ is inclusion-closed, $\widehat{D}_0(t)$ is closed in $L^2(\Omega)$ for all $t \in \mathbb{R}$, $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$ and the second relation in (2.45), the family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$.

Finally, under the assumption (2.46), the set $\cup_{t\le T} R_{L^2}(t)$ is bounded for each $T \in \mathbb{R}$, where the expression of $R_{L^2}(t)$ is given in the statement of Proposition 2.13. Therefore, from Corollary 1.15, we deduce that both families $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ coincide. □

Remark 2.17. *(i) Observe that condition (2.46) is equivalent to*

$$\sup_{s\le 0}\int_{s-1}^{s}\|h(r)\|_*^2dr<\infty.$$

(ii) When $h\in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$ *fulfils condition (2.32) for some* $\mu\in(0,2(\lambda_1 m-\alpha))$, *it holds*

$$\int_{-\infty}^{0}e^{\sigma s}\|h(s)\|_*^2ds<\infty\quad\forall\sigma\in(\mu,2(\lambda_1 m-\alpha)).$$

Therefore, under the assumptions of Theorem 2.16, there exists the corresponding minimal pullback $\mathcal{D}_\sigma^{L^2}$*-attractor* $\mathcal{A}_{\mathcal{D}_\sigma^{L^2}}$ *for any* $\sigma\in(\mu,2(\lambda_1 m-\alpha))$. *Furthermore,* $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)\subset\mathcal{A}_{\mathcal{D}_\sigma^{L^2}}(t)$ *for any* $t\in\mathbb{R}$ *for all* $\sigma\in(\mu,2(\lambda_1 m-\alpha))$, *thanks to Theorem 1.16.*

In fact, if h satisfies (2.46), using the equivalence pointed out in (i), $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t)=\mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)=\mathcal{A}_{\mathcal{D}_\sigma^{L^2}}(t)$ *holds for all* $t\in\mathbb{R}$ *and any* $\sigma\in(\mu,2(\lambda_1 m-\alpha))$.

2.4 Minimal attractors in H^1-norm

In this section we prove the existence of pullback attractors for a dynamical system associated to (2.2) in the phase space $H_0^1(\Omega)$. In addition, we establish some relationships amongst these families and those analysed in Section 2.3.

Observe that from Theorem 2.5, the restriction of U to $\mathbb{R}^2_d\times H_0^1(\Omega)$ defines a process into $H_0^1(\Omega)$. Since no confusion arises, we do not modify the notation and continue denoting this process as U.

The following result ensures that the process U is strong-weak continuous in $H_0^1(\Omega)$.

Proposition 2.18. *Assume that the function a is locally Lipschitz and (2.3) holds,* $f\in C(\mathbb{R})$ *satisfies (2.4) and (2.5), and* $h\in L^2_{loc}(\mathbb{R},L^2(\Omega))$ *and* $l\in L^2(\Omega)$ *are given. Then, the process U is strong-weak continuous in* $H_0^1(\Omega)$.

Proof. Consider $(t,\tau)\in\mathbb{R}^2_d$ fixed and let $\{u_\tau^n\}$ be a sequence of initial data which converges to u_τ strongly in $H_0^1(\Omega)$. Our aim is to prove

$$U(t,\tau)u_\tau^n\rightharpoonup U(t,\tau)u_\tau\quad\text{weakly in } H_0^1(\Omega).\tag{2.47}$$

On the one hand, in Proposition 2.10 we have shown that the map $U(t,\tau)$ is continuous from $L^2(\Omega)$ into itself. Therefore,

$$U(t,\tau)u_\tau^n\to U(t,\tau)u_\tau\quad\text{strongly in } L^2(\Omega).$$

On the other hand, making use of (2.3), (2.4), (2.8) and the Hölder and Cauchy inequalities, we have

$$\|U(t,\tau)u_\tau^n\|_2^2\le\|u_\tau^n\|_2^2+\frac{2C_f^2}{m}\left(\int_\tau^t(|\Omega|+|U(s,\tau)u_\tau^n|_2^2)ds\right)+\frac{1}{m}\int_\tau^t|h(s)|_2^2ds.$$

Observe that it is not difficult to obtain a uniform estimate for $\{U(\cdot,\tau)u_\tau^n\}$ in $L^2(\tau,t;L^2(\Omega))$ using the Gronwall lemma and (2.7). Then, the sequence $\{U(t,\tau)u_\tau^n\}$ is bounded in $H_0^1(\Omega)$. Therefore, by the uniqueness of the limit, (2.47) holds. □

To prove that the process $U : \mathbb{R}^d_2 \times H^1_0(\Omega) \to H^1_0(\Omega)$ is pullback asymptotically compact, we previously establish some uniform estimates of the solutions in a finite-time interval up to t when the initial datum is shifted pullback far enough.

To clarify the statement of the following result, we introduce the next two amounts:

$$\begin{aligned}\rho_1^{ext}(t) &= 1 + \frac{2\beta|\Omega|}{\mu} + \frac{e^{-\mu(t-3)}}{2(m-\alpha\lambda_1^{-1}) - \mu\lambda_1^{-1}} \int_{-\infty}^t e^{\mu\xi}\|h(\xi)\|_*^2 d\xi, \\ \rho_2^{ext}(t) &= \frac{1}{m-\alpha\lambda_1^{-1}} \left(2\beta|\Omega| + \rho_1^{ext}(t) + \frac{1}{m-\alpha\lambda_1^{-1}} \max_{r\in[t-2,t]} \int_{r-1}^r \|h(\xi)\|_*^2 d\xi \right).\end{aligned} \tag{2.48}$$

[The upper script *ext* means that these expressions are estimates, close to those in Lemma 2.14 involving ρ_1 and ρ_2, but in an extended interval, as will be indicated in the proof below.]

Lemma 2.19. *Assume that the function a is locally Lipschitz and (2.3) holds, $f \in C(\mathbb{R})$ fulfils (2.4), (2.5) and (2.30), $l \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies (2.32) for some $\mu \in (0, 2(\lambda_1 m - \alpha))$. Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_2(\widehat{D}, t) < t-3$, such that for any $\tau \le \tau_2(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, it holds*

$$\begin{cases} \|u(r;\tau,u_\tau)\|_2^2 \le \widetilde{\rho}_1(t) \quad \forall r \in [t-2,t], \\ \int_{r-1}^r |-\Delta u(\xi;\tau,u_\tau)|_2^2 d\xi \le \widetilde{\rho}_2(t) \quad \forall r \in [t-1,t], \\ \int_{r-1}^r |u'(\xi;\tau,u_\tau)|_2^2 d\xi \le \widetilde{\rho}_3(t) \quad \forall r \in [t-1,t], \end{cases} \tag{2.49}$$

where, taking into account $\{\rho_i^{ext}\}_{i=1,2}$ from (2.48), the terms $\{\widetilde{\rho}_i\}_{i=1,2,3}$ are given by

$$\widetilde{\rho}_1(t) = \frac{4C_f^2|\Omega|}{m} + \left(1 + \frac{4C_f^2}{\lambda_1 m}\right)\rho_2^{ext}(t) + \frac{2}{m}\max_{r\in[t-2,t]}\int_{r-1}^r |h(\xi)|_2^2 d\xi,$$

$$\widetilde{\rho}_2(t) = \frac{4C_f^2|\Omega|}{m^2} + \frac{1}{m}\widetilde{\rho}_1(t) + \frac{4C_f^2}{\lambda_1 m^2}\rho_2^{ext}(t) + \frac{2}{m^2}\max_{r\in[t-1,t]}\int_{r-1}^r |h(\xi)|_2^2 d\xi,$$

$$\widetilde{\rho}_3(t) = 3(M_{(\rho_1^{ext}(t),l)})^2\widetilde{\rho}_2(t) + 6C_f^2|\Omega| + \frac{6C_f^2}{\lambda_1}\rho_2^{ext}(t) + 3\max_{r\in[t-1,t]}\int_{r-1}^r |h(\xi)|_2^2 d\xi,$$

where $M_{(\rho_1^{ext}(t),l)}$ is a positive constant.

Proof. Analogously as in the proof of Lemma 2.14, we can obtain uniform estimates for the solutions in a longer time-interval. Actually, there exists $\tau_2(\widehat{D}, t) < t-3$ such that for any $\tau \le \tau_2(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, it holds

$$|u(r;\tau,u_\tau)|_2^2 \le \rho_1^{ext}(t) \quad \forall r \in [t-3,t],$$

$$\int_{r-1}^r \|u(\xi;\tau,u_\tau)\|_2^2 d\xi \le \rho_2^{ext}(t) \quad \forall r \in [t-2,t],$$

where $\{\rho_i^{ext}\}_{i=1,2}$ are given in (2.48). Actually, these estimates also hold for the Galerkin approximations $u_n(\cdot;\tau,u_\tau)$ (for short denoted by $u_n(\cdot)$).

From now on, consider fixed $\tau \leq \tau_2(\widehat{D},t)$ and $u_\tau \in D(\tau)$.

Multiplying (2.9) by $\lambda_j\varphi_{nj}(\xi)$, summing from $j=1$ to n, and using (2.3), (2.4) and the Cauchy inequality, we have

$$\frac{d}{d\xi}\|u_n(\xi)\|_2^2+m|-\Delta u_n(\xi)|_2^2 \leq \frac{4C_f^2|\Omega|}{m}+\frac{4C_f^2}{\lambda_1 m}\|u_n(\xi)\|_2^2+\frac{2}{m}|h(\xi)|_2^2 \quad \text{a.e. } \xi>\tau \tag{2.50}$$

Integrating between r and s with $\tau \leq r-1 \leq s \leq r$, we deduce

$$\|u_n(r)\|_2^2 \leq \|u_n(s)\|_2^2 + \frac{4C_f^2|\Omega|}{m} + \frac{4C_f^2}{\lambda_1 m}\int_{r-1}^r \|u_n(\xi)\|_2^2 d\xi + \frac{2}{m}\int_{r-1}^r |h(\xi)|_2^2 d\xi.$$

Integrating the last inequality w.r.t. s between $r-1$ and r,

$$\|u_n(r)\|_2^2 \leq \left(1+\frac{4C_f^2}{\lambda_1 m}\right)\int_{r-1}^r \|u_n(s)\|_2^2 ds + \frac{4C_f^2|\Omega|}{m} + \frac{2}{m}\int_{r-1}^r |h(\xi)|_2^2 d\xi$$

for all $\tau \leq r-1$.

Now, from the estimate on the solutions by ρ_2^{ext} given above,

$$\|u_n(r;\tau,u_\tau)\|_2^2 \leq \widetilde{\rho}_1(t) \quad \forall r \in [t-2,t] \quad \forall n \in \mathbb{N}, \tag{2.51}$$

where $\widetilde{\rho}_1(t)$ is given in the statement. Taking inferior limit in (2.51) and using the well-known fact that u_n converge to $u(\cdot;\tau,u_\tau)$ weakly-star in $L^\infty(t-2,t;H_0^1(\Omega))$ and $u \in C([t-2,t];H_0^1(\Omega))$ (cf. Theorem 2.5), the first inequality in (2.49) holds.

Now, integrating between $r-1$ and r in (2.50), we obtain in particular

$$\begin{aligned}&\int_{r-1}^r |-\Delta u_n(\xi)|_2^2 d\xi \\ &\leq \frac{1}{m}\|u_n(r-1)\|_2^2 + \frac{4C_f^2|\Omega|}{m^2} + \frac{4C_f^2}{\lambda_1 m^2}\int_{r-1}^r \|u_n(\xi)\|_2^2 d\xi + \frac{2}{m^2}\int_{r-1}^r |h(\xi)|_2^2 d\xi,\end{aligned}$$

for all $\tau \leq r-1$.

Therefore,

$$\int_{r-1}^r |-\Delta u_n(\xi)|_2^2 d\xi \leq \widetilde{\rho}_2(t) \quad \forall r \in [t-1,t] \quad \forall n \in \mathbb{N}, \tag{2.52}$$

where $\widetilde{\rho}_2(t)$ is given in the statement. Now, taking inferior limit in (2.52), bearing in mind that u_n converge to $u(\cdot;\tau,u_\tau)$ weakly $L^2(r-1,r;D(-\Delta))$ for all $r \in [t-1,t]$ (cf. Theorem 2.5), the second inequality in (2.49) holds.

On the other hand,

$$\begin{aligned}&\int_{r-1}^r |u_n'(\xi)|_2^2\, d\xi \\ &\leq 3\int_{r-1}^r a(l(u_n(\xi)))^2|-\Delta u_n(\xi)|_2^2 d\xi + 3\int_{r-1}^r |f(u_n(\xi))|_2^2 d\xi + 3\int_{r-1}^r |h(\xi)|_2^2 d\xi,\end{aligned}$$

for all $\tau \le r-1$.

Observe that from the continuity of the function a, the fact that $l \in L^2(\Omega)$ and $|u_n(t)|_2^2 \le \rho_1^{ext}(t)$ for all $r \in [t-3,t]$ and $n \in \mathbb{N}$, we have that there exits a positive constant $M_{(\rho_1^{ext}(t),l)}$ such that

$$a(l(u_n(r))) \le M_{(\rho_1^{ext}(t),l)} \quad \forall r \in [t-3,t] \quad \forall n \in \mathbb{N}.$$

Therefore, from this, (2.4) and the above estimates, it yields

$$\int_{r-1}^{r} |u_n'(\xi)|_2^2 d\xi \le \widetilde{\rho}_3(t) \quad \forall r \in [t-1,t] \quad \forall n \in \mathbb{N},$$

where $\widetilde{\rho}_3(t)$ is given in the statement. Finally, taking into account that u_n' converge to $u'(\cdot;\tau,u_\tau)$ weakly in $L^2(r-1,r;L^2(\Omega))$ for all $r \in [t-1,t]$ (cf. Theorem 2.5), the last inequality in (2.49) holds. □

Now, we introduce additional universes that involve more regularity.

Definition 2.20. *For each $\mu > 0$, $\mathcal{D}_\mu^{L^2,H_0^1}$ denotes the class of all families of nonempty subsets $\widehat{D}_{H_0^1} = \{D(t) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{L^2}$.*

Now, from the existence of a pullback $\mathcal{D}_\mu^{L^2}$-absorbing family (cf. Proposition 2.13) and the regularising effect of the equation (cf. Theorem 2.5), the following result is straightforward.

Proposition 2.21. *Under the assumptions of Lemma 2.19, the family*

$$\widehat{D}_{0,H_0^1} = \{\overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}_\mu^{L^2,H_0^1}$ and for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_3(\widehat{D},t) < t$ such that

$$U(t,\tau)D(\tau) \subset D_{0,H_0^1}(t) \quad \forall \tau \le \tau_3(\widehat{D},t).$$

In particular, the family $\widehat{D}_{0,H_0^1}$ is pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-absorbing for the process $U : \mathbb{R}_d^2 \times H_0^1(\Omega) \to H_0^1(\Omega)$.

Proof. Let us fix $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$. By Proposition 2.13, there exists $\tau_0(\widehat{D},t) < t$ such that

$$|U(t,\tau)u_\tau|_2^2 \le R_{L^2}(t) \quad \forall u_\tau \in D(\tau) \quad \forall \tau \le \tau_0(\widehat{D},t).$$

Moreover, thanks to the regularising effect of the equation, when $u_\tau \in L^2(\Omega)$, it holds that $u(\cdot;\tau,u_\tau) \in C((\tau,\infty);H_0^1(\Omega))$. As a result, $U(t,\tau)u_\tau \in H_0^1(\Omega)$ if $t > \tau$. Therefore, it satisfies

$$U(t,\tau)D(\tau) \subset H_0^1(\Omega) \cap \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \quad \forall \tau \le \tau_0(\widehat{D},t),$$

where $\tau_3(\widehat{D},t) = \tau_0(\widehat{D},t)$.

Finally, as a consequence of $\widehat{D}_0$ belonging to $\mathcal{D}_\mu^{L^2}$, $\widehat{D}_{0,H_0^1} \in \mathcal{D}_\mu^{L^2,H_0^1}$. □

The following result establishes that the process U defined on $H_0^1(\Omega)$ as phase-space is pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-asymptotically compact. To that end, we apply again an energy method analogous to the one we used in Proposition 2.15. We reproduce it here just for the sake of completeness.

Proposition 2.22. *Under the assumptions of Lemma 2.19, the process* $U : \mathbb{R}_d^2 \times H_0^1(\Omega) \to H_0^1(\Omega)$ *is pullback* $\mathcal{D}_\mu^{L^2,H_0^1}$*-asymptotically compact.*

Proof. Let us fix $t \in \mathbb{R}$, a family $\widehat{D}_{H_0^1} \in \mathcal{D}_\mu^{L^2,H_0^1}$, a sequence $\{\tau_n\} \subset (-\infty, t-3]$ with $\tau_n \to -\infty$ and $u_{\tau_n} \in D_{H_0^1}(\tau_n)$ for all n. We will prove that the sequence $\{u(t;\tau_n,u_{\tau_n})\}$ is relatively compact in $H_0^1(\Omega)$. For short, we will denote $u^n(\cdot) = u(\cdot;\tau_n,u_{\tau_n})$.

As a consequence of Lemma 2.19, it holds that there exists $\tau_2(\widehat{D},t) < t-3$, such that $\tau_n \leq \tau_2(\widehat{D},t)$ for all $n \geq n_2$, the sequence $\{u^n\}_{n\geq n_2}$ is bounded in $L^\infty(t-2,t;H_0^1(\Omega)) \cap L^2(t-2,t;D(-\Delta))$, and the sequences $\{-a(l(u^n))\Delta u^n\}_{n\geq n_2}$, $\{f(u^n)\}_{n\geq n_2}$ and $\{(u^n)'\}_{n\geq n_2}$ are bounded in $L^2(t-2,t;L^2(\Omega))$. Then, using the Aubin-Lions lemma, there exists $u \in L^\infty(t-2,t;H_0^1(\Omega)) \cap L^2(t-2,t;D(-\Delta))$ with $u' \in L^2(t-2,t;L^2(\Omega))$, such that for a subsequence (relabeled the same) it holds

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(t-2,t;H_0^1(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^2(t-2,t;D(-\Delta)), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(t-2,t;L^2(\Omega)), \\ u^n \to u \quad \text{strongly in } L^2(t-2,t;H_0^1(\Omega)), \\ u^n(s) \to u(s) \quad \text{strongly in } H_0^1(\Omega) \quad \text{a.e. } s \in (t-2,t), \\ f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^2(t-2,t;L^2(\Omega)), \\ -a(l(u^n))\Delta u^n \rightharpoonup -a(l(u))\Delta u \quad \text{weakly in } L^2(t-2,t;L^2(\Omega)), \end{cases} \tag{2.53}$$

where the last two convergences have been obtained making use of [85, Lemme 1.3, p. 12].

Observe that $u \in C([t-2,t];H_0^1(\Omega))$ and due to (2.53), u satisfies (2.6) in the interval $(t-2,t)$.

Moreover, since $\{(u^n)'\}_{n\geq n_2}$ is bounded in $L^2(t-2,t;L^2(\Omega))$, it satisfies that $\{u^n\}_{n\geq n_2}$ is equicontinuous in $L^2(\Omega)$ on $[t-2,t]$. Indeed, fixed $\varepsilon > 0$ and considering $s_1, s_2 \in [t-2,t]$ with $|s_2 - s_1| < 1$, we have

$$\begin{aligned} |u^n(s_2) - u^n(s_1)|_2^2 &= \left| \int_{s_1}^{s_2} (u^n(\xi))' d\xi \right|_2^2 \\ &\leq \left(\int_{s_1}^{s_2} |(u^n(\xi))'|_2^2 d\xi \right) |s_2 - s_1| \\ &\leq \widetilde{\rho}_3(t)|s_2 - s_1|. \end{aligned}$$

Now, it just suffices to take $\delta_\varepsilon = \min\{\varepsilon^2/\widetilde{\rho}_3(t), 1\}$. From this and taking into account that $\{u^n\}_{n\geq n_2}$ is bounded in $L^\infty(t-2,t;H_0^1(\Omega))$ and the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, applying the Arzela-Ascoli theorem we obtain

$$u^n \to u \quad \text{strongly in } C([t-2,t];L^2(\Omega)). \tag{2.54}$$

On the other hand, using that $\{u^n\}_{n\geq n_2}$ is bounded in $C([t-2,t];H_0^1(\Omega))$, we have that for any sequence $\{s_n\}\subset[t-2,t]$ with $s_n\to s_*$,

$$u^n(s_n)\rightharpoonup u(s_*)\quad\text{weakly in } H_0^1(\Omega), \tag{2.55}$$

where (2.54) has been used to identify the weak limit.

If we prove

$$u^n\to u\quad\text{strongly in } C([t-1,t];H_0^1(\Omega)), \tag{2.56}$$

in particular, we have that the sequence $\{u^n(t)\}$ is relatively compact in $H_0^1(\Omega)$. To that end, we argue by contradiction. We suppose that there exist $\varepsilon>0$, a sequence $\{t_n\}\subset[t-1,t]$, without loss of generality converging to some t_*, with

$$\|u^n(t_n)-u(t_*)\|_2\geq\varepsilon\quad\forall n\geq 1. \tag{2.57}$$

Now, applying (2.3), (2.4) and the Cauchy inequality to the energy equality (2.7), the estimate

$$\|z(s)\|_2^2\leq\|z(r)\|_2^2+\frac{2C_f^2|\Omega|}{m}(s-r)+\frac{2C_f^2}{m}\int_r^s|z(\xi)|_2^2d\xi+\frac{1}{m}\int_r^s|h(\xi)|_2^2d\xi$$

holds with z replaced by u or any u^n for all $t-2\leq r\leq s\leq t$.

Then, we define the following functions

$$J_n(s)=\|u^n(s)\|_2^2-\frac{2C_f^2|\Omega|}{m}s-\frac{2C_f^2}{m}\int_{t-2}^s|u^n(r)|_2^2dr-\frac{1}{m}\int_{t-2}^s|h(r)|_2^2dr,$$
$$J(s)=\|u(s)\|_2^2-\frac{2C_f^2|\Omega|}{m}s-\frac{2C_f^2}{m}\int_{t-2}^s|u(r)|_2^2dr-\frac{1}{m}\int_{t-2}^s|h(r)|_2^2dr.$$

It is clear from the regularity of u and all u^n that these functions are continuous on $[t-2,t]$. In addition, using the above inequality it is not difficult to prove that J and all J_n are non-increasing functions on $[t-2,t]$. Moreover, from (2.53), it holds

$$J_n(s)\to J(s)\quad\text{a.e. } s\in(t-2,t).$$

Hence, there exists a sequence $\{\tilde{t}_k\}\subset(t-2,t_*)$ such that $\tilde{t}_k\to t_*$ when $k\to\infty$ and

$$\lim_{n\to\infty}J_n(\tilde{t}_k)=J(\tilde{t}_k)\quad\forall k\geq 1.$$

Consider $\epsilon>0$ fixed. Since the function J is continuous on $[t-2,t]$, there exists $k(\epsilon)\geq 1$ such that

$$|J(\tilde{t}_k)-J(t_*)|<\frac{\epsilon}{2}\quad\forall k\geq k(\epsilon).$$

Now, we consider $n(\epsilon)\geq 1$ such that

$$t_n\geq\tilde{t}_{k(\epsilon)}\quad\text{and}\quad|J_n(\tilde{t}_{k(\epsilon)})-J(\tilde{t}_{k(\epsilon)})|<\frac{\epsilon}{2}\quad n\geq n(\epsilon).$$

Then, since all the functions J_n are non-increasing, for all $n \geq n(\epsilon)$

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k(\epsilon)}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| + |J(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Then, $\limsup_{n\to\infty} J_n(t_n) \leq J(t_*)$. Thus, it satisfies that $\limsup_{n\to\infty} \|u^n(t_n)\|_2 \leq \|u(t_*)\|_2$ which, together with (2.55) applied to the sequence $\{t_n\}$, allow us to prove that $u^n(t_n)$ converge to $u(t_*)$ strongly in $H_0^1(\Omega)$, in contradiction with (2.57). Therefore, (2.56) holds. □

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}_d^2 \times H_0^1(\Omega) \to H_0^1(\Omega)$.

Theorem 2.23. *Suppose that the function a is locally Lipschitz and (2.3) holds, $f \in C(\mathbb{R})$ fulfils (2.4), (2.5) and (2.30), $l \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ verifies (2.32) for some $\mu \in (0, 2(\lambda_1 m - \alpha))$. Then, there exist the minimal pullback $\mathcal{D}_F^{H_0^1}$-attractor $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ and the minimal pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}$ for the process $U : \mathbb{R}_d^2 \times H_0^1(\Omega) \to H_0^1(\Omega)$. In addition, the following relationship holds*

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) \subset \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}(t) \quad \forall t \in \mathbb{R}, \tag{2.58}$$

In particular, we have the following pullback attraction result in $H_0^1(\Omega)$,

$$\lim_{\tau\to-\infty} \operatorname{dist}_{H_0^1}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R} \quad \forall \widehat{D} \in \mathcal{D}_\mu^{L^2}. \tag{2.59}$$

Finally, if moreover h satisfies

$$\sup_{s\leq 0}\left(e^{-\mu s}\int_{-\infty}^{s} e^{\mu r}|h(r)|_2^2 dr\right) < \infty, \tag{2.60}$$

then the following chain of equalities holds

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}(t)} = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}(t) \quad \forall t \in \mathbb{R},$$

and for any $B \in \mathcal{D}_F^{L^2}$,

$$\lim_{\tau\to-\infty} \operatorname{dist}_{H_0^1}(U(t,\tau)B, \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R}. \tag{2.61}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}$ is a consequence of Corollary 1.15. Indeed, the process U is strong-weak continuous (cf. Proposition 2.18), $\mathcal{D}_F^{H_0^1} \subset \mathcal{D}_\mu^{L^2,H_0^1}$ holds, and the existence of an absorbing family (cf. Proposition 2.21) and the asymptotic compactness (cf. Proposition 2.22) hold.

The chain of inclusions (2.58) follows from Corollary 1.15 and Theorem 1.16. In fact, the equality for all $t \in \mathbb{R}$ between $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}(t)$ is also due to Theorem 1.16, using Proposition 2.21. Then, (2.59) obviously holds.

When h satisfies (2.46), it holds $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)$ for all $t \in \mathbb{R}$ (cf. Theorem 2.16). The equality $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)$ is again due to Theorem 1.16. To that end we need to assume (2.60), an assumption stronger than (2.46), and make use of the first estimate appearing in Lemma 2.19. Therefore, (2.61) is straightforward. □

We conclude this chapter with a complement to Remark 2.17.

Remark 2.24. *Under the assumptions of Theorem 2.23, for any* $\sigma \in (\mu, 2(\lambda_1 m - \alpha))$ *there exists the corresponding minimal pullback* $\mathcal{D}_\sigma^{L^2,H_0^1}$*-attractor,* $\mathcal{A}_{\mathcal{D}_\sigma^{L^2,H_0^1}}$*, and the relationship* $\mathcal{A}_{\mathcal{D}_\sigma^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\sigma^{L^2,H_0^1}}(t)$ *holds for all* $t \in \mathbb{R}$*. In addition, when* h *satisfies (2.60),* $\mathcal{A}_{\mathcal{D}_\sigma^{L^2,H_0^1}}(t) = A_{\mathcal{D}_F^{H_0^1}}(t)$ *for all* $t \in \mathbb{R}$ *and any* $\sigma \in (\mu, 2(\lambda_1 m - \alpha))$*.*

Chapter 3

Non-autonomous nonlocal reaction-diffusion equations

In this chapter, we are interested in studying the long-time behaviour of the solutions of the non-autonomous nonlocal reaction-diffusion equation

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t),$$

where the main difference with respect to the problem analysed in Chapter 2 is that in this case the function f fulfils

$$-\kappa - \alpha_1|s|^p \leq f(s)s \leq \kappa - \alpha_2|s|^p \quad \forall s \in \mathbb{R},$$

where κ, α_1 and α_2 are positive constants and $p > 2$. Although we relax the assumptions on f, now we need to impose smoothness conditions on the domain in order to prove the existence and uniqueness of weak and strong solutions. However, we do not assume any restriction on the dimension of the domain Ω, which is quite useful when researchers want to deal with problems that have dependencies on other variables not only the spatial one. In Chapter 4, however, we do impose restrictions on the dimension of the domain Ω and in return, we do not assume any smoothness condition on Ω.

Next, we focus on analysing the existence of nontrivial stationary solutions under suitable assumptions, making use of a method developed by Chipot and Corrêa in [42] based on a fixed point argument. In a more general framework, a comparison result between the solution to the evolution problem and two stationary solutions (assumed to exist) is given, when the initial datum is ordered w.r.t. the stationary solutions.

Finally, the existence of minimal pullback attractors in the L^2-norm in the frameworks of universes of fixed bounded sets and that given by a tempered growth condition is proved, and some relationships between them are established. Moreover, we prove the existence of minimal pullback attractors in $H_0^1(\Omega)$ in some particular cases and study relationships amongst these new families and those given previously in the L^2-context. To prove the existence of these more regular families, we need to assume that $f(u) \in L^2(\tau, T; L^2(\Omega))$, this way manipulations with $-\Delta u$ make sense, as well as another restriction related to a boundedness of the norm of $f(u)$ in $L^2(\tau, T; L^2(\Omega))$, which is obtained making use of interpolation results together with the regularity imposed on the domain Ω (see (3.54)).

The results of this chapter can be found in [23].

3.1 Setting of the problem and existence results

In this chapter, we analyse the nonlocal reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\tau \in \mathbb{R}$, the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and there exists a positive constant m such that

$$0 < m \le a(s) \quad \forall s \in \mathbb{R}, \tag{3.2}$$

and $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$.

Assume that the function $f \in C(\mathbb{R})$ and there exist positive constants α_1, α_2, η, κ and $p > 2$ such that

$$(f(s) - f(r))(s - r) \le \eta(s - r)^2 \quad \forall s, r \in \mathbb{R}, \tag{3.3}$$

$$-\kappa - \alpha_1|s|^p \le f(s)s \le \kappa - \alpha_2|s|^p \quad \forall s \in \mathbb{R}. \tag{3.4}$$

Observe that the case $p \in [1, 2]$ is not considered here, since the main goals achieved in this chapter have been studied for this particular case in Chapter 2 in a more general framework.

From (3.4) we can deduce that there exists $\beta > 0$ such that

$$|f(s)| \le \beta(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R}. \tag{3.5}$$

Although we weaken the assumptions on f, now we impose smoothness condition on the domain. Namely, in our proofs we require $\Omega \subset \mathbb{R}^N$ to be a bounded open set of class C^k, with $k \ge 2$ such that $k \ge N(p-2)/(2p)$. Observe that even though the domain Ω is smooth, we do not assume any requirement on the dimension N of the domain Ω, unlike what happens in Chapter 4. In that Chapter 4, to study the existence of strong solutions and the regularising effect of the equation, we need to impose strong requirements on either the dimension of the domain Ω or the reaction term, or even both of them since no assumption of regularity is imposed on the domain Ω.

Again as in Chapter 2, we assume that the initial datum $u_\tau \in L^2(\Omega)$ and the non-autonomous term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. Identifying $L^2(\Omega)$ with its dual, the operator l acting on u must be understood as (l, u), but for short we keep the notation $l(u)$.

Definition 3.1. *A weak solution to (3.1) is a function* $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ *for all* $T > \tau$*, with* $u(\tau) = u_\tau$*, and such that for all* $v \in H_0^1(\Omega) \cap L^p(\Omega)$

$$\frac{d}{dt}(u(t), v) + a(l(u(t)))((u(t), v)) = (f(u(t)), v) + \langle h(t), v \rangle, \tag{3.6}$$

where the previous equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$*.*

Observe that if u is a weak solution to (3.1), making use of the continuity of a, $l \in L^2(\Omega)$, (3.5) and (3.6), it holds that $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$ for any $T > \tau$ (where p and q are conjugate exponents). Therefore, $u \in C([\tau, \infty); L^2(\Omega))$. In addition, the initial datum in (3.1) makes sense and the following energy equality holds

$$|u(t)|_2^2 + 2\int_s^t a(l(u(r)))\|u(r)\|_2^2 dr = |u(s)|_2^2 + 2\int_s^t (f(u(r)), u(r)) dr + 2\int_s^t \langle h(r), u(r) \rangle dr \tag{3.7}$$

for all $\tau \leq s \leq t$.

A notion of more regular solution is also suitable for problem (3.1).

Definition 3.2. *A strong solution to (3.1) is a weak solution* u *to (3.1) such that* $u \in L^2(\tau, T; D(-\Delta)) \cap L^\infty(\tau, T; H_0^1(\Omega))$ *for all* $T > \tau$*.*

Regarding $D(-\Delta)$, it holds that $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ thanks to the assumptions made on the domain Ω, since it is a bounded open set of class C^2 at least. Therefore, we will use either the norm of $D(-\Delta)$ or the norm of $H^2(\Omega) \cap H_0^1(\Omega)$, since both are equivalent (see [57, Theorem 4, p. 317] or [100, Theorem 6.16, p. 181]).

Observe that due to the presence of the nonlocal operator in the diffusion term, under the assumptions made, it is not possible to guarantee that the strong solution $u \in C([\tau, T]; H_0^1(\Omega))$ (see Theorem 3.4 below), unlike what happens in reaction-diffusion problems. Nevertheless, every strong u fulfils $u \in C_w([\tau, T]; H_0^1(\Omega))$ (cf. [108, Theorem 2.1, p. 544] or [111, Lemma 3.3, p. 74]).

In this section, we analyse the existence and uniqueness of weak and strong solutions to (3.1) as well as the regularising effect of the equation. Analogously as in Chapter 2, we use the Faedo-Galerkin approximations and compactness arguments. Finally, for the sake of completeness we give a Maximum Principle for (3.1).

First of all, we will prove the existence and uniqueness of weak solutions.

Theorem 3.3. *Assume that the function* a *is locally Lipschitz and satisfies (3.2),* $f \in C(\mathbb{R})$ *fulfils (3.3) and (3.4),* $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ *and* $l \in L^2(\Omega)$*. Then, for each initial datum* $u_\tau \in L^2(\Omega)$*, there exists a unique weak solution to the problem (3.1), denoted by* $u(\cdot; \tau, u_\tau)$ *and fulfilling the energy equality (3.7).*

Proof. We split the proof into two steps.

Step 1. Existence of weak solution. Making use of spectral theory, we deduce that there exists a sequence $\{w_i\}_{i\geq 1}$ of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, which is a Hilbert basis of $L^2(\Omega)$. Observe that thanks to the regularity imposed to the domain Ω, each eigenfunction $w_i \in L^p(\Omega)$.

Now, for each integer $n \geq 1$, consider the function $u_n(t;\tau,u_\tau) = \sum_{j=1}^n \varphi_{nj}(t)w_j$ ($u_n(t)$ for short) which is the local solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t),w_j) + a(l(u_n(t)))((u_n(t),w_j)) = (f(u_n(t)),w_j) + \langle h(t),w_j\rangle, \ t\in(\tau,\infty), \\ (u_n(\tau),w_j) = (u_\tau,w_j), \qquad\qquad j=1,\dots,n. \end{cases} \tag{3.8}$$

To prove this claim we argue analogously as in the proof of Proposition 2.3, making use of [52, Theorem 1.1, p. 43]. For the sake of brevity, we omit the proof.

Multiplying by φ_{nj} in (3.8), summing from $j=1$ to n and using (3.2), we have

$$\frac{d}{dt}|u_n(t)|_2^2 + m\|u_n(t)\|_2^2 \leq 2(f(u_n(t)),u_n(t)) + 2\langle h(t),u_n(t)\rangle \quad \text{a.e. } t\in(\tau,t_n). \tag{3.9}$$

Observe that from (3.4) and the Cauchy inequality, we obtain

$$\begin{aligned} (f(u_n(t)),u_n(t)) &\leq \kappa|\Omega| - \alpha_2|u_n(t)|_p^p, \\ \langle h(t),u_n(t)\rangle &\leq \frac{1}{2m}\|h(t)\|_*^2 + \frac{m}{2}\|u_n(t)\|_2^2. \end{aligned}$$

Taking this into account, from (3.9) we deduce

$$\frac{d}{dt}|u_n(t)|_2^2 + m\|u_n(t)\|_2^2 + 2\alpha_2|u_n(t)|_p^p \leq 2\kappa|\Omega| + \frac{1}{m}\|h(t)\|_*^2 \quad \text{a.e. } t\in(\tau,t_n).$$

Integrating between τ and t with $\tau < t < t_n$, we have

$$\begin{aligned} &|u_n(t)|_2^2 + m\int_\tau^t \|u_n(s)\|_2^2 ds + 2\alpha_2\int_\tau^t |u_n(s)|_p^p ds \\ &\leq |u_\tau|_2^2 + 2\kappa|\Omega|(T-\tau) + \frac{1}{m}\int_\tau^T \|h(s)\|_*^2 ds. \end{aligned}$$

Therefore, $\{u_n\}$ is well defined and bounded in $L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$ for all $T > \tau$. In addition, taking this into account together with the fact that each $u_n \in C([\tau,T];L^2(\Omega))$, we deduce that there exists a positive constant C_∞ such that

$$|u_n(t)|_2 \leq C_\infty \quad \forall t\in[\tau,T] \quad \forall n\geq 1.$$

Then, using that $a \in C(\mathbb{R};\mathbb{R}_+)$ and $l \in L^2(\Omega)$, we have

$$|a(l(u_n(t)))| \leq M_{C_\infty} \quad \forall t\in[\tau,T] \quad \forall n\geq 1. \tag{3.10}$$

Therefore, if fulfils

$$\int_\tau^T |a(l(u_n(t)))|^2 \| - \Delta u_n(t)\|_*^2 dt \leq (M_{C_\infty})^2 \int_\tau^T \|u_n(t)\|_2^2 dt. \tag{3.11}$$

Now, bearing in mind that the sequence $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$, we have that the sequence $\{-a(l(u_n))\Delta u_n\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$.

On the one hand, the sequence $\{f(u_n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$, since making use of (3.5), we have

$$\begin{aligned}\int_\tau^T \int_\Omega |f(u_n(x,t))|^q dx dt &\leq \beta^q \int_\tau^T \int_\Omega (1 + |u_n(x,t)|^{p-1})^q dx dt \\ &\leq 2^{q-1}\beta^q \int_\tau^T (|\Omega| + |u_n(t)|_p^p) dt \\ &\leq 2^{q-1}\beta^q \left(|\Omega|(T-\tau) + \int_\tau^T |u_n(t)|_p^p dt \right).\end{aligned}$$

Then, thanks to the boundedness of $\{u_n\}$ in $L^p(\tau, T; L^p(\Omega))$, it yields that the sequence $\{f(u_n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$.

Finally, to prove that the sequence $\{u_n'\}$ is bounded, we need first to define two additional projection operators related to

$$\begin{array}{rccl} P_n\colon & L^2(\Omega) & \longrightarrow & V_n := \text{span}[w_1, \ldots, w_n] \\ & \phi & \longmapsto & \sum_{j=1}^n (\phi, w_j) w_j. \end{array}$$

The first one is given by

$$\begin{array}{rccl} \widehat{P}_n\colon & H^{-1}(\Omega) & \longrightarrow & H^{-1}(\Omega) \\ & v & \longmapsto & [\phi \in H_0^1(\Omega) \mapsto \langle \widehat{P}_n v, \phi \rangle := \langle v, P_n \phi \rangle]. \end{array}$$

To define the second one, we need to introduce first some notation. We denote $A = -\Delta$ with homogeneous Dirichlet boundary condition, i.e. the isomorphism from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ (also seen as an unbounded operator in $L^2(\Omega)$). Now, we consider the domains of fractional powers of A,

$$D(A^{k/2}) = \{u \in L^2(\Omega) : \sum_{j \geq 1} \lambda_j^k (u, w_j)^2 < \infty\}.$$

Now, we are ready to define the second projection operator, which is given by

$$\begin{array}{rccl} \widetilde{P}_n\colon & L^q(\Omega) & \longrightarrow & D(A^{-k/2}) \\ & v & \longmapsto & [\phi \in D(A^{k/2}) \mapsto \langle \widetilde{P}_n(v), \phi \rangle_{D(A^{-k/2}), D(A^{k/2})} = (v, P_n \phi)]. \end{array}$$

Observe that $\widetilde{P}_n$ and $\widehat{P}_n$ are the continuous extensions in $L^q(\Omega)$ and $H^{-1}(\Omega)$ of P_n, respectively. Therefore, from now on we will denote both projections by P_n making an abuse of notation.

Observe that the sequence $\{P_n f(u_n)\}$ is bounded in $L^q(\tau,T;H^{-k}(\Omega))$ since the sequence $\{f(u_n)\}$ is bounded in $L^q(\tau,T;L^q(\Omega))$. We will show the details for the sake of completeness. It satisfies

$$\begin{aligned}\langle P_n f(u_n(t)), v\rangle_{D(A^{-k/2}),D(A^{k/2})} &= \int_\Omega f(u_n(t))P_n v dx\\ &\le |f(u_n(t))|_q |P_n v|_p\\ &\le C|f(u_n(t))|_q \|P_n v\|_{D(A^{k/2})}\\ &\le C|f(u_n(t))|_q \|v\|_{D(A^{k/2})},\end{aligned}$$

where $C>0$ is the constant of the continuous embedding $D(A^{s/2}) \hookrightarrow L^p(\Omega)$. Notice that $D(A^{k/2}) \hookrightarrow H^k(\Omega) \hookrightarrow L^p(\Omega)$, since Ω is a bounded open set of class C^k, with $k \ge 2$ such that $k \ge N(p-2)/(2p)$ (cf. [100, Proposition 6.18, p. 183], [57, Theorem 5, p. 323]).

Then, we have

$$\int_\tau^T \|P_n f(u_n(t))\|^q_{D(A^{-k/2})} dt \le C^q \int_\tau^T |f(u_n(t))|_q^q dt. \tag{3.12}$$

Therefore, the sequence $\{P_n f(u_n)\}$ is bounded in $L^q(\tau,T;D(A^{-k/2}))$. Bearing in mind that $L^q(\tau,T;D(A^{-k/2})) \hookrightarrow L^q(\tau,T;H^{-k}(\Omega))$ (cf. [100, Proposition 6.19, p. 184]), we have that the sequence $\{P_n f(u_n)\}$ is bounded in $L^q(\tau,T;H^{-k}(\Omega))$,

In addition we have that the sequence $\{P_n h\}$ is bounded in $L^2(\tau,T;H^{-1}(\Omega))$, since

$$\int_\tau^T \|P_n h(t)\|_*^2 dt \le \int_\tau^T \|h(t)\|_*^2 dt. \tag{3.13}$$

Then, bearing in mind (3.11), (3.12), (3.13) and the equality

$$\frac{\partial u_n}{\partial t}(t) - a(l(u_n(t)))\Delta u_n(t) = P_n f(u_n(t)) + P_n h(t) \quad \text{in } H^{-k}(\Omega) \quad \text{a.e. } t\in(\tau,T),$$

it holds that the sequence $\{u_n'\}$ is bounded in $L^q(\tau,T;H^{-k}(\Omega))$. Therefore, from compactness arguments and the Aubin-Lions lemma, there exist a subsequence of $\{u_n\}$ (relabeled the same) and $u \in L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$ with $u' \in L^2(\tau,T;H^{-1}(\Omega)) + L^q(\tau,T;L^q(\Omega))$, such that

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega)),\\ u_n \rightharpoonup u \quad \text{weakly in } L^2(\tau,T;H_0^1(\Omega)),\\ u_n \rightharpoonup u \quad \text{weakly in } L^p(\tau,T;L^p(\Omega)),\\ u_n' \rightharpoonup u' \quad \text{weakly in } L^q(\tau,T;H^{-k}(\Omega)),\\ u_n \to u \quad \text{strongly in } L^2(\tau,T;L^2(\Omega)),\\ a(l(u_n))u_n \rightharpoonup a(l(u))u \quad \text{weakly in } L^2(\tau,T;H_0^1(\Omega)),\\ f(u_n) \rightharpoonup f(u) \quad \text{weakly in } L^q(\tau,T;L^q(\Omega)),\end{cases}$$

for all $T > \tau$. The limits of the sequences $\{f(u_n)\}$ and $\{-a(l(u_n))\Delta u_n\}$ have been obtained by arguing analogously as done in the proof of Theorem 2.4, making use of [85, Lemme 1.3, p. 12].

Then, (3.6) follows taking limit when $n \to \infty$ in (3.8) and bearing in mind that $\cup_{n\in\mathbb{N}} V_n$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$.

Finally, to prove the existence of a weak solution to (3.1), we only need to check that $u(\tau) = u_\tau$, which makes complete sense since $u \in C([\tau, T]; L^2(\Omega))$. To do it we argue analogously as in the proof of Theorem 2.4. Consider $\varphi \in H^1(\tau, T)$ fixed with $\varphi(T) = 0$ and $\varphi(\tau) \neq 0$. Now, we multiply by φ in (3.8), integrate between τ and T, and pass to the limit. Comparing this limiting equation with the expression obtained multiplying (3.6) by φ and integrating between τ and T, we conclude that $u(\tau) = u_\tau$.

Step 2. Uniqueness of weak solution. Let u_1 and u_2 be two weak solutions to (3.1) corresponding to the initial datum u_τ^1, $u_\tau^2 \in L^2(\Omega)$. From the energy equality, we deduce

$$\frac{1}{2}\frac{d}{dt}|u_1(t) - u_2(t)|_2^2 + a(l(u_1(t)))\|u_1(t) - u_2(t)\|_2^2$$
$$= [a(l(u_2(t))) - a(l(u_1(t)))]((u_2(t), u_1(t) - u_2(t))) + (f(u_1(t)) - f(u_2(t)), u_1(t) - u_2(t))$$

a.e. $t \in [\tau, T]$.

Now, using that each solution to (3.1) belongs to $C([\tau, T]; L^2(\Omega))$, we have that $u_i(t) \in S$ for all $t \in [\tau, T]$ and $i = 1, 2$, where S is a bounded subset of $L^2(\Omega)$. In addition, taking into account that $l \in L^2(\Omega)$, we have that $\{l(u_i(t))\}_{t\in[\tau,T]} \subset [-R, R]$ for $i = 1, 2$, for some $R > 0$. Therefore, using (3.2), (3.3), the locally Lipschitz continuity of the function a and the Cauchy inequality (cf. [57, Appendix B, p. 622]), we have

$$\frac{d}{dt}|u_1(t) - u_2(t)|_2^2 \leq \frac{(L_a(R))^2|l|_2^2\|u_2(t)\|_2^2 + 4m\eta}{2m}|u_1(t) - u_2(t)|_2^2 \quad \text{a.e. } t \in (\tau, T),$$

where $L_a(R)$ is the Lipschitz constant of the function a in $[-R, R]$. Then, uniqueness follows. □

Observe that thanks to uniqueness of weak solution to (3.1), the whole sequence $\{u_n\}$ converges to u weakly in $L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ and weakly-star in $L^\infty(\tau, T; L^2(\Omega))$. Similarly, the whole sequence $\{u_n'\}$ converges to u' weakly in $L^q(\tau, T; H^{-k}(\Omega))$.

Now, the existence and uniqueness of strong solutions to (3.1) as well as the regularising effect of the equation will be proved. Recall that the strong solution u of a reaction-diffusion equation belongs to $L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C([\tau, T]; H_0^1(\Omega))$ for all $T > \tau$ (cf. [100, 5]). To obtain this regularity, it is necessary to prove first that $u' \in L^2(\tau, T; L^2(\Omega))$ and later, as a consequence it holds that $f(u) \in L^2(\tau, T; L^2(\Omega))$ and $u \in C([\tau, T]; H_0^1(\Omega))$. However, in this more complex framework, due to the presence of the nonlocal term, we cannot prove directly the regularity of u'. The reason is that it does not seem to provide useful information to multiply by u_n' the equation of the Galerkin approximations to obtain the boundedness of the sequence

$\{u_n'\}$ in $L^2(\tau, T; L^2(\Omega))$. In this case, it seems to be necessary to analyse first the regularity of $f(u)$, which belongs in general to $L^q(\tau, T; L^q(\Omega)$, and as a consequence we will be able to prove that u' belongs to $L^q(\tau, T; L^q(\Omega))$. Observe that this regularity is not enough to prove the continuity of the solution in $H_0^1(\Omega)$ with the strong topology, only with the weak one (cf. [108, Theorem 2.1, p. 544] or [111, Lemma 3.3, p. 74]).

To prove the existence of a more regular solution and the regularising effect of the equation, we need to establish a more regular setting. Instead of assuming that the function f satisfies (3.3), we will suppose that $f \in C^1(\mathbb{R})$ and verifies

$$f'(s) \leq \eta \quad \forall s \in \mathbb{R}. \tag{3.14}$$

Observe that unlike what happens in Chapter 4, to prove this result we do not impose any restriction on dimension of the domain Ω or additional strong assumptions on the function f like (4.13).

Theorem 3.4. *Assume that the function a is locally Lipschitz and fulfils (3.2), $f \in C^1(\mathbb{R})$ satisfies (3.4) and (3.14), $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and $l \in L^2(\Omega)$. Then, for any $u_\tau \in L^2(\Omega)$, the weak solution $u \in L^2(\tau+\varepsilon, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(\tau+\varepsilon, T; H_0^1(\Omega))$ and $u' \in L^q(\tau+\varepsilon, T; L^q(\Omega))$ for every $\varepsilon > 0$ and $T > \tau + \varepsilon$. In addition, if the initial datum $u_\tau \in H_0^1(\Omega)$, then there exists a unique strong solution u to (3.1) with $u' \in L^q(\tau, T; L^q(\Omega))$.*

Proof. We split the proof into two steps.

Step 1. Regularising effect. Under the above assumptions, the existence of a unique weak solution to (3.1) is guaranteed by Theorem 3.3. Now, we will prove that $u \in L^2(\tau+\varepsilon, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(\tau+\varepsilon, T; H_0^1(\Omega))$ for all $T > \tau+\varepsilon > \tau$.

Recall that from the energy equality for the Galerkin approximation u_n, at light of (3.2), we obtained (3.9). Using

$$(f(u_n(t)), u_n(t)) \leq \kappa|\Omega|,$$
$$(h(t), u_n(t)) \leq \frac{1}{2\lambda_1 m}|h(t)|_2^2 + \frac{m}{2}\|u_n(t)\|_2^2,$$

we deduce

$$\frac{d}{dt}|u_n(t)|_2^2 + m\|u_n(t)\|_2^2 \leq 2\kappa|\Omega| + \frac{1}{\lambda_1 m}|h(t)|_2^2 \quad \text{a.e. } t \in (\tau, T).$$

Integrating between τ and T, we obtain

$$|u_n(T)|_2^2 + m\int_\tau^T \|u_n(t)\|_2^2 dt \leq 2\kappa|\Omega|(T-\tau) + \frac{1}{\lambda_1 m}\int_\tau^T |h(t)|_2^2 dt + |u_\tau|_2^2.$$

In particular,

$$\int_\tau^T \|u_n(t)\|_2^2 dt \leq \frac{2\kappa|\Omega|(T-\tau)}{m} + \frac{1}{\lambda_1 m^2}\int_\tau^T |h(t)|_2^2 dt + \frac{1}{m}|u_\tau|_2^2. \tag{3.15}$$

Multiplying by $\lambda_j\varphi_{nj}$ in (3.8), summing from $j=1$ to n, and using (3.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2 + m|-\Delta u_n(t)|_2^2 \leq (f(u_n(t)) - f(0), -\Delta u_n(t)) + (f(0)+h(t), -\Delta u_n(t))$$

a.e. $t \in (\tau, T)$.

Making use of the integration-by-parts formula, (3.14) and the Cauchy inequality, it holds

$$\frac{d}{dt}\|u_n(t)\|_2^2 + m|-\Delta u_n(t)|_2^2 \leq 2\eta\|u_n(t)\|_2^2 + \frac{2}{m}|h(t)|_2^2 + \frac{2|\Omega|(f(0))^2}{m} \quad \text{a.e. } t \in (\tau, T). \tag{3.16}$$

Integrating the previous expression between s and t, where $\tau < s \leq t \leq T$, it holds

$$\begin{aligned} &\|u_n(t)\|_2^2 + m\int_s^t |-\Delta u_n(r)|_2^2 dr \\ &\leq \|u_n(s)\|_2^2 + 2\eta\int_\tau^T \|u_n(r)\|_2^2 dr + \frac{2}{m}\int_\tau^T |h(r)|_2^2 dr + \frac{2|\Omega|(f(0))^2(T-\tau)}{m}. \end{aligned} \tag{3.17}$$

In particular,

$$\|u_n(t)\|_2^2 \leq \|u_n(s)\|_2^2 + 2\eta\int_\tau^T \|u_n(r)\|_2^2 dr + \frac{2}{m}\int_\tau^T |h(r)|_2^2 dr + \frac{2|\Omega|(f(0))^2(T-\tau)}{m}.$$

Integrating in s between τ and t, we obtain

$$\begin{aligned} \|u_n(t)\|_2^2(t-\tau) \leq (1+2\eta T - 2\eta\tau)\int_\tau^T \|u_n(r)\|_2^2 dr + \frac{2(T-\tau)}{m}\int_\tau^T |h(r)|_2^2 dr \\ + \frac{2|\Omega|(f(0))^2(T-\tau)^2}{m}. \end{aligned}$$

Then, taking into account (3.15), it holds that the sequence $\{u_n\}$ is bounded in $L^\infty(\tau+\varepsilon, T; H_0^1(\Omega))$ for all $t \in [\varepsilon+\tau, T]$ with $\varepsilon \in (0, T-\tau)$.

On the other hand, taking $s = \tau+\varepsilon$ and $t = T$ in (3.17), and using that $\{u_n\}$ is bounded in $L^\infty(\tau+\varepsilon, T; H_0^1(\Omega))$, we can deduce that $\{u_n\}$ is bounded in $L^2(\tau+\varepsilon, T; H^2(\Omega)\cap H_0^1(\Omega))$. Therefore, thanks to the uniqueness of weak solution,

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau+\varepsilon, T; H_0^1(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^2(\tau+\varepsilon, T; H^2(\Omega)\cap H_0^1(\Omega)). \end{cases}$$

In addition, using (3.10) and the fact that $\{u_n\}$ is bounded in $L^2(\tau+\varepsilon, T; H^2(\Omega)\cap H_0^1(\Omega))$, it satisfies that $\{-a(l(u_n))\Delta u_n\}$ is bounded in $L^2(\tau+\varepsilon, T; L^2(\Omega))$. On the other hand, using (3.5) and taking into account that the sequence $\{u_n\}$ is bounded in $L^p(\tau, T; L^p(\Omega))$, it holds that $\{f(u_n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$. Therefore, it verifies that $u' \in L^q(\tau+\varepsilon, T; L^q(\Omega))$.

Step 2. Strong solution. Assume that $u_\tau \in H_0^1(\Omega)$. We need to check that $u \in L^2(\tau, T; H^2(\Omega)\cap H_0^1(\Omega)) \cap L^\infty(\tau, T; H_0^1(\Omega))$ and $u' \in L^q(\tau, T; L^q(\Omega))$ for all $T > \tau$.

Integrating (3.16) between τ and $t \in [\tau, T]$, we obtain

$$\begin{aligned} &\|u_n(t)\|_2^2 + m \int_s^t |-\Delta u_n(r)|_2^2 dr \\ &\leq \|u_\tau\|_2^2 + 2\eta \int_\tau^T \|u_n(r)\|_2^2 dr + \frac{2}{m} \int_\tau^T |h(r)|_2^2 dr + \frac{2|\Omega|(f(0))^2(T-\tau)}{m}. \end{aligned}$$

Then, taking into account that $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$ (see (3.15)), we deduce that $\{u_n\}$ is bounded in $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega))$. Thanks to the uniqueness of weak solutions, we have

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau, T; H_0^1(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)). \end{cases}$$

Thus, u is a strong solution in the sense of Definition 3.2.

On the other hand, since

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) \quad \text{in } L^q(\tau, T; L^q(\Omega)),$$

it holds that $u' \in L^q(\tau, T; L^q(\Omega))$. □

Observe that under the assumptions of Theorem 3.4, there exist functions f such that $f(u) \in L^2(\tau, T; L^2(\Omega))$ when u is the strong solution. Therefore, it can be proved that $u \in C([\tau, T]; H_0^1(\Omega))$. For example, if we consider the functions $f(s) = s - s^3$ or $f(s) = -|s|s^\gamma$ with $\gamma \leq 3$ when $N = 3$, it is not difficult to check that f verifies (3.4) and (3.14). In addition, using that $u \in L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(\tau, T; H_0^1(\Omega))$ and the fact that Ω is an open set of class C^2 at least, it holds that $u \in L^8(\tau, T; L^8(\Omega))$ thanks to the Sobolev embeddings and the interpolation results [116, Lemma II.4.1, p. 72]. Hence, $f(u) \in L^2(\tau, T; L^2(\Omega))$ and then, it verifies $u' \in L^2(\tau, T; L^2(\Omega))$. As a result $u \in C([\tau, T]; H_0^1(\Omega))$ (see Section 3.4 below for more details).

Now, we will show the Maximum Principle for (3.1), i.e. we will prove that if the initial datum $u_\tau \geq 0$ a.e. Ω then the solution of (3.1) fulfils that $u(t) \geq 0$ for all $t \geq 0$ under suitable assumptions. This is a natural expected behaviour in a biological framework.

Theorem 3.5. *Assume that the function a is locally Lipschitz and fulfils (3.2), the function $f \in C(\mathbb{R}; \mathbb{R}_+)$ satisfies (3.3) and (3.4), and the non-autonomous term $h \in L^2(\tau, T; H^{-1}(\Omega))$ verifies*

$$\langle h(t), v \rangle \geq 0 \quad a.e.\ t \in (\tau, T) \quad \forall v \in H_0^1(\Omega) \quad \textit{such that} \quad v \geq 0 \quad a.e.\ \Omega.$$

Then, if the initial datum $u_\tau \in L^2(\Omega)$ with $u_\tau \geq 0$ a.e. Ω, the weak solution u to (3.1) fulfils that $u(t) \geq 0$ for all $t \in [\tau, T]$.

Proof. Under the above assumptions, thanks to Theorem 3.3, there exists a unique weak solution u to (3.1). Then we have

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) \quad \text{in } L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega)).$$

Observe that since u is the weak solution to (3.1), u belongs to $L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$. Then, $(-u)^+ \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$. Therefore, we have

$$\begin{aligned} &\int_\tau^t \langle \frac{du}{ds}(s), (-u(s))^+ \rangle ds + \int_\tau^t a(l(u(s)))((u(s), (-u(s))^+))ds \\ &= \int_\tau^t (f(u(s)), (-u(s))^+)ds + \int_\tau^t \langle h(s), (-u(s))^+ \rangle ds \end{aligned} \tag{3.18}$$

for all $t \in [\tau, T]$.

Observe that

$$\begin{aligned} \langle \frac{d}{ds}u(s), (-u(s))^+ \rangle &= -\frac{1}{2}\frac{d}{ds}|(-u(s))^+|_2^2 \quad \text{a.e. } s \in (\tau, t), \\ ((u(s), (-u(s))^+)) &= (\nabla u(s), \nabla(-u(s)^+) = -\int_\Omega |\nabla(-u(s))^+|^2 ds \quad \text{a.e. } s \in (\tau, t), \\ (f(u(s)), (-u(s))^+) &\geq 0 \quad \text{a.e. } s \in (\tau, t), \\ \langle h(s), (-u(s))^+ \rangle &\geq 0 \quad \text{a.e. } s \in (\tau, t). \end{aligned}$$

Therefore, taking this into account, from (3.18) we deduce

$$\frac{1}{2}|(-u(t))^+|_2^2 + \int_\tau^t a(l(u(s)))|\nabla(-u(s))^+|_2^2 ds \leq \frac{1}{2}|(-u(\tau))^+|_2^2 \quad \forall t \in [\tau, T].$$

Observe that $\int_\tau^t a(l(u(s)))|\nabla(-u(s))^+|_2^2 ds \geq 0$ and $(-u_\tau)^+ = 0$ since $u_\tau \geq 0$ a.e. Ω. Then,

$$|(-u(t))^+|_2^2 \leq 0 \quad \forall t \in [\tau, T].$$

Thus, $(-u(t))^+ = 0$, i.e. $u(t) \geq 0$ for all $t \in [\tau, T]$. □

3.2 Analysis of the stationary problem

In Chapter 2, using a corollary of the Brouwer fixed point theorem, we have shown the existence of stationary solutions to (3.1) when the function f is globally Lipschitz (cf. Theorem 2.7). However, when the function f is more general, it is not possible to argue in this way.

In this section, the existence of at least one nontrivial stationary solution to (3.1) is analysed for the particular case $f : \left[0, \frac{1}{b}\right] \to \mathbb{R}$ given by $f(s) = bs - b^3 s^3$ with $b > 0$. Namely, we restrict ourselves to the case in which $b = 1$ for the sake of simplicity. To do that, we make use of a result proved by Chipot & Corrêa in [42] based on a fixed point argument.

Thereupon, a conditional result between the solution to (3.1) and two (assumed to exist) stationary solutions to (3.1) is established in a new setting in the spirit of those appeared in [44, 45]. To do this, we argue similarly to [44, Lemma 4.1].

In what follows, since we are dealing with stationary solutions, we will assume that the function h does not depend on time, i.e. $h \in H^{-1}(\Omega)$.

Now we consider the elliptic problem

$$\begin{cases} -a(l(u))\Delta u = f(u) + h & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

Definition 3.6. *A solution to (3.19) is a function $u^* \in H_0^1(\Omega) \cap L^p(\Omega)$ which fulfils*

$$a(l(u^*))((u^*, v)) = (f(u^*), v) + \langle h, v\rangle \quad \forall v \in H_0^1(\Omega) \cap L^p(\Omega).$$

The analysis of the stationary solutions to (3.1) and their stability is a difficult problem due to the nonlinearity $f(u)$. While many authors have been interested in (3.1) when f does not depend on the function u (for instance cf. [44, 45, 35, 48, 49, 50]), there are few studies in the more complex framework with $f(u)$. Recently, Simsen & Ferreira have analysed a particular case of problem (3.1), which contains a unique stationary solution, the trivial one, and the exponential decay of the solutions of the evolution problem towards the stationary one has been established (cf. [106, Theorem 6]).

Theorem 3.7 (Ferreira & Simsen)**.** *Assume that the function a is globally Lipschitz and satisfies (3.2), $f(s) = g(s) - |s|^{p-2}s$, where g is globally Lipschitz (with Lipschitz constant γ), $g(0) = 0$ and $p \geq 2$, $h \equiv 0$, $l \in L^2(\Omega)$ and $m > \gamma\lambda_1^{-1}$. Then, for each initial datum $u_\tau \in L^2(\Omega)$, the weak solution to (3.1) u fulfils*

$$|u(t)|_2 \leq |u_\tau|_2 e^{-(m\lambda_1 - \gamma)(t-\tau)} \quad \forall t \geq \tau.$$

In particular, this means that in the above situation there exists a unique stationary solution, the trivial one. However, when $h \not\equiv 0$ or f satisfies (3.4) with $\kappa > 0$, it is worth noting that, with the same kind of estimates, the exponential decay of the solution of the evolution problem towards 0 does not necessarily hold. In this more general framework of problem (3.19), the uniqueness of solution is not guaranteed.

Now, making use of a Chipot & Corrêa's result (cf. [42, Theorem 2.1]), the existence of at least one nontrivial solution to (3.19) will be proved. To that end, we need to assume that the function a fulfils not only (3.2), but also

$$0 < m \leq a(s) \leq M \quad \forall s \in \mathbb{R}, \tag{3.20}$$

where M is a positive constant.

Theorem 3.8 (Chipot & Corrêa)**.** *Suppose that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ satisfies (3.20), $\lambda_1 M < 1$, $f(s) = s - s^3$, $h \equiv 0$ and $l \in L^2(\Omega)$. Then, there exists at least one nontrivial solution to problem (3.19). Furthermore, any solution u^* to (3.19) fulfils*

$$\|u^*\|_2 \leq \alpha := \left(\frac{|\Omega|}{4m}\right)^{1/2}. \tag{3.21}$$

Proof. Making use of the Schauder fixed point theorem, we will prove the existence of nontrivial stationary solutions. First, we need first to do some calculations.

Analogously as it was denoted in the proof of Theorem 3.3, we represent by w_1 the first normalized eigenfunction of $-\Delta$, i.e. the solution to the problem

$$\begin{cases} -\Delta w_1 = \lambda_1 w_1 \quad \text{in } \Omega, \\ w_1 \in H_0^1(\Omega), \ w_1(x) \geq 0 \text{ a.e. } x \in \Omega, \ \displaystyle\int_\Omega (w_1(x))^2 dx = 1. \end{cases}$$

Consider ε_0 fixed such that $0 \leq \varepsilon_0 w_1(x) \leq 1$ a.e. $x \in \Omega$ and

$$\lambda_1 M \leq \frac{f(\varepsilon_0 w_1(x))}{\varepsilon_0 w_1(x)} \quad \text{a.e. } x \in \Omega. \tag{3.22}$$

Observe that ε_0 exists since $w_1 \in L^\infty(\Omega)$ and $\lambda_1 M < 1$. In what follows we denote $\tilde{u} = \varepsilon_0 w_1$. From (3.20) and (3.22), we deduce that for all $w \in L^2(\Omega)$

$$-\Delta \tilde{u} = \varepsilon_0 \lambda_1 w_1 \leq \frac{f(\varepsilon_0 w_1)}{M} = \frac{f(\tilde{u})}{M} \leq \frac{f(\tilde{u})}{a(l(w))} \quad \text{a.e. } \Omega.$$

Now we define the closed convex subset of $L^2(\Omega)$

$$K = \{v \in L^2(\Omega) : \varepsilon_0 w_1 \leq v \leq 1 \text{ a.e. } \Omega\}.$$

Then, we are ready to define the map

$$\begin{aligned} T \colon K &\to K \\ w &\mapsto Tw = u^*, \end{aligned}$$

where u^* is the solution to the problem

$$\begin{cases} -\Delta u^* + \dfrac{\mu u^*}{a(l(w))} = \dfrac{g(w)}{a(l(w))}, \\ u^* \in H_0^1(\Omega), \end{cases} \tag{3.23}$$

where $g(s) = f(s) + \mu s$ and μ is a positive constant such $g'(s) \geq 0$ for all $s \in (0, 1)$. Observe that if T has a fixed point, then it is a solution to (3.19).

The application T is well-defined, since $u^* \in K$. Namely, since g is a non-decreasing function in $(0, 1)$ and $w \in K$, it holds

$$\begin{aligned} -\Delta(u^* - 1) + \frac{\mu}{a(l(w))}(u^* - 1) &\leq 0 \quad \text{a.e. } \Omega, \\ -\Delta(\tilde{u} - u^*) + \frac{\mu}{a(l(w))}(\tilde{u} - u^*) &\leq 0 \quad \text{a.e. } \Omega. \end{aligned}$$

In addition, since $\tilde{u}, u^* \in H_0^1(\Omega)$, we have

$$\begin{aligned} u^* - 1 &\leq 0 \quad \text{on } \partial\Omega, \\ \tilde{u} - u^* &\leq 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then, making use of the weak Maximum Principle, we obtain

$$\tilde{u} \leq u^* \leq 1 \quad \text{a.e. } \Omega.$$

Therefore, $u^* \in K$.

To apply the Schauder fixed point Theorem, we need to check that T is continuous and compact. Firstly, we will prove that the application T is continuous. Let $\{w_n\} \subset K$ be such that

$$w_n \to w \quad \text{strongly in } L^2(\Omega). \tag{3.24}$$

Observe that $w \in K$ since K is a closed subset of $L^2(\Omega)$. We want to prove that $\{u_n^*\}$, where $u_n^* = Tw_n$ for all $n \geq 1$, fulfils

$$u_n^* \to u^* \quad \text{strongly in } L^2(\Omega). \tag{3.25}$$

Since u^* and u_n^* are solutions to (3.23) with w and w_n respectively, we have

$$\begin{aligned}((u^* - u_n^*, v)) + \frac{\mu}{a(l(w))}(u^* - u_n^*, v) = \mu \left(\frac{1}{a(l(w_n))} - \frac{1}{a(l(w))} \right) (u_n^*, v) \\ + \left(\frac{g(w)}{a(l(w))} - \frac{g(w_n)}{a(l(w_n))}, v \right),\end{aligned}$$

for all $v \in H_0^1(\Omega)$. Taking $v = u^* - u_n^*$ as a test function, and using (3.20), the fact that the sequence $\{u_n^*\} \subset K$ and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\|u^* - u_n^*\|_2^2 + \frac{\mu}{M}|u^* - u_n^*|_2^2 \leq \mu \left| \frac{1}{a(l(w_n))} - \frac{1}{a(l(w))} \right| |u^* - u_n^*|_2 \\ + \left| \frac{g(w)}{a(l(w))} - \frac{g(w_n)}{a(l(w_n))} \right|_2 |u^* - u_n^*|_2 |u_n^*|_2.\end{aligned}$$

Now, applying the Cauchy inequality, we have

$$\|u^* - u_n^*\|_2^2 \leq \frac{M\mu}{2} \left| \frac{1}{a(l(w_n))} - \frac{1}{a(l(w))} \right|^2 |u_n^*|_2^2 + \frac{M}{2\mu} \left| \frac{g(w)}{a(l(w))} - \frac{g(w_n)}{a(l(w_n))} \right|_2^2.$$

Observe that since $l \in L^2(\Omega)$ and $a \in C(\mathbb{R}; \mathbb{R}_+)$, we deduce

$$a(l(w_n)) \to a(l(w)). \tag{3.26}$$

Now, we will prove

$$g(w_n) \to g(w) \quad \text{strongly in } L^2(\Omega). \tag{3.27}$$

To that end, we only need to prove that

$$f(w_n) \to f(w) \quad \text{strongly in } L^2(\Omega). \tag{3.28}$$

Since $f \in C^1([0,1])$ and (3.24) holds, we deduce

$$f(w_n) \to f(w) \quad \text{a.e. } \Omega.$$

Moreover,

$$|f(w_n(x))|^2 = |f(w_n(x)) - f(0)|^2 = |f'(\xi_{[0,1]})|^2 |w_n(x)|^2 \quad \text{a.e. } \Omega.$$

Therefore, making use of the Lebesgue Dominated Convergence theorem, we deduce (3.28). Then, from (3.26) and (3.27), (3.25) holds.

Finally, we will prove that T is compact. Observe that

$$\sup_{s\in[0,1]} |g(s)| \leq \sup_{s\in[0,1]} |f(s)| + \mu =: C_g,$$

where C_g is a positive constant, since f is continuous and μ is a positive constant. Then, from this and (3.20), it satisfies

$$\begin{aligned} \|u^*\|_2^2 &\leq \frac{1}{a(l(w))}(g(w), u^*) \\ &\leq \frac{1}{m}\int_\Omega |g(w(x))||u^*(x)|dx \\ &\leq \frac{C_g|\Omega|^{1/2}}{\lambda_1^{1/2} m}\|u^*\|_2. \end{aligned}$$

Therefore,

$$\|u^*\|_2 \leq \frac{C_g|\Omega|^{1/2}}{\lambda_1^{1/2} m}.$$

From this and taking into account that the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we deduce that the application T is compact. Therefore, applying the Schauder fixed point theorem, we deduce that there exists at least one solution to (3.19).

In addition, observe that any stationary solution satisfies

$$\begin{aligned} a(l(u^*))\|u^*\|_2^2 &= \int_\Omega \left[(u^*(x))^2 - (u^*(x))^4\right] dx \\ &\leq \frac{|\Omega|}{4}. \end{aligned}$$

Thus, using (3.20), (3.21) holds. □

Now, considering again the general form for the function f and under new suitable assumptions, we show that any stationary solution to (3.1) is positive provided that its existence is guaranteed. To that end, we suppose that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ satisfies (3.2), $f \in C(\mathbb{R};\mathbb{R}_+)$, $h \in H^{-1}(\Omega)$ fulfils

$$h \not\equiv 0, \quad \langle h, v\rangle \geq 0 \quad \forall v \in H_0^1(\Omega) \;\text{ such that }\; v \geq 0 \text{ a.e. } \Omega, \tag{3.29}$$

and $l \in L^2(\Omega)$.

Then, we have the following Maximum Principle for (3.19).

Theorem 3.9. *Under the above assumptions, any solution u^* to (3.19) fulfils that $u^* \geq 0$ a.e. Ω.*

Proof. Since u^* is a stationary solution,

$$a(l(u^*))((u^*, v)) = (f(u^*), v) + \langle h, v\rangle \quad \forall v \in H_0^1(\Omega) \cap L^p(\Omega).$$

Observe that $(-u^*)^+ \in H_0^1(\Omega)$ since $u^* \in H_0^1(\Omega)$. Therefore, taking $v = (-u^*)^+$ as a test function in the above equality, making use of the assumptions made on the functions a, f and h, we have

$$((u^*, (-u^*)^+)) \geq 0.$$

However, we also have

$$((u^*, (-u^*)^+)) = -\int_\Omega |\nabla(-u^*(x))^+|^2 dx.$$

From these two expressions, we deduce that $(-u^*(x))^+ = 0$ a.e. $x \in \Omega$. Therefore, $u^*(x) \geq 0$ a.e. $x \in \Omega$. □

In what follows, we assume that there exist two stationary solutions u_1 and u_2 to the problem (3.1) which satisfy

$$u_1 \leq u_2, \quad u_1 \not\equiv u_2.$$

Furthermore, to define this new setting, we assume that the function a is locally Lipschitz, the function $f \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (3.4) and (3.3) with $\eta = \lambda_1 m$, i.e.

$$(f(r) - f(s))(r - s) \leq \lambda_1 m(r - s)^2 \quad \forall r, s \in \mathbb{R}, \tag{3.30}$$

and

$$l > 0 \quad \text{a.e. } \Omega. \tag{3.31}$$

Finally, we also assume that the function a satisfies

$$a(l(u_2)) \leq a(\xi) \leq a(l(u_1)) \quad \forall \xi \in [l(u_1), l(u_2)]. \tag{3.32}$$

Now we establish a comparison result amongst the weak solution to (3.1) and two (assumed to exist) stationary solutions. The idea of the proof is close to that in [44, Lemma 4.1]. We provide the details for the sake of completeness.

Theorem 3.10. *Assume that the function a is locally Lipschitz, and (3.2) and (3.32) hold, $f \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (3.4) and (3.30), $h \in H^{-1}(\Omega)$ satisfies (3.29), $l \in L^2(\Omega)$ fulfils (3.31). Then, if there exist two ordered stationary solutions $u_1 \leq u_2$ and $u_\tau \in L^2(\Omega)$ satisfies*

$$u_1 \leq u_\tau \leq u_2 \quad \textit{a.e. } \Omega, \tag{3.33}$$

it holds

$$0 \leq u_1 \leq u(t; \tau, u_\tau) \leq u_2 \quad \textit{a.e. } \Omega \quad \forall t \geq \tau. \tag{3.34}$$

Proof. Applying l to (3.33), bearing in mind that l fulfils (3.31), we deduce

$$l(u_1) \leq l(u_\tau) \leq l(u_2).$$

For short, we denote $u(\cdot;\tau,u_\tau)$, the weak solution to (3.1), by $u(\cdot)$. Now we define

$$\sigma = \{t \geq \tau : l(u(s)) \in [l(u_1), l(u_2)] \quad \forall s \in [\tau, t]\},$$

which is nonempty since $\tau \in \sigma$.

In what follows we denote

$$t_0 = \sup \sigma \in [\tau, \infty].$$

Then,

$$l(u(s)) \in [l(u_1), l(u_2)] \quad \forall s \in [\tau, t_0).$$

We split the proof of (3.34) into two steps.

Step 1. Our aim is to prove

$$u_1 \leq u(t) \leq u_2 \quad \text{a.e. } \Omega \quad \forall t \in [\tau, t_0). \tag{3.35}$$

Firstly, we show

$$u(t) \leq u_2 \quad \text{a.e. } \Omega \quad \forall t \in [\tau, t_0). \tag{3.36}$$

Since u is the weak solution to (3.1), u fulfils

$$\langle \frac{du}{dt}(t), v\rangle + a(l(u(t)))((u(t), v)) = (f(u(t)), v) + \langle h, v\rangle \quad \forall v \in H_0^1(\Omega) \cap L^p(\Omega)$$

a.e. $t > \tau$.

Then, introducing $\pm a(l(u(t)))((u_2, v))$ in the previous expression, we obtain

$$\langle \frac{du}{dt}(t), v\rangle + a(l(u(t)))((u(t) - u_2, v)) = (f(u(t)), v) - a(l(u(t)))((u_2, v)) + \langle h, v\rangle$$

a.e. $t > \tau$, for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$.

Since u_2 is a stationary solution to (3.1), it holds

$$-a(l(u(t)))((u_2, v)) = -\frac{a(l(u(t)))}{a(l(u_2))}[(f(u_2), v) + \langle h, v\rangle] \quad \forall v \in H_0^1(\Omega) \cap L^p(\Omega).$$

Therefore, it satisfies

$$\begin{aligned}\langle \frac{du}{dt}(t), v\rangle + a(l(u(t)))((u(t) - u_2, v)) = (f(u(t)), v) - \frac{a(l(u(t)))}{a(l(u_2))}(f(u_2), v) \\ + \frac{a(l(u_2)) - a(l(u(t)))}{a(l(u_2))}\langle h, v\rangle\end{aligned}$$

a.e. $t > \tau$.

Using [41, Lemma 11.2, p. 203], we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|(u(t)-u_2)^+|_2^2 + a(l(u(t)))\|(u(t)-u_2)^+\|_2^2\\ &= (f(u(t)),(u(t)-u_2)^+) - \frac{a(l(u(t)))}{a(l(u_2))}(f(u_2),(u(t)-u_2)^+)\\ &\quad + \frac{a(l(u_2))-a(l(u(t)))}{a(l(u_2))}\langle h,(u(t)-u_2)^+\rangle\end{aligned}$$

a.e. $t > \tau$.

Observe that

$$\frac{a(l(u_2))-a(l(u(t)))}{a(l(u_2))}\langle h,(u(t)-u_2)^+\rangle \leq 0 \quad \text{a.e. } t \in [\tau, t_0),$$

since

- $\langle h,(u(t)-u_2)^+\rangle \geq 0$ because h fulfils (3.29), $(u(t)-u_2)^+ \in H_0^1(\Omega)$ and $(u(t)-u_2)^+ \geq 0$ a.e. Ω,
- $\dfrac{a(l(u_2))-a(l(u(t)))}{a(l(u_2))} \leq 0$ a.e. $t \in [\tau, t_0)$ since the function a fulfils (3.32).

Moreover, observe that

$$-\frac{a(l(u(t)))}{a(l(u_2))}(f(u_2),(u(t)-u_2)^+) \leq -(f(u_2),(u(t)-u_2)^+) \quad \text{a.e. } t \in [\tau, t_0),$$

since

- $(f(u_2),(u(t)-u_2)^+) \geq 0$ because $f \in C(\mathbb{R};\mathbb{R}_+)$,
- $-\dfrac{a(l(u(t)))}{a(l(u_2))} \leq -1$ a.e. $t \in [\tau, t_0)$ since the function a satisfies (3.32).

From this, we deduce

$$\frac{1}{2}\frac{d}{dt}|(u(t)-u_2)^+|_2^2 + a(l(u(t)))\|(u(t)-u_2)^+\|_2^2 \leq (f(u(t))-f(u_2),(u(t)-u_2)^+) \tag{3.37}$$

a.e. $t \in [\tau, t_0)$.

In what follows we denote $\Omega_2 = \{x \in \Omega : u(x,t) \geq u_2(x)\}$. Observe that since f satisfies (3.30), we have

$$\begin{aligned}(f(u(t))-f(u_2),(u(t)-u_2)^+) &= \int_{\Omega_2}(f(u(x,t))-f(u_2(x)))(u(x,t)-u_2(x))dx\\ &\leq \lambda_1 m\int_{\Omega_2}(u(x,t)-u_2(x))^2dx\\ &\leq m\|(u(t)-u_2)^+\|_2^2.\end{aligned} \tag{3.38}$$

Then, making use of (3.2) and (3.38), from (3.37) we deduce

$$\frac{d}{dt}|(u(t)-u_2)^+|_2^2 \leq 0 \quad \text{a.e. } t \in [\tau, t_0),$$

whence (3.36) follows.

Now, we will prove

$$u_1 \leq u(t) \quad \text{a.e. } \Omega \quad \forall t \in [\tau, t_0). \tag{3.39}$$

Analogously to the previous argument, it satisfies

$$\begin{aligned}&\langle \frac{d}{dt}(u_1 - u(t)), v\rangle + a(l(u(t)))((u_1 - u(t), v))\\ =& -(f(u(t)), v) + \frac{a(l(u(t)))}{a(l(u_1))}(f(u_1), v) + \frac{a(l(u(t))) - a(l(u_1))}{a(l(u_1))}\langle h, v\rangle \quad \text{a.e. } t > \tau.\end{aligned}$$

Again, using [41, Lemma 11.2, p. 203], we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|(u_1 - u(t))^+|_2^2 + a(l(u(t)))\|(u_1 - u(t))^+\|_2^2\\ =& -(f(u(t)), (u_1 - u(t))^+) + \frac{a(l(u(t)))}{a(l(u_1))}(f(u_1), (u_1 - u(t))^+)\\ &+ \frac{a(l(u(t))) - a(l(u_1))}{a(l(u_1))}\langle h, (u_1 - u(t))^+\rangle \quad \text{a.e. } t > \tau.\end{aligned}$$

Taking into account (3.2), together with

- $\dfrac{a(l(u(t))) - a(l(u_1))}{a(l(u_1))}\langle h, (u_1 - u(t))^+\rangle \leq 0$ a.e. $t \in [\tau, t_0)$,
- $\dfrac{a(l(u(t)))}{a(l(u_1))}(f(u_1), (u_1 - u(t))^+) \leq (f(u_1), (u_1 - u(t))^+)$ a.e. $t \in [\tau, t_0)$,

we deduce

$$\frac{1}{2}\frac{d}{dt}|(u_1 - u(t))^+|_2^2 + m\|(u_1 - u(t))^+\|_2^2 \leq (f(u_1) - f(u(t)), (u_1 - u(t))^+)$$

a.e. $t \in [\tau, t_0)$.

Analogously as we argued above,

$$(f(u_1) - f(u(t)), (u_1 - u(t))^+) \leq m\|(u_1 - u(t))^+\|_2^2.$$

Therefore,

$$\frac{d}{dt}|(u_1 - u(t))^+|_2^2 \leq 0 \quad \text{a.e. } t \in [\tau, t_0).$$

Using the Gronwall Lemma and taking into account that $(u_1 - u(\tau))^+ = 0$, we obtain (3.39). As a result it holds (3.35).

Step 2. Let us prove that $t_0 = \infty$. We argue by contradiction. Suppose that $t_0 < \infty$. Then, since the function $l(u(\cdot))$ is continuous [observe that $u \in C([\tau, \infty); L^2(\Omega))$], it holds that $l(u(t_0)) \in \{l(u_1), l(u_2)\}$. Therefore, $t_0 \in \sigma = [\tau, t_0]$.

Assume that $l(u(t_0)) = l(u_1)$. Then, let us prove that $u(t_0) \equiv u_1$ a.e. Ω. The relationship between u_1 and $u(t_0)$ is one and only one of the following:

(i) $u(t_0) \equiv u_1$ a.e. Ω.

(ii) $u(t_0) \geq u_1$ a.e. Ω and $u(t_0) \not\equiv u_1$.

(iii) $u(t_0) \leq u_1$ a.e. Ω and $u(t_0) \not\equiv u_1$.

(iv) $u(t_0) > u_1$ in a non-zero measure subset of Ω and $u(t_0) < u_1$ in a non-zero measure subset of Ω.

The relationships (ii) and (iii) are not possible due to (3.31) and $l(u(t_0) - u_1) = 0$. The relationship (iv) is neither possible due to (3.35) and the continuity of the function u. Then, it satisfies that $u(t_0) \equiv u_1$ a.e. Ω.

Analogously, if $l(u(t_0)) = l(u_2)$, then it fulfils that $u(t_0) \equiv u_2$ a.e. Ω.

Therefore, either $u(t_0) = u_1$ or $u(t_0) = u_2$. Taking into account that the problem (3.1) possesses a unique weak solution and the fact of that u_1 and u_2 are stationary solutions (in the weak sense), it holds that either $u(t) = u_1$ or $u(t) = u_2$ for all $t \geq t_0$, which contradicts that $t_0 = \sup \sigma < \infty$. □

3.3 Existence of pullback attractors in $L^2(\Omega)$

Although in the previous section we provide some information concerning the existence and uniqueness of stationary solution and the decay of evolutionary solutions towards this steady state, the result is for some particular choices of f (cf. Theorem 3.7). In Theorem 3.8, we showed that there might exist multiple nontrivial stationary solutions, which in some cases (f independent of u) lead to interesting results (e.g., cf. [44, 45]) comparing evolutionary solutions on intervals. We have extended those results to the case f depending on u (cf. Theorem 3.10).

In this section, we get rid of the special (and somehow restrictive assumptions) imposed in Section 3.2. We aim to obtain more general results concerning the long-time behaviour of the solutions in the initial setting of Section 3.1. Namely, the existence of minimal pullback attractors in $L^2(\Omega)$ is analysed below.

Thanks to Theorem 3.3, we can define a process $U : \mathbb{R}^2_d \times L^2(\Omega) \to L^2(\Omega)$ as

$$U(t, \tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in L^2(\Omega) \quad \forall \tau \leq t, \tag{3.40}$$

where $u(\cdot; \tau, u_\tau)$ denotes the weak solution to (3.1).

In addition, the following result shows that U is continuous from $L^2(\Omega)$ into itself. We omit the proof because it is analogous to the proof of Proposition 2.10.

Proposition 3.11. *Assume that the function a is locally Lipschitz and (3.2) holds, $f \in C(\mathbb{R})$ fulfils (3.3) and (3.4), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, the process U is continuous on $L^2(\Omega)$.*

To define a suitable tempered universe in $\mathcal{P}(L^2(\Omega))$ for our purposes, we first establish the following estimate.

Lemma 3.12. *Under the assumptions made on Proposition 3.11, if $u_\tau \in L^2(\Omega)$, then the solution u to (3.1) satisfies*

$$|u(t)|_2^2 \leq e^{-\mu(t-\tau)}|u_\tau|_2^2 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu t}}{2m-\mu\lambda_1^{-1}} \int_\tau^t e^{\mu s}\|h(s)\|_*^2 ds \quad \forall t \geq \tau$$

for any $\mu \in (0, 2\lambda_1 m)$.

Proof. Applying the Cauchy-Schwartz inequality, (3.2) and (3.4) to the energy equality, we obtain

$$\frac{d}{dt}|u(t)|_2^2 + 2m\|u(t)\|_2^2 \leq 2\kappa|\Omega| + 2\|h(t)\|_*\|u(t)\|_2 \quad \text{a.e. } t \geq \tau.$$

Now, adding $\pm\mu|u(t)|_2^2$, and using the Poincaré and Cauchy inequalities, we have

$$\frac{d}{dt}|u(t)|_2^2 + \mu|u(t)|_2^2 \leq 2\kappa|\Omega| + \frac{1}{2m-\mu\lambda_1^{-1}}\|h(t)\|_*^2 \quad \text{a.e. } t \geq \tau.$$

Multiplying by $e^{\mu t}$ and integrating between τ and t, the result follows. □

Then we are ready to define a suitable tempered universe in $\mathcal{P}(L^2(\Omega))$.

Definition 3.13. *For each $\mu > 0$, $\mathcal{D}_\mu^{L^2}$ denotes the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that*

$$\lim_{\tau \to -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|_2^2 \right) = 0.$$

From the above estimate, if h fulfils a suitable growth condition (see (3.41) [compare to (2.32)]), it is straightforward to conclude the existence of an absorbing family for the tempered universe $\mathcal{D}_\mu^{L^2}$. Namely, we have the following result.

Proposition 3.14. *Under the assumptions of Proposition 3.11, if h also satisfies that there exists some $\mu \in (0, 2\lambda_1 m)$ such that*

$$\int_{-\infty}^0 e^{\mu s}\|h(s)\|_*^2 ds < \infty, \tag{3.41}$$

the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t))$, where

$$R_{L^2}(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu t}}{2m-\mu\lambda_1^{-1}} \int_{-\infty}^t e^{\mu s}\|h(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_\mu^{L^2}$-absorbing for the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$. Moreover, $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$.

Proof. Consider fixed $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$. Thanks to Lemma 3.12 and (3.40), we have

$$|U(t,\tau)u_\tau|_2^2 \le e^{-\mu(t-\tau)}|u_\tau|_2^2 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu t}}{2m-\mu\lambda_1^{-1}}\int_{-\infty}^t e^{\mu s}\|h(s)\|_*^2 ds \tag{3.42}$$

for all $u_\tau \in D(\tau)$ and $\tau \le t$.
Since $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_0(\widehat{D},t) < t$ such that

$$e^{-\mu(t-\tau)}|u_\tau|_2^2 \le 1 \quad \forall u_\tau \in D(\tau) \quad \forall \tau \le \tau_0(\widehat{D},t). \tag{3.43}$$

Therefore, plugging (3.43) in (3.42), we obtain

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \forall \tau \le \tau_0(\widehat{D},t).$$

Finally, using (3.41) it is not difficult to deduce that $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$. □

Then, to prove the existence of the minimal pullback attractor for the process $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$, we only need to check the pullback asymptotic compactness in $L^2(\Omega)$ for the universe $\mathcal{D}_\mu^{L^2}$. To that end, we firstly establish the following result, which is the equivalent to Lemma 2.14 in the setting of this chapter. Observe that the proofs are very close. Nevertheless, we provide the details for the sake of completeness.

Lemma 3.15. *Under the assumptions of Proposition 3.14, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_1(\widehat{D},t) < t-2$ such that, for any $\tau \le \tau_1(\widehat{D},t)$ and any $u_\tau \in D(\tau)$,*

$$\begin{cases} |u(r;\tau,u_\tau)|_2^2 \le \rho_1(t) \quad \forall r \in [t-2,t], \\ \displaystyle\int_{r-1}^r \|u(s;\tau,u_\tau)\|_2^2 ds \le \rho_2(t) \quad \forall r \in [t-1,t], \\ \displaystyle\int_{r-1}^r |u(s;\tau,u_\tau)|_p^p ds \le \frac{m}{2\alpha_2}\rho_2(t) \quad \forall r \in [t-1,t], \end{cases} \tag{3.44}$$

where

$$\rho_1(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu(t-2)}}{2m-\mu\lambda_1^{-1}}\int_{-\infty}^t e^{\mu s}\|h(s)\|_*^2 ds,$$

$$\rho_2(t) = \frac{1}{m}\left(\rho_1(t) + 2\kappa|\Omega| + \frac{1}{m}\max_{r\in[t-1,t]}\int_{r-1}^r \|h(s)\|_*^2 ds\right).$$

Proof. The first inequality in (3.44) as well as the expression of ρ_1 follow by arguing as in the proof of Lemma 3.12, if $\tau \le \tau_1(\widehat{D},t) < t-2$ (far enough pull back in time) due to our choice of tempered universe, taking into account (3.41). Notice that indeed this estimate also holds for the Galerkin approximations, which have already been used in Section 3.1.

For the other two inequalities in (3.44), we will prove them for the Galerkin approximations, and then, passing to the limit, we will obtain the same estimates for the solution.

Multiplying by φ_{nj} in (3.8), summing from $j = 1$ to n, and using (3.2), (3.4) and the Cauchy inequality, we deduce

$$\frac{d}{ds}|u_n(s)|_2^2 + m\|u_n(s)\|_2^2 + 2\alpha_2|u_n(s)|_p^p \leq 2\kappa|\Omega| + \frac{1}{m}\|h(s)\|_*^2 \quad \text{a.e. } s > \tau.$$

Now, integrating the above expression between $r-1$ and r, we have

$$\begin{aligned} &|u_n(r)|_2^2 + m\int_{r-1}^{r}\|u_n(s)\|_2^2 ds + 2\alpha_2\int_{r-1}^{r}|u_n(s)|_p^p ds \\ &\leq |u_n(r-1)|_2^2 + 2\kappa|\Omega| + \frac{1}{m}\int_{r-1}^{r}\|h(s)\|_*^2 ds \end{aligned} \tag{3.45}$$

for all $\tau \leq r-1$.

Then, from (3.45) we obtain for any $n \geq 1$

$$\int_{r-1}^{r}\|u_n(s)\|_2^2 ds \leq \rho_2(t) \quad \forall r \in [t-1,t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_1(\widehat{D},t), \tag{3.46}$$

where $\rho_2(t)$ is given in the statement. Taking inferior limit in (3.46) and using the well-known fact that u_n converge to $u(\cdot;\tau,u_\tau)$ weakly in $L^2(r-1,r;H_0^1(\Omega))$ for all $r \in [t-1,t]$ (cf. Theorem 3.3), the second inequality in (3.44) holds.

In addition, from (3.45) we also deduce that for any $n \geq 1$

$$\int_{r-1}^{r}|u_n(s)|_p^p ds \leq \frac{m}{2\alpha_2}\rho_2(t) \quad \forall r \in [t-1,t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_1(\widehat{D},t).$$

Now, taking inferior limit in the above expression and bearing in mind that u_n converge to $u(\cdot;\tau,u_\tau)$ weakly in $L^p(r-1,r;L^p(\Omega))$ for all $r \in [t-1,t]$ (cf. Theorem 3.3), the last inequality in (3.44) holds. □

Now we will prove that the process U is pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact using an energy method with continuous functions analogous to the one used in the proof of Proposition 2.15.

Proposition 3.16. *Under the assumptions of Proposition 3.14, the process* $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$ *is pullback* $\mathcal{D}_\mu^{L^2}$*-asymptotically compact.*

Proof. Let us fixed $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, a sequence $\{\tau_n\} \subset (-\infty, t-2]$ with $\tau_n \to -\infty$, and $u_{\tau_n} \in D(\tau_n)$ for all n. Our aim is to prove that the sequence $\{u(t;\tau_n,u_{\tau_n})\}$ is relatively compact in $L^2(\Omega)$. For short we denote $u^n(\cdot) = u(\cdot;\tau_n,u_{\tau_n})$.

Thanks to Lemma 3.15 we know that there exists $\tau_1(\widehat{D},t) < t-2$ satisfying that, if $n_1 \geq 1$ is such that $\tau_n \leq \tau_1(\widehat{D},t)$ for all $n \geq n_1$, $\{u^n\}_{n\geq n_1}$ is bounded in $L^\infty(t-2,t;L^2(\Omega)) \cap L^2(t-2,t;H_0^1(\Omega)) \cap L^p(t-2,t;L^p(\Omega))$. Besides, from this, making use of the continuity of the function a and bearing in mind that $l \in L^2(\Omega)$, $\{-a(l(u^n))\Delta u^n\}_{n\geq n_1}$ is bounded in $L^2(t-2,t;H^{-1}(\Omega))$. On the other hand, from (3.5), using that the sequence $\{u_n\}$ is bounded in $L^p(t-2,t;L^p(\Omega))$, we deduce that $\{f(u^n)\}_{n\geq n_1}$ is bounded in $L^q(t-2,t;L^q(\Omega))$. As a consequence of the above uniform estimates, it holds that $\{(u^n)'\}_{n\geq n_1}$ is bounded in $L^2(t-2,t;H^{-1}(\Omega)) + L^q(t-2,t;L^q(\Omega))$. Then, using the Aubin-Lions compactness Lemma, analogously as in the proof of Theorem 3.3, it holds that there exists $u \in L^\infty(t-2,t;L^2(\Omega)) \cap L^2(t-2,t;H_0^1(\Omega)) \cap L^p(t-2,t;L^p(\Omega))$, with $u' \in L^2(t-2,t;H^{-1}(\Omega)) + L^q(t-2,t;L^q(\Omega))$, such that for a subsequence (relabeled the same) it satisfies

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(t-2,t;L^2(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^2(t-2,t;H_0^1(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^p(t-2,t;L^p(\Omega)), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(t-2,t;H^{-1}(\Omega)) + L^q(t-2,t;L^q(\Omega)), \\ u^n \to u \quad \text{strongly in } L^2(t-2,t;L^2(\Omega)), \\ u^n(s) \to u(s) \quad \text{strongly in } L^2(\Omega) \quad \text{a.e. } s \in (t-2,t), \\ f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^q(t-2,t;L^q(\Omega)), \\ -a(l(u^n))\Delta u^n \rightharpoonup -a(l(u))\Delta u \quad \text{weakly in } L^2(t-2,t;H^{-1}(\Omega)), \end{cases} \tag{3.47}$$

where the last two convergences have been obtained arguing in the same way as in the proof of Theorem 2.4. In addition, observe that $u \in C([t-2,t];L^2(\Omega))$ and making use of (3.47), u fulfils (3.6) in the interval $(t-2,t)$.

From (3.47) we can also deduce that $\{u^n\}_{n\geq n_1}$ is equicontinuous in $H^{-1}(\Omega) + L^q(\Omega)$ on $[t-2,t]$. To do this we argue similarly as in the proof of Proposition 2.15.

Moreover, we have that the sequence $\{u^n\}_{n\geq n_1}$ is bounded in $C([t-2,t];L^2(\Omega))$ and the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega) + L^q(\Omega)$ is compact. Therefore, applying the Arzela-Ascoli theorem, we have (for another sequence, relabeled again the same) that

$$u^n \to u \quad \text{strongly in } C([t-2,t];H^{-1}(\Omega) + L^q(\Omega)). \tag{3.48}$$

Thanks to the boundedness of $\{u^n\}_{n\geq n_1}$ in $C([t-2,t];L^2(\Omega))$, for any sequence $\{s_n\} \subset [t-2,t]$ with $s_n \to s_*$, we obtain

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } L^2(\Omega), \tag{3.49}$$

where we have used (3.48) to identify the weak limit.

In this proof, we will show not only that the sequence $\{u^n(t)\}_{n\geq n_1}$ is relatively compact, but also the stronger convergence

$$u^n \to u \quad \text{strongly in } C([t-1,t];L^2(\Omega)). \tag{3.50}$$

We argue by contradiction.

On the one hand, suppose that there exist $\varepsilon > 0$, a sequence $\{t_n\} \subset [t-1,t]$, without loss of generality converging to some t_*, with

$$|u^n(t_n) - u(t_*)|_2 \geq \varepsilon \quad \forall n \geq 1. \tag{3.51}$$

On the other hand, applying the Cauchy inequality, (3.2) and (3.4) to the energy equality (3.7), we deduce

$$|z(s)|_2^2 \leq |z(r)|_2^2 + 2\kappa|\Omega|(s-r) + \frac{1}{2m}\int_r^s \|h(\xi)\|_*^2 d\xi \quad \forall t-2 \leq r \leq s \leq t,$$

where z may be replaced by u or any u^n.

Now we define the following functions

$$J_n(s) = |u^n(s)|_2^2 - 2\kappa|\Omega|s - \frac{1}{2m}\int_{t-2}^s \|h(r)\|_*^2 dr,$$
$$J(s) = |u(s)|_2^2 - 2\kappa|\Omega|s - \frac{1}{2m}\int_{t-2}^s \|h(r)\|_*^2 dr,$$

which are continuous and non-increasing on $[t-2,t]$. Moreover, observe that from (3.47) we obtain

$$J_n(s) \to J(s) \quad \text{a.e. } s \in (t-2,t).$$

Therefore, there exists a sequence $\{\tilde{t}_k\} \subset (t-2,t_*)$ such that $\tilde{t}_k \to t_*$ when $k \to \infty$ and such that the above convergence holds for any $\tilde{t}_k$.

Then consider an arbitrary value $\epsilon > 0$ fixed. Since the function J is continuous on $[t-2,t]$, there exists $k(\epsilon) \geq 1$ such that

$$|J(\tilde{t}_k) - J(t_*)| < \frac{\epsilon}{2} \quad \forall k \geq k(\epsilon).$$

Now consider $n(\epsilon) \geq 1$ such that

$$t_n \geq \tilde{t}_{k(\epsilon)} \quad \text{and} \quad |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| < \frac{\epsilon}{2} \quad \forall n \geq n(\epsilon).$$

Therefore, as the functions J_n are non-increasing, making use of the previous estimates, we have

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k(\epsilon)}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k(\epsilon)}) - J(\tilde{t}_{k(\epsilon)})| + |J(\tilde{t}_{k(\epsilon)}) - J(t_*)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq n(\epsilon). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it yields $\limsup_{n\to\infty} J_n(t_n) \leq J(t_*)$. Therefore, taking into account the expressions of J and all J_n, we deduce that $\limsup_{n\to\infty} |u^n(t_n)|_2 \leq |u(t_*)|_2$. Then, from this, together with (3.49), we conclude that $u^n(t_n)$ converge to $u(t_*)$ strongly in $L^2(\Omega)$, which is contradictory with (3.51). Therefore, (3.50) holds. □

As a consequence, we have the main result of this section. The proof of this result is quite close to the proof of Theorem 2.16. We show the details for the sake of completeness.

Theorem 3.17. *Assume that the function a is locally Lipschitz and (3.2) holds, $f \in C(\mathbb{R})$ satisfies (3.3) and (3.4), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ fulfils condition (3.41) for some $\mu \in (0, 2\lambda_1 m)$ and $l \in L^2(\Omega)$. Then, there exist the minimal pullback $\mathcal{D}_F^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and the minimal pullback $\mathcal{D}_\mu^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ for the process $U : \mathbb{R}^2_d \times L^2(\Omega) \to L^2(\Omega)$. Furthermore, the family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$ and it holds*

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \subset \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$

Besides, if the function h fulfils

$$\sup_{s \le 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu r} \|h(r)\|_*^2 dr \right) < \infty, \tag{3.52}$$

then

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \quad \forall t \in \mathbb{R}.$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$, $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and the first relation between both attractors is a consequence of Corollary 1.15. Indeed, the continuity of the process (cf. Proposition 3.11), the relationship $\mathcal{D}_F^{L^2} \subset \mathcal{D}_\mu^{L^2}$, the existence of an absorbing family (cf. Proposition 3.14) and the asymptotic compactness (cf. Proposition 3.16) hold.

The relation between the family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ and $\widehat{D}_0$ is a direct consequence of Theorem 1.13. In addition, the family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$ since $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$, $D_0(t)$ is a closed subset of $L^2(\Omega)$ for all $t \in \mathbb{R}$ and the universe $\mathcal{D}_\mu^{L^2}$ is inclusion-closed.

Finally, under assumption (3.52), we deduce that $\cup_{t \le T} R_{L^2}(t)$ is bounded for each $T \in \mathbb{R}$, where R_{L^2} is given in Proposition 3.14. Thus, making use of Corollary 1.15, we deduce that both families of attractors coincide. □

3.4 Existence of pullback attractors in $H_0^1(\Omega)$

The goal of this section is to improve the results of the previous one, by establishing attraction in $H_0^1(\Omega)$. In addition, we establish relationships between these new pullback attractors and those analysed in Theorem 3.17.

Under the assumptions made in Section 3.1, namely in Theorem 3.4, we cannot guarantee the existence of a more regular energy equality for strong solutions (cf. Definition 3.2) because in general u' does not belong to $L^2(\tau, T; L^2(\Omega))$. While in reaction-diffusion equations, the regularity of u' can be obtained independently of the regularity of $f(u)$ (cf. [5, Chapter 2, p. 32]), in nonlocal problems like (3.1), u' inherits the regularity of $f(u)$, which in general belongs to $L^q(\tau, T; L^q(\Omega))$.

To guarantee that $f(u) \in L^2(\tau, T; L^2(\Omega))$, it is enough for instance to assume that $|f'(s)| \le C$ for all $s \in \mathbb{R}$, thanks to the mean value theorem, having u the regularity of the weak solutions to (4.1). However, the sublinear case has already been studied with detail in Chapter 2.

In this chapter, to study the asymptotic behaviour of the solutions making use of the theory of attractors, we assume

$$f(u) \in L^2(\tau, T; L^2(\Omega)) \quad \forall u \in L^2(\tau, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(\tau, T; H_0^1(\Omega)). \tag{3.53}$$

Although not every function satisfying (3.4) fulfils (3.53), there exists a wide range of functions which do. For instance, consider the function $f(s) = s - s^3$ when $N = 3$ or $f(s) = -s|s|^\gamma$ where $\gamma \in (0,3]$ when $N = 3$, $\gamma \in (0,2)$ when $N = 4$, and $\gamma \in (0, 4/(N-2)]$ when $N \geq 5$. Observe that using the Sobolev embeddings and interpolation results [116, Lemma II.4.1, p. 72], it can be checked that this last function also satisfies (3.53). In this case, we can deduce some information about the growth of $f(u)$, since

$$\|f(u)\|^2_{L^2(\tau,T;L^2(\Omega))} \leq C_f \|u\|^{2\tilde{b}}_{L^\infty(\tau,T;H^1_0(\Omega))} \|u\|^{2\hat{b}}_{L^2(\tau,T;H^2(\Omega)\cap H^1_0(\Omega))}, \tag{3.54}$$

for some $\hat{b}$, $\tilde{b}$, and $C_f > 0$. Namely, given u belonging to $L^2(\tau,T;H^2(\Omega)\cap H^1_0(\Omega)) \cap L^\infty(\tau,T;H^1_0(\Omega))$, using interpolation [116, Lemma II.4.1, p. 72] and the Sobolev embeddings, we deduce that $\hat{b} = (\gamma+1)\theta$ and $\tilde{b} = (\gamma+1)(1-\theta)$, where $\theta \in [0,1]$ is an interpolation exponent between Sobolev spaces, and

$$\begin{aligned}
\|f(u)\|^2_{L^2(\tau,T;L^2(\Omega))} &= \|u\|^{2\gamma+2}_{L^{2\gamma+2}(\tau,T;L^{2\gamma+2}(\Omega))} \\
&\leq \|u\|^{2\tilde{b}}_{L^\infty(\tau,T;L^{p(N,H^1_0)}(\Omega))} \|u\|^{2\hat{b}}_{L^2(\tau,T;L^{p(N,H^2\cap H^1_0)}(\Omega))} \\
&\leq C^{2\tilde{b}}_{H^1_0} C^{2\hat{b}}_{H^2\cap H^1_0} \|u\|^{2\tilde{b}}_{L^\infty(\tau,T;H^1_0(\Omega))} \|u\|^{2\hat{b}}_{L^2(\tau,T;H^2(\Omega)\cap H^1_0(\Omega))} \\
&= C_f \|u\|^{2\tilde{b}}_{L^\infty(\tau,T;H^1_0(\Omega))} \|u\|^{2\hat{b}}_{L^2(\tau,T;H^2(\Omega)\cap H^1_0(\Omega))},
\end{aligned}$$

where $C_{H^1_0}$ is the constant of the continuous embedding $L^\infty(\tau,T;H^1_0(\Omega)) \hookrightarrow L^\infty(\tau,T;L^{p(N,H^1_0)}(\Omega))$ and $C_{H^2\cap H^1_0}$ is the constant of the embedding $L^2(\tau,T;H^2(\Omega)\cap H^1_0(\Omega)) \hookrightarrow L^2(\tau,T;L^{p(N,H^2\cap H^1_0)}(\Omega))$. Then, assuming that the non-autonomous term $h \in L^2(\tau,T;L^2(\Omega))$, we can deduce that $u' \in L^2(\tau,T;L^2(\Omega))$ for all $T > \tau$. Therefore, considering a more regular initial datum $u_\tau \in H^1_0(\Omega)$, the associated strong solution $u \in C([\tau,\infty);H^1_0(\Omega))$ and the following energy equality holds

$$\|u(t)\|_2^2 + 2\int_s^t a(l(u(r)))|-\Delta u(r)|_2^2 dr = \|u(s)\|_2^2 + 2\int_s^t (f(u(r)) + h(r), -\Delta u(r)) dr, \tag{3.55}$$

for all $\tau \leq s \leq t$.

Observe that we can assume (3.54) thanks to the domain Ω is smooth, since this allows to prove the existence of strong solutions and therefore, we can use this stronger regularity to make the most of the fact that $f(u) \in L^2(\tau,T;L^2(\Omega))$. Namely, we can make use of the Sobolev embedding $L^2(\tau,T;H^2(\Omega)\cap H^1_0(\Omega)) \hookrightarrow L^2(\tau,T;L^{p(N,H^2\cap H^1_0)}(\Omega))$ to deal with a wider range of functions f. Without assuming any smoothness conditions on the domain Ω, as it is analysed in Chapter 4, we need to impose some restrictions on either the reaction term or the dimension of the domain Ω and the reaction term, to prove the asymptotic behaviour of the solutions in $H^1_0(\Omega)$. In addition, in Chapter 4, since we do not impose any smoothness conditions on the domain Ω, we cannot use the cited Sobolev embedding $(L^2(\tau,T;H^2(\Omega)\cap H^1_0(\Omega)) \hookrightarrow L^2(\tau,T;L^{p(N,H^2\cap H^1_0)}(\Omega)))$.

Observe that thanks to Theorem 3.4, the restriction of U to $\mathbb{R}_d^2 \times H_0^1(\Omega)$ defines a process into $H_0^1(\Omega)$. Since no confusion arises, we will not modify the notation and continue denoting this process by U.

Actually, this process defined on $H_0^1(\Omega)$ as phase-space still fulfils properties to apply the results of Chapter 1. The following result shows that the process U is strong-weak continuous in $H_0^1(\Omega)$.

Proposition 3.18. *Suppose that the function a is locally Lipschitz and fulfils (3.2), $f \in C^1(\mathbb{R})$ satisfies (3.4) and (3.14), $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and $l \in L^2(\Omega)$. Then, the process U is strong-weak continuous in $H_0^1(\Omega)$.*

Proof. Consider $(t,\tau) \in \mathbb{R}_d^2$ fixed and let $\{u_\tau^k\}$ be a sequence of initial data which converges to u_τ strongly in $H_0^1(\Omega)$.

On the one hand, by Proposition 3.11, the map $U(t,\tau)$ is continuous from $L^2(\Omega)$ into itself. Therefore,

$$U(t,\tau)u_\tau^k \to U(t,\tau)u_\tau \quad \text{strongly in } L^2(\Omega). \tag{3.56}$$

On the other hand, observe that under the above assumptions we cannot guarantee that $f(u) \in L^2(\tau,t;L^2(\Omega))$, therefore we cannot use the stronger energy equality (3.55). Then, to solve this problem, we use the Galerkin approximations and pass to the limit by compactness arguments.

Multiplying (3.8) by $\lambda_j \varphi_{nj}$, summing from $j=1$ to n, adding $\pm(f(0), -\Delta u_n(t))$ and making use of (3.2), we have

$$\frac{1}{2}\frac{d}{ds}\|u_n(s)\|_2^2 + m|-\Delta u_n(s)|_2^2 \leq (f(u_n(s)) - f(0), -\Delta u_n(s)) + (f(0)+h(s), -\Delta u_n(s))$$

a.e. $s \in (\tau,t)$.
Integrating by parts and using (3.14) and the Cauchy inequality, we deduce

$$\frac{d}{ds}\|u_n(s)\|_2^2 \leq 2\eta\|u_n(s)\|_2^2 + \frac{1}{2m}|f(0)+h(s)|_2^2 \quad \text{a.e. } s \in (\tau,t).$$

Then, integrating between τ and t and applying the Gronwall lemma, we have

$$\|u_n(t)\|_2^2 \leq \left(\|u_\tau^k\|_2^2 + \frac{1}{2m}\int_\tau^t |f(0)+h(s)|_2^2 ds\right) e^{2\eta(t-\tau)}.$$

Now, since the sequence $\{u_n\}$ is bounded in $L^\infty(\tau,t;H_0^1(\Omega))$, $u_n(\cdot;\tau,u_\tau^k) \rightharpoonup u(\cdot;\tau,u_\tau^k)$ weakly in $L^2(\tau,t;H_0^1(\Omega))$ and $u(\cdot;\tau,u_\tau^k) \in C([\tau,t];L^2(\Omega))$ (cf. Theorem 3.3), taking into account [100, Lemma 11.2, p. 288], we have

$$\|U(t,\tau)u_\tau^k\|_2^2 \leq \left(\|u_\tau^k\|_2^2 + \frac{1}{2m}\int_\tau^t |f(0)+h(s)|_2^2 ds\right) e^{2\eta(t-\tau)}.$$

From this, together with (3.56), we obtain

$$U(t,\tau)u_\tau^k \rightharpoonup U(t,\tau)u_\tau \quad \text{weakly in } H_0^1(\Omega).$$

□

The following result, which is analogous to Lemma 2.19, establishes some uniform estimates of the solutions in more regular norms in a finite-time interval up to time t when the initial datum is shifted pullback far enough. To prove it, we will require not only assumption (3.53) but also a specific information of the growth of $f(u)$, namely (3.54) as pointed out above.

Furthermore, to simplify the statement, let us firstly introduce the following two quantities

$$\begin{aligned} \rho_1^{ext}(t) &= 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu(t-3)}}{2m-\mu\lambda_1^{-1}}\int_{-\infty}^{t} e^{\mu\xi}\|h(\xi)\|_*^2 d\xi, \\ \rho_2^{ext}(t) &= \frac{1}{m}\left(\rho_1^{ext}(t) + 2\kappa|\Omega| + \frac{1}{m}\max_{r\in[t-2,t]}\int_{r-1}^{r}\|h(\xi)\|_*^2 d\xi\right). \end{aligned} \tag{3.57}$$

Lemma 3.19. *Under the assumptions of Proposition 3.18, if f also fulfils (3.54) and h satisfies (3.41) for some $\mu \in (0, 2\lambda_1 m)$, then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_2(\widehat{D}, t) < t-3$ such that for any $\tau \leq \tau_2(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, the following estimates hold*

$$\begin{cases} \|u(r;\tau,u_\tau)\|_2^2 \leq \widetilde{\rho}_1(t) \quad \forall r \in [t-2,t], \\ \displaystyle\int_{r-1}^{r} |-\Delta u(\xi;\tau,u_\tau)|_2^2 d\xi \leq \widetilde{\rho}_2(t) \quad \forall r \in [t-1,t], \\ \displaystyle\int_{r-1}^{r} |u'(\xi;\tau,u_\tau)|_2^2 d\xi \leq \widetilde{\rho}_3(t) \quad \forall r \in [t-1,t], \end{cases} \tag{3.58}$$

with

$$\widetilde{\rho}_1(t) = (1+2\eta)\rho_2^{ext}(t) + \frac{1}{m}\max_{r\in[t-2,t]}\int_{r-1}^{r}|f(0)+h(\xi)|_2^2 d\xi,$$

$$\widetilde{\rho}_2(t) = \frac{1}{m}\left(\widetilde{\rho}_1(t) + 2\eta\rho_2^{ext}(t) + \frac{1}{m}\max_{r\in[t-1,t]}\int_{r-1}^{r}|f(0)+h(\xi)|_2^2 d\xi\right),$$

$$\widetilde{\rho}_3(t) = 3(M_{(\rho_1^{ext}(t),l)})^2\widetilde{\rho}_2(t) + 3C_f(\widetilde{\rho}_1(t))^{\tilde{b}}(\widetilde{\rho}_2(t))^{\hat{b}} + 3\max_{r\in[t-1,t]}\int_{r-1}^{r}|h(\xi)|_2^2 d\xi,$$

where $\hat{b}$, $\tilde{b}$, C_f and $M_{(\rho_1^{ext}(t),l)}$ are positive constants.

Proof. Let us firstly observe that, analogously as we argued in Lemma 3.15, we may obtain uniform estimates for solutions in a longer time-interval (useful for our purposes). Namely, there exists $\tau_2(\widehat{D}, t) < t-3$ such that for any $\tau \leq \tau_2(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, we have

$$|u(r;\tau,u_\tau)|_2 \leq \rho_1^{ext}(t) \quad \forall r \in [t-3,t], \tag{3.59}$$

$$\int_{r-1}^{r}\|u(\xi;\tau,u_\tau)\|_2^2 d\xi \leq \rho_2^{ext}(t) \quad \forall r \in [t-2,t],$$

where $\{\rho_i^{ext}\}_{i=1,2}$ are given in (3.57). Observe that these estimates also hold for the Galerkin approximations $u_n(\cdot;\tau,u_\tau)$, which have already been used in Section 3.1.

In addition, from the continuity of the function a, the fact that $l \in L^2(\Omega)$ and the first inequality in (3.59), we deduce that there exits a positive constant $M_{(\rho_1^{ext}(t),l)}$ such that

$$a(l(u_n(r))) \leq M_{(\rho_1^{ext}(t),l)} \quad \forall r \in [t-3,t] \quad \forall n \geq 1. \tag{3.60}$$

Similarly to the proof of Lemma 3.15, we will prove the inequalities in (3.58) for the Galerkin approximations and then, passing to the limit, for the solutions.

Multiplying by $\lambda_j \varphi_{nj}$ in (3.8), summing from $j=1$ to n and making use of (3.2), (3.14), and the Cauchy inequality, we deduce

$$\frac{d}{d\xi}\|u_n(\xi)\|_2^2 + m|-\Delta u_n(\xi)|_2^2 \leq 2\eta\|u_n(\xi)\|_2^2 + \frac{1}{m}|f(0)+h(\xi)|_2^2 \quad \text{a.e. } \xi > \tau. \tag{3.61}$$

Integrating between r and s with $\tau \leq r-1 \leq s \leq r$, we obtain in particular

$$\|u_n(r)\|_2^2 \leq \|u_n(s)\|_2^2 + 2\eta \int_{r-1}^{r} \|u_n(\xi)\|_2 d\xi + \frac{1}{m}\int_{r-1}^{r} |f(0)+h(\xi)|_2^2 d\xi.$$

Integrating the last inequality w.r.t. s between $r-1$ and r, we have

$$\|u_n(r)\|_2^2 \leq (1+2\eta)\int_{r-1}^{r} \|u_n(s)\|_2^2 ds + \frac{1}{m}\int_{r-1}^{r} |f(0)+h(\xi)|_2^2 d\xi$$

for all $\tau \leq r-1$.

Therefore, from the estimate on the solutions by ρ_2^{ext} given above, one deduces that for any $n \geq 1$

$$\|u_n(r;\tau,u_\tau)\|_2^2 \leq \widetilde{\rho}_1(t) \quad \forall r \in [t-2,t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_2(\widehat{D},t), \tag{3.62}$$

where $\widetilde{\rho}_1(t)$ is given in the statement. Now, taking inferior limit in (3.62) and using the well-known fact that u_n converge to $u(\cdot;\tau,u_\tau) \in C([t-2,t];H_0^1(\Omega))$ weakly-star in $L^\infty(t-2,t;H_0^1(\Omega))$ (cf. Theorem 3.4), the first inequality in (3.58) holds.

Now, integrating between $r-1$ and r in (3.61), we obtain in particular

$$\begin{aligned}&\int_{r-1}^{r} |-\Delta u_n(\xi)|_2^2 d\xi \\ &\leq \frac{1}{m}\left(\|u_n(r-1)\|_2^2 + 2\eta\int_{r-1}^{r}\|u_n(\xi)\|_2^2 d\xi + \frac{1}{m}\int_{r-1}^{r}|f(0)+h(\xi)|_2^2 d\xi\right)\end{aligned}$$

for all $\tau \leq r-1$. Then, for any $n \geq 1$

$$\int_{r-1}^{r} |-\Delta u_n(\xi)|_2^2 d\xi \leq \widetilde{\rho}_2(t) \quad \forall r \in [t-1,t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_2(\widehat{D},t), \tag{3.63}$$

where $\widetilde{\rho}_2(t)$ is given in the statement. Then, taking inferior limit in (3.63) and bearing in mind the well-known fact that u_n converge to $u(\cdot;\tau,u_\tau)$ weakly in $L^2(r-1,r;H^2(\Omega)\cap H_0^1(\Omega))$ for all $r \in [t-1,t]$ (cf. Theorem 3.4), the second inequality in (3.58) holds.

Now, taking into account that f satisfies (3.54) and the previous estimates, we have

$$\int_{r-1}^{r} |f(u_n(\xi))|_2^2 d\xi \le C_f(\widetilde{\rho}_1(t))^{\tilde{b}}(\widetilde{\rho}_2(t))^{\hat{b}} \quad \forall r \in [t-1,t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \le \tau_2(\widehat{D},t). \tag{3.64}$$

Finally, since for all $\tau \le r-1$

$$\begin{aligned}&\int_{r-1}^{r} |u_n'(\xi)|_2^2 d\xi \\ \le& 3\int_{r-1}^{r} |-a(l(u_n(\xi)))\Delta u_n(\xi)|_2^2 d\xi + 3\int_{r-1}^{r} |f(u_n(\xi))|_2^2 d\xi + 3\int_{r-1}^{r} |h(\xi)|_2^2 d\xi,\end{aligned}$$

from (3.60), (3.63) and (3.64), we obtain for any $n \ge 1$

$$\int_{r-1}^{r} |u_n'(\xi)|_2^2 d\xi \le \widetilde{\rho}_3(t) \quad \forall r \in [t-1,t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \le \tau_2(\widehat{D},t),$$

where $\widetilde{\rho}_3$ is given in the statement. Then, taking inferior limit in the above expression and taking into account that u_n' converge to $u'(\cdot;\tau,u_\tau)$ weakly in $L^2(r-1,r;L^2(\Omega))$ for all $r \in [t-1,t]$, we obtain the last inequality in (3.58). □

Now, we introduce new universes which involve more regularity.

Definition 3.20. *For each $\mu > 0$, $\mathcal{D}_\mu^{L^2,H_0^1}$ denotes the class of all families of nonempty subsets $\widehat{D}_{H_0^1} = \{D(t) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{L^2}$.*

As a direct consequence of the regularising effect of the equation when the function $h \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ (cf. Theorem 3.4) and the existence of a family pullback $\mathcal{D}_\mu^{L^2}$-absorbing (cf. Proposition 3.14), the existence of an absorbing family in the universe $\mathcal{D}_\mu^{L^2,H_0^1}$ also holds. We omit the proof because it is identical to the proof of Proposition 2.21.

Proposition 3.21. *Suppose that the function a is locally Lipschitz and (3.2) holds, $f \in C^1(\mathbb{R})$ fulfils (3.4) and (3.14), $h \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies condition (3.41) for some $\mu \in (0, 2\lambda_1 m)$ and $l \in L^2(\Omega)$. Then, the family $\widehat{D}_{0,H_0^1} = \{\overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \cap H_0^1(\Omega) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{L^2,H_0^1}$ and for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, there exists $\tau_3(\widehat{D},t) < t$ such that*

$$U(t,\tau)D(\tau) \subset D_{0,H_0^1}(t) \quad \forall \tau \le \tau_3(\widehat{D},t).$$

In particular, the family $\widehat{D}_{0,H_0^1}$ is pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-absorbing for the process $U : \mathbb{R}^2_d \times H_0^1(\Omega) \to H_0^1(\Omega)$.

Now, to prove the pullback asymptotic compactness of U in $H_0^1(\Omega)$ for the universe $\mathcal{D}_\mu^{L^2,H_0^1}$, we apply an energy method similar to the one use to prove Proposition 2.22.

Proposition 3.22. *Under the assumptions of Lemma 3.19, the process $U : \mathbb{R}^2_d \times H_0^1(\Omega) \to H_0^1(\Omega)$ is pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-asymptotically compact.*

Proof. The proof of this result is analogous to the proof of Proposition 2.22. In this case we need to use the estimates (3.58) which appear in the statement of Proposition 3.19. In addition, we have to consider the continuous and non-increasing functions

$$J_n(s) = \|u^n(s)\|_2^2 - 2\eta \int_{t-2}^{s} \|u^n(r)\|_2^2 dr - \frac{1}{2m} \int_{t-2}^{s} |f(0) + h(r)|_2^2 dr,$$
$$J(s) = \|u(s)\|_2^2 - 2\eta \int_{t-2}^{s} \|u(r)\|_2^2 dr - \frac{1}{2m} \int_{t-2}^{s} |f(0) + h(r)|_2^2 dr.$$

□

As a consequence of the previous results, we obtain the main result of this section. The proof of this result is very close to the proof of Theorem 2.23. We show the details for the sake of completeness.

Theorem 3.23. *Assume that the function a is locally Lipschitz and (3.2) holds, $f \in C^1(\mathbb{R})$ fulfils (3.4), (3.14) and (3.54), $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies condition (3.41) for some $\mu \in (0, 2\lambda_1 m)$ and $l \in L^2(\Omega)$. Then, there exist the minimal pullback $\mathcal{D}_F^{H_0^1}$-attractor $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ and the minimal pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}$ for the process $U : \mathbb{R}^2_d \times H_0^1(\Omega) \to H_0^1(\Omega)$. Furthermore, it fulfils*

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) \subset \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}(t) \quad \forall t \in \mathbb{R}. \tag{3.65}$$

In particular, for any $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, the following pullback attraction result in $H_0^1(\Omega)$ holds

$$\lim_{\tau \to -\infty} \operatorname{dist}_{H_0^1}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R}. \tag{3.66}$$

Finally, if the function h also satisfies

$$\sup_{s \le 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu r} |h(r)|^2 dr \right) < \infty, \tag{3.67}$$

then

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}(t) \quad \forall t \in \mathbb{R}. \tag{3.68}$$

Furthermore,

$$\lim_{\tau \to -\infty} \operatorname{dist}_{H_0^1}(U(t,\tau)B, \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R} \quad \forall B \in \mathcal{D}_F^{L^2}. \tag{3.69}$$

Proof. The existence of $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}$ is a consequence of Corollary 1.15, since U is strong-weak continuous in $H_0^1(\Omega)$ (cf. Proposition 3.18), it holds $\mathcal{D}_F^{H_0^1} \subset \mathcal{D}_\mu^{L^2,H_0^1}$, there exists an absorbing family in $\mathcal{D}_\mu^{L^2,H_0^1}$ (cf. Proposition 3.21) and the process U is pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-asymptotically compact (cf. Proposition 3.22).

The chain of inclusions (3.65) follows from Corollary 1.15 and Theorem 1.16. Actually, the equality statement is due to the second part of Theorem 1.16, by using Proposition 3.21. Then, (3.66) is straightforward.

If moreover h satisfies (3.52), we have already proved in Theorem 3.17 the equality $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)$ for all $t \in \mathbb{R}$. Now, in order to obtain (3.68), we assume (3.67), which is a stronger requirement than (3.52). Therefore, the equality $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)$ is again a consequence of Theorem 1.16. Indeed, the solutions are coming into a bounded subset of $H_0^1(\Omega)$ due to the first estimate in Lemma 3.19 by $\widetilde{\rho}_1(t)$ [recall that, analogously as in Remark 2.17 (i), here (3.67) is equivalent to $\sup_{s\le 0}\int_{s-1}^{s}|h(r)|^2 dr < \infty$]. Then, (3.69) obviously holds. □

An immediate consequence of Theorem 3.23 is an improvement in the regularity of the attractor in an autonomous framework. Namely, we have the following

Corollary 3.24. *Suppose that $h \equiv 0$ in (3.1). Under the assumptions made on Theorem 3.23, there exist the global attractors $\mathcal{A}_{L^2}$ and $\mathcal{A}_{H_0^1}$ for the associated autonomous dynamical system in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively, and they coincide. Furthermore,*

$$\|u\|_\infty \le \left(\frac{\kappa}{\alpha_2}\right)^{1/p} \qquad \forall u \in \mathcal{A}_{L^2}. \tag{3.70}$$

Proof. From Theorem 3.23, we deduce the existence of global attractors. In addition, $\mathcal{A}_{L^2} = \mathcal{A}_{H_0^1}$, thanks to the regularising effect of the equation (cf. Theorem 3.4). Finally, the estimate (3.70) follows arguing as in [100, Theorem 11.6, p. 292]. □

Remark 3.25. *Under additional conditions, we may restrict ourselves to study the problem in $\mathcal{C}_+(L^2(\Omega))$, the positive cone of $L^2(\Omega)$. We would redefine suitably new classes of (tempered and non-tempered) families. Observe that assuming that h is a positive function and $f \in C(\mathbb{R};\mathbb{R}_+)$, a Maximum Principle holds (cf. Theorem 3.5). Therefore, U is well-defined from $C_+(L^2(\Omega))$ into itself, which is important if one is dealing with a biological model. Then, all the results from Sections 3.3 and 3.4 can be obtained again analogously, by rearranging the assumptions within this setting.*

Chapter 4

A monotone iterative approach for nonlocal reaction-diffusion equations

This chapter is a natural continuation of Chapter 3 with a different approach since in this case no assumption of smoothness is imposed on the domain $\Omega \subset \mathbb{R}^N$. The elimination of this assumption allows to model real phenomena with more accuracy since they tend to be posed in nonsmooth domains (see [68] for more details).

First, we show the existence and uniqueness of weak solutions making use of the monotonicity method (cf. [85, Chapitre 2]), which has already been used in the reaction-diffusion framework (see [7]), combined with an iterative procedure. Namely in this chapter this method is applied to the local reaction-diffusion equations

$$\frac{\partial u^n}{\partial t} - a(l(u^{n-1}))\Delta u^n = f(u^n) + h(t) \quad \forall n \geq 1,$$

where $u^0 \equiv 0$, fulfilled with homogeneous boundary Dirichlet conditions. Then, making use of compactness arguments, we prove that the limit of the sequence of solutions $\{u^n\}$ is a weak solution to the nonlocal reaction-diffusion problem studied in the previous chapter. Observe that this result is an improvement with regard to the one appearing in Chapter 3 (cf. Theorem 3.3), because without imposing any smoothness on the domain Ω, the existence of weak solutions can be proved. The uniqueness is guaranteed assuming additional requirements on the function a.

Furthermore, the existence of strong solutions and the regularising effect of the equation are also analysed. In this case, since we are not assuming any smoothness restriction on the domain Ω as in Chapter 3, requirements on either the dimension of the domain Ω or the reaction term, or even both of them, are made to prove this result (cf. Theorems 4.8 and 4.10, and Corollary 4.11, respectively).

Our next aim is to study the asymptotic behaviour of the solutions making use of the theory of non-autonomous dynamical systems, namely we study the existence of pullback attractors in the framework of universes in $L^2(\Omega)$ and $H_0^1(\Omega)$. Although these results are not new in this PhD project, because the existence of these families has been proved in Chapter 3 (see Theorems 3.17 and 3.23), the methods applied to prove the asymptotic compactness are. Namely, in the L^2-context we apply the energy method used by Rosa in [101] with some variations due to the presence of the nonlocal operator in the diffusion term. To end this chapter, the pullback asymptotic compactness in $H_0^1(\Omega)$ is analysed applying the flattening property, which was coined by Kloeden & Langa in [77] (for more details see [77, 78, 33, 65]). Again, the assumptions made on the dimension of the domain Ω and the reaction term are imposed in the H^1-context since the domain Ω does not fulfil any smoothness condition as in Chapter 3.

The results of this chapter can be found in [25].

4.1 Statement of the problem. Existence results

In this chapter, we consider the nonlocal reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$ and the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and there exists a positive constant m such that

$$0 < m \leq a(s) \quad \forall s \in \mathbb{R}. \tag{4.2}$$

Furthermore, $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, the function $f \in C(\mathbb{R})$ and there exist positive constants α_1, α_2, η, κ and $p > 2$ such that

$$(f(s) - f(r))(s - r) \leq \eta(s - r)^2 \quad \forall s, r \in \mathbb{R}, \tag{4.3}$$

$$-\kappa - \alpha_1|s|^p \leq f(s)s \leq \kappa - \alpha_2|s|^p \quad \forall s \in \mathbb{R}. \tag{4.4}$$

From (4.4) we can deduce that there exists $\beta > 0$ such that

$$|f(s)| \leq \beta(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R}. \tag{4.5}$$

Analogously to Chapter 3, we continue assuming that $u_\tau \in L^2(\Omega)$ and the non-autonomous term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. In what follows, we identify $L^2(\Omega)$ with its dual. Then, we have the chain of dense and compact embeddings $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. As a consequence of the previous identification, the operator l acting on u must be understood as (l, u), but along this chapter will be denoted by $l(u)$.

Observe that in the setting of problem (4.1), we have not imposed any smoothness condition on the domain Ω unlike what happened in Chapter 3.

Definition 4.1. *A weak solution to (4.1) is a function $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ for all $T > \tau$, with $u(\tau) = u_\tau$, such that for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$*

$$\frac{d}{dt}(u(t), v) + a(l(u(t)))((u(t), v)) = (f(u(t)), v) + \langle h(t), v \rangle, \tag{4.6}$$

where the previous equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

Analogously as in Chapter 3, when u is a weak solution to (4.1) it can be proved, taking into account the continuity of a, $l \in L^2(\Omega)$, (4.5) and (4.6), that $u \in C([\tau,\infty); L^2(\Omega))$ and it holds

$$|u(t)|_2^2+2\int_s^t a(l(u(r)))\|u(r)\|_2^2 dr = |u(s)|_2^2+2\int_s^t (f(u(r)),u(r))dr+2\int_s^t \langle h(r),u(r)\rangle dr \tag{4.7}$$

for all $\tau \leq s \leq t$.

Definition 4.2. *A strong solution to (4.1) is a weak solution u to (4.1) such that $u \in L^\infty(\tau,T;H_0^1(\Omega)) \cap L^2(\tau,T;D(-\Delta))$ for all $T > \tau$.*

In this section we will prove the existence of weak solution to (4.1) combining an iterative procedure and the monotonicity method for solving nonlinear PDEs. Due to the presence of the nonlocal term, we cannot use directly the monotonicity method. We apply this method to a non-autonomous reaction-diffusion equation in which the diffusion term has a viscosity which depends on time, but it does not depend on the unknown. Then through iterations and appropriate estimates, we can prove the existence and uniqueness of weak solutions to (4.1). In the weak-solutions framework, this chapter provides an improvement with regard to the previous one since Ω does not need to fulfil smoothness conditions to guarantee the existence and uniqueness of weak solutions to (4.1). However, in the strong-solutions framework, since Ω is not smooth, we impose additional requirements on either the dimension of Ω or the reaction term or both of them, to guarantee the existence and uniqueness of strong solutions as well as the regularising effect of the equation.

4.1.1 The monotonicity method for solving nonlinear PDEs

In this section we briefly recall the requirements to apply the monotonicity method for solving nonlinear PDEs (see [85, Chapitre 2] for more details).

Consider a separable Hilbert space H, whose norm is denoted by $|\cdot|$. Moreover, suppose given V_i, $i = 1,\dots,m$, with $m \geq 1$, separable and reflexive Banach spaces, such that $\bigcup_{i=1}^m V_i \subset H$, $\bigcap_{i=1}^m V_i$ is dense in H, and $V_i \subset H$ with continuous injection for all $i = 1,\dots,m$.

For all $i = 1,\dots,m$, $\|\cdot\|_i$ and $\|\cdot\|_{*_i}$ denote the norms in V_i and V_i' respectively. By V we represent the space $\bigcap_{i=1} V_i$. In addition, $\langle\cdot,\cdot\rangle$ denotes the duality product between V_i' and V_i for all $i = 1,\dots,m$. Finally, H is identified with its topological dual H' using the Riesz theorem.

Consider $T \in (\tau,\infty)$ fixed and let $B_i : (\tau,T) \times V_i \to V_i'$ be, for $i = 1,\dots,m$, operators, in general nonlinear, such that

A1) The application $t \in (\tau,T) \mapsto B_i(t,v) \in V_i'$ is measurable for each $v \in V$.

A2) Each operator B_i is hemicontinuous, i.e. for all $t \in (\tau,T)$ and for all $u,v,w \in V_i$, the application $\theta \in (\tau,T) \mapsto \langle B_i(t,u+\theta v),w\rangle \in \mathbb{R}$ is continuous.

Suppose also that there exist $1 < p_i < \infty$, $i = 1,\dots,m$, at least one of them greater than or equal to 2, constants $c > 0$, $\alpha > 0$ and $\lambda \geq 0$, and a non-negative function $C \in L^1(\tau,T)$, such that for all $t \in (\tau,T)$ it satisfies

A3) $B_i(t,\cdot)$ is bounded in V_i', i.e.

$$\|B_i(t,v)\|_{*_i} \le c(1+\|v\|_i^{p_i-1}) \quad \forall v \in V_i.$$

A4) $B_i(t,\cdot)$ is monotone, i.e.

$$\langle B_i(t,v) - B_i(t,w), v-w\rangle + \lambda|v-w|^2 \ge 0 \quad \forall v,w \in V_i.$$

A5) $B_i(t,\cdot)$ is coercive, i.e.

$$\langle B_i(t,v), v\rangle + \lambda|v|^2 + C(t) \ge \alpha\|v\|_i^{p_i} \quad \forall v \in V_i.$$

Suppose given functions $h_i \in L^{p_i'}(\tau,T;V_i')$ for $i=1,\dots,m$, and an initial datum $u_\tau \in H$.

In what follows we denote

$$B(t,v) = \sum_{i=1}^m B_i(t,v) \quad \forall v \in V,$$
$$h(t) = \sum_{i=1}^m h_i(t).$$

Now we consider the following problem

$$\begin{cases} u \in \bigcap_{i=1}^m L^{p_i}(\tau,T;V_i) \quad \forall T > \tau, \\ u'(t) + B(t,u(t)) = h(t) \quad \text{in } \mathcal{D}'(\tau,T;V'), \\ u(\tau) = u_\tau. \end{cases} \tag{4.8}$$

Then we have the following result (see [85, Théorème 1.4, p. 168]).

Theorem 4.3. *Under the above assumptions there exists a unique solution u to (4.8). In addition, this solution fulfils*

$$u \in C([\tau,T];H) \quad \textit{and} \quad u' \in \sum_{i=1}^m L^{p_i'}(\tau,T;V_i').$$

Remark 4.4. *To prove Theorem 4.3, it can be used any numerable family formed by linearly independent elements such that the vector space generated by this family is dense in V.*

4.1.2 Existence and uniqueness of weak solutions

In this section we are going to apply the method stated in Section 4.1.1 to a non-autonomous reaction-diffusion equation whose diffusion term is composed by the

Laplacian and a viscosity term which depends only on time. Namely, for each $n \geq 1$, we denote by u^n the weak solution to

$$
(\mathrm{P}_n) \begin{cases} u \in L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]; L^2(\Omega)), \\ \dfrac{d}{dt}(u(t), v) + a(l(u^{n-1}(t)))((u(t), v)) = (f(u(t)), v) + \langle h(t), v \rangle, \\ u(\tau) = u_\tau, \end{cases}
$$

where $u^0 \equiv 0$ and u^n is the solution to (P_n) if $n \geq 1$. Observe that the equation in (P_n) must be understood in the sense of $\mathcal{D}'(\tau, \infty)$ for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$.

Theorem 4.5. *Suppose that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ satisfies (4.2), $f \in C(\mathbb{R})$ fulfils (4.3) and (4.4), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, for any $u_\tau \in L^2(\Omega)$ there exists a unique weak solution u^n to (P_n) for all $n \geq 1$.*

Proof. The existence and uniqueness of solution to problem (P_n) is due to Theorem 4.3. Namely, take $H = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $p_1 = 2$, $V_2 = L^p(\Omega)$ and $p_2 = p$, and define

$$
\begin{aligned}
B_1(t, v) &= -a(l(u^{n-1}(t)))\Delta v \quad \forall v \in H_0^1(\Omega) \quad \forall t \in (\tau, T), \\
B_2(t, v) &= -f(v) \quad \forall v \in L^p(\Omega) \quad \forall t \in (\tau, T), \\
h_1(t) &= h(t) \quad \forall t \in (\tau, T), \\
h_2(t) &= 0 \quad \forall t \in (\tau, T).
\end{aligned}
$$

Then, making use of the fact that $a \in C(\mathbb{R}; \mathbb{R}_+)$ satisfies (4.2) and $l \in L^2(\Omega)$, we have that B_1 fulfils A1)-A5). Analogously, using $f \in C(\mathbb{R})$ fulfils (4.3) and (4.4), it is not difficult to check that B_2 satisfies A1)-A5). As a result, there exists a unique solution to (P_n) for all $n \geq 1$. $\square$

Now we are ready to prove the existence of weak solutions to (4.1). Observe that this result has been proved without assuming any smoothness condition on the domain Ω.

Theorem 4.6. *Suppose that the function a is locally Lipschitz and satisfies (4.2), $f \in C(\mathbb{R})$ fulfils (4.3) and (4.4), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, for each $u_\tau \in L^2(\Omega)$, there exists a unique weak solution to problem (4.1), which is denoted by $u(\cdot) = u(\cdot; \tau, u_\tau)$. Moreover, this solution behaves continuously in $L^2(\Omega)$ w.r.t. initial data.*

Proof. We split the proof into two steps.

Step 1. Existence of weak solution. Consider $u^0 \equiv 0$, and defining u^n the solution to (P_n) (cf. Theorem 4.5), we have

$$
\frac{1}{2}\frac{d}{dt}|u^n(t)|_2^2 + a(l(u^{n-1}))\|u^n(t)\|_2^2 = (f(u^n(t)), u^n(t)) + \langle h(t), u^n(t) \rangle \quad \text{a.e. } t \in (\tau, T).
$$

Now, making use of (4.2) and (4.4), we obtain

$$\frac{d}{dt}|u^n(t)|_2^2 + m\|u^n(t)\|_2^2 + 2\alpha_2|u^n(t)|_p^p \leq 2\kappa|\Omega| + \frac{1}{m}\|h(t)\|_*^2 \quad \text{a.e. } t \in (\tau, T).$$

Integrating between τ and $t \in [\tau, T]$,

$$|u^n(t)|_2^2 + m\int_\tau^t \|u^n(s)\|_2^2 ds + 2\alpha_2\int_\tau^t |u^n(s)|_p^p ds \leq |u_\tau|_2^2 + 2k|\Omega|(T-\tau) + \frac{1}{m}\int_\tau^T \|h(s)\|_*^2 ds.$$

Therefore, the sequence $\{u^n\}$ is bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$. From this, we deduce that there exists a positive constant C_∞ such that

$$|u^n(t)|_2 \leq C_\infty \quad \forall t \in [\tau, T] \quad \forall n \geq 1.$$

Then, using that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and $l \in L^2(\Omega)$, there exists a positive constant M_{C_∞} such that

$$a(l(u^{n-1}(t))) \leq M_{C_\infty} \quad \forall t \in [\tau, T] \quad \forall n \geq 1.$$

Therefore, $\{-a(l(u^{n-1}))\Delta u^n\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$. Moreover, making use of the boundedness of $\{u_n\}$ in $L^p(\tau, T; L^p(\Omega))$ and (4.5), we obtain that the sequence $\{f(u^n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$ (where p and q are conjugate exponents). Taking this into account, we deduce that the sequence $\{(u^n)'\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$.

Therefore, using the Aubin-Lions lemma, there exist a subsequence of $\{u^n\}$ (relabeled the same), a function $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ with $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$ such that

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau, T; L^2(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^2(\tau, T; H_0^1(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^p(\tau, T; L^p(\Omega)), \\ u^n \to u \quad \text{strongly in } L^2(\tau, T; L^2(\Omega)), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega)), \\ f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^q(\tau, T; L^q(\Omega)), \\ -a(l(u^{n-1}))\Delta u^n \rightharpoonup -a(l(u))\Delta u \quad \text{weakly in } L^2(\tau, T; H^{-1}(\Omega)), \end{cases} \tag{4.9}$$

where the limits of the sequences $\{f(u^n)\}$ and $\{-a(l(u^n))\Delta u^n\}$ have been obtained applying [85, Lemme 1.3, p. 12] (see Theorem 2.4 for more details).

Thereupon, we will show that u fulfils (4.6) for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$. Consider fixed $T > \tau$ and $\varphi \in \mathcal{D}(\tau, T)$. Since u^n is a solution to (P$_n$), it satisfies for all $n \geq 1$

$$\int_\tau^T (u^n(t), v)\varphi'(t)dt + \int_\tau^T a(l(u^{n-1}(t)))\langle -\Delta u^n(t), v\rangle\varphi(t)dt = \int_\tau^T \langle f(u^n(t)) + h(t), v\rangle\varphi(t)dt.$$

Taking limit when $n \to \infty$ in the previous expression and making use of (4.9), (4.6) holds.

Finally, to prove the existence of a weak solution to (4.1), we only need to check that $u(\tau) = u_\tau$. Observe that this equality makes complete sense since $u \in C([\tau, T]; L^2(\Omega))$ (cf. [56, Théorème 2, p. 575]). To do this, consider fixed $v \in H_0^1(\Omega) \cap L^p(\Omega)$ and $\varphi \in H^1(\tau, T)$, with $\varphi(T) = 0$ and $\varphi(\tau) \neq 0$. Since u^n is a weak solution to (P$_n$), it holds

$$\frac{d}{dt}(u^n(t), v) + a(l(u^{n-1}(t)))((u^n(t), v)) = (f(u^n(t)), v) + \langle h(t), v\rangle \quad \text{a.e. } t \in (\tau, T).$$

Now, multiplying by φ in the previous expression and integrating between τ and T, we have

$$\begin{aligned} &- (u_\tau, v)\varphi(\tau) + \int_\tau^T a(l(u^{n-1}(t)))((u^n(t), v))\varphi(t)dt \\ = &\int_\tau^T (f(u^n(t)), v)\varphi(t)dt + \int_\tau^T \langle h(t), v\rangle\varphi(t)dt. \end{aligned}$$

Thereupon, taking limit when $n \to \infty$ and considering (4.9), we obtain

$$\begin{aligned} &- (u_\tau, v)\varphi(\tau) + \int_\tau^T a(l(u(t)))((u(t), v))\varphi(t)dt \\ = &\int_\tau^T (f(u(t)), v)\varphi(t)dt + \int_\tau^T \langle h(t), v\rangle\varphi(t)dt. \end{aligned} \tag{4.10}$$

Otherwise, we deduce from (4.6)

$$\begin{aligned} &- (u(\tau), v)\varphi(\tau) + \int_\tau^T a(l(u(t)))((u(t), v))\varphi(t)dt \\ = &\int_\tau^T (f(u(t)), v)\varphi(t)dt + \int_\tau^T \langle h(t), v\rangle\varphi(t)dt. \end{aligned}$$

Comparing this with (4.10), we deduce that $(u(\tau), v)\varphi(\tau) = (u_\tau, v)\varphi(\tau)$. Finally, since $\varphi(\tau) \neq 0$ and $H_0^1(\Omega) \cap L^p(\Omega)$ is dense in $L^2(\Omega)$, the equality $u(\tau) = u_\tau$ holds.

Step 2. Uniqueness of weak solution and continuity w.r.t. initial data. This has already been proved in Theorem 3.3, namely Step 2. □

Remark 4.7. *(i) Thanks to the uniqueness of weak solution to (4.1), if fulfils that the whole sequence* $\{u^n\}$ *converges to* u *weakly in* $L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ *and weakly-star in* $L^\infty(\tau, T; L^2(\Omega))$*. Analogously, the whole sequence* $\{(u^n)'\}$ *converges to* u' *weakly in* $L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$*.*

(ii) The previous result can be extended to more general diffusion terms like $a(l(u))Au$*, where* $A = \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial}{\partial x_j}\right)$*, with* $b_{ij} \in L^\infty(\Omega)$ *for all* $i, j = 1 \ldots N$ *and* $\sum_{i,j=1}^N b_{ij}(x)\xi_i\xi_j \geq \zeta|\xi|^2$*, where* $\zeta > 0$*.*

4.1.3 Strong solutions and regularising effect

In this section, the existence and uniqueness of strong solutions and the regularising effect of the equation are proved without assuming any smoothness condition on the domain Ω as in Chapter 3. To do this, it will not be enough to assume $u_\tau \in H_0^1(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ like in Chapter 2. To prove the existence of strong solutions to (4.1), we do not take an arbitrary Hilbert basis of $L^2(\Omega)$ dense in $H_0^1(\Omega) \cap L^p(\Omega)$, we use the eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, denoted by $\{w_i\}$, to make the most of the compactness arguments. To that end, since we are not assuming that Ω is regular, we impose some requirements on the dimension N of the domain Ω. Namely, we assume that

$$N \leq \frac{2p}{p-2}. \tag{4.11}$$

Then, the continuous embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ holds. Therefore, the eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ can be used as a basis to prove the existence of weak solutions to (4.1).

Now we denote $V_n = \operatorname{span}[w_1, \ldots, w_n]$ for all $n \geq 1$ and for each integer $n \geq 1$, we represent by $u_n(t) = u_n(t; \tau, u_\tau)$ the Galerkin approximation of the solution $u(t, \tau; u_\tau)$ to (4.1), which is given by

$$u_n(t) = \sum_{j=1}^{n} \varphi_{nj}(t) w_j,$$

and it is the local solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) + a(l(u_n(t)))((u_n(t), w_j)) = (f(u_n(t)), w_j) + (h(t), w_j), & t \in (\tau, \infty), \\ (u_n(\tau), w_j) = (u_\tau, w_j), & j = 1, \ldots, n. \end{cases} \tag{4.12}$$

Then we are ready to establish the existence of strong solutions and the regularising effect of the equation. We omit the proof because it is identical to that of Theorem 3.4.

Theorem 4.8. *Suppose that Ω is a bounded open subset of $\mathbb{R}^N$ with N fulfilling (4.11), the function a is locally Lipschitz and satisfies (4.2), $f \in C^1(\mathbb{R})$ fulfils (4.3) and (4.4), $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and $l \in L^2(\Omega)$. Then for each $u_\tau \in L^2(\Omega)$, there exists a unique weak solution u to problem (4.1) which belongs to $L^2(\tau + \varepsilon, T; D(-\Delta)) \cap C((\tau, T]; H_0^1(\Omega))$ for every $\varepsilon > 0$ and $T > \tau + \varepsilon$. In addition, if $u_\tau \in H_0^1(\Omega)$, the existence of a strong solution u to (4.1) is guaranteed with $u' \in L^q(\tau, T; L^q(\Omega))$.*

Another possible choice to prove the regularising effect of the equation and the existence of strong solutions without supposing either requirements on the domain Ω (cf. Chapter 3) or restrictions on the dimension of the domain, like in the previous result, is to assume restrictive conditions on the function f which guarantee that $f(u) \in L^2(\tau, T; L^2(\Omega))$ for all $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega))$. To that end, we assume that $f \in C(\mathbb{R})$ such that

$$|f(s)| \leq C(1 + |s|^{\gamma+1}) \quad \forall s \in \mathbb{R}, \tag{4.13}$$

where $\gamma = 2/N$ if $N \geq 3$. The fact that $f(u) \in L^2(\tau,T;L^2(\Omega))$ is obtained applying interpolation results (cf. [116, Lemma II.4.1, p. 72]) to the Sobolev spaces $L^\infty(\tau,T;L^2(\Omega))$ and $L^2(\tau,T;H_0^1(\Omega))$. Namely, when $u \in L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega))$, we obtain

$$\begin{aligned}\int_\tau^T\int_\Omega |f(u(x,t))|^2 dxdt &\leq 2\int_\tau^T\int_\Omega \left(C^2+|u(x,t)|^{2\gamma+2}\right) dxdt \\ &\leq 2C^2|\Omega|(T-\tau)+2C^2\|u\|^{2\hat{b}}_{L^\infty(\tau,T;L^2(\Omega))}\|u\|^{2\tilde{b}}_{L^2(\tau,T;L^{p(N,H_0^1)})} \\ &\leq 2C^2\left[|\Omega|(T-\tau)+(C_I(N))^{2\tilde{b}}\|u\|^{2\hat{b}}_{L^\infty(\tau,T;L^2(\Omega))}\|u\|^{2\tilde{b}}_{L^2(\tau,T;H_0^1(\Omega))}\right],\end{aligned} \tag{4.14}$$

where $\hat{b} = (1-\theta)(\gamma+1)$, $\tilde{b} = \theta(\gamma+1)$, $C_I(N)$ is the constant of the continuous embedding of $H_0^1(\Omega)$ into $L^{\mathfrak{p}}$-spaces and $\theta \in [0,1]$. Observe that it is not necessary to impose smoothness conditions on the domain Ω to make use of the cited Sobolev embeddings (cf. [20, Remarque 21, p.173]).

Remark 4.9. *(i) When $N = 1,2$, γ can be any positive value since the solution $u \in L^\infty(\tau,T;H_0^1(\Omega)) \hookrightarrow L^s(\tau,T;L^s(\Omega))$ with $s = \infty$ when $N = 1$ and $s < \infty$ when $N = 2$. Therefore, the norm of u in $L^{2\gamma+2}(\tau,T;L^{2\gamma+2}(\Omega))$ can be bounded making use of the more regular spaces $L^\infty(\tau,T;L^\infty(\Omega)$ or $L^s(\tau,T;L^s(\Omega))$ with $s < \infty$.*
(ii) Observe that there exist functions which are not all sublinear that fulfil (4.13). For example, it can be considered $f(s) = -s\sqrt{|s|}$ when $N = 4$.

Theorem 4.10. *Suppose that the function a is locally Lipschitz and satisfies (4.2), $f \in C(\mathbb{R})$ fulfils (4.3), (4.4) and (4.13), $h \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ and $l \in L^2(\Omega)$. Then for each $u_\tau \in L^2(\Omega)$, there exists a unique weak solution u to problem (4.1) which belongs to $L^2(\tau+\varepsilon,T;D(-\Delta)) \cap C((\tau,T];H_0^1(\Omega))$ for every $\varepsilon > 0$ and $T > \tau+\varepsilon$. In addition, if $u_\tau \in H_0^1(\Omega)$, the existence of a strong solution u to (4.1) holds. In addition, $u \in C([\tau,\infty);H_0^1(\Omega))$ and it fulfils the energy equality*

$$\|u(t)\|_2^2+2\int_s^t a(l(u(r)))|-\Delta u(r)|_2^2dr = \|u(s)\|_2^2+2\int_s^t (f(u(r))+h(r),-\Delta u(r))dr \tag{4.15}$$

for all $\tau \leq s \leq t$.

Proof. We split the proof into two steps. In the first one, we will show the regularising effect of the equation. Finally, in the second step, we will prove the existence of strong solutions together with the strong energy equality (4.15).

Step 1. Regularising effect. Analogously to what it was done in Theorem 2.4, it proves that the sequence $\{u_n\}$, which is bounded in $L^2(\tau,T;H_0^1(\Omega)) \cap L^\infty(\tau,T;L^2(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$, converges to u, the weak solution to (4.1). In this case, we have to take into account that thanks to (4.13), $\{f(u_n)\}$ is bounded in $L^2(\tau,T;L^2(\Omega))$ (see (4.14) above). Thereupon, we will prove that u belongs to $L^2(\tau+\varepsilon,T;D(-\Delta)) \cap C((\tau,T];H_0^1(\Omega))$ for all $T > \tau+\varepsilon > \tau$.

Multiplying by φ_{nj} in (4.12) and summing from $j = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + a(l(u_n(t)))\|u_n(t)\|_2^2 = (f(u_n(t)),u_n(t)) + (h(t),u_n(t)) \quad \text{a.e. } t \in (\tau,T). \tag{4.16}$$

Applying (4.2) together with

$$
\begin{aligned}
(f(u_n(t)), u_n(t)) &\le \kappa|\Omega|,\\
(h(t), u_n(t)) &\le \frac{1}{2\lambda_1 m}|h(t)|_2^2 + \frac{m}{2}\|u_n(t)\|_2^2,
\end{aligned}
$$

to (4.16), it yields

$$
\frac{d}{dt}|u_n(t)|_2^2 + m\|u_n(t)\|_2^2 \le 2\kappa|\Omega| + \frac{1}{\lambda_1 m}|h(t)|_2^2 \quad \text{a.e. } t \in (\tau, T).
$$

Integrating between τ and T, in particular we obtain

$$
\int_\tau^T \|u_n(t)\|_2^2 dt \le \frac{2\kappa|\Omega|(T-\tau)}{m} + \frac{1}{\lambda_1 m^2}\int_\tau^T |h(t)|_2^2 dt + \frac{1}{m}|u_\tau|_2^2. \tag{4.17}
$$

Multiplying by $\lambda_j \varphi_{nj}$ in (4.12), summing from $j = 1$ to n, and making use of (4.2) and the Cauchy inequality (cf. [57, Appendix B, p. 622]), we obtain

$$
\frac{d}{dt}\|u_n(t)\|_2^2 + m|-\Delta u_n(t)|_2^2 \le \frac{2}{m}|f(u_n(t))|_2^2 + \frac{2}{m}|h(t)|_2^2 \quad \text{a.e. } t \in (\tau, T). \tag{4.18}
$$

Now integrating this expression between s and t, where $\tau < s \le t \le T$, we have

$$
\|u_n(t)\|_2^2 + m\int_s^t |-\Delta u_n(r)|_2^2 dr \le \|u_n(s)\|_2^2 + \frac{2}{m}\int_\tau^T |f(u_n(r))|_2^2 dr + \frac{2}{m}\int_\tau^T |h(r)|_2^2 dr. \tag{4.19}
$$

In particular,

$$
\|u_n(t)\|_2^2 \le \|u_n(s)\|_2^2 + \frac{2}{m}\int_\tau^T |f(u_n(r))|_2^2 dr + \frac{2}{m}\int_\tau^T |h(r)|_2^2 dr.
$$

Integrating the previous expression w.r.t. s between τ and t, bearing in mind that $\{f(u_n)\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$ (cf. (4.14)) and (4.17), we obtain that $\{u_n\}$ is bounded in $L^\infty(\tau+\varepsilon, T; H_0^1(\Omega))$ with $\varepsilon > 0$. Taking $s = \tau+\varepsilon$ and $t = T$ in (4.19), we deduce that $\{u_n\}$ is bounded in $L^2(\tau+\varepsilon, T; D(-\Delta))$, thanks to the boundedness of $\{u_n\}$ in $L^\infty(\tau+\varepsilon, T; H_0^1(\Omega))$. As a consequence, it is not difficult to prove that $\{u_n'\}$ is bounded in $L^2(\tau+\varepsilon, T; L^2(\Omega))$. Therefore, thanks to the uniqueness of a weak solution, we deduce

$$
\begin{aligned}
u_n &\rightharpoonup u \quad \text{weakly in } L^2(\tau+\varepsilon, T; D(-\Delta)),\\
u_n' &\rightharpoonup u' \quad \text{weakly in } L^2(\tau+\varepsilon, T; L^2(\Omega)).
\end{aligned}
$$

Then, since $u \in L^2(\tau+\varepsilon, T; D(-\Delta))$ and $u' \in L^2(\tau+\varepsilon, T; L^2(\Omega))$ for any $\varepsilon \in (0, T-\tau)$, it satisfies that $u \in C((\tau, T]; H_0^1(\Omega))$.

Step 2. Strong solutions. Now, assuming that $u_\tau \in H_0^1(\Omega)$, we will prove that the weak solution u belongs to $L^2(\tau, T; D(-\Delta)) \cap C([\tau, T]; H_0^1(\Omega))$. Multiplying by $\lambda_j \varphi_{nj}$ in (4.12), summing from $j = 1$ to n, using (4.2) and the Cauchy inequality, we deduce (4.18). Integrating between τ and $t \in [\tau, T]$, we have

$$\|u_n(t)\|_2^2 + m \int_\tau^t |-\Delta u_n(s)|_2^2 ds \leq \|u_\tau\|_2^2 + \frac{2}{m} \int_\tau^T |f(u_n(s))|_2^2 ds + \frac{2}{m} \int_\tau^T |h(s)|_2^2 ds.$$

Since $\{f(u_n)\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$, the sequence $\{u_n\}$ is bounded in $L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$. As a consequence, we have that $\{-a(l(u_n))\Delta u_n\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$.

Now, we define the projection operator

$$\begin{array}{rccl} P_n\colon & L^2(\Omega) & \longrightarrow & V_n \\ & \phi & \longmapsto & \sum_{j=1}^n (\phi, w_j) w_j. \end{array}$$

Since $\{w_j : j \geq 1\}$ is a special basis, P_n is non-expansive in $L^2(\Omega)$. Therefore, $\{P_n h\}$ and $\{P_n f(u_n)\}$ are bounded in $L^2(\tau, T; L^2(\Omega))$. As a consequence, the sequence $\{u_n'\}$ is bounded in $L^2(\tau, T; L^2(\Omega))$. Hence, as the weak solution is unique, it holds

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } L^2(\tau, T; D(-\Delta)), \\ u_n' &\rightharpoonup u' \quad \text{weakly in } L^2(\tau, T; L^2(\Omega)). \end{aligned}$$

Then, since $u \in L^2(\tau, T; D(-\Delta))$ and $u' \in L^2(\tau, T; L^2(\Omega))$, it satisfies that $u \in C([\tau, T]; H_0^1(\Omega))$ and (4.15) ([100, Theorem 7.2, p. 191]) . □

Corollary 4.11. *The thesis of Theorem 4.10 also holds when the assumption (4.13) is weakened by taking $\gamma = 2/(N-2)$ in (4.13) as long as $f \in C^1(\mathbb{R})$ and the dimension of Ω fulfils $3 \leq N \leq 2p/(p-2)$.*

4.2 Pullback attraction in L^2-norm

This section is devoted to studying the asymptotic behaviour of the solutions of (4.1). Namely, under the assumptions made on Section 4.1.2, we prove the existence of pullback attractors in the phase space $L^2(\Omega)$. Although this result is not new in this PhD project, since the existence of these families has been proved in Theorem 3.17, the method applied to prove the pullback asymptotic compactness is, since until now we have used energy methods that make use of continuous and non-increasing functions (see Propositions 2.15, 2.22, 3.16 and 3.22 for more details). Now, in this chapter we apply the energy method used by Rosa in [101] with some variations due to the presence of the nonlocal operator in the diffusion term.

Thanks to Theorem 4.6, we can define a process $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$ as

$$U(t, \tau) u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in L^2(\Omega) \quad \forall \tau \leq t,$$

where $u(\cdot; \tau, u_\tau)$ denotes the weak solution to (4.1).

Now, we show that the process U is continuous on $L^2(\Omega)$. We omit the proof because it is straightforward, since the solution behaves continuously in $L^2(\Omega)$ w.r.t. initial data (cf. Theorem 4.6).

Proposition 4.12. *Suppose that the function a is locally Lipschitz and fulfils (4.2), $f \in C(\mathbb{R})$ satisfies (4.3) and (4.4), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, for any pair $(t,\tau) \in \mathbb{R}^2_d$, the map $U(t,\tau)$ is continuous from $L^2(\Omega)$ into itself.*

Then we have the following result, which will be essential to build a suitable tempered universe in $\mathcal{P}(L^2(\Omega))$. Observe that this result has already been proved in Lemma 3.12.

Lemma 4.13. *Under the assumptions of Proposition 4.12, if the initial datum u_τ belongs to $L^2(\Omega)$, the solution u to (4.1) fulfils for all $\mu \in (0, 2m\lambda_1)$*

$$|u(t)|_2^2 \le e^{-\mu(t-\tau)}|u_\tau|_2^2 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu t}}{2m - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu s}\|h(s)\|_*^2 ds \quad \forall t \ge \tau. \tag{4.20}$$

Now, we define the following tempered universe in $\mathcal{P}(L^2(\Omega))$.

Definition 4.14. *The class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that*

$$\lim_{\tau \to -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|_2^2 \right) = 0 \tag{4.21}$$

is denoted by $\mathcal{D}_\mu^{L^2}$ for all $\mu > 0$.

Again, according to the notation in Chapter 1, we denote by $\mathcal{D}_F^{L^2}$ the universe of families (parameterized in time but constant for all $t \in \mathbb{R}$) of fixed nonempty bounded subsets of $L^2(\Omega)$.

Now, if h fulfils a suitable growth condition, the existence of an absorbing family is guaranteed. This result has already been proved in Proposition 3.14.

Proposition 4.15. *Under the assumptions of Proposition 4.12, if for some $\mu \in (0, 2m\lambda_1)$ the function $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ also fulfils*

$$\int_{-\infty}^0 e^{\mu s}\|h(s)\|_*^2 ds < \infty, \tag{4.22}$$

the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t))$, where

$$R_{L^2}(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu t}}{2m - \mu\lambda_1^{-1}} \int_{-\infty}^t e^{\mu s}\|h(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_\mu^{L^2}$-absorbing for the process $U : \mathbb{R}^2_d \times L^2(\Omega) \to L^2(\Omega)$ and belongs to $\mathcal{D}_\mu^{L^2}$.

Then, to prove the existence of minimal pullback attractors in $L^2(\Omega)$, we only need to check that the process U is pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact. First, we prove the following continuity result.

Proposition 4.16. *Under the assumptions of Proposition 4.12, if* $\{u_\tau^n\} \subset L^2(\Omega)$ *satisfies that* $u_\tau^n \rightharpoonup u_\tau$ *weakly in* $L^2(\Omega)$*, then for all* $T > \tau$

$$\begin{cases} U(t,\tau)u_\tau^n \rightharpoonup U(t,\tau)u_\tau \quad \text{weakly in } L^2(\Omega) \quad \forall t \in [\tau,T], \\ U(\cdot,\tau)u_\tau^n \rightharpoonup U(\cdot,\tau)u_\tau \quad \text{weakly in } L^2(\tau,T;H_0^1(\Omega)), \\ U(\cdot,\tau)u_\tau^n \rightharpoonup U(\cdot,\tau)u_\tau \quad \text{weakly in } L^p(\tau,T;L^p(\Omega)), \\ U(\cdot,\tau)u_\tau^n \rightarrow U(\cdot,\tau)u_\tau \quad \text{strongly in } L^2(\tau,T;L^2(\Omega)), \\ a(l(U(\cdot,\tau)u_\tau^n))U(\cdot,\tau)u_\tau^n \rightharpoonup a(l(U(\cdot,\tau)u_\tau))U(\cdot,\tau)u_\tau \text{ weakly in } L^2(\tau,T;H_0^1(\Omega)), \\ \sqrt{a(l(U(\cdot,\tau)u_\tau^n))}U(\cdot,\tau)u_\tau^n \rightharpoonup \sqrt{a(l(U(\cdot,\tau)u_\tau))}U(\cdot,\tau)u_\tau \text{ weakly in } L^2(\tau,T;H_0^1(\Omega)), \\ f(U(\cdot,\tau)u_\tau^n) \rightharpoonup f(U(\cdot,\tau)u_\tau) \quad \text{weakly in } L^q(\tau,T;L^q(\Omega)). \end{cases} \tag{4.23}$$

Proof. Consider $(T,\tau) \in \mathbb{R}_d^2$ fixed. For short, we denote by $u^n(t) = U(t,\tau)u_\tau^n$ and $u(t) = U(t,\tau)u_\tau$. Then, from the energy equality (4.7), applying (4.2), (4.4) and the Cauchy inequality, we obtain

$$|u^n(t)|_2^2 + m\int_\tau^t \|u^n(r)\|_2^2 dr + 2\alpha_2 \int_\tau^t |u^n(r)|_p^p dr \leq |u_\tau^n|_2^2 + 2\kappa|\Omega|(T-\tau) + \frac{1}{m}\int_\tau^T \|h(r)\|_*^2 dr.$$

Therefore, $\{u^n\}$ is bounded in $L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega)) \cap C([\tau,T];L^2(\Omega))$ for all $T > \tau$. From this we deduce that there exists a positive constant C_∞ such that

$$|u^n(t)|_2 \leq C_\infty \quad \forall t \in [\tau,T] \quad \forall n \geq 1.$$

Then, using that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ and $l \in L^2(\Omega)$, there exists a positive constant M_{C_∞} such that

$$a(l(u^n(t))) \leq M_{C_\infty} \quad \forall t \in [\tau,T] \quad \forall n \geq 1. \tag{4.24}$$

Therefore, $\{\sqrt{a(l(u^n))}u^n\}$ and $\{a(l(u^n))u^n\}$ are bounded in $L^2(\tau,T;H_0^1(\Omega))$. On the other hand, using (4.5) and taking into account the boundedness of $\{u^n\}$ in $L^p(\tau,T;L^p(\Omega))$, it satisfies that $\{f(u^n)\}$ is bounded in $L^q(\tau,T;L^q(\Omega))$. Besides, $\{u^n(T)\}$ is bounded in $L^2(\Omega)$. Therefore, as a consequence of the previous estimates and using the Aubin-Lions Lemma, there exist a subsequence of $\{u^n\}$ (relabeled the same) and functions $v \in L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$,

$\xi \in L^2(\Omega)$, $\chi_3 \in L^q(\tau,T;L^q(\Omega))$, and χ_1 and $\chi_2 \in L^2(\tau,T;H_0^1(\Omega))$, such that

$$\begin{cases} u^n \overset{*}{\rightharpoonup} v & \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega)),\\ u^n \rightharpoonup v & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)),\\ u^n \rightharpoonup v & \text{weakly in } L^p(\tau,T;L^p(\Omega)),\\ u^n(T) \rightharpoonup \xi & \text{weakly in } L^2(\Omega),\\ a(l(u^n))u^n \rightharpoonup \chi_1 & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)),\\ \sqrt{a(l(u^n))}u^n \rightharpoonup \chi_2 & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)),\\ f(u^n) \rightharpoonup \chi_3 & \text{weakly in } L^q(\tau,T;L^q(\Omega)). \end{cases} \tag{4.25}$$

From this, taking into account the following equality

$$\frac{du^n}{dt}(t) = a(l(u^n(t)))\Delta u^n(t) + f(u^n(t)) + h(t) \quad \text{in } H^{-1}(\Omega)+L^q(\Omega) \quad \text{a.e. } t\in(\tau,T), \tag{4.26}$$

it is a standard matter to prove that we can pick an element in the equivalence class of v satisfying

$$v(t) = u_\tau + \int_\tau^t (\Delta\chi_1(r)+\chi_3(r)+h(r))dr \quad \text{in } H^{-1}(\Omega)+L^q(\Omega) \quad \forall t\in[\tau,T]. \tag{4.27}$$

To prove that $\xi = v(T)$, $\chi_1 = a(l(v))v$, $\chi_2 = \sqrt{a(l(v))}v$ and $\chi_3 = f(v)$, we will argue similarly to [101, 7]. Consider $w \in H_0^1(\Omega)\cap L^p(\Omega)$ fixed. From (4.26) we deduce

$$(u^n(T),w) = (u_\tau^n,w) + \int_\tau^T \langle a(l(u^n(t)))\Delta u^n(t) + f(u^n(t)) + h(t), w\rangle dt.$$

Taking limit when $n\to\infty$ in the previous expression and using (4.25), we have

$$(\xi,w) = (u_\tau,w) + \int_\tau^T \langle \Delta\chi_1(t)+\chi_3(t)+h(t), w\rangle dt.$$

Then, from the above expression and (4.27), we obtain that $\xi = v(T)$.

Now, from (4.6) we deduce

$$\frac{d}{dt}(u^n(t),w) = -a(l(u^n(t)))((u^n(t),w)) + (f(u^n(t)),w) + \langle h(t),w\rangle \quad \text{a.e. } t\in(\tau,T).$$

Integrating the previous equality between t and $t+b$, with $b\in(0,T-\tau)$, $t\in(\tau,T-b)$, and using (4.24) and the Hölder inequality, we have

$$\begin{aligned} &(u^n(t+b)-u^n(t),w)\\ &\le M_{C_\infty}\int_t^{t+b}\|u^n(r)\|_2\|w\|_2dr + \int_t^{t+b}|f(u^n(r))|_q|w|_pdr + \int_t^{t+b}\|h(r)\|_*\|w\|_2dr\\ &\le b^{1/2}\|w\|_2(M_{C_\infty}\|u^n\|_{L^2(\tau,T;H_0^1(\Omega))} + \|h\|_{L^2(\tau,T;H^{-1}(\Omega))}) + b^{1/p}|w|_p\|f(u^n)\|_{L^q(\tau,T;L^q(\Omega))}. \end{aligned}$$

Since $\{u^n\}$ is bounded in $L^2(\tau,T;H_0^1(\Omega))$ and $\{f(u^n)\}$ is bounded in $L^q(\tau,T;L^q(\Omega))$, there exists a positive constant C such that

$$(u^n(t+b)-u^n(t),w)\le C(b^{1/2}+b^{1/p})(\|w\|_2+|w|_p)\quad \forall t\in(\tau,T-b).$$

Taking in the previous inequality $w=u^n(t+b)-u^n(t)\in H_0^1(\Omega)\cap L^p(\Omega)$ a.e. $t\in(\tau,T-b)$, we obtain

$$|u^n(t+b)-u^n(t)|_2^2\le C(b^{1/2}+b^{1/p})(\|u^n(t+b)-u^n(t)\|_2+|u^n(t+b)-u^n(t)|_p)$$

a.e. $t\in(\tau,T-b)$.

Now, integrating between τ and $T-b$, we have

$$\int_\tau^{T-b}|u^n(t+b)-u^n(t)|_2^2dt\le 2C(b^{1/2}+b^{1/p})\left(\int_\tau^T\|u^n(t)\|_2dt+\int_\tau^T|u^n(t)|_pdt\right).$$

Thereupon, using the Hölder inequality,

$$\begin{aligned}&\int_\tau^{T-b}|u^n(t+b)-u^n(t)|_2^2dt\\&\le 2C(b^{1/2}+b^{1/p})\left((T-\tau)^{1/2}\|u^n\|_{L^2(\tau,T;H_0^1(\Omega))}+(T-\tau)^{1/q}\|u^n\|_{L^p(\tau,T;L^p(\Omega))}\right).\end{aligned}$$

We conclude that there exists a positive constant $\overline{C}(T)$ such that

$$\int_\tau^{T-b}|u^n(t+b)-u^n(t)|_2^2dt\le\overline{C}(T)(b^{1/2}+b^{1/p})\quad\forall n\ge1\quad\forall b\in(0,T-\tau).$$

Therefore,

$$\lim_{b\to0}\sup_n\int_\tau^{T-b}|u^n(t+b)-u^n(t)|_2^2dt=0. \tag{4.28}$$

In addition, taking into account that $\{u^n\}$ is bounded in $L^\infty(\tau,T;L^2(\Omega))$, it satisfies

$$\lim_{b\to0}\sup_n\left(\int_\tau^{\tau+b}|u^n(t)|_2^2dt+\int_{T-b}^T|u^n(t)|_2^2dt\right)=0. \tag{4.29}$$

Then, since the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ is compact and taking into account (4.28) and (4.29), applying [110, Theorem 13.2, p. 97] and [110, Remark 13.1, p. 100], we obtain that the sequence $\{u^n\}$ is relatively compact in $L^2(\tau,T;L^2(\Omega))$. Then, making use of [85, Lemme 1.3, p. 12], it holds

$$\begin{gathered}a(l(u^n))u^n\rightharpoonup a(l(v))v\quad\text{weakly in } L^2(\tau,T;H_0^1(\Omega)),\\ \sqrt{a(l(u^n))}u^n\rightharpoonup\sqrt{a(l(v))}v\quad\text{weakly in } L^2(\tau,T;H_0^1(\Omega)),\\ f(u^n)\rightharpoonup f(v)\quad\text{weakly in } L^q(\tau,T;L^q(\Omega)).\end{gathered}$$

Therefore, by the uniqueness of the limit, $\chi_1=a(l(v))v$, $\chi_2=\sqrt{a(l(v))}v$ and $\chi_3=f(v)$.

Then, from (4.27), we deduce

$$v(t) = u_\tau + \int_\tau^t (a(l(v(s)))\Delta v(s) + f(v(s)) + h(s))ds \quad \forall t \in [\tau, T].$$

As a consequence, since the weak solution to (4.1) is unique, $v(t) = u(t)$ holds for all $t \in [\tau, T]$. Therefore, (4.23) holds for the whole sequence $\{u_\tau^n\}$. □

Now, we are ready to prove the pullback asymptotic compactness.

Proposition 4.17. *Under the assumptions of Proposition 4.15, the process U is pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact.*

Proof. We will argue similarly to [7, 101].

Let us fix $t \in \mathbb{R}$, a family $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and $u_{\tau_n} \in D(\tau_n)$ for all n. Let us prove that $\{U(t, \tau_n)u_{\tau_n}\}$ is relatively compact in $L^2(\Omega)$.

As the family $\widehat{D}_0$ is pullback $\mathcal{D}_\mu^{L^2}$-absorbing, for each integer $k \geq 0$, there exists a $\tau(\widehat{D}, k) < t - k$ such that

$$U(t-k, \tau)D(\tau) \subset D_0(t-k) \quad \forall \tau \leq \tau(\widehat{D}, k). \tag{4.30}$$

By a diagonal procedure, it is not difficult to conclude from (4.30) that there exist $\{(\tau_{n'}, u_{\tau_{n'}})\} \subset \{(\tau_n, u_{\tau_n})\}$ and $\{v_k : k \geq 0\} \subset L^2(\Omega)$ such that for all $k \geq 0$, $v_k \in D_0(t-k)$ and

$$U(t-k, \tau_{n'})u_{\tau_{n'}} \rightharpoonup v_k \quad \text{weakly in } L^2(\Omega). \tag{4.31}$$

From this we deduce

$$|v_0|_2 \leq \liminf_{n' \to \infty} |U(t, \tau_{n'})u_{\tau_{n'}}|_2.$$

If we prove that

$$\limsup_{n' \to \infty} |U(t, \tau_{n'})u_{\tau_{n'}}|_2 \leq |v_0|_2, \tag{4.32}$$

then we will have proved that the sequence $\{U(t, \tau_{n'})u_{\tau_{n'}}\}$ converges to v_0 strongly in $L^2(\Omega)$.

Observe that

$$\begin{aligned} &\frac{d}{dt}|U(t,\tau)u_\tau|_2^2 + 2a(l(U(t,\tau)u_\tau))\|U(t,\tau)u_\tau\|_2^2 \\ &= 2(f(U(t,\tau)u_\tau), U(t,\tau)u_\tau) + 2\langle h(t), U(t,\tau)u_\tau\rangle \end{aligned}$$

a.e. $t \geq \tau$.

Thereupon, multiplying by e^t and integrating between τ and t, we obtain

$$\begin{aligned} &|U(t,\tau)u_\tau|_2^2 \\ &= e^{-(t-\tau)}|u_\tau|_2^2 + \int_\tau^t e^{-(t-r)}|U(r,\tau)u_\tau|_2^2 dr - 2\int_\tau^t e^{-(t-r)}a(l(U(r,\tau)u_\tau))\|U(r,\tau)u_\tau\|_2^2 dr \\ &\quad + 2\int_\tau^t e^{-(t-r)}(f(U(r,\tau)u_\tau), U(r,\tau)u_\tau)dr + 2\int_\tau^t e^{-(t-r)}\langle h(r), U(r,\tau)u_\tau\rangle dr. \end{aligned} \tag{4.33}$$

Taking into account the previous equality, we have for all $k \geq 0$ and all $\tau_{n'} \leq t-k$

$$\begin{aligned}
&|U(t,\tau_{n'})u_{\tau_{n'}}|_2^2 \\
=&\ |U(t,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}|_2^2 \\
=&\ e^{-k}|U(t-k,\tau_{n'})u_{\tau_{n'}}|_2^2 + \int_{t-k}^{t} e^{-(t-r)}|U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}|_2^2 dr \\
&- 2\int_{t-k}^{t} e^{-(t-r)} a(l(U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}))\|U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\|_2^2 dr \\
&+ 2\int_{t-k}^{t} e^{-(t-r)}(f(U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}), U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}})dr \\
&+ 2\int_{t-k}^{t} e^{-(t-r)}\langle h(r), U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\rangle dr.
\end{aligned} \tag{4.34}$$

Now, all the terms in the right hand side will be estimated.

From (4.30), we deduce

$$\limsup_{n'\to\infty}(e^{-k}|U(t-k,\tau_{n'})u_{\tau_{n'}}|_2^2) \leq e^{-k}R_{L^2}(t-k) \quad \forall k \geq 0,$$

where R_{L^2} is given in the statement of Proposition 4.15.

On the other hand, since $e^{-(t-\cdot)}h \in L^2(t-k,t;H^{-1}(\Omega))$, using (4.23) and (4.31) we deduce

$$\lim_{n'\to\infty}\int_{t-k}^{t} e^{-(t-r)}\langle h(r), U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\rangle dr = \int_{t-k}^{t} e^{-(t-r)}\langle h(r), U(r,t-k)v_k\rangle dr.$$

Again, considering (4.23) and (4.31), we obtain

$$\lim_{n'\to\infty}\int_{t-k}^{t} e^{-(t-r)}|U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}|_2^2 dr = \int_{t-k}^{t} e^{-(t-r)}|U(r,t-k)v_k|_2^2 dr.$$

Next, we will prove

$$\begin{aligned}
&\limsup_{n'\to\infty} -2\int_{t-k}^{t} e^{-(t-r)} a(l(U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}))\|U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\|_2^2 dr \\
\leq&\ -2\int_{t-k}^{t} e^{-(t-r)} a(l(U(r,t-k)v_k))\|U(r,t-k)v_k\|_2^2 dr.
\end{aligned}$$

Indeed, observe that the sequence $\{\sqrt{a(l(U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}))}U(\cdot,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\}$ converges to $\sqrt{a(l(U(\cdot,t-k)v_k))}U(\cdot,t-k)v_k$ weakly in $L^2(t-k,t;H_0^1(\Omega))$ thanks to (4.23) and (4.31). Therefore, using the lower semicontinuity of the norm, we have

$$\begin{aligned}
&2\int_{t-k}^{t} e^{-(t-r)} a(l(U(r,t-k)v_k))\|U(r,t-k)v_k\|_2^2 dr \\
\leq&\ \liminf_{n'\to\infty} 2\int_{t-k}^{t} e^{-(t-r)} a(l(U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}))\|U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}\|_2^2 dr.
\end{aligned}$$

Finally, we will prove

$$\limsup_{n'\to\infty}\int_{t-k}^{t} e^{-(t-r)}(f(U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}),U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}})dr$$
$$\leq \int_{t-k}^{t} e^{-(t-r)}(f(U(r,t-k)v_k),U(r,t-k)v_k)dr.$$

For short, analogously to [7], we denote

$$\begin{aligned} A_{k,n'}(r) &:= U(r,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}}, \\ B_k(r) &:= U(r,t-k)v_k. \end{aligned}$$

Then, it satisfies

$$\begin{aligned} &\liminf_{n'\to\infty} -\int_{t-k}^{t} e^{-(t-r)}(f(A_{k,n'}(r)),A_{k,n'}(r))dr \\ \geq &\liminf_{n'\to\infty}\int_{t-k}^{t} e^{-(t-r)}(f(B_k(r))-f(A_{k,n'}(r)),A_{k,n'}(r)-B_k(r))dr \\ &+\liminf_{n'\to\infty} -\int_{t-k}^{t} e^{-(t-r)}(f(B_k(r)),A_{k,n'}(r))dr \\ &+\int_{t-k}^{t} e^{-(t-r)}(f(B_k(r)),B_k(r))dr \\ &+\liminf_{n'\to\infty} -\int_{t-k}^{t} e^{-(t-r)}(f(A_{k,n'}(r)),B_k(r))dr. \end{aligned}$$

Using (4.3), it follows

$$\begin{aligned} &\liminf_{n'\to\infty} -\int_{t-k}^{t} e^{-(t-r)}(f(A_{k,n'}(r)),A_{k,n'}(r))dr \\ \geq\ &\liminf_{n'\to\infty} -\eta\int_{t-k}^{t} e^{-(t-r)}|A_{k,n'}(r)-B_k(r)|_2^2dr \\ &+\liminf_{n'\to\infty} -\int_{t-k}^{t} e^{-(t-r)}(f(B_k(r)),A_{k,n'}(r))dr \\ &+\int_{t-k}^{t} e^{-(t-r)}(f(B_k(r)),B_k(r))dr \\ &+\liminf_{n'\to\infty} -\int_{t-k}^{t} e^{-(t-r)}(f(A_{k,n'}(r)),B_k(r))dr. \end{aligned}$$

Taking into account (4.23) and (4.31), from the above inequality

$$\liminf_{n'\to\infty} -\int_{t-k}^{t} e^{-(t-r)}(f(A_{k,n'}(r)),A_{k,n'}(r))dr \geq -\int_{t-k}^{t} e^{-(t-r)}(f(B_k(r)),B_k(r))dr.$$

Then, applying the previous estimates to (4.34), we deduce

$$\begin{aligned}\limsup_{n'\to\infty} |U(t,\tau_{n'})u_{\tau_{n'}}|_2^2 \leq e^{-k}R_{L^2}(t-k) &+ \int_{t-k}^{t} e^{-(t-r)}|U(r,t-k)v_k|_2^2 dr \\ &- 2\int_{t-k}^{t} e^{-(t-r)}a(l(U(r,t-k)v_k))\|U(r,t-k)v_k\|_2^2 dr \\ &+ 2\int_{t-k}^{t} e^{-(t-r)}(f(U(r,t-k)v_k),U(r,t-k)v_k)dr \\ &+ 2\int_{t-k}^{t} e^{-(t-r)}\langle h(r),U(r,t-k)v_k\rangle dr. \end{aligned} \tag{4.35}$$

Observe that taking into account the first convergence of (4.23) and (4.31), we obtain

$$\begin{aligned} v_0 &= \text{weak-}\lim_{n'\to\infty} U(t,\tau_{n'})u_{\tau_{n'}} \\ &= \text{weak-}\lim_{n'\to\infty} U(t,t-k)U(t-k,\tau_{n'})u_{\tau_{n'}} \\ &= U(t,t-k)\left(\text{weak-}\lim_{n'\to\infty} U(t-k,\tau_{n'})u_{\tau_{n'}}\right) \\ &= U(t,t-k)v_k. \end{aligned}$$

Now, from this and (4.33), we deduce

$$\begin{aligned} |v_0|_2^2 =& |U(t,t-k)v_k|_2^2 \\ =& e^{-k}|v_k|_2^2 + \int_{t-k}^{t} e^{-(t-r)}|U(r,t-k)v_k|_2^2 dr \\ &- 2\int_{t-k}^{t} e^{-(t-r)}a(l(U(r,t-k)v_k))\|U(r,t-k)v_k\|_2^2 dr \\ &+ 2\int_{t-k}^{t} e^{-(t-r)}(f(U(r,t-k)v_k),U(r,t-k)v_k)dr \\ &+ 2\int_{t-k}^{t} e^{-(t-r)}\langle h(r),U(r,t-k)v_k\rangle dr. \end{aligned}$$

Then, comparing the above expression with (4.35), we deduce

$$\begin{aligned}\limsup_{n'\to\infty} |U(t,\tau_{n'})u_{\tau_{n'}}|_2^2 &\leq R_{L^2}(t-k)e^{-k} + |v_0|_2^2 - e^{-k}|v_k|_2^2 \\ &\leq R_{L^2}(t-k)e^{-k} + |v_0|_2^2, \end{aligned}$$

for all $k \geq 0$. As a result, (4.32) holds. □

Now we can establish the main result of this section. We omit the proof because it is analogous to those of Theorems 2.16 and 3.17.

Theorem 4.18. *Assume that the function a is locally Lipschitz and (4.2) holds, $f \in C(\mathbb{R})$ satisfies (4.3) and (4.4), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ fulfils condition (4.22)* *for some* $\mu \in (0, 2m\lambda_1)$ *and* $l \in L^2(\Omega)$. *Then, there exist the minimal pullback* $\mathcal{D}_F^{L^2}$*-attractor* $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ *and the minimal pullback* $\mathcal{D}_\mu^{L^2}$*-attractor* $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ *for the process* $U : \mathbb{R}_d^2 \times L^2(\Omega) \to L^2(\Omega)$. *Besides, the family* $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ *belongs to* $\mathcal{D}_\mu^{L^2}$, *and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \subset \overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$

In addition, if h *also satisfies*

$$\sup_{s\le 0}\left(e^{-\mu s}\int_{-\infty}^{s} e^{\mu r}\|h(r)\|_*^2 dr\right) < \infty,$$

then both attractors coincide, i.e. $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)$ *for all* $t \in \mathbb{R}$.

4.3 Pullback attraction in H^1-norm

In this section, we will improve the results given in the previous section by proving attraction in $H_0^1(\Omega)$ and establish relationships amongst these new pullback attractors and those given in Section 4.2 without assuming any condition of regularity on the domain Ω as done in Chapter 3. To do this, we state the results in the setting of Theorem 4.10. Observe that although it is possible to prove the existence of strong solutions assuming that $N \le 2p/(p-2)$ and without making any additional requirement on f (cf. Theorem 4.8), to study the asymptotic behaviour of the solution in H^1-norm, we need to make use of stronger energy equalities and the continuity of the solution u in $H_0^1(\Omega)$. To that end, we need that $u' \in L^2(\tau, T; L^2(\Omega))$. Due to the nonlinearity created by the nonlocal operator in the diffusion term, we cannot analyse the regularity of u' directly (multiplying the equation by u'), but u' inherits the regularity of $a(l(u))\Delta u + f(u) + h$. Therefore, $f(u)$ must belong to $L^2(\tau, T; L^2(\Omega))$. To that end, it is essential in this section to assume (4.13).

Observe that the main result of this section, the existence of pullback attractors in $H_0^1(\Omega)$, is not new in this PhD project, since the existence of these families has been proved in Theorem 3.23 under different assumptions. Nevertheless, the energy method applied here to prove the pullback asymptotic compactness is, since we make use of the flattening property.

In the setting of Theorem 4.10, the restriction of U to $\mathbb{R}_d^2 \times H_0^1(\Omega)$ defines a process into $H_0^1(\Omega)$. Since no confusion arises, we will not modify the notation and continue denoting this process by U.

Making use of the results studied in Chapter 1, we prove the existence of pullback attractors in $H_0^1(\Omega)$. First of all, the process U is strong-weak in $H_0^1(\Omega)$.

Proposition 4.19. *Assume that the function* a *is locally Lipschitz and (4.2) holds,* $f \in C(\mathbb{R})$ *fulfils (4.3), (4.4) and (4.13),* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *and* $l \in L^2(\Omega)$. *Then, the process* U *is strong-weak continuous in* $H_0^1(\Omega)$.

Proof. Consider $(t,\tau) \in \mathbb{R}^2_d$ fixed. Let $\{u^n_\tau\} \subset H^1_0(\Omega)$ be a sequence which converges to u_τ strongly in $H^1_0(\Omega)$.

By Proposition 4.12, the map $U(t,\tau)$ is continuous from $L^2(\Omega)$ into itself, therefore

$$U(t,\tau)u^n_\tau \to U(t,\tau)u_\tau \quad \text{strongly in } L^2(\Omega).$$

On the other hand, from (4.7) using (4.2), (4.4) and the Cauchy inequality, we obtain

$$\|U(\cdot,\tau)u^n_\tau\|^2_{L^\infty(\tau,t;L^2(\Omega))} \leq 2\kappa|\Omega|(t-\tau) + \frac{1}{\lambda_1 m}\|h\|^2_{L^2(\tau,t;L^2(\Omega))} + |u^n_\tau|^2_2,$$
$$\int_\tau^t \|U(s,\tau)u^n_\tau\|^2_2 ds \leq \frac{2\kappa|\Omega|(t-\tau)}{m} + \frac{1}{\lambda_1 m^2}\|h\|^2_{L^2(\tau,t;L^2(\Omega))} + \frac{1}{m}|u^n_\tau|^2_2.$$

Now, applying (4.2), the Cauchy inequality, (4.13) and interpolation results (see (4.14)) to the energy equality (4.15), we deduce

$$\begin{aligned} &\|U(t,\tau)u^n_\tau\|^2_2 \\ &\leq \|u^n_\tau\|^2_2 + \frac{1}{m}\|h\|^2_{L^2(\tau,t;L^2(\Omega))} + \frac{2C^2|\Omega|(T-\tau)}{m} \\ &\quad + \frac{2C^2(C_I(N))^{2\tilde{b}}}{m}\|U(\cdot,\tau)u^n_\tau\|^{2\hat{b}}_{L^\infty(\tau,t;L^2(\Omega))}\|U(\cdot,\tau)u^n_\tau\|^{2\tilde{b}}_{L^2(\tau,t;H^1_0(\Omega))}, \end{aligned}$$

where $C_I(N)$ is the constant of the continuous embedding of $H^1_0(\Omega)$ into $L^{\mathfrak{p}}$-spaces, $\hat{b} = (1-\theta)(\gamma+1)$ and $\tilde{b} = \theta(\gamma+1)$, with $\gamma = 2/N$ when $N \geq 3$ and $\theta \in [0,1]$.

Then, thanks to the previous estimates, $\{U(t,\tau)u^n_\tau\}$ is bounded in $H^1_0(\Omega)$. Thus, by the uniqueness of the limit, we obtain

$$U(t,\tau)u^n_\tau \rightharpoonup U(t,\tau)u_\tau \quad \text{weakly in } H^1_0(\Omega).$$

□

The following lemma is essential to prove the pullback flattening property. We establish uniform estimates of the solutions in a finite-time interval up to t when the initial datum is shifted pullback far enough. The idea of the proof is similar to the proofs of Lemmas 3.15 and 3.19. We provide the details for the sake of completeness.

Lemma 4.20. *Under the assumptions of Proposition 4.19, if $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies (4.22) for some $\mu \in (0, 2\lambda_1 m)$, then for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^{L^2}_\mu$, there exists $\tau_1(\widehat{D}, t) < t-2$ such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$*

$$\begin{cases} |u(r;\tau,u_\tau)|^2_2 \leq \rho_1(t) \quad \forall r \in [t-2,t], \\ \displaystyle\int_{r-1}^r \|u(\xi;\tau,u_\tau)\|^2_2 d\xi \leq \rho_2(t) \quad \forall r \in [t-1,t], \\ \|u(r;\tau,u_\tau)\|^2_2 \leq \rho_3(t) \quad \forall r \in [t-1,t], \end{cases} \tag{4.36}$$

where

$$\rho_1(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{e^{-\mu(t-2)}}{2m - \lambda_1^{-1}\mu}\int_{-\infty}^{t} e^{\mu\xi}\|h(\xi)\|_*^2 d\xi,$$

$$\rho_2(t) = \frac{2\kappa|\Omega|}{m} + \frac{1}{m}\rho_1(t) + \frac{1}{m^2}\max_{r\in[t-1,t]}\int_{r-1}^{r}\|h(\xi)\|_*^2 d\xi,$$

$$\rho_3(t) = \rho_2(t) + \frac{2C^2|\Omega|}{m} + \frac{2C^2(C_I(N))^{2\tilde{b}}}{m}(\rho_1(t))^{\hat{b}}(\rho_2(t))^{\tilde{b}} + \frac{1}{m}\max_{r\in[t-1,t]}\int_{r-1}^{r}|h(\xi)|_2^2 d\xi,$$

where $\hat{b} = (1-\theta)(\gamma+1)$, $\tilde{b} = \theta(\gamma+1)$, $C_I(N)$ *is the constant of the continuous embedding of* $H_0^1(\Omega)$ *into* $L^{\mathfrak{p}}$*-spaces and* $\theta \in [0,1]$.

Proof. From (4.21) we deduce that there exists $\tau_1(\widehat{D}, t) < t-2$ such that

$$e^{-\mu(t-2)}e^{\mu\tau}|u_\tau|_2^2 \le 1 \quad \forall u_\tau \in D(\tau) \quad \forall \tau \le \tau_1(\widehat{D}, t).$$

The first inequality of (4.36) follows directly from (4.20) using the previous estimate. Making use of this estimate together with the energy equality (4.7), the second inequality follows arguing analogously as it was done in Lemma 3.15. Observe that these two estimates also holds for the Galerkin approximations.

Finally, we will prove the third inequality of (4.36). Consider fixed $\tau \le \tau_1(\widehat{D}, t)$ and $u_\tau \in D(\tau)$. From (4.15) for the Galerkin approximations, making use of (4.2) and the Cauchy inequality, we deduce

$$\|u_n(r;\tau,u_\tau)\|_2^2 \le \|u_n(s;\tau,u_\tau)\|_2^2 + \frac{1}{m}\int_{r-1}^{r}|f(u_n(\xi))|_2^2 ds + \frac{1}{m}\int_{r-1}^{r}|h(\xi)|_2^2 ds$$

with $\tau \le r-1 \le s \le r$.

Integrating the previous inequality w.r.t. s on $[r-1, r]$ and taking into account (4.14), we obtain

$$\begin{aligned}\|u_n(r;\tau,u_\tau)\|_2^2 \le &\int_{r-1}^{r}\|u_n(s;\tau,u_\tau)\|_2^2 ds + \frac{1}{m}\int_{r-1}^{r}|h(\xi)|_2^2 d\xi + \frac{2C^2|\Omega|}{m}\\ &+ \frac{2C^2(C_I(N))^{2\tilde{b}}}{m}\|u_n(\cdot;\tau,u_\tau)\|_{L^\infty(r-1,r;L^2(\Omega))}^{2\hat{b}}\|u_n(\cdot;\tau,u_\tau)\|_{L^2(r-1,r;H_0^1(\Omega))}^{2\tilde{b}}\end{aligned}$$

for all $\tau \le r-1$.

Now, applying the two first inequalities of (4.36) to the previous expression we have

$$\|u_n(r;\tau,u_\tau)\|_2^2 \le \rho_3(t) \quad \forall r \in [t-1, t],$$

where $\rho_3(t)$ is given in the statement. Taking inferior limit in the above expression and using the well-known fact that u_n converge to $u(\cdot;\tau,u_\tau)$ weakly-star in $L^\infty(t-1, t; H_0^1(\Omega))$ (cf. Theorem 4.10), the third inequality of (4.36) holds. □

Thereupon, we introduce the following universe in $\mathcal{P}(H_0^1(\Omega))$.

Definition 4.21. *For each* $\mu > 0$, $\mathcal{D}_\mu^{L^2,H_0^1}$ *denotes the class of all families of nonempty subsets* $\widehat{D}_{H_0^1} = \{D(t) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$, *where* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{L^2}$.

Observe that $\mathcal{D}_F^{H_0^1} \subset \mathcal{D}_\mu^{L^2,H_0^1}$ and $\mathcal{D}_\mu^{L^2,H_0^1}$ is inclusion-closed.

The existence of a pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-absorbing family follows directly from the regularising effect of the equation (cf. Theorem 4.10) and the existence of a pullback $\mathcal{D}_\mu^{L^2}$-absorbing family (cf. Proposition 4.15). We omit the proof of this result because it is identical to that of Proposition 2.21.

Proposition 4.22. *Under the assumptions of Proposition 4.19, if* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *also fulfils condition (4.22) for some* $\mu \in (0, 2\lambda_1 m)$, *then, the family*

$$\widehat{D}_{0,H_0^1} = \{\overline{B}_{L^2}(0, R_{L^2}^{1/2}(t)) \cap H_0^1(\Omega) : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}_\mu^{L^2,H_0^1}$ *and for any* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, *there exists* $\tau_2(\widehat{D}, t) < t$ *such that*

$$U(t,\tau)D(\tau) \subset D_{0,H_0^1}(t) \quad \forall \tau \leq \tau_2(\widehat{D}, t).$$

In particular, the family $\widehat{D}_{0,H_0^1}$ *is pullback* $\mathcal{D}_\mu^{L^2,H_0^1}$*-absorbing for the process* $U : \mathbb{R}^2_d \times H_0^1(\Omega) \to H_0^1(\Omega)$.

Thereupon, we will prove that the process $U : \mathbb{R}^2_d \times H_0^1(\Omega) \to H_0^1(\Omega)$ satisfies the pullback $\widehat{D}_{0,H_0^1}$-flattening property. In fact, we will prove that U fulfils the pullback $\widehat{D}_{H_0^1}$-flattening property for any $\widehat{D}_{H_0^1} \in \mathcal{D}_\mu^{L^2,H_0^1}$.

We will also use the following result, whose proof is analogous to that of [78, Lemma 12].

Lemma 4.23. *If* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *satisfies condition*

$$\int_{-\infty}^0 e^{\mu s}|h(s)|^2 ds < \infty \tag{4.37}$$

for some $\mu \in (0, 2m\lambda_1)$, *then for any* $t \in \mathbb{R}$

$$\lim_{\rho \to \infty} e^{-\rho t} \int_{-\infty}^t e^{\rho s}|h(s)|_2^2 ds = 0.$$

Then we have the following result (the idea of the proof is close to that in [65, Proposition 31]).

Proposition 4.24. *Under the assumptions of Proposition 4.19, if* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *also fulfils (4.37) for some* $\mu(0, 2\lambda_1 m)$, *then, for any* $\varepsilon > 0$ *and* $t \in \mathbb{R}$, *there exists* $n(\varepsilon, t) \in \mathbb{N}$ *such that for any* $\widehat{D} \in \mathcal{D}_\mu^{L^2}$, *the projection* $P_n : H_0^1(\Omega) \to V_n$ *satisfies*

a) $\{P_n U(t,\tau)D(\tau) : \tau \leq \tau_1(\widehat{D}, t)\}$ *is bounded in* $H_0^1(\Omega)$,

b) for all $\tau \leq \tau_1(\widehat{D}, t)$ *and* $u_\tau \in D(\tau)$, *it fulfils that* $\|(I - P_n)U(t,\tau)u_\tau\|_2 < \varepsilon$, *where $\tau_1(\widehat{D},t)$ is given in Lemma 4.20.*

In particular, the process U on $H_0^1(\Omega)$ satisfies the pullback $\widehat{D}_{H_0^1}$-flattening property for any $\widehat{D}_{H_0^1} \in \mathcal{D}_\mu^{L^2,H_0^1}$.

Proof. Let us fix $\varepsilon > 0$, $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$.

The first property given in the statement follows directly from the fact that P_n is non-expansive in $H_0^1(\Omega)$, since $\{w_j : j \geq 1\}$ is a special basis (see Section 4.1.3) for more details) and the third inequality of (4.36).

Thereupon, we will prove the second property. To that end consider fixed $\tau \leq \tau_1(\widehat{D},t)$, $u_\tau \in D(\tau)$, and define $u(r) := U(r,\tau)u_\tau$ and $q_n(r) := u(r) - P_n u(r)$. Then, using (4.2) and the Cauchy inequality, from (4.15) we deduce for each $n \geq 1$

$$\frac{d}{dr}\|q_n(r)\|_2^2 + m|-\Delta q_n(r)|_2^2 \leq \frac{2}{m}|f(u(r))|_2^2 + \frac{2}{m}|h(r)|_2^2 \quad \text{a.e. } r \in (t-1,t).$$

Since $|-\Delta q_n(r)|_2^2 \geq \lambda_{n+1}\|q_n(r)\|_2^2$, from the above inequality we deduce

$$\frac{d}{dr}\|q_n(r)\|_2^2 + \lambda_{n+1}m\|q_n(r)\|_2^2 \leq \frac{2}{m}|f(u(r))|_2^2 + \frac{2}{m}|h(r)|_2^2 \quad \text{a.e. } r \in (t-1,t). \tag{4.38}$$

Now multiplying by $e^{m\lambda_{n+1}r}$ in (4.38), integrating between $t-1$ and t, making use of (4.13) and (4.36), we have

$$\begin{aligned} e^{m\lambda_{n+1}t}\|q_n(t)\|_2^2 \leq& e^{m\lambda_{n+1}(t-1)}\|q_n(t-1)\|_2^2 + \frac{2}{m}\int_{t-1}^t e^{m\lambda_{n+1}r}|h(r)|_2^2 dr \\ &+ \frac{4C^2 e^{m\lambda_{n+1}t}}{m^2\lambda_{n+1}}\left[|\Omega| + (C_N^2\rho_3(t))^{(\gamma+1)}\right] \\ \leq& e^{m\lambda_{n+1}(t-1)}\rho_3(t) + \frac{2}{m}\int_{-\infty}^t e^{m\lambda_{n+1}r}|h(r)|_2^2 dr \\ &+ \frac{4C^2 e^{m\lambda_{n+1}t}}{m^2\lambda_{n+1}}\left[|\Omega| + (C_N^2\rho_3(t))^{(\gamma+1)}\right], \end{aligned}$$

where C_N is the constant of the continuous embedding of $H_0^1(\Omega)$ into $L^{2\gamma+2}(\Omega)$, with $\gamma = 2/N$ when $N \geq 3$.

Then, applying Lemma 4.23 and taking into account that $\lambda_n \to \infty$ as $n \to \infty$, there exists $n = n(\varepsilon,t) \in \mathbb{N}$ such that the second property holds. □

From the above result, the asymptotic compactness in the H^1-norm yields (see Proposition 1.18 for more details).

Proposition 4.25. *Under the assumptions of Proposition 4.24, the process U on $H_0^1(\Omega)$ is pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-asymptotically compact.*

As a consequence of the previous results, we obtain the existence of minimal pullback attractors for the process U on $H_0^1(\Omega)$. Relationships amongst these new attractors and those given in Theorem 4.18 are also established. We omit the proof because it is similar to those of Theorems 2.23 and 3.23.

Theorem 4.26. *Assume that the function a is locally Lipschitz and (4.2) holds, $f \in C(\mathbb{R})$ satisfies (4.3), (4.4) and (4.13), $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ fulfils condition (4.37) for some $\mu \in (0, 2\lambda_1 m)$ and $l \in L^2(\Omega)$. Then, there exist the minimal pullback $\mathcal{D}_F^{H_0^1}$-attractor $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ and the minimal pullback $\mathcal{D}_\mu^{L^2,H_0^1}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}$ for the process $U : \mathbb{R}^2_d \times H_0^1(\Omega) \to H_0^1(\Omega)$. In addition, the following relationships holds*

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) \subset \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}(t) \quad \forall t \in \mathbb{R},$$

where $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ are respectively the minimal pullback $\mathcal{D}_F^{L^2}$-attractor and the minimal pullback $\mathcal{D}_\mu^{L^2}$-attractor for the process $U : \mathbb{R}^2_d \times L^2(\Omega) \to L^2(\Omega)$, whose existence is guaranteed by Theorem 4.18. In particular, the following pullback attraction result holds

$$\lim_{\tau \to -\infty} \operatorname{dist}_{H_0^1}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R} \quad \forall \widehat{D} \in \mathcal{D}_\mu^{L^2}.$$

Finally, if h also satisfies

$$\sup_{s \leq 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu r} |h(r)|_2^2 dr \right) < \infty,$$

then

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2,H_0^1}}(t) \quad \forall t \in \mathbb{R}.$$

Furthermore, in this case, for any $B \in \mathcal{D}_F^{L^2}$

$$\lim_{\tau \to -\infty} \operatorname{dist}_{H_0^1}(U(t,\tau)B, \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R}.$$

As a consequence of the previous result and as a complement of Corollary 4.11, we have the following result.

Corollary 4.27. *Assume that the function a is locally Lipschitz and (4.2) holds, $f \in C^1(\mathbb{R})$ satisfies (4.3), (4.4) and (4.13) with $\gamma = 2/(N-2)$ if $3 \leq N \leq 2p/(p-2)$, $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ fulfils condition (4.37) for some $\mu \in (0, 2\lambda_1 m)$ and $l \in L^2(\Omega)$. Then, the thesis of Theorem 4.26 holds.*

Remark 4.28. *Observe that if $N = 1, 2$, Corollary 4.27 holds without assuming any restriction on the positive value γ (see Remark 4.9 (i) for more details).*

Chapter 5

Abstract results on the theory of multi-valued processes and pullback attractors

In the last few decades, many authors have been interested in analysing problems without uniqueness of solution because it allows to weaken the assumptions on the nonlinear functions which appears in the equation.

In addition, for a wide range of problems, such as differential inclusions, variational inequalities, control infinite-dimensional systems and some partial differential equations such as the three-dimensional Navier-Stokes equations, the uniqueness of solution is not guaranteed. It is interesting to analyse what the asymptotic behaviour of the solutions of this type of problems is and to do it, many authors make use of the theory of multi-valued dynamical systems. For instance, in [96] Melnik & Valero study the existence of the compact global attractor for differential inclusions in Hilbert spaces. In [6], Anguiano et al. analyse the existence of pullback attractors for non-autonomous reaction-diffusion equations in some unbounded domains. For problems where terms with delay appear there exist also papers in this multi-valued framework. For example, in [89] Marín-Rubio studies the existence of attractors corresponding to a general class of parameterized delay differential equations posed in potentially different state spaces.

In this chapter, we briefly recall some abstract results on multi-valued non-autonomous dynamical systems. Concepts such as multi-valued process, pullback absorbing family for a universe $\mathcal{D}$ or pullback asymptotic compactness, amongst others, are recalled. In addition, some properties of the omega-limit set are analysed in this new framework together with a result which guarantees the existence of minimal pullback attractors. Besides, relationships between these families are established.

All the results of this chapter can be found in [96, 89, 27, 94, 6, 5].

5.1 Basic concepts

Let (X, d_X) be a metric space.

Definition 5.1. *A multi-valued process (also called multi-valued non-autonomous dynamical system) U on X is a family of mappings $U(t,\tau) : X \mapsto \mathcal{P}(X)$ for any pair $(t,\tau) \in \mathbb{R}^2_d$, such that*

(i) $U(\tau,\tau)x = \{x\} \quad \forall \tau \in \mathbb{R} \quad \forall x \in X.$

(ii) $U(t,\tau)x \subset U(t,s)(U(s,\tau)x) \quad \forall \tau \le s \le t \quad \forall x \in X,$ *where*

$$U(t,\tau)W := \bigcup_{y\in W} U(t,\tau)y \; \forall W \subset X.$$

Observe that if the relationship given in (ii) is an equality instead of an inclusion, the multi-valued process U is called *strict*.

Definition 5.2. *A multi-valued process U on X is upper semicontinuous if the mapping $U(t,\tau)$ is upper semicontinuous from X into $\mathcal{P}(X)$ for all $(t,\tau) \in \mathbb{R}^2_d$, i.e. for any $x \in X$ and for every neighborhood $\mathcal{N}$ in X of the set $U(t,\tau)x$, there exists a value $\varepsilon > 0$ such that $U(t,\tau)y \subset \mathcal{N}$ provided that $d_X(x,y) < \varepsilon$.*

Consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 5.3. *A multi-valued process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ such that $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, it fulfils that any sequence $\{y_n\}$ is relatively compact in X, where $y_n \in U(t,\tau_n)x_n$ for all n.*

Again, analogously as in Chapter 1, we denote the omega-limit set of the family $\widehat{D}_0$ by

$$\Lambda(\widehat{D}_0, t) = \bigcap_{s\le t} \overline{\bigcup_{\tau \le s} U(t,\tau)D_0(\tau)}^X \quad \forall t \in \mathbb{R}.$$

Lemma 5.4 (**Sequential characterisation of the omega-limit set**)**.** *It holds that $y \in \Lambda(\widehat{D}_0, t)$ if and only if there exist sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{y_n\} \subset X$ such that $\tau_n \to -\infty$, $y_n \in U(t,\tau_n)D_0(\tau_n)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} y_n = y$.*

The following lemma will be very helpful to prove the existence of a pullback $\mathcal{D}$-attractor for a multi-valued process U in Section 5.2. This result was proved in [5, Lemma 3.9] and it is a generalization of [29, Theorem 6, Lemma 8]. We will show the proof for the sake of completeness.

Lemma 5.5. *If the multi-valued process U is pullback $\widehat{D}_0$-asymptotically compact, then the omega-limit set $\Lambda(\widehat{D}_0, t)$ is nonempty, compact and*

$$\lim_{\tau\to-\infty} \mathrm{dist}_X(U(t,\tau)D_0(\tau), \Lambda(\widehat{D}_0, t)) = 0 \quad \forall t \in \mathbb{R}. \tag{5.1}$$

In addition, the family $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$ *is minimal in the sense that if there exists a family of closed sets* $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ *which fulfils*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D_0(\tau), C(t)) = 0,$$

then $\Lambda(\widehat{D}_0, t) \subset C(t)$ *for all* $t \in \mathbb{R}$. *Moreover, if the multi-valued process* U *is upper semicontinuous with closed values, it holds*

$$\Lambda(\widehat{D}_0, t) \subset U(t,\tau)\Lambda(\widehat{D}_0, \tau) \quad \forall (t,\tau) \in \mathbb{R}^2_d. \tag{5.2}$$

Proof. Consider $t \in \mathbb{R}$ fixed.

First of all, we will check that the set $\Lambda(\widehat{D}_0, t)$ is nonempty. This is straightforward. Let $\{y_n\}$ be a sequence such that $y_n \in U(t, \tau_n)D_0(\tau_n)$ for all $n \in \mathbb{N}$ and $\{\tau_n\} \subset (-\infty, t]$ converging to $-\infty$. As the multi-valued process U is pullback $\widehat{D}_0$-asymptotically compact, there exists a convergent subsequence of $\{y_n\}$, i.e. $y_{n_k} \to y \in \Lambda(\widehat{D}_0, t)$. Therefore, $\Lambda(\widehat{D}_0, t)$ is nonempty.

Now, we will show that the set $\Lambda(\widehat{D}_0, t)$ is compact. Since it is closed, we only need to check that it is relatively compact. To do this, consider given a sequence $\{y_n\} \subset \Lambda(\widehat{D}_0, t)$. Making use of Lemma 5.4, for each $y_n \in \Lambda(\widehat{D}_0, t)$, there exist $\tau_n \to -\infty$ and $z_n \in U(t, \tau_n)D_0(\tau_n)$, such that

$$d_X(y_n, z_n) < \frac{1}{n}. \tag{5.3}$$

Then, as the process U is pullback $\widehat{D}_0$-asymptotically compact, there exists a convergent subsequence of $\{z_n\}$. Therefore, making use of (5.3), the sequence $\{y_n\}$ is relatively compact.

The next step consists in proving (5.1). To do this, we argue by contradiction. Assume that there exist $\varepsilon > 0$ and a sequence $\{y_n\}$ with $y_n \in U(t, \tau_n)D_0(\tau_n)$ for all $n \in \mathbb{N}$ and $\{\tau_n\} \subset (-\infty, t]$ converging to $-\infty$, such that

$$d_X(y_n, \Lambda(\widehat{D}_0, t)) > \varepsilon \quad \forall n \in \mathbb{N}. \tag{5.4}$$

Now, since the multi-valued process U is pullback $\widehat{D}_0$-asymptotically compact, there exists a subsequence of $\{y_n\}$ (relabeled the same) such that $y_n \to y \in \Lambda(\widehat{D}_0, t)$ (see Lemma 5.4), which is a contradiction with (5.4).

In addition, the family $\{\Lambda(\widehat{D}_0, t) : t \in \mathbb{R}\}$ is minimal. Assume that $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ is a family of closed sets which fulfils

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D_0(\tau), C(t)) = 0. \tag{5.5}$$

Consider given $y \in \Lambda(\widehat{D}_0, t)$, we will show that $y \in C(t)$. On the one hand, from Lemma 5.4, we deduce that there exist sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{y_n\} \subset X$ with $\tau_n \to -\infty$ and $y_n \in U(t, \tau_n)D_0(\tau_n)$ for all $n \in \mathbb{N}$, such that $y_n \to y$. On the other hand, from (5.5) we deduce

$$\lim_{n \to \infty} \operatorname{dist}_X(U(t,\tau_n)D_0(\tau_n), C(t)) = 0.$$

Therefore, $y \in C(t)$ since $C(t)$ is a closed subset of X.

Finally, assuming that the multi-valued process U is upper semicontinuous with closed values, (5.2) is proved. Consider $(t,\tau) \in \mathbb{R}^2_d$ fixed. Given $y \in \Lambda(\widehat{D}_0, t)$, there exists a sequence $\{y_n\} \subset X$ with $y_n \in U(t, \tau_n + \tau)x_n$ for all $n \in \mathbb{N}$, where $x_n \in D_0(\tau_n + \tau)$ and $\tau_n \subset (-\infty, 0]$ with $\tau_n \to -\infty$, such that $y_n \to y$. Observe that $U(t, \tau_n+\tau)x_n \subset U(t,\tau)U(\tau, \tau_n+\tau)x_n$, therefore, $y_n \in U(t,\tau)z_n$, where $z_n \in U(\tau, \tau_n+\tau)x_n$ for all n. Then, as the multi-valued process U is pullback $\widehat{D}_0$-asymptotically compact, there exists a subsequence of $\{z_n\}$ (relabeled the same) such that

$$z_n \to z \in \Lambda(\widehat{D}_0, \tau).$$

Finally, using that the multi-valued process U is upper semicontinuous with closed values, we deduce that $y \in U(t,\tau)z \subset U(t,\tau)\Lambda(\widehat{D}_0, \tau)$. □

From now on, consider a *universe* $\mathcal{D}$, that is a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 5.6. *A universe $\mathcal{D}$ is inclusion-closed if given $\widehat{D} \in \mathcal{D}$ and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all $t \in \mathbb{R}$, it holds that $\widehat{D}' \in \mathcal{D}$.*

Definition 5.7. *The family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ is said to be pullback $\mathcal{D}$-absorbing for a multi-valued process U if for every $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, there exists $\tau(\widehat{D}, t) \le t$ such that*

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \forall \tau \le \tau(\widehat{D}, t).$$

Proposition 5.8. *If the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the multi-valued process U, then*

$$\Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t) \quad \forall \widehat{D} \in \mathcal{D} \quad \forall t \in \mathbb{R}.$$

In addition, if the family $\widehat{D}_0 \in \mathcal{D}$, then

$$\Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^X \quad \forall t \in \mathbb{R}. \tag{5.6}$$

Proof. Consider fixed a family $\widehat{D} \in \mathcal{D}$, $t \in \mathbb{R}$ and $y \in \Lambda(\widehat{D}, t)$. Making use of Lemma 5.4, there exist two sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{y_n\} \subset X$, such that $\tau_n \to -\infty$, $y_n \in U(t, \tau_n)D(\tau_n)$ for all $n \in \mathbb{N}$, and $y_n \to y$.

Taking into account that the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the multi-valued process U, for any $k \in \mathbb{N}$, there exists $\tau_{n_k} \in \{\tau_n\}$ with $\tau_{n_k} \le t - k$ such that $U(t-k, \tau_{n_k})D(\tau_{n_k}) \subset D_0(t-k)$. Then, since

$$y_{n_k} \in U(t, \tau_{n_k})D(\tau_{n_k}) \subset U(t, t-k)U(t-k, \tau_{n_k})D(\tau_{n_k}) \subset U(t, t-k)D_0(t-k),$$

and $y_{n_k} \to y$, by Lemma 5.4, $y \in \Lambda(\widehat{D}_0, t)$.

Finally, to prove (5.6), assume that $\widehat{D}_0 \in \mathcal{D}$ and consider $t \in \mathbb{R}$ fixed. Making use of Lemma 5.4, it holds that given $y \in \Lambda(\widehat{D}_0, t)$, there exist $\{\tau_n\} \subset (-\infty, t]$ and

$\{y_n\} \subset X$ such that $\tau_n \to -\infty$, $y_n \in U(t,\tau_n)D_0(\tau_n)$ for all $n \in \mathbb{N}$, and $y_n \to y$. Then, taking into account that the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing for the multi-valued process U, there exists $n_1 \in \mathbb{N}$ such that $y_n \in D_0(t)$ for all $n \geq n_1$. Therefore, $y \in \overline{D_0(t)}^X$. □

Proposition 5.9. *Assume that the family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing and the multi-valued process U is pullback $\widehat{D}_0$-asymptotically . Then, the multi-valued process U is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$, i.e. U is pullback $\mathcal{D}$-asymptotically compact.*

Proof. Consider fixed $\widehat{D} \in \mathcal{D}$, $t \in \mathbb{R}$, $\{\tau_n\} \subset (-\infty, t]$ and $\{y_n\} \subset X$, such that $\tau_n \to -\infty$ and $y_n \in U(t,\tau_n)D(\tau_n)$ for all $n \in \mathbb{N}$. Our aim is to prove that the sequence $\{y_n\}$ is relatively compact in X.

Since the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing, for any $k \in \mathbb{N}$, there exists $\tau_{n_k} \in \{\tau_n\}$ such that $\tau_{n_k} \leq t-k$ and $U(t-k,\tau_{n_k})D(\tau_{n_k}) \subset D_0(t-k)$. Then, bearing this in mind together with item (ii) of Definition 5.1, we have

$$y_{n_k} \in U(t,\tau_{n_k})D(\tau_{n_k}) \subset U(t,t-k)U(t-k,\tau_{n_k})D(\tau_{n_k}) \subset U(t,t-k)D_0(t-k).$$

Finally, taking into account that the process U is pullback $\widehat{D}_0$-asymptotically compact, we deduce that there exists a convergent subsequence of $\{y_{n_k}\}$. □

5.2 Existence and relationships between pullback attractors

Definition 5.10. *A pullback $\mathcal{D}$-attractor for a multi-valued process U on X is a family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ such that*

1. *for any $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X;*
2. *$\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, i.e.*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0 \quad \forall \widehat{D} \in \mathcal{D} \quad \forall t \in \mathbb{R},$$

3. *$\mathcal{A}_{\mathcal{D}}$ is negatively invariant, i.e.*

$$\mathcal{A}_{\mathcal{D}}(t) \subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) \quad \forall (t,\tau) \in \mathbb{R}^2_d.$$

A pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}}$ is said to be minimal if it satisfies that if there exists another family of closed sets $\widehat{C} = \{C(t) : t \in \mathbb{R}\}$ such that it is pullback $\mathcal{D}$-attracting, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Observe that pullback attractors are not unique in general (cf. [93]); however, the minimal pullback attractor is. Therefore, in the sense of minimality, one recovers uniqueness of pullback attractor.

The following result ensures the existence of a pullback $\mathcal{D}$-attractor for a multi-valued process U (see also [27, 94, 5]).

Theorem 5.11. *Assume that U is an upper semicontinuous multi-valued process with closed values, $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a pullback $\mathcal{D}$-absorbing family and also suppose that the process U is pullback $\widehat{D}_0$-asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ given by*

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t)}^X \quad \forall t \in \mathbb{R}$$

is the minimal pullback $\mathcal{D}$-attractor. In addition, when $\widehat{D}_0 \in \mathcal{D}$, $\mathcal{A}_{\mathcal{D}}(t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$. Finally, if $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ and U is strict, $\mathcal{A}_{\mathcal{D}}$ is invariant under the multi-valued process U, i.e.

$$\mathcal{A}_{\mathcal{D}}(t) = U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) \quad \forall (t,\tau) \in \mathbb{R}^2_d.$$

Proof. First of all, we will show that for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X. Since the family $\widehat{D}_0$ is pullback $\mathcal{D}$-absorbing and the process U is pullback $\widehat{D}_0$-asymptotically compact, making use of Proposition 5.9, it holds that the process U is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$. Therefore, from Lemma 5.5, we deduce that the set $\Lambda(\widehat{D}, t)$ is nonempty and compact for any $\widehat{D} \in \mathcal{D}$ and for all $t \in \mathbb{R}$. Now, applying Lemma 5.8, we have

$$\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t) \subset \Lambda(\widehat{D}_0, t).$$

Since the set $\Lambda(\widehat{D}_0, t)$ is compact (see Lemma 5.5), the set $\mathcal{A}_{\mathcal{D}}(t)$ is nonempty and compact for all $t \in \mathbb{R}$.

The family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ is pullback $\mathcal{D}$-attracting, since (5.1) holds for all $\widehat{D} \in \mathcal{D}$. In addition, it is minimal as a consequence of Lemma 5.5.

Finally, to prove the existence of the minimal pullback $\mathcal{D}$-attractor, we only need to check the negative invariance of the family $\mathcal{A}_{\mathcal{D}}$. To do this, we will use that the multi-valued process U is upper semicontinuos with closed values. Namely, from Lemma 5.5, using (5.2), we deduce

$$\begin{aligned}\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t) &\subset \bigcup_{\widehat{D}\in\mathcal{D}} U(t,\tau)\Lambda(\widehat{D}, \tau) = \bigcup_{\widehat{D}\in\mathcal{D}} \bigcup_{x_0\in\Lambda(\widehat{D},\tau)} U(t,\tau)x_0 \\ &= \bigcup_{x_0\in\bigcup_{\widehat{D}\in\mathcal{D}}\Lambda(\widehat{D},\tau)} U(t,\tau)x_0 = U(t,\tau) \bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, \tau).\end{aligned}$$

Therefore, $\mathcal{A}_{\mathcal{D}}(t) \subset U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)$ for all $(t,\tau) \in \mathbb{R}^2_d$. Then, the existence of the minimal pullback $\mathcal{D}$-attractor is guaranteed.

In addition, if the family $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) \subset \overline{D_0(t)}^X$ for all $t \in \mathbb{R}$ thanks to (5.6).

Finally, we will check that the family $\mathcal{A}_{\mathcal{D}}$ is invariant. Consider $(t,r) \in \mathbb{R}^2_d$ fixed. Since $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ and $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting, given $\varepsilon > 0$, there exists $T(\varepsilon, t, r) < 0$ such that

$$\operatorname{dist}_X(U(t, r+\tau)\mathcal{A}_{\mathcal{D}}(r+\tau), \mathcal{A}_{\mathcal{D}}(t)) < \varepsilon \quad \forall \tau \le T(\varepsilon, t, r).$$

Observe that using that the multi-valued process U is strict, we have

$$U(t,r)\mathcal{A}_{\mathcal{D}}(r) \subset U(t,r)U(r,r+\tau)\mathcal{A}_{\mathcal{D}}(r+\tau) = U(t,r+\tau)\mathcal{A}_{\mathcal{D}}(r+\tau) \quad \forall \tau \leq 0.$$

Therefore,

$$\text{dist}_X(U(t,r)\mathcal{A}_{\mathcal{D}}(r), \mathcal{A}_{\mathcal{D}}(t)) < \varepsilon \quad \forall \varepsilon > 0.$$

□

Remark 5.12. *(i) If $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$ and the universe $\mathcal{D}$ is inclusion-closed, then the family $\mathcal{A}_{\mathcal{D}}$ belongs to $\mathcal{D}$.*
(ii) If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, it is the unique family of closed subsets in $\mathcal{D}$ that fulfils properties 1-3 in Definition 5.10.

Again, analogously as in Chapter 1, we denote by $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = B : t \in \mathbb{R}\}$, where B is a fixed nonempty bounded subset of X.

Now, we establish some relationships between pullback attractors (for more details see [94, Corollaries 2 and 3]).

Corollary 5.13. *Under the assumptions of Theorem 5.11, if $\mathcal{D}_F^X \subset \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}_F^X} = \{\mathcal{A}_{\mathcal{D}_F^X}(t) : t \in \mathbb{R}\}$, where*

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \overline{\bigcup_{B \text{ bounded}} \Lambda(B,t)}^X \quad \forall t \in \mathbb{R},$$

is the minimal pullback $\mathcal{D}_F^X$-attractor for the multi-valued process U and the following relationship holds

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_D(t) \quad \forall t \in \mathbb{R}.$$

In addition, if there exists $T \in \mathbb{R}$ such that the set $\bigcup_{t \leq T} D_0(t)$ is bounded in X, then

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \quad \forall t \leq T.$$

Analogously as in Theorem 1.16, we have the following result which allows to compare two attractors for a process. We omit the proof because it is very close to that of Theorem 1.16.

Theorem 5.14. *Assume that $\{(X_i, d_{X_i})\}_{i=1,2}$ are two metric spaces such that $X_1 \subset X_2$ with continuous injection, $\mathcal{D}_i$ is a universe in $\mathcal{P}(X_i)$ for $i = 1,2$, and $\mathcal{D}_1 \subset \mathcal{D}_2$. Suppose that U is a multi-valued map that acts as a multi-valued process in both cases, i.e. $U : \mathbb{R}_d^2 \times X_i \to \mathcal{P}(X_i)$ for $i = 1,2$ is a multi-valued process. For each $t \in \mathbb{R}$,*

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i} \quad i = 1,2,$$

where the subscript i in the symbol of the omega-limit set Λ_i is used to denote the dependence on the respective topology. Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

In addition, if

(i) $\mathcal{A}_1(t)$ *is a compact subset of* X_1 *for all* $t \in \mathbb{R}$*,*

(ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ *and* $t \in \mathbb{R}$*, there exist a family* $\widehat{D}_1 \in \mathcal{D}_1$ *and a* $t^*_{\widehat{D}_1}$ *such that* U *is pullback* $\widehat{D}_1$*-asymptotically compact, and for any* $s \leq t^*_{\widehat{D}_1}$ *there exists a* $\tau_s < s$ *such that*

$$U(s,\tau)D_2(\tau) \subset D_1(s) \quad \forall \tau \leq \tau_s,$$

then $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ *for all* $t \in \mathbb{R}$*.*

Chapter 6

Nonlocal reaction-diffusion equations without uniqueness

In Chapters 2, 3 and 4 we have analysed nonlocal problems with uniqueness of solution. Namely, we have studied the non-autonomous nonlocal parabolic equation

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t),$$

fulfilled with zero Dirichlet boundary conditions. To guarantee the uniqueness of solution we have assumed the function a is locally Lipschitz. In this chapter we get rid of this assumption to deal with nonlocal problems in a multi-valued framework.

Our main goal is to show the existence of minimal pullback attractors in the phase spaces $L^2(\Omega)$ and $H_0^1(\Omega)$. Since the uniqueness of a solution is not guaranteed, to analyse the asymptotic behaviour of the solutions of the evolution problem we will use the abstract results of the theory of non-autonomous multi-valued dynamical systems analysed in Chapter 5.

In addition to proving the existence of attractors, we will study their upper semicontinuous behaviour in L^2 and H^1-norms. Many authors have been interested in studying this robustness property in different frameworks. For instance, in the random context, it is studied by Caraballo et al. in [30]. There, the upper semicontinuity w.r.t. a parameter is proved for two problems, reaction-diffusion and Navier-Stokes equations, both with a small random perturbation involving additive noise. In [34], the study of this property allows Carvalho et al. prove that diffusively coupled abstract semilinear parabolic systems synchronize. Later, Arrieta et al. prove in [13] the upper semicontinuity for attractors associated to a nonlinear second-order parabolic equation for which the diffusion coefficient was large in a subdomain of Ω. In a multi-valued framework, in [89] the upper semicontinuous behaviour of a family of attractors related to a general class of parameterized delay differential equations posed in potentially different state spaces is studied by Marín-Rubio.

The results of this chapter can be found in [22] and [26].

6.1 Setting of the problem and existence result

Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded open set and $\tau \in \mathbb{R}$. Then, we consider the following perturbed non-autonomous nonlocal reaction-diffusion problem

$$(P_\varepsilon)\begin{cases} \dfrac{du}{dt} - (1-\varepsilon)a(l(u))\Delta u = f(u) + \varepsilon h(t) \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases}$$

where $\varepsilon \in [0,1)$, the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and there exists a positive constant m such that

$$0 < m \leq a(s) \quad \forall s \in \mathbb{R}, \tag{6.1}$$

and $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$. The function $f \in C(\mathbb{R})$ and fulfils

$$-\kappa - \alpha_1 |s|^p \leq f(s)s \leq \kappa - \alpha_2 |s|^p \quad \forall s \in \mathbb{R}, \tag{6.2}$$

where α_1, α_2 and κ are positive constants and $p \geq 2$.
Observe that from (6.2), it follows that there exists a constant $\beta > 0$ such that

$$|f(s)| \leq \beta(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R}. \tag{6.3}$$

Analogously to Chapter 4, $u_\tau \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. From now on, we identify $L^2(\Omega)$ with its dual. Therefore, the chain of compact and dense embedding $H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ holds. Observe that as a result of the previous identification, $l(u)$ is understood as (l, u), but for short it is denoted by $l(u)$.

Now we are going to analyse the existence of weak solutions to (P_ε).

Definition 6.1. *A weak solution to the problem (P_ε) is a function u that belongs to $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ for all $T > \tau$, with $u(\tau) = u_\tau$, and such that for all $v \in H^1_0(\Omega) \cap L^p(\Omega)$*

$$\frac{d}{dt}(u(t), v) + (1-\varepsilon)a(l(u(t)))((u(t), v)) = (f(u(t)), v) + \varepsilon\langle h(t), v\rangle, \tag{6.4}$$

where the previous equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

When u is a weak solution to (P_ε), making use of the continuity of a, $l \in L^2(\Omega)$, (6.3) and (6.4), we deduce that $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$ for any $T > \tau$ (where p and q are conjugate exponents). Therefore, $u \in C([\tau, \infty); L^2(\Omega))$ and the initial datum in (P_ε) makes sense. Furthermore, the following energy equality holds

$$\begin{aligned} &|u(t)|_2^2 + 2(1-\varepsilon)\int_s^t a(l(u(r)))\|u(r)\|_2^2 dr \\ &= |u(s)|_2^2 + 2\int_s^t (f(u(r)), u(r))dr + 2\varepsilon \int_s^t \langle h(r), u(r)\rangle dr \end{aligned} \tag{6.5}$$

for all $\tau \leq s \leq t$.

Now, the existence of weak solution to (P_ε) is proved. It is worth noting that no assumption of smoothness on Ω is imposed.

Theorem 6.2. *Assume that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (6.1), $\varepsilon \in [0,1)$, $f \in C(\mathbb{R})$ satisfies (6.2), $h \in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, for any $\tau \in \mathbb{R}$ and any $u_\tau \in L^2(\Omega)$, there exists at least one weak solution to (P_ε).*

Proof. To prove this result we use the Galerkin approximations. Let $\{w_j : j \geq 1\} \subset H^1_0(\Omega) \cap L^p(\Omega)$ be a Hilbert basis of $L^2(\Omega)$ such that $\bigcup_{n\in\mathbb{N}} V_n$, where $V_n := \text{span}\{w_1,\dots,w_n\}$, is dense in $H^1_0(\Omega)\cap L^p(\Omega)$. Now, consider $T > \tau$ fixed. For each integer $n \geq 1$, we denote by $u_n(t;\tau,u_\tau) = \sum_{j=1}^n \varphi_{nj}(t)w_j$ (for short denoted by $u_n(t)$) a local solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t),w_j)+(1-\varepsilon)a(l(u_n(t)))((u_n(t),w_j))=\langle f(u_n(t))+\varepsilon h(t),w_j\rangle & t\in(\tau,\infty),\\ (u_n(\tau),w_j)=(u_\tau,w_j), & j=1,\dots,n, \end{cases} \tag{6.6}$$

Multiplying (6.6) by $\varphi_{nj}(t)$, summing from $j = 1$ to n and making use of (6.1), we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + (1-\varepsilon)m\|u_n(t)\|_2^2 \leq (f(u_n(t)),u_n(t)) + \varepsilon\langle h(t),u_n(t)\rangle \quad \text{a.e. } t\in(\tau,t_n), \tag{6.7}$$

where (τ,t_n) is an interval of existence of solutions to (6.6) by the Carathéodory Theorem (cf. [52, Theorem 1.1, p. 43]).

From (6.2),

$$(f(u_n(t)),u_n(t)) \leq \kappa|\Omega| - \alpha_2|u_n(t)|_p^p.$$

On the other hand, using the Cauchy inequality and taking into account that $\varepsilon\in[0,1)$, we have

$$\langle h(t),u_n(t)\rangle \leq \frac{1}{2(1-\varepsilon)m}\|h(t)\|_*^2 + \frac{(1-\varepsilon)m}{2}\|u_n(t)\|_2^2.$$

Therefore, applying these two inequalities to (6.7) we obtain

$$\frac{d}{dt}|u_n(t)|_2^2 + (1-\varepsilon)m\|u_n(t)\|_2^2 + 2\alpha_2|u_n(t)|_p^p \leq 2\kappa|\Omega| + \frac{\varepsilon}{(1-\varepsilon)m}\|h(t)\|_*^2$$

a.e. $t\in(\tau,t_n)$. Now, integrating between τ and $t\in(\tau,t_n)$, we deduce

$$\begin{aligned} |u_n(t)|_2^2 &+ (1-\varepsilon)m\int_\tau^t \|u_n(s)\|_2^2ds + 2\alpha_2\int_\tau^t |u_n(s)|_p^p ds \\ &\leq |u_\tau|_2^2 + 2\kappa\Omega(T-\tau) + \frac{\varepsilon}{(1-\varepsilon)m}\int_\tau^T \|h(s)\|_*^2 ds. \end{aligned}$$

From the above a priori estimate, we deduce that solutions to (6.6) are defined in the whole interval $[\tau,T]$ and the sequence $\{u_n\}$ is bounded in $L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H^1_0(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$. From this, bearing in mind that for all $n\in\mathbb{N}$ each $u_n \in C[\tau,T];L^2(\Omega)$, it holds that there exists a positive constant $C_\infty > 0$ such that

$$|u_n(t)|_2 \leq C_\infty \quad \forall t\in[\tau,T] \quad \forall n\geq 1.$$

Now making use of the continuity of the function a and $l \in L^2(\Omega)$, we deduce that there exists a positive constant $M_{C_\infty} > 0$ such that

$$a(l(u_n(t))) \leq M_{C_\infty} \quad \forall t \in [\tau, T] \quad \forall n \geq 1. \tag{6.8}$$

Taking this into account together with the boundedness of $\{u_n\}$ in $L^2(\tau, T; H_0^1(\Omega))$, we obtain that the sequence $\{-a(l(u_n))\Delta u_n\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$.

On the other hand, using (6.3) and the boundedness of $\{u_n\}$ in $L^p(\tau, T; L^p(\Omega))$, we deduce that $\{f(u_n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$.

Thus, we deduce that there exist a function $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$, $\xi_1 \in L^q(\tau, T; L^q(\Omega))$, $\xi_2 \in L^2(\tau, T; H^{-1}(\Omega))$ and a subsequence of $\{u_n\}$ (relabeled the same) such that

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau, T; L^2(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^2(\tau, T; H_0^1(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^p(\tau, T; L^p(\Omega)), \\ f(u_n) \rightharpoonup \xi_1 & \text{weakly in } L^q(\tau, T; L^q(\Omega)), \\ -a(l(u_n))\Delta u_n \rightharpoonup \xi_2 & \text{weakly in } L^2(\tau, T; H^{-1}(\Omega)). \end{cases} \tag{6.9}$$

To prove that $\xi_1 = f(u)$ and $\xi_2 = -a(l(u))\Delta u$, we will use similar arguments to the ones used in the proof of Proposition 4.16. Consider $w \in V_n$ fixed. Integrating in (6.6) between t and $t+b$, with $b \in (0, T-\tau)$ and $t \in (\tau, T-b)$, and using (6.8) and the Hölder inequality, we obtain

$$\begin{aligned} &(u_n(t+b) - u_n(t), w) \\ &\leq (1-\varepsilon) M_{C_\infty} \int_t^{t+b} \|u_n(s)\|_2 \|w\|_2 ds + \varepsilon \int_t^{t+b} \|h(s)\|_* \|w\|_2 ds + \int_t^{t+b} |f(u_n(s))|_q |w|_p ds \\ &\leq b^{1/2} \|w\|_2 \left[(1-\varepsilon) M_{C_\infty} \|u_n\|_{L^2(\tau,T;H_0^1(\Omega))} + \varepsilon \|h\|_{L^2(\tau,T;H^{-1}(\Omega))} \right] \\ &\quad + b^{1/p} |w|_p \|f(u_n)\|_{L^q(\tau,T;L^q(\Omega))}. \end{aligned}$$

Since $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$ and $\{f(u_n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$, there exists $C_\varepsilon > 0$ such that

$$(u_n(t+b) - u_n(t), w) \leq C_\varepsilon (b^{1/2} + b^{1/p})(\|w\|_2 + |w|_p).$$

Then, taking in the previous inequality $w = u_n(t+b) - u_n(t)$, we have

$$|u_n(t+b) - u_n(t)|_2^2 \leq C_\varepsilon (b^{1/2} + b^{1/p})(\|u_n(t+b) - u_n(t)\|_2 + |u_n(t+b) - u_n(t)|_p)$$

a.e. $t \in (\tau, T-b)$.

Now, integrating between τ and $T-b$, we have

$$\int_\tau^{T-b} |u_n(t+b) - u_n(t)|_2^2 dt \leq 2C_\varepsilon (b^{1/2} + b^{1/p}) \left(\int_\tau^T \|u_n(r)\|_2 dr + \int_\tau^T |u_n(r)|_p dr \right).$$

Then, applying the Hölder inequality in the previous expression, we obtain

$$\int_\tau^{T-b} |u_n(t+b) - u_n(t)|_2^2 dt$$
$$\leq 2C_\varepsilon(b^{1/2} + b^{1/p}) \left((T-\tau)^{1/2} \|u_n\|_{L^2(\tau,T;H_0^1(\Omega))} + (T-\tau)^{1/q} \|u_n\|_{L^p(\tau,T;L^p(\Omega))} \right).$$

As a result of the previous estimates, there exists $\overline{C}_\varepsilon(T) > 0$ such that

$$\int_\tau^{T-b} |u_n(t+b) - u_n(t)|_2^2 dt \leq \overline{C}_\varepsilon(T)(b^{1/2} + b^{1/p}) \quad \forall n \geq 1 \quad \forall b \in (0, T-\tau).$$

Therefore,

$$\lim_{b\to 0} \sup_n \int_\tau^{T-b} |u_n(t+b) - u_n(t)|_2^2 dt = 0. \tag{6.10}$$

In addition, taking into account that $\{u_n\}$ is bounded in $L^\infty(\tau, T; L^2(\Omega))$, it is not difficult to check that

$$\lim_{b\to 0} \sup_n \left(\int_\tau^{\tau+b} |u_n(t)|_2^2 dt + \int_{T-b}^T |u_n(t)|_2^2 dt \right) = 0. \tag{6.11}$$

Then, since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and taking into account (6.10) and (6.11), applying [110, Theorem 13.2, p. 97] and [110, Remark 13.1, p. 100], we obtain that the sequence $\{u_n\}$ is relatively compact in $L^2(\tau, T; L^2(\Omega))$. From this, making use of [85, Lemme 1.3, p. 12] and arguing as in the proof of Theorem 2.4, we identify ξ_1 and ξ_2 in (6.9). Namely, it has

$$f(u_n) \rightharpoonup f(u) \quad \text{weakly in } L^q(\tau, T; L^q(\Omega)), \tag{6.12}$$
$$a(l(u_n))u_n \rightharpoonup a(l(u))u \quad \text{weakly in } L^2(\tau, T; H_0^1(\Omega)). \tag{6.13}$$

Now, we are ready to prove (6.4). Consider fixed n, $\varphi \in \mathcal{D}(\tau, T)$ and $w \in V_n$ Then, integrating (6.6) between τ and T, for all $\mu > n$ we obtain

$$-\int_\tau^T (u_\mu(t), w)\varphi'(t)dt + (1-\varepsilon)\int_\tau^T a(l(u_\mu(t)))\langle -\Delta u_\mu(t), w\rangle \varphi(t)dt$$
$$= \int_\tau^T (f(u_\mu(t)), w)\varphi(t)dt + \varepsilon \int_\tau^T \langle h(t), w\rangle \varphi(t)dt.$$

Taking limit when $\mu \to \infty$, making use of (6.9), (6.12) and (6.13), it yields

$$-\int_\tau^T (u(t), w)\varphi'(t)dt + (1-\varepsilon)\int_\tau^T a(l(u(t)))\langle -\Delta u(t), w\rangle \varphi(t)dt$$
$$= \int_\tau^T (f(u(t)), w)\varphi(t)dt + \varepsilon \int_\tau^T \langle h(t), w\rangle \varphi(t)dt$$

for all $w \in H_0^1(\Omega) \cap L^p(\Omega)$, since $\bigcup_{n\in\mathbb{N}} V_n$ is dense in $H_0^1(\Omega) \cap L^p(\Omega)$. Therefore, (6.4) holds.

In addition, as a result of the previous equality,

$$\frac{\partial u}{\partial t} - (1-\varepsilon)a(l(u))\Delta u = f(u) + \varepsilon h \quad \text{in } \mathcal{D}'(\tau, T; H^{-1}(\Omega) + L^q(\Omega)),$$

and thanks to the regularity of $f(u)$, $-a(l(u))\Delta u$ and h, we deduce that $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$. As a consequence, $u \in C([\tau, T]; L^2(\Omega))$ and it makes complete sense to check that $u(\tau) = u_\tau$.

On the one hand, consider fixed n, $\varphi \in H^1(\tau, T)$ such that $\varphi(T) = 0$ and $\varphi(\tau) \neq 0$ and $w \in V_n$. From (6.6), we deduce for all $\mu > n$

$$\begin{aligned} &-(u_\tau, w)\varphi(\tau) - \int_\tau^T (u_\mu(t), w)\varphi'(t)dt + (1-\varepsilon)\int_\tau^T a(l(u_\mu(t)))((u_\mu(t), w))\varphi(t)dt \\ &= \int_\tau^T (f(u_\mu(t)), w)\varphi(t)dt + \varepsilon \int_\tau^T \langle h(t), w\rangle \varphi(t)dt. \end{aligned}$$

Now, taking limit when $\mu \to \infty$, we obtain

$$\begin{aligned} &-(u_\tau, w)\varphi(\tau) - \int_\tau^T (u(t), w)\varphi'(t)dt + (1-\varepsilon)\int_\tau^T a(l(u(t)))((u(t), w))\varphi(t)dt \\ &= \int_\tau^T (f(u(t)), w)\varphi(t)dt + \varepsilon \int_\tau^T \langle h(t), w\rangle \varphi(t)dt. \end{aligned} \tag{6.14}$$

On the other hand, in light of (6.4) we have

$$\begin{aligned} &-(u(\tau), w)\varphi(\tau) - \int_\tau^T (u(t), w)\varphi'(t)dt + (1-\varepsilon)\int_\tau^T a(l(u(t)))((u(t), w))\varphi(t)dt \\ &= \int_\tau^T (f(u(t)), w)\varphi(t)dt + \varepsilon \int_\tau^T \langle h(t), w\rangle \varphi(t)dt. \end{aligned}$$

Then, comparing (6.14) with this last expression, $(u_\tau, w)\varphi(\tau) = (u(\tau), w)\varphi(\tau)$ holds. Therefore, since $\varphi(\tau) \neq 0$ and $\{w_j\}$ is a Hilbert basis of $L^2(\Omega)$, we deduce that $u(\tau) = u_\tau$.

We have obtained a weak solution on an arbitrary finite time interval $[\tau, T]$. Now, we may repeat this argument on an interval of the form $[T, T+1]$, then on $[T+1, T+2]$, etcetera. This way, concatenating these solutions we finally obtain a weak solution well-defined globally in time. □

6.2 Minimal pullback attractors in $L^2(\Omega)$

In this section, we want to study the long-time behaviour of the solutions to (P_ε) in $L^2(\Omega)$ making use of the results of pullback attractors analysed in Chapter 5.

In what follows, $\Phi^\varepsilon(\tau, u_\tau)$ denotes the set of weak solutions to (P_ε) in $[\tau, \infty)$ with initial datum $u_\tau \in L^2(\Omega)$.

Now, we define the multi-valued map $U^\varepsilon : \mathbb{R}^2_d \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ by

$$U^\varepsilon(t, \tau)u_\tau = \{u(t) : u \in \Phi^\varepsilon(\tau, u_\tau)\} \quad \forall u_\tau \in L^2(\Omega) \quad \forall \tau \leq t.$$

Firstly, we show that the multi-valued map U^ε is a strict multi-valued process. Roughly speaking, this is a consequence of the translation and concatenation properties of the weak solutions. We will show the proof for the sake of completeness.

Lemma 6.3. *Assume that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ and (6.1) holds, $\varepsilon \in [0,1)$, $f \in C(\mathbb{R})$ satisfies (6.2), $h \in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Then, the multi-valued map U^ε is a strict multi-valued process on $L^2(\Omega)$ for all $\varepsilon \in [0,1)$.*

Proof. Consider $\varepsilon \in [0,1)$ fixed. The multi-valued map U^ε is well-defined because every weak solution u belongs to $C([\tau,T];L^2(\Omega))$ (cf. Theorem 6.2).

In addition, observe that $U^\varepsilon(\tau,\tau)u_\tau = \{u(\tau) : u \in \Phi^\varepsilon(\tau,u_\tau)\}$. Therefore, according to Definition 5.1, to prove that U^ε is a multi-valued process we only need to check

$$U^\varepsilon(t,\tau)u_\tau \subset U^\varepsilon(t,s)U^\varepsilon(s,\tau)u_\tau \quad \forall \tau \le s \le t \quad \forall u_\tau \in L^2(\Omega). \tag{6.15}$$

Consider fixed $(t,\tau) \in \mathbb{R}^2_d$ and $u_\tau \in L^2(\Omega)$. Given $\phi \in U^\varepsilon(t,\tau)u_\tau$, there exists $u \in \Phi^\varepsilon(\tau,u_\tau)$ such that $u(t) = \phi$. Observe that when $s \ge \tau$, $u(s) \in U^\varepsilon(s,\tau)u_\tau$. Then, since $U^\varepsilon(t,s)u(s) = \{z(t) : z \in \Phi^\varepsilon(s,u(s))\}$, we have

$$\phi = u(t) \in U^\varepsilon(t,s)u(s) \subset U^\varepsilon(t,s)U^\varepsilon(s,\tau)u_\tau.$$

Therefore, (6.15) holds.

Finally, we will check that in fact the multi-valued process U^ε is strict. To do this, consider given $\phi \in U^\varepsilon(t,s)U^\varepsilon(s,\tau)u_\tau$. Then, there exists a solution u to (P_ε) such that $u(t) = \phi$ and $u(s) = z(s)$, where z is another solution to (P_ε) which fulfils that $z(\tau) = u_\tau$.

Now, we define

$$y(r) := \begin{cases} z(r) & \text{if} \quad \tau \le r \le s, \\ u(r) & \text{if} \quad s \le r \le t. \end{cases}$$

First of all, we will show that the function y is a weak solution to (P_ε). Taking into account the regularity of z and u, it holds that $y \in L^p(\tau,T;L^p(\Omega)) \cap L^2(\tau,T;H^1_0(\Omega))$ $\cap C([\tau,T];L^2(\Omega))$. Therefore, it makes complete sense to define its distributional derivative as follows

$$y'(r) := \begin{cases} z'(r) & \text{if} \quad \tau \le r < s, \\ u'(r) & \text{if} \quad s \le r \le t. \end{cases}$$

Now, fix $v \in H^1_0(\Omega) \cap L^p(\Omega)$ and $\varphi \in \mathcal{D}(\tau,t)$. Then, it holds

$$\begin{aligned}
&\langle \langle y'(r), \varphi(r)\rangle_{\mathcal{D}'(\tau,t;H^{-1}(\Omega)+L^q(\Omega)),\mathcal{D}(\tau,t)}, v\rangle \\
=&\langle -\int_\tau^t y(r)\varphi'(r)dr, v\rangle \\
=&-\int_\tau^t (y(r),v)\varphi'(r)dr \\
=&-\int_\tau^s (z(r),v)\varphi'(r)dr - \int_s^t (u(r),v)\varphi'(r)dr \\
=&\int_\tau^s \langle z'(r),v\rangle\varphi(r)dr + \int_s^t \langle u'(r),v\rangle\varphi(r)dr.
\end{aligned}$$

Then, $y' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$. Finally, we need to check that the function y fulfils (6.4), i.e. given $\varphi \in \mathcal{D}(\tau, t)$, we need to prove that

$$
\begin{aligned}
&-\int_\tau^t (y(s), v)\varphi'(s)ds + (1-\varepsilon)\int_\tau^t a(l(y(s)))((y(s), v))\varphi(s)ds \\
&= \int_\tau^t (f(y(s)), v)\varphi(s)ds + \varepsilon\int_\tau^t \langle h(s), v\rangle\varphi(s)ds.
\end{aligned} \tag{6.16}
$$

Observe that if $\text{supp}(\varphi) \subset (\tau, s)$, (6.16) holds since z is a weak solution to (P_ε) in (τ, s). Analogously, if $\text{supp}(\varphi) \subset (s, t)$, it satisfies (6.16) because u is a weak solution to (P_ε) in (s, t). Let us prove (6.16) when $\text{supp}(\varphi) \not\subset (\tau, s)$ and $\text{supp}(\varphi) \not\subset (s, t)$. Consider $\varphi \in \mathcal{D}(\tau, t)$ fixed.

On the one hand, since u is a weak solution to (P_ε) in (s, t), u fulfils (6.4) in $\mathcal{D}'(s, t)$. Then, since $\varphi \in \mathcal{D}(\tau, t)$, we have

$$
\begin{aligned}
&-(u(s), v)\varphi(s) - \int_s^t (u(r), v)\varphi'(r)dr + (1-\varepsilon)\int_s^t a(l(u(r)))((u(r), v))\varphi(r)dr \\
&= \int_s^t (f(u(r)), v)\varphi(r)dr + \varepsilon\int_s^t \langle h(r), v\rangle\varphi(r)dr.
\end{aligned}
$$

Analogously, as z is a weak solution to (P_ε) in (τ, s), z satisfies (6.4) in $\mathcal{D}'(\tau, s)$. Therefore, making use of the fact that $\varphi \in \mathcal{D}(\tau, t)$, it yields

$$
\begin{aligned}
&(z(s), v)\varphi(s) - \int_\tau^s (z(r), v)\varphi'(r)dr + (1-\varepsilon)\int_\tau^s a(l(z(r)))((z(r), v))\varphi(r)dr \\
&= \int_\tau^s (f(z(r)), v)\varphi(r)dr + \varepsilon\int_\tau^s \langle h(r), v\rangle\varphi(r)dr.
\end{aligned}
$$

On the other hand, $u(s) = z(s)$ holds.

Then, taking this into account, (6.16) holds. Therefore, the multi-valued process U^ε is strict. □

Remark 6.4. *When $\varepsilon = 0$, $U^0(t, \tau) = \mathcal{S}(t-\tau)$ for all $(t, \tau) \in \mathbb{R}^2_d$, where $\mathcal{S}$ is the multi-valued semiflow associated to the weak solutions of the autonomous problem (P_0). In what follows, we also keep the notation Φ^0 for the set of solutions to (P_0).*

The following result is crucial to show that the multi-valued process U^ε is upper semicontinuous with closed values for all $\varepsilon \in [0, 1)$. To prove this continuity result, we use the energy method applied in Propositions 2.15 and 3.16 to analyse the pullback asymptotic compactness.

Proposition 6.5. *Under the assumptions of Lemma 6.3, consider a sequence $\{u_\tau^n\} \subset L^2(\Omega)$ such that $u_\tau^n \to u_\tau$ strongly in $L^2(\Omega)$. Then for any sequence $\{u^n\}$ with $u^n \in \Phi^\varepsilon(\tau, u_\tau^n)$ for all $n \geq 1$, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in \Phi^\varepsilon(\tau, u_\tau)$ such that*

$$
u^n(t) \to u(t) \quad \text{strongly in } L^2(\Omega) \quad \forall t \geq \tau. \tag{6.17}
$$

Proof. Consider $\tau < T$ fixed. In view of the energy equality and (6.1), we deduce

$$\frac{1}{2}\frac{d}{dt}|u^n(t)|_2^2 + (1-\varepsilon)m\|u^n(t)\|_2^2 \le (f(u^n(t)), u^n(t)) + \varepsilon\langle h(t), u^n(t)\rangle \quad \text{a.e. } t \in (\tau, T).$$

Then, bearing in mind

$$(f(u^n(t)), u^n(t)) \le \kappa|\Omega| - \alpha_2|u^n(t)|_p^p,$$
$$\varepsilon\langle h(t), u^n(t)\rangle \le \frac{\varepsilon^2\|h(t)\|_*^2}{2(1-\varepsilon)m} + \frac{(1-\varepsilon)m}{2}\|u^n(t)\|_2^2,$$

we have

$$\frac{d}{dt}|u^n(t)|_2^2 + (1-\varepsilon)m\|u^n(t)\|_2^2 + 2\alpha_2|u^n(t)|_p^p \le 2\kappa|\Omega| + \frac{\varepsilon^2}{(1-\varepsilon)m}\|h(t)\|_*^2 \quad \text{a.e. } t > \tau.$$

Integrating between τ and $t \in (\tau, T]$,

$$|u^n(t)|_2^2 + (1-\varepsilon)m\int_\tau^t \|u^n(s)\|_2^2 ds + 2\alpha_2\int_\tau^t |u^n(s)|_p^p ds$$
$$\le |u_\tau^n|_2^2 + 2\kappa|\Omega|(T-\tau) + \frac{\varepsilon^2}{(1-\varepsilon)m}\int_\tau^T \|h(s)\|_*^2 ds.$$

From the previous inequality, we obtain that the sequence $\{u^n\}$ is bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$. Taking this into account together with the fact that each $u^n \in C([\tau, T]; L^2(\Omega))$, we deduce that there exists a positive constant $C_\infty > 0$ such that

$$|u^n(t)|_2 \le C_\infty \quad \forall t \in [\tau, T] \quad \forall n \ge 1.$$

Now, since the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and $l \in L^2(\Omega)$, there exists a positive constant $M_{C_\infty} > 0$ such that

$$a(l(u^n(t))) \le M_{C_\infty} \quad \forall t \in [\tau, T] \quad \forall n \ge 1.$$

Therefore, as the sequence $\{u_n\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$, we deduce that the sequence $\{-a(l(u^n))\Delta u^n\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$. In addition, the sequence $\{f(u^n)\}$ is bounded in $L^q(\tau, T; L^q(\Omega))$, thanks to (6.3) and the boundedness of $\{u^n\}$ in $L^p(\tau, T; L^p(\Omega))$. As a consequence, the sequence $\{(u^n)'\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$. Then, applying the Aubin-Lions lemma, there exist a subsequence of $\{u^n\}$ (relabeled the same) and an element $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$ with $u' \in L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$, such that

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau, T; L^2(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^2(\tau, T; H_0^1(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^p(\tau, T; L^p(\Omega)), \\ u^n \to u \quad \text{strongly in } L^2(\tau, T; L^2(\Omega)), \\ u^n(s) \to u(s) \quad \text{strongly in } L^2(\Omega) \quad \text{a.e. } (\tau, T), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega)), \\ f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^q(\tau, T; L^q(\Omega)), \\ -a(l(u^n))\Delta u^n \rightharpoonup -a(l(u))\Delta u \quad \text{weakly in } L^2(\tau, T; H^{-1}(\Omega)), \end{cases} \tag{6.18}$$

where the limits of the last two convergences have been obtained applying [85, Lemme 1.3, p. 12], as done in the proof of Theorem 2.4.

From (6.18), we deduce that u fulfils (6.4) in the interval (τ, T). Moreover, since $u \in C([\tau, T]; L^2(\Omega))$, a similar argument to the one used in the proof of Theorem 6.2 yields $u(\tau) = u_\tau$. Therefore, $u \in \Phi^\varepsilon(\tau, u_\tau)$.

Finally, we will show the convergence (6.17). First of all, we have that the sequence $\{u^n\}$ is equicontinuous in $H^{-1}(\Omega) + L^q(\Omega)$ on $[\tau, T]$, thanks to the boundedness of $\{(u^n)'\}$ in $L^2(\tau, T; H^{-1}(\Omega)) + L^q(\tau, T; L^q(\Omega))$. Then, since the sequence $\{u^n\}$ is bounded in $C([\tau, T]; L^2(\Omega))$ and the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega) + L^q(\Omega)$ is compact, the Arzela-Ascoli theorem implies (for another subsequence, relabeled again the same) the following convergence

$$u^n \to u \quad \text{strongly in } C([\tau, T]; H^{-1}(\Omega) + L^q(\Omega)). \tag{6.19}$$

Now, making use of the boundedness of $\{u^n\}$ in $C([\tau, T]; L^2(\Omega))$, we deduce

$$u^n(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega) \quad \forall t \in [\tau, T], \tag{6.20}$$

where we have used (6.19) to identify the weak limit.

On the other hand, making use of (6.5), the following estimate holds

$$|z(t)|_2^2 \leq |z(s)|_2^2 + 2\kappa|\Omega|(t-s) + \frac{\varepsilon^2}{2(1-\varepsilon)m} \int_s^t \|h(\theta)\|_*^2 d\theta \quad \forall \tau \leq s \leq t \leq T, \tag{6.21}$$

with z replaced by either u or any u^n.

Now, we define the following functions

$$J_n(t) = |u^n(t)|_2^2 - 2\kappa|\Omega|t - \frac{\varepsilon^2}{2(1-\varepsilon)m} \int_\tau^t \|h(r)\|_*^2 dr,$$
$$J(t) = |u(t)|_2^2 - 2\kappa|\Omega|t - \frac{\varepsilon^2}{2(1-\varepsilon)m} \int_\tau^t \|h(r)\|_*^2 dr.$$

As a result of the regularity of u and all u^n, the functions J and all J_n are continuous on $[\tau, T]$. Further, it is not difficult to check using (6.21) that these functions are non-increasing on $[\tau, T]$. In addition, from (6.18), we deduce

$$J_n(t) \to J(t) \quad \text{a.e. } t \in (\tau, T).$$

In fact, it can be proved

$$J_n(t) \to J(t) \quad \forall t \in [\tau, T]. \tag{6.22}$$

To do this, consider $t_0 \in (\tau, T]$ fixed. Let $\{t_m\}_{m \geq 1} \subset (\tau, T)$ be a sequence such that $J_n(t_m) \to J(t_m)$ for all $m \geq 1$ and $t_m \uparrow t_0$. Now, fix $\varepsilon > 0$. Then, there exist $m(\varepsilon) \geq 1$ and $n(\varepsilon) \geq 1$ such that

$$|J(t_m) - J(t_0)| < \frac{\varepsilon}{2} \quad \forall m \geq m(\varepsilon),$$
$$|J_n(t_{m(\varepsilon)}) - J(t_{m(\varepsilon)})| < \frac{\varepsilon}{2} \quad \forall n \geq n(\varepsilon).$$

Taking this into account together with the non-increasing character of all J_n on $[\tau, T]$, we obtain

$$\begin{aligned} J_n(t_0) - J(t_0) &= J_n(t_0) - J_n(t_{m(\varepsilon)}) + J_n(t_{m(\varepsilon)}) - J(t_{m(\varepsilon)}) + J(t_{m(\varepsilon)}) - J(t_0) \\ &\leq |J_n(t_{m(\varepsilon)}) - J(t_{m(\varepsilon)})| + |J(t_{m(\varepsilon)}) - J(t_0)| \\ &< \varepsilon \quad \forall n \geq n(\varepsilon). \end{aligned}$$

Then, bearing in mind the definitions of J and J_n, we deduce

$$\limsup_{n\to\infty} |u^n(t_0)|_2^2 \leq |u(t_0)|_2^2.$$

From this and (6.20) in $t = t_0$, not only does (6.22) hold, but also (6.17) in $[\tau, T]$. Successive repetitions of this procedure in $[\tau, T+1]$, $[\tau, T+2]$, and so on, and a diagonal argument, yield (6.17) for all $t \geq \tau$ for a suitable subsequence. □

The following result establishes that the multi-valued process U^ε is upper-semicontinuous with closed values for all $\varepsilon \in [0, 1)$.

Proposition 6.6. *Under the assumptions of Lemma 6.3, the multi-valued process U^ε is upper semicontinuous with closed values for all $\varepsilon \in [0, 1)$.*

Proof. Fix $\varepsilon \in [0, 1)$.

Firstly, we will show that the multi-valued process U^ε is upper semicontinuous. To prove it, we argue by contradiction. Suppose that there exist $t \geq \tau$, $u_\tau \in L^2(\Omega)$, a neighbourhood $\mathcal{N}$ of $U^\varepsilon(t, \tau)u_\tau$, and a sequence $\{y_n\}$ such that $y_n \in U^\varepsilon(t, \tau)u_\tau^n$, where $u_\tau^n \to u_\tau$ in $L^2(\Omega)$ and $y_n \notin \mathcal{N}$ for all $n \geq 1$.
Observe that since $y_n \in U^\varepsilon(t, \tau)u_\tau^n$, there exists $u^n \in \Phi^\varepsilon(\tau, u_\tau^n)$ such that $u^n(t) = y_n$. In addition, making use of Proposition 6.5, since $u_\tau^n \to u_\tau$ strongly in $L^2(\Omega)$, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in \Phi^\varepsilon(\tau, u_\tau)$ such that (6.17) holds. Therefore, there exists a subsequence of $\{y_n\}$ (relabeled the same) such that $y_n \to u(t)$ strongly in $L^2(\Omega)$, which contradicts the fact that $y_n \notin \mathcal{N}$ for all $n \in \mathbb{N}$. Then, the multi-valued process U^ε is upper semicontinuous.

Finally, we will show that the multi-valued process U^ε has closed values, i.e. the set $U^\varepsilon(t, \tau)u_\tau$ is closed in $L^2(\Omega)$ for any $u_\tau \in L^2(\Omega)$ and $(t, \tau) \in \mathbb{R}_d^2$. Fix $(t, \tau) \in \mathbb{R}_d^2$ and $u_\tau \in L^2(\Omega)$. Consider a sequence $\{u^n(t)\} \subset U^\varepsilon(t, \tau)u_\tau$ converging strongly in $L^2(\Omega)$. Then, $u^n(\tau) = u_\tau$ for all $n \in \mathbb{N}$. Since $u^n(\tau) \to u_\tau$ strongly in $L^2(\Omega)$, making use of Proposition 6.5, we deduce that there exists a subsequence of $\{u^n(t)\}$ (relabeled the same) such that $u^n(t) \to u(t) \in U^\varepsilon(t, \tau)u_\tau$ strongly in $L^2(\Omega)$. This concludes the proof. □

The next result will be used to define a universe in $\mathcal{P}(L^2(\Omega))$ that will be appropriate for our purposes. The idea of the proof is close to those of Lemmas 2.11 and 3.12.

Proposition 6.7. *Under the assumptions of Lemma 6.3, if $u_\tau \in L^2(\Omega)$, then every solution u to (P_ε) fulfils*

$$|u(t)|_2^2 \leq e^{-\mu_\varepsilon(t-\tau)}|u_\tau|_2^2 + \frac{2\kappa|\Omega|}{\mu_\varepsilon} + \frac{\varepsilon^2 e^{-\mu_\varepsilon t}}{2(1-\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon} \int_\tau^t e^{\mu_\varepsilon s} \|h(s)\|_*^2 ds \quad \forall t \geq \tau \tag{6.23}$$

for any $\mu_\varepsilon \in (0, 2(1-\varepsilon)\lambda_1 m)$.

Proof. Using (6.1) and (6.2) in the energy equality, we obtain

$$\frac{d}{dt}|u(t)|_2^2 + 2(1-\varepsilon)m\|u(t)\|_2^2 \le 2\kappa|\Omega| + 2\varepsilon\|h(t)\|_*\|u(t)\|_2 \quad \text{a.e. } t \ge \tau.$$

Adding $\pm\mu_\varepsilon|u(t)|_2^2$, multiplying by $e^{\mu_\varepsilon t}$, and using the Cauchy inequality, we deduce

$$\frac{d}{dt}(e^{\mu_\varepsilon t}|u(t)|_2^2) \le 2\kappa|\Omega|e^{\mu_\varepsilon t} + \frac{\varepsilon^2 e^{\mu_\varepsilon t}}{2(1-\varepsilon)m - \mu_\varepsilon\lambda_1^{-1}}\|h(t)\|_*^2 \quad \text{a.e. } t \ge \tau.$$

Integrating the previous expression between τ and t, (6.23) holds. □

We are now able to define a suitable tempered universe in $\mathcal{P}(L^2(\Omega))$.

Definition 6.8. *For each* $\mu > 0$, $\mathcal{D}_\mu^{L^2}$ *denotes the class of all families of nonempty subsets* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ *such that*

$$\lim_{\tau\to-\infty}\left(e^{\mu\tau}\sup_{v\in D(\tau)}|v|_2^2\right) = 0.$$

The following result shows the existence of a pullback absorbing family. For that, we need to assume that there exist $\varepsilon_0 \in (0,1)$ and $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$ such that the function h fulfils

$$\int_{-\infty}^0 e^{\mu_{\varepsilon_0}s}\|h(s)\|_*^2 ds < \infty. \tag{6.24}$$

Actually, once that such a couple $(\varepsilon_0, \mu_{\varepsilon_0})$ exists, then for any $\varepsilon \in [0, \varepsilon_0)$ it is possible to obtain the previous estimate for some $\mu_\varepsilon \in (0, 2(1-\varepsilon)\lambda_1 m)$. Indeed, it is enough to use $\mu_\varepsilon = \mu_{\varepsilon_0}$.

Observe that the proof of this result is very close to those of Propositions 2.13 and 3.14.

Proposition 6.9. *Under the assumptions of Lemma 6.3, if the function h also fulfils condition (6.24) for some* $\varepsilon_0 \in (0,1)$ *and* $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$, *then for any* $\varepsilon \in [0, \varepsilon_0]$, *the family* $\widehat{D}_0^\varepsilon = \{D_0^\varepsilon(t) : t \in \mathbb{R}\}$ *defined by* $D_0^\varepsilon(t) = \overline{B}_{L^2}(0, (R_{L^2}^\varepsilon(t))^{1/2})$, *where*

$$R_{L^2}^\varepsilon(t) = 1 + \frac{2\kappa|\Omega|}{\mu_\varepsilon} + \frac{\varepsilon^2 e^{-\mu_\varepsilon t}}{2(1-\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon}\int_{-\infty}^t e^{\mu_\varepsilon s}\|h(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$*-absorbing for the multi-valued process* $U^\varepsilon : \mathbb{R}_d^2 \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$. *Moreover,* $\widehat{D}_0^\varepsilon \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$.

Proof. Fix $\varepsilon \in [0, \varepsilon_0]$, $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$. From Proposition 6.7, taking into account condition (6.24), we obtain

$$|u(t)|_2^2 \le e^{-\mu_\varepsilon(t-\tau)}|u_\tau|_2^2 + \frac{2\kappa|\Omega|}{\mu_\varepsilon} + \frac{\varepsilon^2 e^{-\mu_\varepsilon t}}{2(1-\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon}\int_{-\infty}^t e^{\mu_\varepsilon s}\|h(s)\|_*^2 ds,$$

for all $u \in \Phi^\varepsilon(\tau, u_\tau)$, $u_\tau \in D(\tau)$ and $\tau \leq t$.

Now, since $\widehat{D} \in \mathcal{D}^{L^2}_{\mu_\varepsilon}$, there exists $\tau_0(\widehat{D}, t) < t$ such that

$$e^{-\mu_\varepsilon(t-\tau)}|u_\tau|_2^2 \leq 1 \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_0(\widehat{D}, t).$$

Therefore, taking this into account, we deduce

$$|u(t)|_2^2 \leq R^\varepsilon_{L^2}(t) \quad \forall u \in \Phi^\varepsilon(\tau, u_\tau) \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_0(\widehat{D}, t),$$

where the expression of $R^\varepsilon_{L^2}$ is given in the statement. □

Finally, we only need to check the pullback $\mathcal{D}^{L^2}_{\mu_\varepsilon}$-asymptotic compactness. To that end, we firstly establish the following result, which is the equivalent to Lemmas 2.14 and 3.15 in the setting of this chapter. Observe that the proofs are very close. Nevertheless, we provide the details for the sake of completeness.

Lemma 6.10. *Under the assumptions of Proposition 6.9, for any $\varepsilon \in [0, \varepsilon_0]$, $t \in \mathbb{R}$, and $\widehat{D} \in \mathcal{D}^{L^2}_{\mu_\varepsilon}$, there exists $\tau_1(\widehat{D}, t) < t - 2$, such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, the solutions to (P_ε) satisfy*

$$\begin{cases} |u(r;\tau,u_\tau)|_2^2 \leq \rho_1^\varepsilon(t) \quad \forall r \in [t-2, t], \\ \displaystyle\int_{r-1}^r \|u(s;\tau,u_\tau)\|_2^2 ds \leq \rho_2^\varepsilon(t) \quad \forall r \in [t-1, t], \\ \displaystyle\int_{r-1}^r |u(s;\tau,u_\tau)|_p^p ds \leq \frac{(1-\varepsilon)m}{2\alpha_2}\rho_2^\varepsilon(t) \quad \forall r \in [t-1, t], \end{cases} \tag{6.25}$$

where

$$\rho_1^\varepsilon(t) = 1 + \frac{2\kappa|\Omega|}{\mu_\varepsilon} + \frac{\varepsilon^2 e^{-\mu_\varepsilon(t-2)}}{2(1-\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon}\int_{-\infty}^t e^{\mu_\varepsilon s}\|h(s)\|_*^2 ds,$$

$$\rho_2^\varepsilon(t) = \frac{1}{(1-\varepsilon)m}\left(\rho_1^\varepsilon(t) + 2\kappa|\Omega| + \frac{\varepsilon^2}{(1-\varepsilon)m}\max_{r\in[t-1,t]}\int_{r-1}^r \|h(s)\|_*^2 ds\right).$$

Proof. Let $\tau_1(\widehat{D}, t) < t - 2$ such that

$$e^{-\mu(t-2)}e^{\mu\tau}|u_\tau|_2^2 \leq 1 \quad \forall u_\tau \in D(\tau) \quad \forall \tau \leq \tau_1(\widehat{D}, t).$$

Consider fixed $\tau \leq \tau_1(\widehat{D}, t)$ and $u_\tau \in D(\tau)$.

The first inequality of (6.25) follows from (6.23), (6.24) and the increasing character of the exponential.

Now, we will prove the two last inequalities simultaneously. Using the energy equality and (6.1), we have

$$\frac{1}{2}\frac{d}{ds}|u(s)|_2^2 + (1-\varepsilon)m\|u(s)\|_2^2 \leq (f(u(s)), u(s)) + \varepsilon\langle h(s), u(s)\rangle \quad \text{a.e. } s > \tau.$$

Applying (6.2) and the Cauchy inequality,

$$\frac{d}{ds}|u(s)|_2^2 + (1-\varepsilon)m\|u(s)\|_2^2 + 2\alpha_2|u(s)|_p^p \leq 2\kappa|\Omega| + \frac{\varepsilon^2}{(1-\varepsilon)m}\|h(s)\|_*^2 \quad \text{a.e. } s > \tau.$$

Then, integrating between $r-1$ and r with $r \in [t-1, t]$, it holds

$$\begin{aligned} &|u(r)|_2^2 + (1-\varepsilon)m\int_{r-1}^{r}\|u(s)\|_2^2 ds + 2\alpha_2\int_{r-1}^{r}|u(s)|_p^p ds \\ &\leq |u(r-1)|_2^2 + 2\kappa|\Omega| + \frac{\varepsilon^2}{(1-\varepsilon)m}\int_{r-1}^{r}\|h(s)\|_*^2 ds. \end{aligned}$$

In particular, from above and the first inequality in (6.25), we conclude the proof. □

In the following result, making use of the previous estimates (cf. (6.25)), we show that the multi-valued process U^ε is pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$-asymptotically compact for any ε small enough (namely, for $\varepsilon \leq \varepsilon_0$, after (6.24)). To do it, we make use of continuous and non-increasing functions, in a similar way to which it is done in the proof of Proposition 6.5.

Proposition 6.11. *Under the assumptions of Proposition 6.9, for any $\varepsilon \in [0, \varepsilon_0]$, the multi-valued process U^ε is pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$-asymptotically compact.*

Proof. We omit the proof of this result because it is analogous to the proofs of Propositions 2.15 and 3.16. The main difference is that in this case we need to use these continuous and non-increasing functions on $[t-2, t]$:

$$J_n(s) = |u^n(s)|_2^2 - 2\kappa|\Omega|s - \frac{\varepsilon}{(1-\varepsilon)m}\int_{t-2}^{s}\|h(r)\|_*^2 dr,$$
$$J(s) = |u(s)|_2^2 - 2\kappa|\Omega|s - \frac{\varepsilon}{(1-\varepsilon)m}\int_{t-2}^{s}\|h(r)\|_*^2 dr.$$

□

As a consequence of above results, we obtain the existence of minimal pullback attractors for the multi-valued process $U^\varepsilon : \mathbb{R}_d^2 \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ [compare to Theorems 2.16 and 3.17].

Theorem 6.12. *Assume that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and (6.1) holds, $f \in C(\mathbb{R})$ fulfils (6.2), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ satisfies condition (6.24) for some $\varepsilon_0 \in (0,1)$ and $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$, and $l \in L^2(\Omega)$. Then, for all the multi-valued processes U^ε with $\varepsilon \in (0, \varepsilon_0]$, there exist the minimal pullback $\mathcal{D}_F^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and the minimal pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$-attractor $\mathcal{A}^\varepsilon_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}$, which is U^ε-invariant.*

Furthermore, the family $\mathcal{A}^\varepsilon_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}$ belongs to $\mathcal{D}_{\mu_\varepsilon}^{L^2}$ and it holds

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}^\varepsilon_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}(t) \subset \overline{B}_{L^2}(0, (R^\varepsilon_{L^2}(t))^{1/2}) \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in (0, \varepsilon_0].$$

Moreover, if there exists some $\mu_{\widetilde{\varepsilon}_0}$ for some $\widetilde{\varepsilon}_0 \in (0, \varepsilon_0]$ such that h fulfils

$$\sup_{s\leq 0}\left(e^{-\mu_{\widetilde{\varepsilon}_0}s}\int_{-\infty}^{s} e^{\mu_{\widetilde{\varepsilon}_0}\theta}\|h(\theta)\|_*^2 d\theta\right) < \infty, \tag{6.26}$$

then

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}^{\varepsilon}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}(t) \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in (0, \widetilde{\varepsilon}_0]. \tag{6.27}$$

Proof. The existence of the minimal pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$-attractor $\mathcal{A}^{\varepsilon}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}$ and the minimal pullback $\mathcal{D}_F^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and the relationship between them are guaranteed thanks to Corollary 5.13. Namely, the upper semicontinuity of the multi-valued process with closed values (cf. Proposition 6.6), the relation $\mathcal{D}_F^{L^2} \subset \mathcal{D}_{\mu_\varepsilon}^{L^2}$, the existence of a pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$-absorbing family (cf. Proposition 6.9) and the pullback $\mathcal{D}_{\mu_\varepsilon}^{L^2}$-asymptotic compactness in the L^2-norm (cf. Proposition 6.11) hold.

Further, as a consequence of Theorem 5.11, the relationship $\mathcal{A}^{\varepsilon}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}(t) \subset \overline{D_0^{\varepsilon}(t)}^{L^2}$ holds for all $t \in \mathbb{R}$. In fact, taking this relation into account together with the facts that $\widehat{D}_0^{\varepsilon} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$, the set $D_0^{\varepsilon}(t)$ is closed for all $t \in \mathbb{R}$ and the universe $\mathcal{D}_{\mu_\varepsilon}^{L^2}$ is inclusion-closed, it fulfils that the family $\mathcal{A}^{\varepsilon}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$. In addition, from this, and bearing in mind that U^{ε} is a strict multi-valued process (cf. Lemma 6.3), we deduce that the family $\mathcal{A}^{\varepsilon}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}$ is invariant under the multi-valued process U^{ε}.

Finally, observe that thanks to (6.26), for each $\varepsilon \in (0, \widetilde{\varepsilon}_0)$ and $T \in \mathbb{R}$ the set $\cup_{t\leq T} R^{\varepsilon}_{L^2}(t)$ is bounded, where $R^{\varepsilon}_{L^2}$ given in the statement of Proposition 6.9. Then, from Corollary 5.13, (6.27) follows. □

Remark 6.13. *The above results also holds for the autonomous problem (P_0). Namely, the global compact attractor $\mathcal{A}^0_{L^2}$ in $L^2(\Omega)$ for the multi-valued semiflow $\mathcal{S}$ (cf. Remark 6.4) exists and it can be seen as pullback attractor not only for the universe $\mathcal{D}_F^{L^2}$ but also for the tempered universe $\mathcal{D}_{\mu_0}^{L^2}$ with $\mu_0 = 2\lambda_1 m$ (cf. Propositions 6.7 and 6.9). Indeed, $\mathcal{A}^0_{\mathcal{D}_{\mu_0}^{L^2}}(t) = \mathcal{A}^0_{L^2}$ for all $t \in \mathbb{R}$.*

6.3 Upper semicontinuous behaviour of attractors in L^2-norm

In this section, we will study the upper semicontinuous behaviour of the attractors $\mathcal{A}^{\varepsilon}_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}(t)$ as $\varepsilon \to 0$ for all $t \in \mathbb{R}$. Namely, we will show that this family of attractors converges upper semicontinuously to the global compact attractor $\mathcal{A}^0_{L^2}$ of the multi-valued semiflow $\mathcal{S}$ associated to problem (P_0). To do this, we will argue by contradiction and make use of the following sequential continuity result in the spirit of [9, Theorem 7].

Theorem 6.14. *Assume that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ and (6.1) holds, $f \in C(\mathbb{R})$ satisfies (6.2), $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ and $l \in L^2(\Omega)$. Consider also sequences $\{\varepsilon_n\} \subset (0, 1)$ with $\lim_n \varepsilon_n = 0$ and $\{u_\tau^n\} \subset L^2(\Omega)$ such that $u_\tau^n \rightharpoonup u_\tau$ weakly in $L^2(\Omega)$.*

Then, there exist a subsequence of $\{u_\tau^n\}$ (relabeled the same), a sequence $\{u^{\varepsilon_n}\}$ with $u^{\varepsilon_n} \in \Phi^{\varepsilon_n}(\tau, u_\tau^n)$, and $u^0 \in \Phi^0(\tau, u_\tau)$ such that for all $T > \tau$

$$\begin{cases} u^{\varepsilon_n} \overset{*}{\rightharpoonup} u^0 & \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega)), \\ u^{\varepsilon_n} \rightharpoonup u^0 & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)), \\ u^{\varepsilon_n} \rightharpoonup u^0 & \text{weakly in } L^p(\tau,T;L^p(\Omega)), \\ u^{\varepsilon_n} \to u^0 & \text{strongly in } L^2(\tau,T;L^2(\Omega)), \\ f(u^{\varepsilon_n}) \rightharpoonup f(u^0) & \text{weakly in } L^q(\tau,T;L^q(\Omega)), \\ -a(l(u^{\varepsilon_n}))\Delta u^{\varepsilon_n} \rightharpoonup -a(l(u^0))\Delta u^0 & \text{weakly in } L^2(\tau,T;H^{-1}(\Omega)), \\ (u^{\varepsilon_n})' \rightharpoonup (u^0)' & \text{weakly in } L^2(\tau,T;H^{-1}(\Omega)) + L^q(\tau,T;L^q(\Omega)), \\ u^{\varepsilon_n}(t) \to u^0(t) & \text{strongly in } L^2(\Omega) \text{ for all } t > \tau. \end{cases} \tag{6.28}$$

Proof. We split the proof into two steps. In the first one, we will show all the convergences in (6.28) except the last one, which will be proved in Step 2.

Step 1. Let $\{u_\tau^n\}$ be a sequence such that $u_\tau^n \rightharpoonup u_\tau$ weakly in $L^2(\Omega)$. We will prove the convergences in the interval $(\tau, \tau+1)$. Using the exact same arguments, the convergences can be proved in intervals of the form $(\tau, \tau+2)$, $(\tau, \tau+3)$, etcetera. A diagonal argument then shows that all the convergences (but the last one) hold in the interval (τ, T), for all $T > \tau$.

Since each u^{ε_n} is a weak solution to $(\mathrm{P}_{\varepsilon_n})$ in $[\tau, \tau+1]$, making use of the energy equality and (6.1), we obtain

$$\frac{1}{2}\frac{d}{dt}|u^{\varepsilon_n}(t)|_2^2 + (1-\varepsilon_n)m\|u^{\varepsilon_n}(t)\|_2^2 \le (f(u^{\varepsilon_n}(t)), u^{\varepsilon_n}(t)) + \varepsilon_n\langle h(t), u^{\varepsilon_n}(t)\rangle$$

a.e. $t \in [\tau, \tau+1]$.

Now, define $\gamma := \max_n\{\varepsilon_n\} \in (0,1)$. Then, taking this into account together with (6.2) and the Cauchy inequality, we have

$$\frac{d}{dt}|u^{\varepsilon_n}(t)|_2^2 + (1-\gamma)m\|u^{\varepsilon_n}(t)\|_2^2 + 2\alpha_2|u^{\varepsilon_n}(t)|_p^p \le 2\kappa|\Omega| + \frac{\varepsilon_n^2\|h(t)\|_*^2}{(1-\gamma)m} \quad \text{a.e. } t \in [\tau, \tau+1].$$

Therefore, the sequence $\{u^{\varepsilon_n}\}$ is bounded in $L^\infty(\tau, \tau+1; L^2(\Omega)) \cap L^2(\tau, \tau+1; H_0^1(\Omega)) \cap L^p(\tau, \tau+1; L^p(\Omega))$. In addition, using the boundedness of $\{u^{\varepsilon_n}\}$ in $L^\infty(\tau, \tau+1; L^2(\Omega))$ and the fact that $u^{\varepsilon_n} \in C([\tau, \tau+1]; L^2(\Omega))$ for all n, it holds

$$|u^{\varepsilon_n}(t)|_2 \le C_\infty \quad \forall t \in [\tau, \tau+1] \quad \forall n \ge 1,$$

where C_∞ is a positive constant independent of ε_n. Now, since $l \in L^2(\Omega)$ and $a \in C(\mathbb{R}; \mathbb{R}_+)$, there exists $M_{C_\infty} > 0$ such that

$$a(l(u^{\varepsilon_n}(t))) \le M_{C_\infty} \quad \forall t \in [\tau, \tau+1] \quad \forall n \ge 1.$$

This, together with the fact that $\{u^{\varepsilon_n}\}$ is bounded in $L^2(\tau, \tau+1; H_0^1(\Omega))$, implies that $\{-a(l(u^{\varepsilon_n}))\Delta u^{\varepsilon_n}\}$ is bounded in $L^2(\tau, \tau+1; H^{-1}(\Omega))$. Finally, from (6.3) and the boundedness of $\{u^{\varepsilon_n}\}$ in $L^p(\tau, \tau+1; L^p(\Omega))$, we deduce that $\{f(u^{\varepsilon_n})\}$ is bounded in

$L^q(\tau,\tau+1;L^q(\Omega))$. Finally, bearing in mind the previous estimates and the following equality

$$\frac{\partial u^{\varepsilon_n}}{\partial t}=(1-\varepsilon_n)a(l(u^{\varepsilon_n}))\Delta u^{\varepsilon_n}+f(u^{\varepsilon_n})+\varepsilon_n h \quad \text{in } \mathcal{D}'(\tau,\tau+1;H^{-1}(\Omega)+L^q(\Omega)),$$

we obtain that $\{(u^{\varepsilon_n})'\}$ is bounded in $L^2(\tau,\tau+1;H^{-1}(\Omega))+L^q(\tau,\tau+1;L^q(\Omega))$.

Using the Aubin-Lions lemma, there exist a subsequence of $\{u^{\varepsilon_n}\}$ (relabeled the same), $u^0\in L^2(\tau,\tau+1;H_0^1(\Omega))\cap L^p(\tau,\tau+1;L^p(\Omega))\cap L^\infty(\tau,\tau+1;L^2(\Omega))$ with $(u^0)'\in L^2(\tau,\tau+1;H^{-1}(\Omega))+L^q(\tau,\tau+1;L^q(\Omega))$, such that

$$\begin{cases} u^{\varepsilon_n}\overset{*}{\rightharpoonup}u^0 \quad \text{weakly-star in } L^\infty(\tau,\tau+1;L^2(\Omega)),\\ u^{\varepsilon_n}\rightharpoonup u^0 \quad \text{weakly in } L^2(\tau,\tau+1;H_0^1(\Omega)),\\ u^{\varepsilon_n}\rightharpoonup u^0 \quad \text{weakly in } L^p(\tau,\tau+1;L^p(\Omega)),\\ u^{\varepsilon_n}\to u^0 \quad \text{strongly in } L^2(\tau,\tau+1;L^2(\Omega)),\\ (u^{\varepsilon_n})'\rightharpoonup (u^0)' \quad \text{weakly in } L^2(\tau,\tau+1;H^{-1}(\Omega))+L^q(\tau,\tau+1;L^q(\Omega)),\\ f(u^{\varepsilon_n})\rightharpoonup f(u^0) \quad \text{weakly in } L^q(\tau,\tau+1;L^q(\Omega)),\\ -a(l(u^{\varepsilon_n}))\Delta u^{\varepsilon_n}\rightharpoonup -a(l(u^0))\Delta u^0 \quad \text{weakly in } L^2(\tau,\tau+1;H^{-1}(\Omega)), \end{cases} \tag{6.29}$$

where the limits of the last two convergences have been obtained applying [85, Lemme 1.3, p. 12]. In addition, since $u^{\varepsilon_n}\to u^0$ strongly in $L^2(\tau,\tau+1;L^2(\Omega))$, there exists a subsequence of $\{u^{\varepsilon_n}\}$ (relabeled the same) such that

$$u^{\varepsilon_n}(t)\to u^0(t) \quad \text{strongly in } L^2(\Omega) \text{ a.e. } t\in(\tau,\tau+1). \tag{6.30}$$

Let us show now that $u^0(t)\in U^0(t,\tau)u_\tau$ for $t\in[\tau,\tau+1]$.

Consider fixed $v\in H_0^1(\Omega)\cap L^p(\Omega)$ and $\varphi\in\mathcal{D}(\tau,\tau+1)$. Then, since u^{ε_n} is a weak solution to (P_{ε_n}), we have

$$\begin{aligned} &-\int_\tau^{\tau+1}(u^{\varepsilon_n}(t),v)\varphi'(t)dt+(1-\varepsilon_n)\int_\tau^{\tau+1}a(l(u^{\varepsilon_n}(t)))((u^{\varepsilon_n}(t),v))\varphi(t)dt\\ &=\int_\tau^{\tau+1}(f(u^{\varepsilon_n}(t)),v)\varphi(t)dt+\varepsilon_n\int_\tau^{\tau+1}\langle h(t),v\rangle\varphi(t)dt. \end{aligned}$$

Using (6.29) and the fact that $\varepsilon_n\to 0$ as $n\to\infty$, we obtain

$$-\int_\tau^{\tau+1}(u^0(t),v)\varphi'(t)dt+\int_\tau^{\tau+1}a(l(u^0(t)))((u^0(t),v))\varphi(t)dt=\int_\tau^{\tau+1}(f(u^0(t)),v)\varphi(t)dt.$$

Therefore, u^0 fulfils (6.4) when $\varepsilon=0$. To prove that u^0 is a weak solution to (P_0) we only need to check that $u^0(\tau)=u_\tau$. To do this we argue as in the proof of Theorem 6.2. Consider fixed $v\in H_0^1(\Omega)\cap L^p(\Omega)$ and $\varphi\in H^1(\tau,\tau+1)$, with $\varphi(\tau+1)=0$ and $\varphi(\tau)\neq 0$, as test elements in the problems (P_{ε_n}) in $[\tau,\tau+1]$. Then, after passing to the limit, considering (6.29) and the fact that $u_\tau^n\rightharpoonup u_\tau$ weakly in $L^2(\Omega)$, we deduce

$$\begin{aligned} &-(u_\tau,v)\varphi(\tau)-\int_\tau^{\tau+1}(u^0(t),v)\varphi'(t)dt+\int_\tau^{\tau+1}a(l(u^0(t)))((u^0(t),v))\varphi(t)dt\\ &=\int_\tau^{\tau+1}(f(u^0(t)),v)\varphi(t)dt. \end{aligned}$$

On the other hand, the same test element $v\varphi$ in (P_0), after integration between τ and $\tau+1$, yields

$$-(u^0(\tau),v)\varphi(\tau) - \int_\tau^{\tau+1}(u^0(t),v)\varphi'(t)dt + \int_\tau^{\tau+1} a(l(u^0(t)))((u^0(t),v))\varphi(t)dt$$
$$= \int_\tau^{\tau+1}(f(u^0(t)),v)\varphi(t)dt.$$

Comparing both expressions, since $\varphi(\tau)\neq 0$ and $H_0^1(\Omega)\cap L^p(\Omega)$ is dense in $L^2(\Omega)$, $u^0(\tau)=u_\tau$ holds.

Step 2: In this step we complete the proof by showing the last convergence in (6.28).

Consider $t>\tau$ fixed. Then, the energy equality (6.5), making use of (6.1), (6.2) and the Cauchy inequality, implies

$$|u^{\varepsilon_n}(s)|_2^2 \leq |u^{\varepsilon_n}(r)|_2^2 + 2\kappa|\Omega|(s-r) + \frac{\varepsilon_n^2}{2(1-\varepsilon_n)m}\int_r^s \|h(\theta)\|_*^2 d\theta \quad \forall \tau\leq r\leq s\leq t.$$

Analogously, u^0 satisfies

$$|u^0(s)|_2^2 \leq |u^0(r)|_2^2 + 2\kappa|\Omega|(s-r) \quad \forall\tau\leq r\leq s\leq t.$$

Then, we have the following continuous and non-increasing functions on $[\tau,t]$

$$J_{\varepsilon_n}(s) = |u^{\varepsilon_n}(s)|_2^2 - 2\kappa|\Omega|s - \frac{\varepsilon_n^2}{2(1-\varepsilon_n)m}\int_\tau^s \|h(r)\|_*^2 dr,$$
$$J_0(s) = |u^0(s)|_2^2 - 2\kappa|\Omega|s.$$

On the one hand, making use of (6.30), it can be proved analogously as in the proof of Proposition 6.5 the following convergence

$$J_{\varepsilon_n}(s)\to J_0(s) \quad \forall s\in(\tau,t].$$

From this, we deduce

$$\lim_{n\to\infty}|u^{\varepsilon_n}(s)|_2^2 = |u^0(s)|_2^2 \quad \forall s\in(\tau,t]. \tag{6.31}$$

On the other hand, taking into account the sequences $\{u^{\varepsilon_n}\}$ and $\{(u^{\varepsilon_n})'\}$ are bounded in $C([\tau,t];L^2(\Omega))$ and $L^2(\tau,t;H^{-1}(\Omega))+L^q(\tau,t;L^q(\Omega))$ respectively, and the compactness of the embedding $L^2(\Omega)\hookrightarrow H^{-1}(\Omega)+L^q(\Omega)$, the Arzela-Ascoli theorem implies that

$$u^{\varepsilon_n}\to u^0 \quad \text{strongly in } C([\tau,t];H^{-1}(\Omega)+L^q(\Omega)). \tag{6.32}$$

Since $\{u^{\varepsilon_n}\}$ is bounded in $C([\tau,t];L^2(\Omega))$, we obtain

$$u^{\varepsilon_n}(s)\rightharpoonup u^0(s) \quad \text{weakly in } L^2(\Omega) \quad \forall s\in[\tau,t],$$

where (6.32) has been used to identify the weak limit. The previous expression, combined with (6.31), concludes the proof. □

Now, we are ready to prove the upper semicontinuous convergence of the attractors $\mathcal{A}^{\varepsilon}_{\mathcal{D}^{L^2}_{\mu_\varepsilon}}(t)$ to $\mathcal{A}^0_{L^2}$ as $\varepsilon \to 0$ for all $t \in \mathbb{R}$.

Theorem 6.15. *Assume that the function $a \in C(\mathbb{R};\mathbb{R}^+)$ and (6.1) holds, $f \in C(\mathbb{R})$ satisfies (6.2), there exist $\varepsilon_0 \in (0,1)$ and $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$ such that $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$ fulfils (6.24) and $l \in L^2(\Omega)$. Then, the family $\{\mathcal{A}^{\varepsilon}_{\mathcal{D}^{L^2}_{\mu_\varepsilon}}(t)\}_{\varepsilon\in(0,\varepsilon_0]}$ converges upper semicontinuously to $\mathcal{A}^0_{L^2}$ as $\varepsilon \to 0$, i.e.*

$$\lim_{\varepsilon\to 0} \operatorname{dist}_{L^2}(\mathcal{A}^{\varepsilon}_{\mathcal{D}^{L^2}_{\mu_\varepsilon}}(t), \mathcal{A}^0_{L^2}) = 0 \quad \forall t \in \mathbb{R}. \tag{6.33}$$

Proof. The convergence (6.33) will be proved arguing by contradiction. Assume that

$$\operatorname{dist}_{L^2}(\mathcal{A}^{\varepsilon_n}_{\mathcal{D}^{L^2}_{\mu_{\varepsilon_n}}}(t), \mathcal{A}^0_{L^2}) > \delta \quad \forall n \geq 1,$$

for some $\delta > 0$, $t \in \mathbb{R}$, and some sequence $\{\varepsilon_n\}_{n\geq 1} \subset (0, \varepsilon_0]$ with $\lim_{n\to\infty} \varepsilon_n = 0$.

From above, taking into account the negative invariance of the pullback attractors, there exists $\{u^{\varepsilon_n}\}$, a sequence of such solutions to (P_{ε_n}) with $u^{\varepsilon_n}(t) \in \mathcal{A}^{\varepsilon_n}_{\mathcal{D}^{L^2}_{\mu_{\varepsilon_n}}}(t)$, which fulfils

$$d_{L^2}(u^{\varepsilon_n}(t), \mathcal{A}^0_{L^2}) > \delta \quad \forall n \geq 1. \tag{6.34}$$

Observe that since $\mathcal{A}^{\varepsilon_n}_{\mathcal{D}^{L^2}_{\mu_{\varepsilon_n}}}(t) \subset D^{\varepsilon_n}_0(t)$ for all n and $t \in \mathbb{R}$ (cf. Theorem 6.12), and $D^{\varepsilon_n}_0(t) \subset D^{\varepsilon_0}_0(t)$ (when $\mu_{\varepsilon_n} = \mu_{\varepsilon_0}$) for all n, we have

$$\mathcal{A}^{\varepsilon_n}_{\mathcal{D}^{L^2}_{\mu_{\varepsilon_n}}}(t) \subset D^{\varepsilon_0}_0(t) \quad \forall t \in \mathbb{R} \quad \forall n \geq 1. \tag{6.35}$$

On the other hand, the pullback $\mathcal{D}^{L^2}_{\mu_{\varepsilon_0}}$-absorbing family $\widehat{D}^{\varepsilon_0}_0$ belongs to $\mathcal{D}^{L^2}_{2\lambda_1 m}$ (since $\mu_{\varepsilon_0} < 2\lambda_1 m$). Therefore, there exists $\tau(t, \widehat{D}^{\varepsilon_0}_0, \delta) < t$ such that

$$\operatorname{dist}_{L^2}(U^0(t,\tau)D^{\varepsilon_0}_0(\tau), \mathcal{A}^0_{L^2}) \leq \frac{\delta}{2} \quad \forall \tau \leq \tau(t, \widehat{D}^{\varepsilon_0}_0, \delta). \tag{6.36}$$

From the uniform boundedness of all the pullback attractors at time $\tau(t, \widehat{D}^{\varepsilon_0}_0, \delta)$ (cf. (6.35)), the sequence $\{u^{\varepsilon_n}(\tau(t, \widehat{D}^{\varepsilon_0}_0, \delta))\}$ is bounded and satisfies (up to subsequence) that

$$u^{\varepsilon_n}(\tau(t, \widehat{D}^{\varepsilon_0}_0, \delta)) \rightharpoonup u_\tau \quad \text{weakly in } L^2(\Omega).$$

Theorem 6.14 then shows the existence of $u^0 \in \Phi^0(\tau, u_\tau)$ and a subsequence of $\{\varepsilon_n\}_{n\geq 1}$ (relabeled the same) such that (6.28) holds in $(\tau(t, \widehat{D}^{\varepsilon_0}_0, \delta), t)$. In particular, the last convergence in (6.28) at time t implies that there exists $n_0 \geq 1$ such that

$$|u^{\varepsilon_n}(t) - u^0(t)| \leq \frac{\delta}{2} \quad \forall n \geq n_0. \tag{6.37}$$

However, in light of (6.36) and (6.37), we deduce

$$\begin{aligned} d_{L^2}(u^{\varepsilon_n}(t), \mathcal{A}^0_{L^2}) &\leq d_{L^2}(u^{\varepsilon_n}(t), u^0(t)) + d_{L^2}(u^0(t), \mathcal{A}^0_{L^2}) \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad \forall n \geq n_0, \end{aligned}$$

which results in contradiction with (6.34). □

6.4 Regularity results

All the results analysed along this chapter can be improved by establishing the existence of strong solutions as well as attraction in $H_0^1(\Omega)$. To that end, we make the same assumptions as in Chapter 3. First of all, we assume that Ω is an open bounded set of class C^k, with $k \geq 2$ such that $k \geq N(p-2)/(2p)$. In addition, we also suppose that the function $f \in C^1(\mathbb{R})$ fulfils

$$f'(s) \leq \eta \quad \forall s \in \mathbb{R}, \tag{6.38}$$

where $\eta > 0$.

Then, we have the following definition.

Definition 6.16. *A strong solution to (P_ε) is a weak solution u to (P_ε) such that $u \in L^\infty(\tau, T; H_0^1(\Omega)) \cap L^2(\tau, T; D(-\Delta))$ for all $T > \tau$.*

Observe that analogously as it was done in Chapter 3, it fulfils that $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, thanks to the assumptions made on the domain Ω. Therefore, in what follows we will use either the norm of $D(-\Delta)$ of the norm of $H^2(\Omega) \cap H_0^1(\Omega)$.

Now we will show the regularising effect of the equation and the existence of strong solutions to (P_ε) making use of an argument of a posteriori regularity.

Theorem 6.17. *Assume that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ and (6.1) holds, the function $f \in C^1(\mathbb{R})$ fulfils (6.2) and (6.38), $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and $l \in L^2(\Omega)$. Then, for any $u_\tau \in L^2(\Omega)$, each weak solution u to (P_ε)satisfies that $u \in L^\infty(\tau+\epsilon, T; H_0^1(\Omega)) \cap L^2(\tau+\epsilon, T; H^2(\Omega) \cap H_0^1(\Omega))$ for every $\epsilon > 0$ and $T > \tau+\epsilon$. In addition, if the initial datum $u_\tau \in H_0^1(\Omega)$, then the weak solutions to (P_ε) are in fact strong solutions.*

Proof. Fix a weak solution $u \in \Phi^\varepsilon(\tau, u_\tau)$ to (P_ε). Then, we consider the problem

$$(P_{\varepsilon,u}) \begin{cases} \dfrac{dy}{dt} - (1-\varepsilon)a(l(u))\Delta y = f(y) + \varepsilon h(t) & \text{in } \Omega \times (\tau, \infty) \\ y = 0 \quad \text{on } \partial\Omega \times (\tau, \infty) \\ y(x, \tau) = u_\tau(x) \quad \text{in } \Omega. \end{cases}$$

Observe that there exists a unique solution to ($\mathrm{P}_{\varepsilon,u}$) thanks to the monotonicity of the Laplacian and the assumption (6.38) made on f (cf. [85, Chapitre II]). Thus, more regular (a posteriori) estimates as well as using the Galerkin approximations make complete sense. Moreover, it holds that $y = u$, since u is a solution to (P_ε) and ($\mathrm{P}_{\varepsilon,u}$) possesses a unique solution.

Now, making use of spectral theory, consider a sequence $\{w_i\}_{i\geq 1}$ of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, which is a Hilbert basis of $L^2(\Omega)$. For each integer $n \geq 1$, we define the function $u_n(t; \tau, u_\tau) = \sum_{j=1}^n \varphi_{nj}(t) w_j$ ($u_n(t)$ for short), which is the local solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) + (1-\varepsilon)a(l(u(t)))((u_n(t), w_j)) = (f(u_n(t)) + \varepsilon h(t), w_j) & t \in (\tau, \infty), \\ (u_n(\tau), w_j) = (u_\tau, w_j), & j = 1, \ldots, n. \end{cases} \tag{6.39}$$

Multiplying (6.39) by $\varphi_{nj}(t)$ and summing from $j=1$ until n, we deduce

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2+(1-\varepsilon)a(l(u(t)))\|u_n(t)\|_2=(f(u_n(t)),u_n(t))+\varepsilon(h(t),u_n(t))$$

a.e. $t\in(\tau,T)$.

Integrating the previous expression between τ and T, and making use of (6.1) and (6.2), we obtain

$$|u_n(T)|_2^2+2(1-\varepsilon)m\int_\tau^T\|u_n(t)\|_2^2dt\le|u_\tau|_2^2+2(T-\tau)\kappa|\Omega|+2\varepsilon\int_\tau^T(h(t),u_n(t))dt.$$

Using the Cauchy inequality,

$$\int_\tau^T\|u_n(t)\|_2^2dt\le\frac{1}{(1-\varepsilon)m}|u_\tau|_2^2+\frac{2(T-\tau)\kappa|\Omega|}{(1-\varepsilon)m}+\frac{1}{(1-\varepsilon)^2m^2}\int_\tau^T|h(t)|_2^2dt. \quad (6.40)$$

On the other hand, multiplying (6.39) by $\lambda_j\varphi_{nj}(t)$, summing from $j=1$ until n and using (6.1), we have

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2+(1-\varepsilon)m|-\Delta u_n(t)|_2^2\le(f(u_n(t)),-\Delta u_n(t))+\varepsilon(h(t),-\Delta u_n(t))$$

a.e. $t\in(\tau,T)$.

Applying (6.38) and the Cauchy inequality, it holds

$$\frac{d}{dt}\|u_n(t)\|_2^2+(1-\varepsilon)m|-\Delta u_n(t)|_2^2\le2\eta\|u_n(t)\|_2^2+\frac{|f(0)+\varepsilon h(t)|_2^2}{(1-\varepsilon)m}$$

a.e. $t\in(\tau,T)$.

Integrating between s and t, with $\tau<s\le t\le T$, we obtain

$$\begin{aligned}&\|u_n(t)\|_2^2+(1-\varepsilon)m\int_s^t|-\Delta u_n(r)|_2^2dr\\ \le&\|u_n(s)\|_2^2+2\eta\int_\tau^T\|u_n(r)\|_2^2dr+\frac{1}{(1-\varepsilon)m}\int_\tau^T|f(0)+\varepsilon h(r)|_2^2dr.\end{aligned} \quad (6.41)$$

Now, integrating w.r.t. s between τ and t, we have in particular

$$(t-\tau)\|u_n(t)\|_2^2\le(1+2\eta(T-\tau))\int_\tau^T\|u_n(r)\|_2^2dr+\frac{T-\tau}{(1-\varepsilon)m}\int_\tau^T|f(0)+\varepsilon h(r)|_2^2dr$$

for all $t\in[\epsilon+\tau,T]$ with $\epsilon\in(0,T-\tau)$.

Then, using (6.40), we deduce that the sequence $\{u_n\}$ is bounded in $L^\infty(\tau+\epsilon,T;H_0^1(\Omega))$. Now, taking $s=\tau+\varepsilon$ and $t=T$ in (6.41), it holds that sequence $\{u_n\}$ is bounded in $L^2(\tau+\epsilon,T;H^2(\Omega)\cap H_0^1(\Omega))$. Therefore, by the uniqueness of weak solution to $(\mathrm{P}_{\varepsilon,u})$, it fulfils

$$\begin{cases}u_n\overset{*}{\rightharpoonup}u & \text{weakly-star in } L^\infty(\tau+\varepsilon,T;H_0^1(\Omega)),\\ u_n\rightharpoonup u & \text{weakly in } L^2(\tau+\varepsilon,T;H^2(\Omega)\cap H_0^1(\Omega)).\end{cases}$$

In addition, if $u_\tau\in H_0^1(\Omega)$, it fulfils that in fact u is a strong solution to (P_ε). □

Observe that analogously as it was done in Chapter 3, if we also assume that

$$f(u) \in L^2(\tau,T;L^2(\Omega)) \quad \forall u \in L^\infty(\tau,T;H_0^1(\Omega)) \cap L^2(\tau,T;H^2(\Omega)\cap H_0^1(\Omega)). \quad (6.42)$$

Then, since $u' \in L^2(\tau,T;L^2(\Omega))$, $u \in C([\tau,T];H_0^1(\Omega))$ and the following energy equality holds

$$\begin{aligned} &\|u(t)\|_2^2 + 2(1-\varepsilon)\int_s^t a(l(u(r)))|-\Delta u(r)|_2^2 dr \\ =&\|u(s)\|_2^2 + 2\int_s^t (f(u(r)) + \varepsilon h(r), -\Delta u(r))dr \end{aligned}$$

for all $\tau \leq s \leq t$.

Thanks to Theorem 6.17, the restriction of U^ε to $\mathbb{R}_d^2 \times H_0^1(\Omega)$ defines a multi-valued process into $\mathcal{P}(H_0^1(\Omega))$. Since no confusion arises, we will not modify the notation and continue denoting this process by U^ε.

However, to prove the existence of pullback attractors in $H_0^1(\Omega)$, it is not enough for f to fulfil (6.42), but also, it is necessary to assume

$$\|f(u)\|^2_{L^2(\tau,T;L^2(\Omega))} \leq C_f \|u\|^{2\tilde{b}}_{L^\infty(\tau,T;H_0^1(\Omega))} \|u\|^{2\hat{b}}_{L^2(\tau,T;H^2(\Omega)\cap H_0^1(\Omega))}, \quad (6.43)$$

where $\tilde{b} = (\gamma+1)(1-\theta)$, $\hat{b} = (\gamma+1)\theta$, $\theta \in [0,1]$, $f(s) = -s|s|^\gamma$ where $\gamma \in (0,3]$ when $N = 3$, $\gamma \in (0,2)$ when $N = 4$, and $\gamma \in (0, 4/(N-2)]$ when $N \geq 5$, and C_f is a positive constant related to the constants of the continuos embedding used to obtain this estimate. Observe that this assumption has been made in Section 3.4, using the regularity of the strong solutions together with interpolation results (cf. [116, Lemma II.4.1, p. 72]).

The first requirement to show the existence of attractors is to prove that the multi-valued process U^ε is upper semicontinuous with closed values in $H_0^1(\Omega)$. To that end, we use the following result.

Proposition 6.18. *Under the assumptions of Theorem 6.17, if f also fulfils (6.43) and $\{u_\tau^n\} \subset H_0^1(\Omega)$ is a sequence of initial data such that $u_\tau^n \to u_\tau$ strongly in $H_0^1(\Omega)$, then, for any sequence $\{u^n\}$ with $u^n \in \Phi^\varepsilon(\tau, u_\tau^n)$ for all $n \geq 1$, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in \Phi^\varepsilon(\tau, u_\tau)$ such that*

$$u^n(t) \to u(t) \quad \textit{strongly in } H_0^1(\Omega) \quad \forall t \geq \tau.$$

Proof. The proof of this result is similar to the one of Proposition 6.5. In this case we need to consider the continuous and non-increasing functions in $[\tau, T]$

$$\begin{aligned} J_n(s) &= \|u^n(s)\|_2^2 - 2\eta\int_{t-2}^s \|u^n(r)\|_2^2 dr - \frac{1}{2(1-\varepsilon)m}\int_{t-2}^s |f(0)+\varepsilon h(r)|_2^2 dr, \\ J(s) &= \|u(s)\|_2^2 - 2\eta\int_{t-2}^s \|u(r)\|_2^2 dr - \frac{1}{2(1-\varepsilon)m}\int_{t-2}^s |f(0)+\varepsilon h(r)|_2^2 dr. \end{aligned}$$

□

Now, we have the following result (see Proposition 6.6 for a similar proof).

Proposition 6.19. *Under the assumptions of Proposition 6.18, the multi-valued process* $U^\varepsilon : \mathbb{R}^2_d \times H^1_0(\Omega) \to H^1_0(\Omega)$ *is upper semicontinuous with closed values for all* $\varepsilon \in [0,1)$.

As a consequence of the regularising effect of the equation (cf. Theorem 6.17) and the existence of a pullback $\mathcal{D}^{L^2}_{\mu_\varepsilon}$-absorbing family (cf. Proposition 6.9), the existence of a pullback $\mathcal{D}^{L^2,H^1_0}_{\mu_\varepsilon}$-absorbing family is guaranteed (cf. Proposition 2.21 for a proof).

Proposition 6.20. *Under the assumptions of Theorem 6.17, if* f *also fulfils (6.42) and* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *satisfies condition (6.24) for some* $\varepsilon_0 \in [0,1)$ *and* $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$*, then, the family* $\widehat{D}^\varepsilon_{0,H^1_0} = \{\overline{B}_{L^2}(0, (R^\varepsilon_{L^2}(t))^{1/2}) \cap H^1_0(\Omega) : t \in \mathbb{R}\} \in \mathcal{D}^{L^2,H^1_0}_{\mu_\varepsilon}$ *and for any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}^{L^2}_{\mu_\varepsilon}$*, there exists* $\tau_2(\widehat{D}, t) < t$ *such that*

$$U^\varepsilon(t,\tau)D(\tau) \subset D^\varepsilon_{0,H^1_0}(t) \quad \forall \tau \le \tau_2(\widehat{D}, t).$$

In particular, the family $\widehat{D}^\varepsilon_{0,H^1_0}$ *is pullback* $\mathcal{D}^{L^2,H^1_0}_{\mu_\varepsilon}$*-absorbing for the process* U^ε : $\mathbb{R}^2_d \times H^1_0(\Omega) \to H^1_0(\Omega)$.

To prove that the process $U^\varepsilon : \mathbb{R}^d_2 \times H^1_0(\Omega) \to H^1_0(\Omega)$ is pullback asymptotically compact, we previously establish some uniform estimates of the solutions in a finite-time interval up to t when the initial datum is shifted pullback far enough.

To clarify the statement of the following result, we introduce the next two amounts:

$$[(\rho^\varepsilon_1)^{ext}](t) = 1 + \frac{2\kappa|\Omega|}{\mu_\varepsilon} + \frac{\varepsilon^2 e^{-\mu_\varepsilon(t-3)}}{2(1-\varepsilon)m - \lambda_1^{-1}\mu_\varepsilon}\int_{-\infty}^t e^{\mu_\varepsilon s}\|h(s)\|^2_* ds,$$

$$[(\rho^\varepsilon_2)^{ext}](t) = \frac{1}{(1-\varepsilon)m}\left([(\rho^\varepsilon_1)^{ext}](t) + 2\kappa|\Omega| + \frac{\varepsilon^2}{(1-\varepsilon)m}\max_{r\in[t-2,t]}\int_{r-1}^r \|h(s)\|^2_* ds\right). \tag{6.44}$$

Then, we are ready for the following result. The idea of the proof is close to the proofs of Lemmas 2.19 and 3.19.

Lemma 6.21. *Under the assumptions of Theorem 6.17, if* f *also fulfils (6.43) and* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *satisfies condition (6.24) for some* $\varepsilon_0 \in [0,1)$ *and* $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$*, then, for any* $\varepsilon \in [0, \varepsilon_0]$, $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}^{L^2}_{\mu_\varepsilon}$*, there exists* $\tau_3(\widehat{D}, t) < t - 3$ *such that for any* $\tau \le \tau_3(\widehat{D}, t)$ *and any* $u_\tau \in D(\tau)$*, the following estimates hold*

$$\begin{cases} \|u(r;\tau,u_\tau)\|_2^2 \le \tilde{\rho}^\varepsilon_1(t) \quad \forall r \in [t-2,t], \\ \displaystyle\int_{r-1}^r |-\Delta u(s;\tau,u_\tau)|_2^2 ds \le \tilde{\rho}^\varepsilon_2(t) \quad \forall r \in [t-1,t], \\ \displaystyle\int_{r-1}^r |u'(s;\tau,u_\tau)|_2^2 ds \le \tilde{\rho}^\varepsilon_3(t) \quad \forall r \in [t-1,t], \end{cases}$$

where, taking into account $\{[(\rho_i^\varepsilon)^{ext}]\}_{i=1,2}$ *from (6.44), the terms* $\{\tilde{\rho}_i^\varepsilon\}_{i=1,2,3}$ *are given by*

$$\tilde{\rho}_1^\varepsilon(t) = (1+2\eta)[(\rho_2^\varepsilon)^{ext}](t) + \frac{1}{(1-\varepsilon)m}\max_{r\in[t-2,t]}\int_{r-1}^{r}|f(0)+\varepsilon h(s)|_2^2 ds,$$

$$\tilde{\rho}_2^\varepsilon(t) = \frac{1}{(1-\varepsilon)m}\left(\tilde{\rho}_1^\varepsilon(t) + 2\eta[(\rho_2^\varepsilon)^{ext}](t) + \frac{1}{(1-\varepsilon)m}\max_{r\in[t-1,t]}\int_{r-1}^{r}|f(0)+\varepsilon h(s)|_2^2 ds\right),$$

$$\tilde{\rho}_3^\varepsilon(t) = 3(M_{([(\rho_1^\varepsilon)^{ext}](t),l)})^2(1-\varepsilon)^2\tilde{\rho}_2^\varepsilon(t) + 3C_f(\tilde{\rho}_1^\varepsilon(t))^{\tilde{b}}(\tilde{\rho}_2^\varepsilon(t))^{\hat{b}} + 3\varepsilon^2\max_{r\in[t-1,t]}\int_{r-1}^{r}|h(s)|_2^2 ds,$$

with $\hat{b}$, $\tilde{b}$, C_f *and* $M_{([(\rho_1^\varepsilon)^{ext}](t),l)}$, *positive constants.*

Now, to prove the pullback asymptotic compactness of U^ε in $H_0^1(\Omega)$ for the universe $\mathcal{D}_{\mu_\varepsilon}^{L^2,H_0^1}$, we apply an energy method similar to the one use to prove Proposition 6.11.

Proposition 6.22. *Under the assumptions of Lemma 6.21, the multi-valued process* $U^\varepsilon : \mathbb{R}_d^2 \times H_0^1(\Omega) \to \mathcal{P}(H_0^1(\Omega))$ *is pullback* $\mathcal{D}_{\mu_\varepsilon}^{L^2,H_0^1}$*-asymptotically compact.*

Proof. The proof of this result is analogous to that of Proposition 2.22. The main differences are that in this case we need to use the estimates which appear in the statement of Proposition 6.21. Furthermore, we have to make use of the continuous and non-increasing functions

$$J_n(s) = \|u^n(s)\|_2^2 - 2\eta\int_{t-2}^{s}\|u^n(r)\|_2^2 dr - \frac{1}{2(1-\varepsilon)m}\int_{t-2}^{s}|f(0)+\varepsilon h(r)|_2^2 dr,$$
$$J(s) = \|u(s)\|_2^2 - 2\eta\int_{t-2}^{s}\|u(r)\|_2^2 dr - \frac{1}{2(1-\varepsilon)m}\int_{t-2}^{s}|f(0)+\varepsilon h(r)|_2^2 dr.$$

□

The following result shows the existence of pullback attractors in $H_0^1(\Omega)$ as well as some relationships between them. We omit the proof because it is similar to the ones done in Theorems 2.23 and 3.23.

Theorem 6.23. *Assume that the function a is locally Lipschitz and (6.1) holds,* $f \in C^1(\mathbb{R})$ *fulfils (6.2), (6.38) and (6.43),* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *and there exist* $\varepsilon_0 \in (0,1)$ *and* $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$ *such that (6.24) holds, and* $l \in L^2(\Omega)$*. Then, there exist the minimal pullback* $\mathcal{D}_F^{H_0^1}$*-attractor* $\mathcal{A}_{\mathcal{D}_F^{H_0^1}}$ *and the minimal pullback* $\mathcal{D}_{\mu_\varepsilon}^{L^2,H_0^1}$*-attractor* $\mathcal{A}^\varepsilon_{\mathcal{D}_{\mu_\varepsilon}^{L^2,H_0^1}}$ *for the multi-valued process* $U^\varepsilon : \mathbb{R}_d^2 \times H_0^1(\Omega) \to \mathcal{P}(H_0^1(\Omega))$*. Furthermore, it fulfils*

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) \subset \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}^\varepsilon_{\mathcal{D}_{\mu_\varepsilon}^{L^2}}(t) = \mathcal{A}^\varepsilon_{\mathcal{D}_{\mu_\varepsilon}^{L^2,H_0^1}}(t) \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{6.45}$$

In particular, for any $\widehat{D} \in \mathcal{D}_{\mu_\varepsilon}^{L^2}$, *the following pullback attraction result in* $H_0^1(\Omega)$ *holds*

$$\lim_{\tau \to -\infty} \operatorname{dist}_{H_0^1}(U^\varepsilon(t,\tau)D(\tau), \mathcal{A}^\varepsilon_{\mathcal{D}^{L^2}_{\mu_\varepsilon}}(t)) = 0 \quad \forall t \in \mathbb{R}.$$

Finally, if there exists some $\mu_{\widetilde{\varepsilon}_0}$ *for some* $\widetilde{\varepsilon}_0 \in (0, \varepsilon_0]$ *such that* h *fulfils*

$$\sup_{s \le 0} \left(e^{-\mu_{\widetilde{\varepsilon}_0} s} \int_{-\infty}^{s} e^{\mu_{\widetilde{\varepsilon}_0} r} |h(r)|^2 dr \right) < \infty,$$

then

$$\mathcal{A}_{\mathcal{D}_F^{H_0^1}}(t) = \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}^\varepsilon_{\mathcal{D}^{L^2}_{\mu}}(t) = \mathcal{A}^\varepsilon_{\mathcal{D}_\mu^{L^2, H_0^1}}(t) \quad \forall t \in \mathbb{R} \quad \forall \varepsilon \in (0, \widetilde{\varepsilon}_0].$$

In addition, again a result of pullback attraction holds

$$\lim_{\tau \to -\infty} \operatorname{dist}_{H_0^1}(U^\varepsilon(t,\tau)B, \mathcal{A}_{\mathcal{D}_F^{L^2}}(t)) = 0 \quad \forall t \in \mathbb{R} \quad \forall B \in \mathcal{D}_F^{L^2}.$$

Remark 6.24. *The case* $\varepsilon = 0$ *can be deduced from the above results, but more simply because when* $\varepsilon = 0$ *the problem* (P_0) *is autonomous. Therefore, there exists the compact global attractor* $\mathcal{A}^0_{H_0^1}$ *in* $H_0^1(\Omega)$*, which can be seen as pullback attractor for the universes* $\mathcal{D}_F^{H_0^1}$ *and* $\mathcal{D}_{\mu_0}^{L^2, H_0^1}$ *with* $\mu_0 = 2\lambda_1 m$*. Namely,* $\mathcal{A}^0_{\mathcal{D}_{\mu_0}^{L^2, H_0^1}}(t) = \mathcal{A}^0_{H_0^1}$ *for all* $t \in \mathbb{R}$.

Finally, the upper semicontinuous behaviour of the attractors $\{\mathcal{A}^\varepsilon_{\mathcal{D}^{L^2}_{\mu_\varepsilon}}(t)\}_{\varepsilon \in (0,\varepsilon_0]}$ as $\varepsilon \to 0$ for all $t \in \mathbb{R}$ is analysed. Analogously as it was done in Section 6.3, to prove this property we need the following continuity result.

Theorem 6.25. *Assume that the function* $a \in C(\mathbb{R}; \mathbb{R}_+)$ *and (6.1) holds,* $f \in C^1(\mathbb{R})$ *fulfils (6.2), (6.38) and (6.43),* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *and* $l \in L^2(\Omega)$*. Consider also sequences* $\{\varepsilon_n\} \subset (0,1)$ *with* $\lim_n \varepsilon_n = 0$ *and* $\{u_\tau^n\} \subset L^2(\Omega)$ *such that* $u_\tau^n \rightharpoonup u_\tau$ *weakly in* $L^2(\Omega)$*. Then, there exist a subsequence of* $\{u_\tau^n\}$ *(relabeled the same), a sequence* $\{u^{\varepsilon_n}\}$ *with* $u^{\varepsilon_n} \in \Phi^{\varepsilon_n}(\tau, u_\tau^n)$*, and* $u^0 \in \Phi^0(\tau, u_\tau)$ *such that*

$$u^{\varepsilon_n}(t) \to u^0(t) \quad \text{strongly in } H_0^1(\Omega) \text{ for all } t > \tau.$$

Proof. The proof of this result is analogous to the proof of Proposition 6.14. The main difference is that in this case we need to use the regularising effect of the equation (cf. Theorem 6.17) and the following continuous and non-increasing functions in $[\tau + \delta, T]$ with $\delta > 0$:

$$J_{\varepsilon_n}(s) = \|u^{\varepsilon_n}(s)\|_2^2 - 2\eta \int_{\tau+\delta}^{s} \|u^{\varepsilon_n}(r)\|_2^2 dr - \frac{1}{2(1-\varepsilon_n)m} \int_{\tau+\delta}^{s} |f(0) + \varepsilon_n h(r)|_2^2 dr,$$

$$J_0(s) = \|u^0(s)\|_2^2 - 2\eta \int_{\tau+\delta}^{s} \|u^0(r)\|_2^2 dr - \frac{(f(0))^2 |\Omega| [s - (\tau+\delta)]}{2m}.$$

□

Then, we are ready to prove the upper semicontinuous behaviour of attractors in $H_0^1(\Omega)$.

Theorem 6.26. *Assume that the function a is locally Lipschitz and (6.1) holds, $f \in C^1(\mathbb{R})$ fulfils (6.2), (6.38) and (6.43), $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and there exist $\varepsilon_0 \in (0,1)$ and $\mu_{\varepsilon_0} \in (0, 2(1-\varepsilon_0)\lambda_1 m)$ such that (6.24), and $l \in L^2(\Omega)$. Then, the family $\{\mathcal{A}^{\varepsilon}_{\mathcal{D}^{L^2}_{\mu_\varepsilon}}(t)\}_{\varepsilon \in (0,\varepsilon_0]}$ converges upper semicontinuously to $\mathcal{A}^0_{L^2}$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$, i.e.*

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{H_0^1}(\mathcal{A}^{\varepsilon}_{\mathcal{D}^{L^2}_{\mu_\varepsilon}}(t), \mathcal{A}^0_{L^2}) = 0 \quad \forall t \in \mathbb{R}.$$

Proof. The proof of this result is analogous to the proof of Proposition 6.15. In this case, we have to make use of the regularising effect of the equation (cf. Theorem 6.17) and the fact that $\mathcal{A}^0_{L^2} = \mathcal{A}^0_{H_0^1}$ thanks to the cited effect. □

Corollary 6.27. *As a consequence of Theorem 6.17, Theorem 6.23 (namely the chain of inclusions (6.45)) and Theorem 6.26, it also holds*

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{H_0^1}(\mathcal{A}^{\varepsilon}_{\mathcal{D}^{L^2, H_0^1}_{\mu_\varepsilon}}(t), \mathcal{A}^0_{H_0^1}) = 0 \quad \forall t \in \mathbb{R}.$$

Remark 6.28. *All the results analysed in this chapter hold for the more general family of equations*

$$\frac{du}{dt} - g_1(\varepsilon) a(l(u)) \Delta u = g_2(\varepsilon) f(u) + g_3(\varepsilon) h(t),$$

where g_1, g_2 and g_3 are continuous functions with values in $[0,1]$ and such that $\lim_{\varepsilon \to 0} g_3(\varepsilon) = 0$, $\lim_{\varepsilon \to 0} g_1(\varepsilon) = \lim_{\varepsilon \to 0} g_2(\varepsilon) = 1$. Other generalisations are also possible, like replacing the continuity assumption on g_1, g_2 and g_3 given above by monotonicity.

Chapter 7

A nonlocal p-Laplacian equation

In Chapters 2, 3, 4 and 6, we have analysed nonlocal parabolic problems in which the Laplacian belongs to the diffusion term. In this chapter, we are going to generalise the diffusion making use of the p-Laplacian.

The p-Laplacian operator appears in wide range of scientific fields, for instance, in Fluid Dynamics (e.g. flow through porous media), Nonlinear Elasticity, Glaciology and Image Restoration (cf. [102, 39, 4, 103]).

In [47], Chipot & Savistka have analysed a nonlocal problem for the p-Laplacian, studying the existence and uniqueness of weak solutions as well as the existence of global minimizers associated to an energy functional. On the other hand, in the last decade several authors have been interested in proving the existence of global attractors for the local p-Laplacian problem (for more detail cf. [118, 121, 99, 107]).

In this chapter, we will combine these two features. Namely, we prove the existence of the global attractor for a nonlocal problem for the p-Laplacian. Firstly, we analyse the existence of weak solutions arguing in the same line as done in [47] by Chipot & Savistka, making use of a change of variable and compactness arguments. Next, we study the asymptotic behaviour of the solutions. We prove the existence of the global attractor in the phase space $L^2(\Omega)$ in a multi-valued framework, since under the assumptions made on the function a we cannot guarantee the uniqueness of a weak solution. The main difficulty in proving the existence of this object, the global attractor, relies on showing the asymptotic compactness. To that end, we prove the existence of an absorbing set in $W_0^{1,p}(\Omega)$ and use the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$. Observe that the reason why the study has been done in the autonomous framework is to avoid complex notation and make clearer the idea of how to combine the already used techniques with a change of variables that removes the nonlocal term from the diffusion. However, the same ideas can be extended to deal with the non-autonomous case.

The results of this chapter can be found in [24].

7.1 Statement of the problem. Existence result

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Consider the following problem for a nonlocal p-Laplacian equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta_p u = f \quad \text{in } \Omega \times (0, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{7.1}$$

where $p \geq 2$ and the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils

$$0 < m \leq a(s) \quad \forall s \in \mathbb{R}. \tag{7.2}$$

Moreover, $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$, $f \in W^{-1,q}(\Omega)$ (where q is the conjugate exponent of p) and the initial datum $u_0 \in L^2(\Omega)$.

From now on, we identify $L^2(\Omega)$ with its dual. Therefore, the chain of compact and dense embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,q}(\Omega)$ holds. Observe that thanks to the previous identification, $l(u)$ is in fact (l, u). However, we keep the notation used along this thesis $l(u)$.

Before analysing the existence of weak solutions, we would like to recall that the p-Laplacian operator is a one-to-one mapping from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$, given by

$$\langle -\Delta_p u, v\rangle = (|\nabla u|^{p-2}\nabla u, \nabla v) \quad \forall u, v \in W_0^{1,p}(\Omega),$$

where for short we are denoting $(|\nabla u|^{p-2}\nabla u, \nabla v) = \sum_{i=1}^N (|\partial_i u|^{p-2}\partial_i u, \partial_i v)$.

Definition 7.1. *A weak solution to (7.1) is a function $u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ for all $T > 0$, with $u(0) = u_0$, such that*

$$\frac{d}{dt}(u(t), v) + a(l(u(t)))\langle -\Delta_p u(t), v\rangle = \langle f, v\rangle \quad \forall v \in W_0^{1,p}(\Omega), \tag{7.3}$$

where the previous equation must be understood in the sense of $\mathcal{D}'(0, \infty)$.

Observe that if u is a weak solution to (7.1), making use of the continuity of a, $l \in L^2(\Omega)$, and (7.3), it is straightforward to check that $u' \in L^q(0, T; W^{-1,q}(\Omega))$ for any $T > 0$, and therefore, $u \in C([0, \infty); L^2(\Omega))$. Then, the initial datum in (7.1) makes complete sense and the following energy equality holds

$$|u(t)|_2^2 + 2\int_s^t a(l(u(r)))\|u(r)\|_p^p dr = |u(s)|_2^2 + 2\int_s^t \langle f, u(r)\rangle dr \tag{7.4}$$

for all $0 \leq s \leq t$.

Now, the existence of weak solutions to (7.1) is analysed. To do it, we use the Galerkin approximations, a change of variable (see (7.8) below) which has been already used by Chipot and his collaborators (cf. [49, 47]) and compactness arguments. Furthermore, in this result we also study a regularising property of the solutions to (7.1).

Theorem 7.2. *Assume that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (7.2), $f \in W^{-1,q}(\Omega)$ and $l \in L^2(\Omega)$. Then, for each $u_0 \in L^2(\Omega)$, there exists at least a weak solution to (7.1).*

Furthermore, if $f \in L^2(\Omega)$, for every $\varepsilon > 0$ and $T > \varepsilon$, any solution u fulfils that $u \in C_w([\varepsilon, T]; W_0^{1,p}(\Omega))$. In fact, if the initial condition $u_0 \in W_0^{1,p}(\Omega)$, then $u \in C_w([0,T]; W_0^{1,p}(\Omega))$.

Proof. We split the proof into two steps. In Step 1, we will prove the existence of weak solutions to (7.1). Next, the regularizing property will be analysed in Step 2.

Step 1. Existence of weak solutions.

We will prove the existence of a weak solution to (7.1) in an interval $[0,\widetilde{T}]$ (to be specified later). An inductive concatenation procedure will provide the desired global solution. We split the proof into three steps.

Step 1.1: Galerkin approximations, a priori estimates and compactness arguments.

Consider a special basis of $L^2(\Omega)$ composed by elements $\{w_j\} \subset H_0^s(\Omega)$ with $s \geq (2p + N(p-2))/(2p)$ in the sense of [85, p. 161]. Therefore, thanks to the assumption made on s, $H_0^s(\Omega) \subset W_0^{1,p}(\Omega)$.

In what follows, we denote by $V_n := \text{span}[w_1, \ldots, w_n]$. Observe that the set $\bigcup_{n\in\mathbb{N}} V_n$ is dense in $W_0^{1,p}(\Omega)$.

Consider an arbitrary positive value $T > 0$ fixed. For each $n \in \mathbb{N}$, the function $u_n(t; 0, u_0) = \sum_{j=1}^n \varphi_{nj}(t) w_j$ (for short denoted $u_n(t)$), is a local solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) + a(l(u_n(t)))(|\nabla u_n(t)|^{p-2}\nabla u_n(t), \nabla w_j) = \langle f, w_j\rangle, & t \in (0,T), \\ (u_n(0), w_j) = (u_0, w_j), & j = 1, \ldots, n, \end{cases} \tag{7.5}$$

in some interval $[0, t_n)$, thanks to the Caratheodory theorem [52, Theorem 1.1, p. 43]. Our aim is to prove the existence of global solution.

Multiplying in (7.5) by $\varphi_{nj}(t)$, summing from $j = 1$ to n and using (7.2), we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + m\|u_n(t)\|_p^p \leq \langle f, u_n(t)\rangle \quad a.e.\ \ t \in (0, t_n). \tag{7.6}$$

Observe that using the Young inequality, we deduce

$$\langle f, u_n(t)\rangle \leq \|f\|_* \|u_n(t)\|_p \leq \frac{1}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q + \frac{m}{2}\|u_n(t)\|_p^p.$$

Taking this into account, from (7.6) we obtain

$$\frac{d}{dt}|u_n(t)|_2^2 + m\|u_n(t)\|_p^p \leq \frac{2}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q \quad a.e.\ \ t \in (0, t_n).$$

Now, integrating between 0 and $t < t_n$, we obtain

$$|u_n(t)|_2^2 + m\int_0^t \|u_n(s)\|_p^p ds \leq |u_0|_2^2 + \frac{2T}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q.$$

Therefore, the Gronwall lemma implies that $\{u_n\}$ is well-defined and bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Bearing this in mind, the sequence $\{-\Delta_p u_n\}$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$.

Now, we define

$$\begin{array}{rlcl} P_n\colon & H^{-s}(\Omega) & \longrightarrow & V_n \\ & f & \longmapsto & P_n f := \sum_{j=1}^n w_j \langle f, w_j\rangle_{H^{-s},H_0^s}, \end{array}$$

which is the continuous extension of the projector P_n defined as

$$\begin{array}{rlcl} P_n\colon & L^2(\Omega) & \longrightarrow & V_n \\ & f & \longmapsto & \sum_{j=1}^n (f,w_j)w_j. \end{array}$$

Then,

$$\frac{\partial u_n}{\partial t} = a(l(u_n))\Delta_p u_n + P_n f \quad \text{in } \mathcal{D}'(0,T;H^{-s}(\Omega)).$$

Therefore, the sequence $\{u_n'\}$ is bounded in $L^q(0,T;H^{-s}(\Omega))$. Moreover, taking into account the fact that $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (7.2), we deduce that the sequence $\{f/a(l(u_n))\}$ is bounded in $L^\infty(0,T;W^{-1,q}(\Omega))$.

Thus, making use of the Aubin-Lions lemma and the Dominated Convergence theorem, there exist a subsequence of $\{u_n\}$ (relabeled the same), $u \in L^\infty(0,T;L^2(\Omega))$ $\cap L^p(0,T;W_0^{1,p}(\Omega))$ with $u' \in L^q(0,T;H^{-s}(\Omega))$ and $\xi \in L^q(0,T;W^{-1,q}(\Omega))$, such that

$$\begin{cases} u_n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)), \\ u_n \rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ u_n \to u \quad \text{strongly in } L^p(0,T;L^p(\Omega)), \\ a(l(u_n)) \overset{*}{\rightharpoonup} a(l(u)) \quad \text{weakly-star in } L^\infty(0,T), \\ -\Delta_p u_n \rightharpoonup \xi \quad \text{weakly in } L^q(0,T;W^{-1,q}(\Omega)), \\ u_n' \rightharpoonup u' \quad \text{weakly in } L^q(0,T;H^{-s}(\Omega)), \\ \dfrac{f}{a(l(u_n))} \to \dfrac{f}{a(l(u))} \quad \text{strongly in } L^s(0,T;W^{-1,q}(\Omega)) \quad \forall s \in [1,\infty). \end{cases} \tag{7.7}$$

With these convergences, we cannot obtain directly the existence of weak solutions to (7.1) due to the presence of the nonlocal operator in front of the p-Laplacian. The main reason is that the p-Laplacian is not a linear operator. Unlike what happens with the Laplacian in the previous chapters, in this case it is not enough to use [85, Lemme 1.3, p. 12] to deal with the nonlinear term. In this chapter, we are going to remove the nonlocal term in front of the p-Laplacian and apply monotonicity arguments (cf. [85]) to identify ξ with $-\Delta_p u$.

Step 1.2: Local diffusion problems through a change of variable.

As it was done in [49, 47], we can obtain formally a local diffusion problem by rescaling the time. Namely, we put

$$\alpha(t) = \int_0^t a(l(u(s)))ds, \tag{7.8}$$

where u is formally the solution to (7.1). Then, the change of variable $u(x,t) = \omega(x,\alpha(t))$ leads to the problem

$$\begin{cases} \omega_s(\alpha(t)) - \Delta_p\omega(\alpha(t)) = \dfrac{f}{a(l(\omega(\alpha(t))))} \quad \text{in } \Omega \times (0,T), \\ \omega = 0 \quad \text{on } \partial\Omega \times (0,T), \\ \omega(x,\alpha(0)) = u_0(x) \quad \text{in } \Omega. \end{cases}$$

Observe that the previous problem can be rewritten as follows

$$\begin{cases} \omega_t - \Delta_p\omega = \dfrac{f}{a(l(\omega))} \quad \text{in } \Omega \times (0,\alpha(T)), \\ \omega = 0 \quad \text{on } \partial\Omega \times (0,\alpha(T)), \\ \omega(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{7.9}$$

To deal with this problem not only formally but rigorously, we consider a sequence of the Galerkin approximation problems associated to (7.5) and the corresponding rescaled times

$$\alpha_n(t) := \int_0^t a(l(u_n(s)))ds.$$

The new unknown $\omega_n(t) = \sum_{j=1}^n \widetilde{\varphi}_{nj}(t)w_j$ satisfies that $\omega_n(x,\alpha_n(t)) = u_n(x,t)$ and solves

$$\begin{cases} \dfrac{d}{dt}(\omega_n(t),w_j) + \langle -\Delta_p\omega_n(t),w_j\rangle = \dfrac{\langle f,w_j\rangle}{a(l(\omega_n(t)))}, \ t \in (0,\alpha_n(T)), \\ (\omega_n(0),w_j) = (u_0,w_j), \qquad j = 1,\dots,n. \end{cases} \tag{7.10}$$

Observe that problems (7.10) can be set in common time-interval $(0,mT)$ for all n thanks to (7.2). Moreover, if $\varphi \in \mathcal{D}(0,mT)$, then $\varphi \in \mathcal{D}(0,\alpha_n(T))$ and $\varphi(\alpha_n(\cdot)) \in W_0^{1,p}(0,T)$ for all n.

From (7.5), we deduce

$$\begin{aligned} &\int_0^T \left(\int_\Omega -u_n(x,t)v(x)dx\right)\varphi'(\alpha_n(t))a(l(u_n(t)))dt \\ &\quad + \int_0^T \left(\int_\Omega |\nabla u_n(x,t)|^{p-2}\nabla u_n(x,t)\nabla v(x)dx\right)a(l(u_n(t)))\varphi(\alpha_n(t))dt \\ &= \int_0^T \langle f,v\rangle\varphi(\alpha_n(t))dt, \end{aligned} \tag{7.11}$$

for all $v \in V_n$.

Since $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and each $u_n \in C([0,T];L^2(\Omega))$, there exists a positive constant $C_\infty > 0$ such that

$$|u_n(t)|_2 \le C_\infty \quad \forall t \in [0,T] \quad \forall n \ge 1.$$

From this, bearing in mind that $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (7.2) and $l \in L^2(\Omega)$, there exists a positive constant $M_{C_\infty} > 0$ such that

$$0 < m \le a(l(u_n(t))) \le M_{C_\infty} \quad \forall t \in [0,T] \quad \forall n \ge 1.$$

Now, replacing $u_n(x,t)$ by $\omega_n(x,\alpha_n(t))$ in (7.11) and using [49, Lemma 2.2], it holds

$$\begin{aligned}&\int_0^{\alpha_n(T)}\left(\int_\Omega -\omega_n(x,t)v(x)dx\right)\varphi'(t)dt\\&+\int_0^{\alpha_n(T)}\left(\int_\Omega |\nabla\omega_n(x,t)|^{p-2}\nabla\omega_n(x,t)\nabla v(x)dx\right)\varphi(t)dt\\=&\int_0^{\alpha_n(T)}\frac{\langle f,v\rangle}{a(l(\omega_n(t)))}\varphi(t)dt,\end{aligned}$$

for all $v\in V_n$.

Since $\operatorname{supp}(\varphi)\subset(0,mT)$ and $0<mT\le\alpha_n(T)$ for all $n\ge 1$, all integrals above can be considered in $(0,mT)$. Then, taking limit when $n\to\infty$ and bearing in mind (7.7), we deduce

$$\int_0^{mT}\left(\int_\Omega -\omega(x,t)v(x)dx\right)\varphi'(t)dt+\int_0^{mT}\langle\widehat{\xi}(t),v\rangle\varphi(t)dt=\int_0^{mT}\frac{\langle f,v\rangle}{a(l(\omega(t)))}\varphi(t)dt,$$

where

$$\widehat{\xi}(x,\alpha(t))=\xi(x,t)\quad\text{a.e. } t\in(0,\alpha^{-1}(mT)).$$

Therefore,

$$\omega'(t)+\widehat{\xi}(t)=\frac{f}{a(l(\omega(t)))}\quad\text{in } W^{-1,q}(\Omega)\quad\text{a.e. } t\in(0,mT).\tag{7.12}$$

Step 1.3: Monotonicity argument for the limiting equation.

In this step, we are going to check that $\widehat{\xi}$ coincides with $-\Delta_p\omega$ making use of monotonicity and compactness arguments applied to (7.10).

Making use of (7.12), the following energy equality holds

$$\frac{1}{2}\frac{d}{dt}|\omega(t)|_2^2+\langle\widehat{\xi}(t),\omega(t)\rangle=\frac{\langle f,\omega(t)\rangle}{a(l(\omega(t)))}\quad\text{a.e. } t\in(0,mT).$$

Therefore, integrating in the previous expression between 0 and mT, we have

$$\int_0^{mT}\langle\widehat{\xi}(t),\omega(t)\rangle dt=\int_0^{mT}\frac{\langle f,\omega(t)\rangle}{a(l(\omega(t)))}dt+\frac{|\omega(0)|_2^2}{2}-\frac{|\omega(mT)|_2^2}{2}.\tag{7.13}$$

Claim 1: The equality $\omega(0)=u_0$ holds.

Consider fixed $\varphi\in W^{1,p}(0,mT)$ with $\varphi(0)\neq 0$ and $\varphi(mT)=0$, and $v\in V_n$.

On the one hand, from (7.12), we obtain

$$-(\omega(0),v)\varphi(0)-\int_0^{mT}(\omega(t),v)\varphi'(t)dt+\int_0^{mT}\langle\widehat{\xi}(t),v\rangle\varphi(t)dt=\int_0^{mT}\frac{\langle f,v\rangle}{a(l(\omega(t)))}\varphi(t)dt.\tag{7.14}$$

On the other hand, from (7.10), multiplying by φ and integrating between 0 and mT, we deduce

$$\begin{aligned}&-(u_0,v)\varphi(0)-\int_0^{mT}(\omega_n(t),v)\varphi'(t)dt+\int_0^{mT}(|\nabla\omega_n(t)|^{p-2}\nabla\omega_n(t),\nabla v)\varphi(t)dt\\&=\int_0^{mT}\frac{\langle f,v\rangle}{a(l(\omega_n(t)))}\varphi(t)dt,\end{aligned}$$

for all $v\in V_n$.

Now, taking limit when $n\to\infty$ in the previous expression and making use of (7.7), bearing in mind (7.14), we deduce $\omega(0)=u_0$.

Claim 2: It fulfils

$$\omega_n(mT)\rightharpoonup\omega(mT)\quad\text{weakly in } L^2(\Omega). \tag{7.15}$$

Integrating (7.12) between 0 and mT, we obtain

$$\omega(mT)=\omega(0)+\int_0^{mT}\left(\widehat{\xi}(t)+\frac{f}{a(l(\omega(t)))}\right)dt\quad\text{in } W^{-1,q}(\Omega).$$

On the other hand, from (7.10), integrating between 0 and mT, we have

$$(\omega_n(mT),v)=(u_0,v)+\int_0^{mT}\left[(|\nabla\omega_n(t)|^{p-2}\nabla\omega_n(t),\nabla v)+\langle\frac{f}{a(l(\omega_n(t)))},v\rangle\right]dt,$$

for all $v\in V_n$. Then, taking limit when $n\to\infty$ and making use of (7.7), (7.15) holds.

Claim 3: Identification of $\widehat{\xi}$ as $-\Delta_p\omega$.

Multiplying (7.10) by $\widetilde{\varphi}_{nj}(t)$, summing from $j=1$ until n, and integrating between 0 and mT, we obtain

$$\frac{|\omega_n(mT)|_2^2}{2}+\int_0^{mT}\|\omega_n(t)\|_p^p dt=\frac{|u_0|_2^2}{2}+\int_0^{mT}\frac{\langle f,\omega_n(t)\rangle}{a(l(\omega_n(t)))}dt$$

Taking limit when $n\to\infty$ in the previous expression, making use of (7.7) and (7.15), we deduce

$$\limsup_{n\to\infty}\int_0^{mT}\|\omega_n(t)\|_p^p dt\le\int_0^{mT}\frac{\langle f,\omega(t)\rangle}{a(l(\omega(t)))}dt+\frac{|u_0|_2^2}{2}-\frac{|\omega(mT)|_2^2}{2}. \tag{7.16}$$

Now, consider $v\in L^p(0,mT;W_0^{1,p}(\Omega))$. Then, from the well-known inequality

$$\int_0^{mT}\int_\Omega(|\nabla\omega_n(t)|^{p-2}\nabla\omega_n(t)-|\nabla v(t)|^{p-2}\nabla v(t))\nabla(\omega_n(t)-v(t))dxdt\ge 0,$$

combined with (7.7) and (7.16), it yields

$$\begin{aligned}&\int_0^{mT}\frac{\langle f,\omega(t)\rangle}{a(l(\omega(t)))}dt+\frac{|u_0|_2^2}{2}-\frac{|\omega(mT)|_2^2}{2}-\int_0^{mT}\langle\widehat{\xi}(t),v(t)\rangle dt\\&-\int_0^{mT}\int_\Omega|\nabla v(t)|^{p-2}\nabla v(t)\nabla(\omega(t)-v(t))dxdt\ge 0.\end{aligned}$$

Then, taking into account (7.13), from the previous inequality we deduce for all $v \in L^p(0, mT; W_0^{1,p}(\Omega))$

$$\int_0^{mT} \left[\langle \widehat{\xi}(t), \omega(t) - v(t) \rangle - (|\nabla v(t)|^{p-2} \nabla v(t), \nabla(\omega(t) - v(t))) \right] dt \geq 0.$$

Then, taking $v = \omega - \delta z$ with $\delta > 0$ and $z \in L^p(0, mT; W_0^{1,p}(\Omega))$, we conclude

$$\int_0^{mT} \langle \widehat{\xi}(t) - \nabla \cdot |\nabla(\omega(t) - \delta z(t))|^{p-2} \nabla(\omega(t) - \delta z(t)), z(t) \rangle dt \geq 0.$$

Since $\delta > 0$ is arbitrary, we deduce that $\widehat{\xi}(x,t) = -\Delta_p \omega(x,t)$ a.e. $t \in (0, mT)$ (in particular $\xi(x,t) = -\Delta_p u(x,t)$ a.e. $t \in (0, \alpha^{-1}(mT))$). Thus, ω solves (7.9) in $(0, mT)$. Then, $u(x,t) = \omega(x, \alpha(t))$ is a solution to (7.1) in $[0, \widetilde{T}]$ with $\widetilde{T} = \alpha^{-1}(mT)$. Applying the same arguments to intervals of the form $[k\widetilde{T}, (k+1)\widetilde{T}]$ with $k \in \mathbb{N}$ and concatenation, we obtain a global solution to (7.1).

Step 2. Regularising effect. Assume that $f \in L^2(\Omega)$, and consider fixed an arbitrary value $T > 0$ and a solution to (7.1) denoted by $u(\cdot; 0, u_0)$. Observe that the problem

$$(P_u) \begin{cases} \dfrac{\partial y}{\partial t} - a(l(u)) \Delta_p y = f \quad \text{in } \Omega \times (0, T), \\ y = 0 \quad \text{on } \partial\Omega \times (0, T), \\ y(x, 0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

possesses a unique solution because of the monotonicity of the p-Laplacian (cf. [85, Chapitre II]). Therefore, more regular (a posteriori) estimates as well as using the Galerkin approximations make complete sense. In addition, observe that since u is a solution to (7.1), by the uniqueness of solution to (P_u), it follows that $y = u$.

Then, we consider the Galerkin formulation associated to problem (P_u)

$$\begin{cases} \dfrac{d}{dt}(\hat{u}_n(t), w_j) + a(l(u))(|\nabla \hat{u}_n(t)|^{p-2} \nabla \hat{u}_n(t), \nabla w_j) = (f, w_j) \quad \text{a.e. } t \in (0, T), \\ (\hat{u}_n(0), w_j) = (u_0, w_j), \qquad j = 1, \dots, n, \end{cases} \tag{7.17}$$

with $\hat{u}_n(t; 0, u_0) = \sum_{j=1}^n \widehat{\varphi}_{nj}(t) w_j$, which is denoted by $\hat{u}_n(t)$ in what follows.

Multiplying (7.17) by $\widehat{\varphi}_{nj}(t)$, summing from $j = 1$ until n and making use of (7.2), we have

$$\frac{1}{2} \frac{d}{dt} |\hat{u}_n(t)|_2^2 + m \|\hat{u}_n(t)\|_p^p \leq \langle f, \hat{u}_n(t) \rangle \quad \text{a.e. } t \in (0, T).$$

Using the Young inequality, we deduce

$$\langle f, \hat{u}_n(t) \rangle \leq \|f\|_* \|\hat{u}_n(t)\|_p \leq \frac{1}{q} \left(\frac{2}{mp} \right)^{\frac{q}{p}} \|f\|_*^q + \frac{m}{2} \|\hat{u}_n(t)\|_p^p.$$

Taking this into account,

$$\frac{d}{dt} |\hat{u}_n(t)|_2^2 + m \|\hat{u}_n(t)\|_p^p \leq \frac{2}{q} \left(\frac{2}{mp} \right)^{\frac{q}{p}} \|f\|_*^q \quad \text{a.e. } t \in (0, T).$$

Integrating between 0 and T, we obtain in particular

$$\int_0^T \|\hat{u}_n(t)\|_p^p dt \le \frac{|u_0|_2^2}{m} + \frac{2T}{qm}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q. \tag{7.18}$$

Now, multiplying (7.17) by $\widehat{\varphi}'_{nj}(t)/a(l(u(t)))$ and summing from $j=1$ until n, we have

$$\frac{|\hat{u}'_n(t)|_2^2}{a(l(u(t)))} + \frac{1}{p}\frac{d}{dt}\|\hat{u}_n(t)\|_p^p = \frac{(f,\hat{u}'_n(t))}{a(l(u(t)))} \quad \text{a.e. } t \in (0,T).$$

Then, making use of the Cauchy inequality and (7.2), we deduce

$$\frac{1}{p}\frac{d}{dt}\|\hat{u}_n(t)\|_p^p \le \frac{|f|_2^2}{4m} \quad \text{a.e. } t \in (0,T).$$

Now, integrating between s and t, with $0 < s \le t \le T$,

$$\|\hat{u}_n(t)\|_p^p \le \|\hat{u}_n(s)\|_p^p + \frac{pT}{4m}|f|_2^2.$$

Integrating w.r.t. s between 0 and t,

$$t\|\hat{u}_n(t)\|_p^p \le \int_0^T \|\hat{u}_n(s)\|_p^p ds + \frac{pT^2}{4m}|f|_2^2.$$

Therefore,

$$\|\hat{u}_n(t)\|_p^p \le \frac{1}{\varepsilon}\int_0^T \|\hat{u}_n(s)\|_p^p ds + \frac{pT^2}{4\varepsilon m}|f|_2^2$$

for all $t \in [\varepsilon, T]$ with $\varepsilon \in (0,T)$. From this, taking into account (7.18), we deduce that the sequence $\{\hat{u}_n\}$ is bounded in $L^\infty(\varepsilon, T; W_0^{1,p}(\Omega))$. By the uniqueness of solution, the whole sequence

$$\hat{u}_n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\varepsilon, T; W_0^{1,p}(\Omega)).$$

In addition, since $u \in C([0,T]; L^2(\Omega))$, it holds that $u \in C_w([\varepsilon,T]; W_0^{1,p}(\Omega))$ (cf. [108, Theorem 2.1, p. 544] or [111, Lemma 3.3, p. 74]).

The case in which the initial datum u_0 belongs to $W_0^{1,p}(\Omega)$ allows to simplify the above estimates in a standard way and the solution $u \in C_w([0,T]; W_0^{1,p}(\Omega))$. □

Now, we have an equivalent result to Theorem 7.2 when the operator l is allowed to belong to a less regular space, namely $L^q(\Omega)$, and the function a fulfils an additional restriction. The proof is analogous to the previous one with minor changes, therefore it is omitted.

Corollary 7.3. *Assume that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils*

$$0 < m \le a(s) \le M \quad \forall s \in \mathbb{R}, \tag{7.19}$$

where M is a positive constant, $f \in W^{-1,q}(\Omega)$ and $l \in L^q(\Omega)$. Then, for each $u_0 \in L^2(\Omega)$, there exists at least a weak solution to (7.1). In addition, when $f \in L^2(\Omega)$, for every $\varepsilon > 0$ and $T > \varepsilon$, any solution u fulfils that $u \in C_w([\varepsilon,T]; W_0^{1,p}(\Omega))$. In fact, if the initial condition $u_0 \in W_0^{1,p}(\Omega)$, the above regularity holds for $\varepsilon = 0$.

7.2 Compact global attractor in $L^2(\Omega)$

In Chapters 2, 3, 4 and 6, we have applied the theory of attractors, which has been analysed in Chapters 1 and 5, to non-autonomous problems to prove the existence of minimal pullback attractors. In this chapter we are studying a problem in an autonomous setting. In this more straightforward framework, we are going to use results analysed in Chapter 5, since the theory of non-autonomous dynamical systems is a generalisation of the abstract results on autonomous dynamical systems (cf. for more details [96, 74, 75], cf. [28] in a random setting). While in non-autonomous problems, the concept of attraction is understood making the initial time go to $-\infty$, in the autonomous context this concept is understood making the current time go to ∞.

The main goal of this section is to study the asymptotic behaviour of the solutions to (7.1) analysing the existence of the compact global attractor in $L^2(\Omega)$. To guarantee the existence of this object, we need to prove that the multi-valued semiflow $\mathcal{S}$ is asymptotically compact amongst other requirements (for more detail cf. Theorem 5.11). To do this, we build an absorbing set in $W_0^{1,p}(\Omega)$ (cf. Proposition 7.8) and make use of the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.

In what follows, analogously as it was done in Chapter 6, we denote by $\Phi(u_0)$ the set of solutions to (7.1) in $[0,\infty)$ with initial datum $u_0 \in L^2(\Omega)$.

Now, thanks to Theorem 7.2, we can define a multi-valued map $\mathcal{S} : \mathbb{R}_+ \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ as

$$\mathcal{S}(t)u_0 = \{u(t) : u \in \Phi(u_0)\} \quad \forall u_0 \in L^2(\Omega) \quad \forall t \geq 0. \tag{7.20}$$

Then we have the following result, whose proof is analogous to the one given for Lemma 6.3, taking into account that $\mathcal{S}(t) = U(t,0)$ for all $t \in \mathbb{R}_+$.

Lemma 7.4. *Assume that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ fulfils (7.2), $f \in W^{-1,q}(\Omega)$ and $l \in L^2(\Omega)$. Then, the multi-valued map $\mathcal{S}$ defined in (7.20) is a strict multi-valued semiflow in $L^2(\Omega)$.*

Now, to study more properties of the multi-valued semiflow $\mathcal{S}$, we need the following result. To prove it, we argue as in the proof of Proposition 6.5, using the continuity of the solutions and energy equalities.

Lemma 7.5. *Under the assumptions of Lemma 7.4, consider a sequence of initial data $\{u_0^n\} \subset L^2(\Omega)$ such that $u_0^n \to u_0$ strongly in $L^2(\Omega)$. Then, for any sequence $\{u^n\}$ where $u^n \in \Phi(u_0^n)$, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in \Phi(u_0)$, such that*

$$u^n(t) \to u(t) \quad \textit{strongly in } L^2(\Omega) \quad \forall t \geq 0. \tag{7.21}$$

Proof. Consider $T > 0$ fixed. Applying (7.2) to the energy equality, it holds

$$\frac{1}{2}\frac{d}{dt}|u^n(t)|_2^2 + m\|u^n(t)\|_p^p \leq \langle f, u^n(t)\rangle \quad \text{a.e. } t \in (0,T).$$

Using the Young inequality, we deduce

$$\langle f, u^n(t)\rangle \leq \frac{1}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q + \frac{m}{2}\|u^n(t)\|_p^p.$$

Therefore,

$$\frac{d}{dt}|u^n(t)|_2^2 + m\|u^n(t)\|_p^p \leq \frac{2}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q \quad \text{a.e. } t \in (0,T).$$

Integrating the previous expression between 0 and $t \in (0,T)$, we have

$$|u_n(t)|_2^2 + m\int_0^t \|u^n(s)\|_p^p ds \leq |u_0|_2^2 + \frac{2T}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q.$$

Thus, the sequence $\{u^n\}$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Taking into account the boundedness of $\{u^n\}$ in $C([0,T];L^2(\Omega))$, there exists a positive constant $C_\infty > 0$ such that

$$|u^n(t)|_2 \leq C_\infty \quad \forall t \in [0,T] \quad \forall n \geq 1.$$

From this, making use of the continuity of the function a and the fact that $l \in L^2(\Omega)$, there exists a positive constant $M_{C_\infty} > 0$ such that

$$a(l(u^n(t))) \leq M_{C_\infty} \quad \forall t \in [0,T] \quad \forall n \geq 1.$$

Then, taking this into account together with the boundedness of the sequence $\{u^n\}$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we deduce that the sequence $\{-a(l(u^n))\Delta_p u^n\}$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$. Therefore, $\{(u^n)'\}$ is bounded in $L^q(0,T;W^{-1,q}(\Omega))$. Now, applying the Aubin Lions lemma, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ with $u' \in L^q(0,T;W^{-1,q}(\Omega))$, such that

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)), \\ u^n \rightharpoonup u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \\ u^n \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \\ u^n(t) \to u(t) \quad \text{strongly in } L^2(\Omega) \quad \text{a.e. } t \in (0,T), \\ (u^n)' \rightharpoonup u' \quad \text{weakly in } L^q(0,T;W^{-1,q}(\Omega)), \\ -a(l(u^n))\Delta_p u^n \rightharpoonup -a(l(u))\Delta_p u \quad \text{weakly in } L^q(0,T;W^{-1,q}(\Omega)), \end{cases}$$

where the last convergence has been obtained arguing as in the proof of the existence of solution (cf. Theorem 7.2). Indeed, in that way we deduce that u solves (7.1) with $u(0) = u_0$.

The next goal will be to prove (7.21). We split the proof into two steps.

Step 1. Our aim is to prove that

$$u^n(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega) \quad \forall t \in [0,T]. \tag{7.22}$$

To that end, we apply the Arzela-Ascoli theorem. Observe that the sequence $\{u^n\}$ is equicontinuous in $W^{-1,q}(\Omega)$ on $[0,T]$ thanks to the boundedness of $\{(u^n)'\}$ in $L^q(0,T;W^{-1,q}(\Omega))$. Namely, fixed $\varepsilon>0$ and considering $s_1,s_2\in[0,T]$, it holds

$$\begin{aligned}\|u^n(s_2)-u^n(s_1)\|_* &\le \sup_{v\in W_0^{1,p}(\Omega)/\|v\|_p=1}\left|\langle\int_{s_1}^{s_2}(u^n(\theta))'d\theta,v\rangle\right|\\ &\le \int_{s_1}^{s_2}\|(u^n(\theta))'\|_*d\theta\\ &\le \|(u^n)'\|_{L^q(0,T;W^{-1,q}(\Omega))}|s_2-s_1|^{1/p}.\end{aligned}$$

In addition, since the sequence $\{u^n\}$ is bounded in $C([0,T];L^2(\Omega))$ and the embedding $L^2(\Omega)\hookrightarrow W^{-1,q}(\Omega)$ is compact, by the Arzela-Ascoli theorem,

$$u^n\to u \quad \text{strongly in } C([0,T];W^{-1,q}(\Omega)).$$

Taking this into account together with the fact that the sequence $\{u^n\}$ is bounded in $C([0,T];L^2(\Omega))$, (7.22) holds.

Step 2. The aim of this step is to prove

$$\limsup_{n\to\infty}|u^n(t)|_2\le|u(t)|_2 \quad \forall t\in[0,T]. \tag{7.23}$$

From the energy equality (7.4), using (7.2) and the Young inequality, we obtain

$$|z(t)|_2^2\le|z(s)|_2^2+(t-s)\frac{2}{q}\left(\frac{1}{mp}\right)^{\frac{q}{p}}\|f\|_*^q \quad \forall 0\le s\le t\le T,$$

where z is replaced by u or any u^n.

Now, we define the following continuous and non-increasing functions on $[0,T]$

$$\begin{aligned}J_n(t)&=|u^n(t)|_2^2-\frac{2t}{q}\left(\frac{1}{mp}\right)^{\frac{q}{p}}\|f\|_*^q,\\ J(t)&=|u(t)|_2^2-\frac{2t}{q}\left(\frac{1}{mp}\right)^{\frac{q}{p}}\|f\|_*^q.\end{aligned}$$

Observe that since

$$u^n(t)\to u(t) \quad \text{strongly in } L^2(\Omega) \quad \text{a.e. } t\in(0,T),$$

we have

$$J_n(t)\to J(t) \quad \text{a.e. } t\in(0,T). \tag{7.24}$$

In fact, making use of the continuity of the functional J on $[0,T]$, the non-increasing character of the function J_n on $[0,T]$, together with (7.24), we obtain

$$J_n(t)\to J(t) \quad \forall t\in(0,T).$$

Therefore, (7.23) holds.

Then, from (7.22) and (7.23) we deduce

$$u^n(t)\to u(t) \quad \forall t\in[0,T].$$

A diagonal procedure allows now to conclude (7.21). □

Now, we are ready to prove that the multi-valued semiflow $\mathcal{S}$ is upper-semicontinuous with closed values.

Proposition 7.6. *Under the assumptions of Lemma 7.4, the multi-valued semiflow $\mathcal{S}$ is upper semicontinuous with closed values.*

Proof. The proof of this result is analogous to that of Proposition 6.6. We reproduce it in the autonomous framework for the sake of completeness.

First, we will prove that the multi-valued semiflow $\mathcal{S}$ is upper semicontinuous. To that end, we argue by contradiction. Suppose that there exist $t \in \mathbb{R}_+$, $u_0 \in L^2(\Omega)$, a neighbourhood $\mathcal{N}$ of $\mathcal{S}(t)u_0$, and a sequence $\{y_n\}$ which fulfils that each $y_n \in \mathcal{S}(t)u_0^n$, where $u_0^n \to u_0$ strongly in $L^2(\Omega)$ and $y_n \notin \mathcal{N}$ for all $n \geq 1$.

Since $y_n \in \mathcal{S}(t)u_0^n$ for all n, there exists $u^n \in \Phi(u_0^n)$ such that $y_n = u^n(t)$. In addition, as $u_0^n \to u_0$ strongly in $L^2(\Omega)$, applying Lemma 7.5, there exists a subsequence of $\{u^n(t)\}$ (relabeled the same) which converges to $u(t) \in \mathcal{S}(t)u_0$. This is contradictory because $y_n \notin \mathcal{N}$ for all $n \geq 1$.

Finally, using again Lemma 7.5, it is straightforward to check that the multi-valued semiflow $\mathcal{S}$ has closed values. □

Now, we show the existence of an absorbing set in $L^2(\Omega)$.

Proposition 7.7. *Under the assumptions of Lemma 7.4, the set $\overline{B}_{L^2}(0, R_1)$, where*

$$R_1^2 = 1 + \frac{p-2}{p}\left(\frac{2C_I^p}{p}\right)^{\frac{2}{p-2}} + \frac{1}{\mu_* q}\left(\frac{2^p}{p(2m-\mu_*)}\right)^{\frac{q}{p}} \|f\|_*^q,$$

is an absorbing set for the multi-valued semiflow $\mathcal{S} : \mathbb{R}_+ \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$, where C_I is the constant of the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ and $\mu_* = (2^{p+1}m)/(q+2^p)$.

Proof. Consider fixed a nonempty bounded subset B of $L^2(\Omega)$, $u_0 \in B$ and $u \in \Phi(u_0)$. Observe, there exists $b > 0$ such that $B \subset \overline{B}_{L^2}(0, b)$.

Our aim is to prove that there exists $t(B) > 0$ such that

$$|u(t)|_2 \leq R_1 \quad \forall t \geq t(B), \tag{7.25}$$

where R_1^2 is given in the statement.

From the energy equality, making use of (7.2), we have

$$\frac{d}{dt}|u(t)|_2^2 + 2m\|u(t)\|_p^p \leq 2\langle f, u(t)\rangle \quad \text{a.e. } t > 0.$$

Now, adding $\pm\mu|u(t)|^2$ (with $\mu \in (0, 2m)$), multiplying by $e^{\mu t}$ and making use of

$$|u(t)|_2^2 \leq \frac{(p-2)}{p}\left(\frac{2C_I^p}{p}\right)^{\frac{2}{p-2}} + \|u(t)\|_p^p,$$

we deduce

$$\frac{d}{dt}(e^{\mu t}|u(t)|_2^2) \leq \left[\frac{\mu(p-2)}{p}\left(\frac{2C_I^p}{p}\right)^{\frac{2}{p-2}} + \frac{1}{q}\left(\frac{2^p}{p(2m-\mu)}\right)^{\frac{q}{p}} \|f\|_*^q\right] e^{\mu t},$$

where C_I is the constant of the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$.
Integrating between 0 and t, we obtain

$$|u(t)|_2^2 \leq |u_0|_2^2 e^{-\mu t} + \frac{(p-2)}{p}\left(\frac{2C_I^p}{p}\right)^{\frac{2}{p-2}} + \frac{1}{\mu_* q}\left(\frac{2^p}{p(2m-\mu_*)}\right)^{\frac{q}{p}} \|f\|_*^q, \qquad (7.26)$$

where $\mu_* = (2^{p+1}m)/(q+2^p)$.

Observe that there exists $t(B) := \max\{0, \frac{2}{\mu}\ln b\}$ such that

$$|u_0|_2^2 e^{-\mu t} \leq 1 \quad \forall t \geq t(B).$$

Therefore, taking this into account, from (7.26), (7.25) holds. □

Now, assuming that f is more regular, we make the most of the additional regularity of any solution to (7.1) (cf. Theorem 7.2) and the existence of an absorbing set in $W_0^{1,p}(\Omega)$ for $\mathcal{S}$ is established. As a consequence, the asymptotic compactness of the multi-valued semiflow $\mathcal{S} : \mathbb{R}_+ \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ follows.

Proposition 7.8. *Under the assumptions of Lemma 7.4, if $f \in L^2(\Omega)$, the set $\overline{B}_{W_0^{1,p}}(0, R_2)$, where*

$$R_2^p = \frac{R_1^2}{m} + \frac{2}{mq}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q + \frac{p}{4m}|f|_2^2,$$

is an absorbing set for the multi-valued semiflow $\mathcal{S} : \mathbb{R}_+ \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$, where R_1 is given in Proposition 7.7.

Proof. Consider fixed a nonempty bounded subset B of $L^2(\Omega)$, $u_0 \in B$ and $u \in \Phi(u_0)$.

Our aim is to prove that there exists $t'(B) > 0$ such that

$$\|u(t)\|_p \leq R_2 \quad \forall t \geq t'(B), \qquad (7.27)$$

where R_2^p is given in the statement.

To that end, we argue as in the proof of Theorem 7.2, namely as it was done in Step 2. We use the Galerkin formulation associated to problem (P_u), whose unique solution is u, and prove (7.27) for the Galerkin approximations $\hat{u}_n$. Then, applying compactness arguments, (7.27) is shown for the solution u to (P_u).

Multiplying (7.17) by $\widehat{\varphi}_{nj}(t)$ and summing from $j=1$ to n, making use of (7.2) and the Young inequality, we deduce

$$\frac{d}{dt}|\hat{u}_n(t)|_2^2 + m\|\hat{u}_n(t)\|_p^p \leq \frac{2}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q \quad \text{a.e. } t > 0.$$

Now, integrating between $t-1$ and t,

$$|\hat{u}_n(t)|_2^2 + m\int_{t-1}^{t} \|\hat{u}_n(s)\|_p^p ds \leq |\hat{u}_n(t-1)|_2^2 + \frac{2}{q}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q. \qquad (7.28)$$

Observe that similarly to it was done in the proof of Proposition 7.7, it holds

$$|\hat{u}_n(t)|_2 \leq R_1 \quad \forall t \geq t(B) \quad \forall n \geq 1,$$

where R_1 and $t(B)$ are given in Proposition 7.7. Then, taking this into account, from (7.28), we have

$$\int_{t-1}^{t} \|\hat{u}_n(s)\|_p^p ds \leq \frac{R_1^2}{m} + \frac{2}{mq}\left(\frac{2}{mp}\right)^{\frac{q}{p}} \|f\|_*^q \quad \forall t \geq t'(B) := t(B) + 1. \tag{7.29}$$

On the other hand, multiplying (7.17) by $\widehat{\varphi}'_{nj}(t)/a(l(u(t)))$ and summing from $j = 1$ until n, it holds

$$\frac{|\hat{u}'_n(t)|_2^2}{a(l(u(t)))} + \frac{1}{p}\frac{d}{dt}\|\hat{u}_n(t)\|_p^p = \frac{(f, \hat{u}'_n(t))}{a(l(u(t)))} \quad \text{a.e. } t > 0.$$

Applying the Cauchy inequality and (7.2) to the above expression, it holds

$$\frac{1}{p}\frac{d}{dt}\|\hat{u}_n(t)\|_p^p \leq \frac{|f|_2^2}{4m} \quad \text{a.e. } t > 0.$$

Now, integrating between r and t, with $t - 1 \leq r \leq t$,

$$\|\hat{u}_n(t)\|_p^p \leq \|\hat{u}_n(r)\|_p^p + \frac{p}{4m}|f|_2^2.$$

Then, integrating w.r.t. r between $t - 1$ and t, we have

$$\|\hat{u}_n(t)\|_p^p \leq \int_{t-1}^{t} \|\hat{u}_n(r)\|_p^p dr + \frac{p}{4m}|f|_2^2.$$

Making use of (7.29), from the previous expression we deduce

$$\|\hat{u}_n(t)\|_p^p \leq R_2^p \quad \forall t \geq t'(B).$$

Therefore, the sequence $\{\hat{u}_n\}$ is bounded in $L^\infty(t'(B), \infty; W_0^{1,p}(\Omega))$. In particular, for any $T > t'(B)$, $\{\hat{u}_n\}$ converges to u weakly in $L^p(t'(B), T; W_0^{1,p}(\Omega))$, since u is the unique solution to (P_u). As $u \in C([t'(B), \infty); L^2(\Omega))$, making use of [100, Lemma 11.2], (7.27) holds. □

To conclude, we have the main result of this section, the existence of the compact global attractor in $L^2(\Omega)$.

Theorem 7.9. *Assume that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (7.2), and both f and l belong to $L^2(\Omega)$. Then, there exists the compact global attractor $\mathcal{A}$, which is invariant and is given by*

$$\mathcal{A} = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \mathcal{S}(s)\overline{B}_{W_0^{1,p}}(0, R_2)}. \tag{7.30}$$

Proof. The multi-valued semiflow $\mathcal{S}$ is upper semicontinuous with closed values thanks to Proposition 7.6. In addition, the existence of an absorbing set in $L^2(\Omega)$ is guaranteed by Proposition 7.7. Therefore, according to Theorem 5.11, to prove the existence of the compact global attractor, we only need to check that the multi-valued semiflow $\mathcal{S}$ is asymptotically compact. This is immediate thanks to Proposition 7.8 and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$. Therefore, by Theorem 5.11, the existence of the compact global attractor $\mathcal{A}$, given by (7.30), holds.

In addition, since the multi-valued semiflow $\mathcal{S}$ is strict (cf. Lemma 7.4), $\mathcal{A}$ is invariant (cf. Theorem 5.11). □

As a straightforward consequence, we obtain the following generalised result ensuring the existence of attractor under a weaker assumption on l.

Corollary 7.10. *Assume that the function $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfils (7.19), $f \in L^2(\Omega)$ and $l \in L^q(\Omega)$. Then, the thesis of Theorem 7.9 hold.*

Current and future research

This PhD project has focused on variational techniques that have been applied to parabolic problems with nonlocal diffusion.

Along this process, some improvements have been obtained as well as some weakenings of the assumptions on certain terms of the problem in return of some restrictions or stronger conditions in other parts.

During this study, we have encountered some technical difficulties and some unfinished projects whose analysis most certainly will be of great interest to us in the future.

Below we show a brief non-exhaustive list of some of these questions and open problems that we find interesting.

The study of parabolic problems with nonlocal diffusion in unbounded domains is still an open problem. We have already proved the existence and uniqueness of weak solutions for nonlocal reaction-diffusion equations in this kind of domains. In fact, we have obtained the existence of a pullback absorbing family in $L^2(\Omega)$. However, several difficulties arise when we try to prove the pullback asymptotic compactness in order to show the existence of pullback attractors in $L^2(\Omega)$. For instance, when we try to apply the method used in [101], we cannot build the scalar product that plays the essential role in the proof of this property due to the fact that the nonlocal operator appears in the diffusion term.

In Chapters 6 and 7, we have analysed multivalued problems. An interesting feature that we would like to study is the Kneser property, which consists in proving that the set of values reached by the solutions at each instant is compact and connected (see [112, 71, 72] for more details). It would be even more enriching to study this property for nonlocal reaction-diffusion equations in unbounded domains as it was done for local reaction-diffusion systems in [98] by Morillas & Valero or in [10] by Anguiano, Morillas and Valero.

Furthermore, we also want to analyse nonlocal problems with delay terms because of their importance in real applications (cf. [86, 117]). From a biological point of view, they can help to study the behaviour of species better because we take into account not only the present but also the history of the population.

In the framework of attractors, we plan to prove the existence of uniform attractors associated to the nonlocal problems analysed during this PhD project as a different approach to study the asymptotic behaviour of the solutions.

In addition to dealing with new problems, we want to improve the current results. For instance, we are determined to continue working on the elliptic problem

$$\begin{cases} -a(l(u))\Delta u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

During a stay with Prof. Chipot at the Zurich University, we analysed interesting results related to the above problem and we would like to make more progress in that line because there is not much known about the possible relations between the solution to the evolution problem associated to the above one and the stationary solutions. In this complex elliptic framework, we also intend to study stationary problems with more than one nonlocal term like the one analysed by Alves and Covei [1], in which the existence of solutions is proved making use of the sub-supersolution method.

Furthermore, we want to weaken the assumptions made on f in Chapters 3 and 4 that guarantee that $f(u) \in L^2(\tau, T; L^2(\Omega))$ (cf. (3.54) and (4.13) respectively).

We would also like to extend our results to more general operators as done in Chapter 7. In addition, we also plan to study the asymptotic behaviour of the solutions associated to local nonlocal problems (cf. [2, 3]).

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