THE DISTRIBUTION OF ZEROS OF GENERAL Q-POLYNOMIALS. [1]

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Abstract

A general system of q-orthogonal polynomials is defined by means of its three-term recurrence relation. This system encompasses many of the known families of q-polynomials, among them the q-analog of the classical orthogonal polynomials. The asymptotic density of zeros of the system is shown to be a simple and compact expression of the parameters which characterize the asymptotic behavior of the coefficients of the recurrence relation. This result is applied to specific classes of polynomials known by the names q-Hahn, q-Kravchuk, q-Racah, q-Askey & Wilson, Al Salam-Carlitz and the celebrated q-little and q-big Jacobi.

1 Introduction.

In the last decade an increasing interest on the so called q-orthogonal polynomials (or basic orthogonal polynomials) is observed (for a review see [1], [2] and [3]). The reason is not only of purely intrinsic nature but also because of the so many applications in several areas of Mathematics (e.g., continued fractions, eulerian series, theta functions, elliptic functions,...; see for instance [4] and [5]) and Physics (e.g., angular momentum [6] and [7] and its q-analog [8]-[11], q-Shrödinger equation [12] and q-harmonic oscillators [13]-[19]). Moreover, it is well known the connection between the representation theory of quantum algebras (Clebsch-Gordan coefficients, 3j and 6j symbols) and the q-orthogonal polynomials, (see [20], [21] (Vol. III), [22], [23], [24]), and the important role that these q-algebras play in physical applications (see for instance [26]-[31] and references therein).

However, the distribution of zeros of these polynomials remains practically unknown to the best of our information. The present paper continues, corrects and considerably extends the investigation of the asymptotic behavior of zeros of the q-polynomials initiated by one of us [32]. This is done by the consideration of a general system of q-polynomials which include most of the

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q-polynomials encountered in the literature and then, study of its distribution density of zeros as well as the corresponding asymptotic limit.

The method of proof used is very straightforward. It is based on an explicit formula for the moments-around-the-origin of the discrete density of zeros of a polynomial with a given degree in terms of the coefficients of the three term recurrence relation [37], as described in Lemma 1 given below. This method was previously employed to normal (non-q) polynomials where recurrence coefficients are given by means of a rational function of the degree [38], as well as to corresponding Jacobi matrices [39] encountered in quantum mechanical description of some physical systems.

The paper is structured as follows. Firstly, in section 2, one introduces a general set of q-polynomials $\{P_n(x)_q\}_{n=0}^N$ by means of its three-term recurrence relation. Section 3 contains the main results which refer to the discrete density of zeros (i.e. the number of zeros per unit of zero interval) of the polynomial $P_n(x)_q$, n being a sufficiently large value, and to its asymptotical limit (i.e., when $n \to \infty$). Both discrete and asymptotic densities of zeros are supposed to be characterized by the knowledge of all their moments. These results are given in the form of four theorems. Theorem 1 gives the behavior of the moments of the discrete density of zeros in terms of the parameters defining the recurrence relation. The asymptotic density of zeros is given by Theorems 2, 3 and 4 in a similar way.

Proofs and detailed discussion of these theorems are contained in Sections 4 and 5 respectively. The utmost effort has been concentrated on searching for an appropiate asymptotic density of zeros to obtain as much information as possible about the asymptotic distribution of zeros of the new polynomials. Finally, Section 6 contains application of theorems 1, 2, 3 and 4 formulated in section 3 to several known families of q-polynomials.

2 The general system of q-orthogonal polynomials.

The general system of q-orthogonal polynomials $\{P_n(x)_q\}_{n=0}^N$ is defined by the recurrence relation

$$
\begin{gathered}
P_n(x) = (x - a_n)P_{n-1}(x) - b_{n-1}^2 P_{n-2}(x) \\
P_{-1}(x) = 0, \quad P_0(x) = 1, \quad n \geq 1
\end{gathered}
\tag{1}
$$

with the coefficients a_n and b_{n-1}^2 given by

$$
\begin{aligned}
a_n &= \frac{\sum_{m=0}^{A} \left(\sum_{i=0}^{g_m} \alpha_i^{(m)} n^{g_m - i} \right) q^{d_m n}}{\sum_{m=0}^{A'} \left(\sum_{i=0}^{h_m} \beta_i^{(m)} n^{h_m - i} \right) q^{e_m n}} \equiv \frac{a_n^{num}}{a_n^{den}} \\
b_n^2 &= \frac{\sum_{m=0}^{B} \left(\sum_{i=0}^{k_m} \theta_i^{(m)} n^{k_m - i} \right) q^{f_m n}}{\sum_{m=0}^{B'} \left(\sum_{i=0}^{l_m} \gamma_i^{(m)} n^{l_m - i} \right) q^{s_m n}} \equiv \frac{(b_n^{num})^2}{(b_n^{den})^2}
\end{aligned}
\tag{2}
$$

where q is an arbitrary positive real number bigger than 1. Further, the following general requeriments on the real parameters defining a_n and b_n^2 will be assumed:

1. All members of the sequence $\{\beta_i^{(m)};\ 0 \le i \le h_m\}_{m=0}^{A'}$, $\{\gamma_i^{(m)};\ 0 \le i \le l_m\}_{m=0}^{B'}$ do not vanish simultaneously. So we assure a_n and b_n^2 not to be infinite for all n.

2. The parameters $\{\theta_i^{(m)};\ 0 \le i \le k_m\}_{m=0}^{B}$ and $\{\gamma_i^{(m)};\ 0 \le i \le l_m\}_{m=0}^{B'}$ are such that $b_n^2 > 0$ for $n \ge 1$. Then Favard's theorem assures the orthogonality of the polynomials $\{P_n(x)_q\}_{n=0}^{N}$.

3. The following inequalities are verified:

$$q^{d_0} > q^{d_1} > \ldots > q^{d_A};\quad q^{e_0} > q^{e_1} > \ldots > q^{e_{A'}}$$
$$q^{f_0} > q^{f_1} > \ldots > q^{f_B};\quad q^{s_0} > q^{s_1} > \ldots > q^{s_{B'}} \tag{3}$$

and

$$g_0 > g_1 > \ldots > g_m;\quad h_0 > h_1 > \ldots > h_m$$
$$k_0 > k_1 > \ldots > k_m;\quad l_0 > l_1 > \ldots > l_m \tag{4}$$

Conditions (3) and (4) do not obviously imply any loss of generality. Here one should also point out that the polynomials discussed in reference [32] are instances of the polynomials (1)-(2) corresponding to the values $g_m = k_m = h_m = e_m = l_m = s_m = 0$ for all m.

3 Main Results.

Before collecting the main results of this work, let us describe Lemma 1 which is the basic tool to find them.

Lemma 1 *Let $\{P_N(x)\}$ be a system of polynomials $P_N(x)$ defined by the recurrence relation (1), which is characterized by the secuences of numbers $\{a_n\}$ and $\{b_n\}$. Let the quantities*

$$\mu_0 = N,\quad \mu_m'^{(N)} = \int_a^b x^m \rho_N(x)\,dx,\quad m = 1,2,...,N \tag{5}$$

be the non-normalized-to-unity spectral moments of the polynomials $P_N(x)$, i.e., the moments around the origin of the discrete density of zeros $\rho_N(x)$, defined by

$$\rho_N(x) = \sum_{i=1}^{N} \delta(x - x_{N,i}), \tag{6}$$

$\{x_{N,i},\ i = 1,2,...,N\}$ being the zeros of that polynomial. It is fulfilled that

$$\mu_m'^{(N)} = \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \sum_{i=1}^{N-t} a_i^{r_1'} b_i^{2r_1} a_{i+1}^{r_2'} b_{i+1}^{2r_2} \ldots b_{i+j-1}^{2r_j} a_{i+j}^{r_{j+1}'}, \tag{7}$$

for $m = 1,2,...,N$. The sumation $\sum_{(m)}$ runs over all partitions $(r_1', r_1, ..., r_{j+1}')$ of the number m such that

1. *$R' + 2R = m$, where R and R' denote the sums $R = \sum_{i=1}^{j} r_i$ and by $R' = \sum_{i=1}^{j-1} r_i'$, or*

$$\sum_{i=1}^{j-1} r_i' + 2\sum_{i=1}^{j} r_i = m \tag{8}$$

2. If $r_s = 0,\ 1 < s < j$*, then* $r_k = r'_k = 0$ *for each* $k > s$ *and*

3. $j = \dfrac{m}{2}$ *or* $j = \dfrac{m-1}{2}$ *for* m *even or odd respectively.*

The factorial coefficient F *are defined by*

$$F(r'_1, r_1, r'_2, ..., r'_{p-1}, r_{p-1}, r'_p) = m\frac{(r'_1 + r_1 - 1)!}{r'_1! r_1!}\left[\prod_{i=2}^{p-1} \frac{(r_{i-1} + r'_i + r_i - 1)!}{(r_{i-1} - 1)! r_i! r'_i!}\right] \frac{(r_{p-1} + r'_p - 1)!}{(r_{p-1} - 1)! r'_p!}, \quad (9)$$

with the convention $r_0 = r_p = 1$*. For the evaluation of these coefficients, we must take into account the following convenction*

$$F(r'_1, r_1, r'_2, r_2..., r'_{p-1}, 0, 0) = F(r'_1, r_1, r'_2, r_2..., r'_{p-1})$$

In (7), t *denotes the number of non-vanishing* r_i *which are involved in each partition of* m*.*

This Lemma was initially found in a context of Jacobi matrices [37]-[38]. Just to understand the practical use of the Lemma, let us give the first three spectral moments

$$\mu'_1 = \sum_{i=1}^{N} a_i,$$

$$\mu'_2 = \sum_{i=1}^{N} a_i^2 + 2\sum_{i=1}^{N-1} b_i^2, \quad (10)$$

$$\mu'_3 = \sum_{i=1}^{N} a_i^3 + 3\sum_{i=1}^{N-1} b_i^2(a_i + a_{i+1}).$$

In the following, the main results of this work are collected in the form of four theorems. The first of them refers to the discrete density of zeros (6) of the polynomials defined by (1)-(2) and the other three concern with the asymptotic density of zeros, i.e., when the degree of the polynomial tends towards infinity. Throughout the paper the symbol $\sim$ means *behaves as.*

Theorem 1 *Let* $P_N(x)_q$*, very large* N*, be a polynomial defined by the expressions (1)-(4). The moments* $\{\mu'^{(N)}_m; m = 1, 2, ..., N\}$ *of the non-normalized density of zeros* $\rho_N(x) = \sum_{i=1}^N \delta(x - x_{N,i})$ *of the polynomial* $P_N(x)_q$ *have the following behavior*

1. If $d_0 - e_0 = \frac{1}{2}(f_0 - s_0) = 0$*, three cases occur:*

(a) If $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$*. Then*

$$\mu'^{(N)}_m \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m N^{(g_0-h_0)m+1}. \quad (11)$$

(b) If $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$*. Then*

$$\mu'^{(N)}_m \sim \sum_{(m)} F(r'_1, r_1, ..., r'_{j+1}) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N^{\frac{1}{2}(k_0-l_0)m+1}. \quad (12)$$

(c) *If* $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$. *Then*

$$\mu_m'^{(N)} \sim \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} N^{\frac{1}{2}(k_0-l_0)m+1}. \tag{13}$$

2. *If* $d_0 - e_0 \neq 0$ *and/or* $f_0 - s_0 \neq 0$, *two cases occur:*

(a) i. *If* $d_0 - e_0 < 0$ *and* $f_0 - s_0 < 0$ *in such a way that* $\Omega_1 \neq 0$. *Then*

$$\mu_m'^{(N)} \sim \sum_{(m)} \frac{F(r_1', r_1, ..., r_{j+1}')}{q^{-\Omega_2}(ln\, q)^M} \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{d^M}{d\Omega_1^M}\left(\frac{q^{\Omega_1}}{1-q^{\Omega_1}}\right), \tag{14}$$

where $\dfrac{d^M}{d\Omega_1^M}$ *denotes the* M derivative with respect to Ω_1.

ii. *If* $d_0 - e_0 = 0$ *and* $f_0 - s_0 < 0$ *and* $g_0 - h_0 = k_0 - l_0 = 0$. *Then*

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', 0, ..., 0, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} N. \tag{15}$$

iii. *If* $d_0 - e_0 < 0$ *and* $f_0 - s_0 = 0$ *and* $g_0 - h_0 = k_0 - l_0 = 0$. *Then*

$$\mu_m'^{(N)} \sim \sum_{(m)} F(0, r_1, ..., r_j, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N. \tag{16}$$

(b) *If* $d_0 - e_0 > 0$ *and/or* $f_0 - s_0 > 0$, *three different subcases may occur, namely:*

i. $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$. *Then*

$$\mu_m'^{(N)} \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{m} \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} N^{(g_0-h_0)m}. \tag{17}$$

ii. *If* $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$. *Then three diferent types still come up:*

A. *If* $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$, *then*

$$\mu_m'^{(N)} \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{m} \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} N^{(g_0-h_0)m}. \tag{18}$$

B. *If* $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$, *then*

$$\begin{aligned}\mu_m'^{(N)} \sim & \sum_{(m)} F(r_1', r_1, ..., r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \times \\ & \times \frac{q^{\Omega_2+m(N+1-t)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} N^{m(g_0-h_0)}.\end{aligned} \tag{19}$$

C. *If* $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, *then*

$$\mu_m'^{(N)} \sim \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{(d_0-e_0)mN}}{q^{(d_0-e_0)m}-1} N^{\frac{1}{2}(k_0-l_0)m}. \tag{20}$$

iii. $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$. *Then*

$$\mu_m'^{(N)} \sim \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{\frac{1}{2}(f_0-s_0)mN}}{q^{\frac{1}{2}(f_0-s_0)m}-1} N^{\frac{1}{2}(f_0-s_0)m}. \tag{21}$$

The sumation $\sum_{(m)}$ *and the parameter* t *are as defined in Lemma 1. Besides, the parameters* Ω_1, Ω_2 *and* M *are as follows:*

$$\Omega_1 = [(d_0 - e_0) - \frac{1}{2}(f_0 - s_0)]R' + \frac{m}{2}(f_0 - s_0) \tag{22}$$

$$\Omega_2 = (d_0 - e_0)\sum_{k=1}^{j} kr'_{k+1} + 2(f_0 - s_0)\sum_{k=1}^{j-1} kr_{k+1} \tag{23}$$

$$M = [(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)]R' + \frac{m}{2}(k_0 - l_0) \tag{24}$$

The proof of this theorem is shown in Section 4.

Theorem 2 *Let* $P_N(x)_q$ *be a polynomial defined as in Theorem 1 with the aditional condition* $(d_0 - e_0) = \frac{1}{2}(f_0 - s_0) = 0$. *(i.e. case 1)*

Let $\rho(x)$, $\rho_1^*(x)$ *and* $\rho_2^*(x)$ *be the asymptotic (i.e. when* $N \to \infty$*) densities of zeros of the polynomial* $P_N(x)_q$ *defined by*

$$\begin{aligned} \rho(x) &= \lim_{N\to\infty} \rho_N(x), \\ \rho_1^*(x) &= \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^{(g_0-h_0)}}\right), \\ \rho_2^*(x) &= \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^{\frac{1}{2}(k_0-l_0)}}\right) \end{aligned} \tag{25}$$

and their corresponding moments are as follows:

$$\begin{aligned} \mu_m' &= \lim_{N\to\infty} \mu_m'^{(N)}, \\ \mu_m^*(1) &= \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N^{(g_0-h_0)m}}, \\ \mu_m^*(2) &= \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N^{(k_0-l_0)\frac{m}{2}}} \end{aligned} \tag{26}$$

for $m = 0, 1, 2, \ldots$ *respectively. Here* $\rho_N(x)$ *denotes the (discrete) density of zeros of the polynomial* $P_N(x)_q$. *It turns out that*

$$\mu_m' = \infty, \quad m \geq 0 \tag{27}$$

and

1. *If* $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$. *Then*

$$\mu_m^*(1) = \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m, \quad m \geq 0 \tag{28}$$

2. If $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$*. Then*

$$\mu_m^*(2) = \sum_{(m)} F(r_1', r_1, ..., r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R}, \quad m \geq 0 \tag{29}$$

3. If $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$*. Then*

$$\mu_m^*(2) = \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}}, \quad m \geq 0. \tag{30}$$

Here the coefficients F *and the symbol of summation* $\sum_{(m)}$ *are as in Theorem 1.*

Theorem 3 *Let* $P_N(x)_q$ *be a polynomial defined as in Theorem 1 with the aditional condition* $(d_0 - e_0) \leq 0$ *and* $\frac{1}{2}(f_0 - s_0) \leq 0$*. (i.e. subcase 2a)*

Let $\rho(x)$ *and* $\rho_1(x)$ *be the asymptotic densities of zeros of the polynomial* $P_N(x)_q$ *defined by*

$$\rho(x) = \lim_{N\to\infty} \rho_N(x); \qquad \rho_1(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N(x) \tag{31}$$

and their corresponding moments are as follows:

$$\mu_m' = \lim_{N\to\infty} \mu_m'^{(N)}; \qquad \mu_m'(1) = \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N} \tag{32}$$

for $m \geq 0$*, respectively. It turns out that:*

1. If $d_0 - e_0 < 0$ *and* $f_0 - s_0 < 0$ *in such a way that* $\Omega_1 \neq 0$*. Then*

$$\mu_m' = \sum_{(m)} \frac{F(r_1', r_1, ..., r_{j+1}')}{q^{-\Omega_2}(ln\, q)^M} \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{d^M}{d\Omega_1^M}\left(\frac{q^{\Omega_1}}{1-q^{\Omega_1}}\right), \tag{33}$$

and

$$\mu_0'(1) = 1, \quad \mu_m'(1) = 0, \quad m \geq 1. \tag{34}$$

2. If $d_0 - e_0 = 0$ *and* $f_0 - s_0 < 0$ *and* $g_0 - h_0 = k_0 - l_0 = 0$*. Then*

$$\mu_m' = \infty, \quad m \geq 0 \tag{35}$$

$$\mu_m'(1) = \begin{cases} 1 & m = 0 \\ \sum_{(m)} F(r_1', 0, ..., 0, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} & m \geq 1. \end{cases} \tag{36}$$

3. If $d_0 - e_0 < 0$ *and* $f_0 - s_0 = 0$ *and* $g_0 - h_0 = k_0 - l_0 = 0$*. Then*

$$\mu_m' = \infty, \quad m \geq 0 \tag{37}$$

$$\mu_m'(1) = \begin{cases} 1 & m = 0 \\ \sum_{(m)} F(0, r_1, 0, ..., r_j, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} & m \geq 1. \end{cases} \tag{38}$$

Here the coefficients F and the symbol of summation $\sum_{(m)}$ and the parameters Ω_1, Ω_2 and M are as in Theorem 1.

Theorem 4 *Let $P_N(x)_q$ be a polynomial defined as in Theorem 1 with the aditional condition $(d_0 - e_0) > 0$ and/or $\frac{1}{2}(f_0 - s_0) > 0$. (i.e. subcase 2b)*

Let $\rho(x)$, $\rho_1^{**}(x)$, $\rho_2^{**}(x)$, $\rho_3^{**}(x)$, $\rho_1^{++}(x)$, $\rho_2^{++}(x)$ and $\rho_3^{++}(x)$ be the asymptotic densities of zeros of the polynomial $P_N(x)_q$ given by

$$\rho(x) = \lim_{N\to\infty} \rho_N(x), \tag{39}$$

$$\begin{aligned}
\rho_1^{**}(x) &= \lim_{N\to\infty} \rho_N \left(\frac{xq^{-(d_0-e_0)N}}{N^{(g_0-h_0)}} \right), \\
\rho_2^{**}(x) &= \lim_{N\to\infty} \rho_N \left(\frac{xq^{-(d_0-e_0)N}}{N^{\frac{1}{2}(k_0-l_0)}} \right), \\
\rho_3^{**}(x) &= \lim_{N\to\infty} \rho_N \left(\frac{xq^{-\frac{1}{2}(f_0-s_0)N}}{N^{\frac{1}{2}(k_0-l_0)}} \right),
\end{aligned} \tag{40}$$

$$\begin{aligned}
\rho_1^{++}(x) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \rho_N \left(\frac{xq^{-(d_0-e_0-1)N}}{N^{(g_0-h_0)}} \right), \\
\rho_2^{++}(x) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \rho_N \left(\frac{xq^{-(d_0-e_0-1)N}}{N^{\frac{1}{2}(k_0-l_0)}} \right), \\
\rho_3^{++}(x) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \rho_N \left(\frac{xq^{-\frac{1}{2}(f_0-s_0-2)N}}{N^{\frac{1}{2}(k_0-l_0)}} \right),
\end{aligned} \tag{41}$$

and their corresponding moments are as follows:

$$\mu'_m = \lim_{N\to\infty} \mu'^{(N)}_m \tag{42}$$

$$\begin{aligned}
\mu_m^{**}(1) &= \lim_{N\to\infty} \frac{\mu'^{(N)}_m}{N^{(g_0-h_0)} q^{(d_0-e_0)mN}} \\
\mu_m^{**}(2) &= \lim_{N\to\infty} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{(d_0-e_0)mN}} \\
\mu_m^{**}(3) &= \lim_{N\to\infty} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{\frac{1}{2}(f_0-s_0)mN}}
\end{aligned} \tag{43}$$

$$\begin{aligned}
\mu_m^{++}(1) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \frac{\mu'^{(N)}_m}{N^{(g_0-h_0)} q^{(d_0-e_0-1)mN}} \\
\mu_m^{++}(2) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{(d_0-e_0-1)mN}} \\
\mu_m^{++}(3) &= \lim_{N\to\infty} \frac{(m)_q}{(mN)_q} \frac{\mu'^{(N)}_m}{N^{\frac{1}{2}(k_0-l_0)} q^{\frac{1}{2}(f_0-s_0-2)mN}}
\end{aligned} \tag{44}$$

for $m \geq 0$, respectively, and where symbol $(n)_q$ denotes the q-basic number

$$(n)_q = \frac{q^n - 1}{q - 1}, \tag{45}$$

related with the q-numbers $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ *by formula* $(n)_q = q^{\frac{n-1}{2}}[n]_{q^{\frac{1}{2}}}$. *It turns out that*

$$\mu'_m = \infty, \quad m \geq 0 \tag{46}$$

and

1. $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$. *Then*

$$\mu^{**}_m(1) = \begin{cases} \infty & m = 0 \\ \left[\dfrac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \dfrac{q^{m(d_0-e_0)}}{q^{m(d_0-e_0)} - 1} & m \geq 1. \end{cases} \tag{47}$$

Also,

$$\mu^{++}_m(1) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu^{**}_m(1) & m \geq 1 \end{cases} \tag{48}$$

2. *If* $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$. *Then three diferent situation come up:*

   (a) *If* $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$. *Then the moments* $\mu^{**}_m(1)$ *and* $\mu^{++}_m(1)$ *have the same values as in the previous case., i.e., as formulas (47) and (48).*

   (b) *If* $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$, *then*

$$\mu^{**}_m(1) = \begin{cases} \infty & m = 0 \\ \displaystyle\sum_{(m)} F(r'_1, r_1, ..., r'_{j+1}) \left[\dfrac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\dfrac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \dfrac{q^{\Omega_2 + m(1-t)(d_0-e_0)}}{q^{m(d_0-e_0)} - 1} & m \geq 1 \end{cases} \tag{49}$$

Also,

$$\mu^{++}_m(1) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu^{**}_m(1) & m \geq 1. \end{cases} \tag{50}$$

   (c) *If* $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, *then*

$$\mu^{**}_m(2) = \begin{cases} \infty & m = 0 \\ \left[\dfrac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \dfrac{1}{q^{(d_0-e_0)m} - 1} & m \geq 1 \end{cases} \tag{51}$$

Also,

$$\mu^{++}_m(2) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu^{**}_m(2) & m \geq 1 \end{cases} \tag{52}$$

3. $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$. *Then*

$$\mu^{**}_m(3) = \begin{cases} \infty & m = 0 \\ \left[\dfrac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \dfrac{1}{q^{\frac{1}{2}(f_0-s_0)m} - 1} & m \geq 1 \end{cases} \tag{53}$$

Also,

$$\mu_m^{++}(3) = \begin{cases} 1 & m = 0 \\ (q^m - 1)\mu_m^{**}(3) & m \geq 1 \end{cases} \tag{54}$$

Here the coefficients F and the symbol of summation $\sum_{(m)}$ and the parameters Ω_1, Ω_2 and M are as in Theorem 1.

It is important to make the following observation. To get as much information as possible about the asymptotic distribution of zeros when the moments μ'_m of the conventional asymptotic density of zeros $\rho(x) = \lim_{N\to\infty} \rho_N(x)$ diverge, it is often used in theorems 2, 3 and 4 a normalization factor D, i.e., it is usually defined an asymptotic density of zeros of the form

$$f(x) = \lim_{N\to\infty} C\rho_N(Dx). \tag{55}$$

where the factors C and D are choosen so that the moments μ_m of $f(x)$ given by

$$\mu_m = \lim_{N\to\infty} CD^m \mu_m'^{(N)} \tag{56}$$

are finite [40]. This is the great advantage of the densities of $f(x) - type$. The scaling factor D turns out to be a function of N and/or q^N. A detailed analysis of this procedure is done in Section 5.

4 Determining the discrete density of zeros.

Here Theorem 1 will be proved. Let us consider the polynomial $P_N(x)_q$, N being a very large number, defined by the expressions (1)-(4), i.e., that

$$P_N(x) = (x - a_N)P_{N-1}(x) - b_{N-1}^2 P_{N-2}(x), \tag{57}$$

where a_N and b_N^2 are the values of a_n and b_n^2 given by Eq.(2) for $n = N$. Firstly, let us find what are the N-dominant terms in the expressions (2) for a_N and b_{N-1}^2. Replacing n by N in Eq. (2) and taking into account that

$$\begin{aligned} &\sum_{m=0}^{A} \left(\sum_{i=0}^{g_m} \alpha_i^{(m)} N^{g_m - i} \right) q^{d_m N} \sim \left(\sum_{i=0}^{g_0} \alpha_i^{(0)} N^{g_0 - i} \right) q^{d_0 N} \sim \alpha_0^{(0)} N^{g_0} q^{d_0 N}, \\ &\sum_{m=0}^{A'} \left(\sum_{i=0}^{h_m} \beta_i^{(m)} N^{h_m - i} \right) q^{e_m N} \sim \left(\sum_{i=0}^{h_0} \beta_i^{(0)} N^{h_0 - i} \right) q^{e_0 N} \sim \beta_0^{(0)} N^{h_0} q^{e_0 N} \end{aligned} \tag{58}$$

one easily obtains that

$$a_N \sim \frac{\alpha_0^{(0)}}{\beta_0^{(0)}} N^{g_0 - h_0} q^{(e_0 - d_0)N} \tag{59}$$

and in a similar way one easily obtains that

$$b_N^2 \sim \frac{\theta_0^{(0)}}{\gamma_0^{(0)}} N^{k_0 - l_0} q^{(f_0 - s_0)N}. \tag{60}$$

The symbol $\sim$ means, as alredy pointed out, *behaves with N as.* To get (58) the conditions (3) and (4) have been used. Remark that, taking into account Eqs. (59)-(60), Eq. (2) may be written as

$$a_n = \frac{\alpha_0^{(0)}}{\beta_0^{(0)}} n^{(g_0-h_0)} q^{(e_0-d_0)n} + O(n^{g_0-h_0-1} q^{(e_0-d_0)n})$$
$$b_n^2 = \frac{\theta_0^{(0)}}{\gamma_0^{(0)}} n^{(k_0-l_0)} q^{(f_0-s_0)n} + O(n^{k_0-l_0-1} q^{(f_0-s_0)n}), \tag{61}$$

for $n \geq 1$. To calculate the discrete density of zeros $\rho_N(x)$ of the polynomial $P_N(x)_q$, one first assumes it may be characterized by the knowledge of all its moments $\{\mu_m'^{(N)},\ m = 0, 1, 2, ..., N\}$ defined by (5).

Taking the values (61) of a_n and b_n^2 into Eq. (7), one obtains for $\mu_m'^{(N)}$ the following values:

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \times$$
$$\times \sum_{i=1}^{N-t} \left[\prod_{k=0}^{j-1} (i+k)^{(g_0-h_0)r_{k+1}' + (k_0-l_0)r_{k+1}}\right] (i+j)^{(g_0-h_0)r_{j+1}'} q^{\Omega_2 + i\Omega_1} \tag{62}$$

If we take in Eq. (62) the dominat term then it reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} q^{\Omega_2} \sum_{i=1}^{N-t} i^M q^{i\Omega_1} \tag{63}$$

with the following notations

$$R = \sum_{i=1}^{j} r_i, \quad R' = \sum_{i=1}^{j-1} r_i'$$

and

$$\Omega_1 = (d_0 - e_0)R' + (f_0 - s_0)R$$
$$\Omega_2 = (d_0 - e_0)\sum_{k=1}^{j} k r_{k+1}' + 2(f_0 - s_0)\sum_{k=1}^{j-1} k r_{k+1} \tag{64}$$
$$M = (g_0 - h_0)R' + (k_0 - l_0)R$$

One should notice that, because of relation (8), $R' + 2R = m$ and consequentely the parameters Ω_1 and M may be written in the form

$$\Omega_1 = [(d_0 - e_0) - \frac{1}{2}(f_0 - s_0)]R' + \frac{m}{2}(f_0 - s_0), \tag{65}$$
$$M = [(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)]R' + \frac{m}{2}(k_0 - l_0), \tag{66}$$

which are the expressions (22) and (24) given in the previous Section.

To go further one has to perform the $i - summation$ in Eq. (63). In doing that two cases appear when one analyzes the expression (65) of Ω_1:

1. $d_0 - e_0 = \frac{1}{2}(f_0 - s_0) = 0$
2. $d_0 - e_0 \neq 0$ and/or $\frac{1}{2}(f_0 - s_0) \neq 0$

Let us see how Eq. (63) gets simplified in each case.

Case 1: $d_0 - e_0 = \frac{1}{2}(f_0 - s_0) = 0.$

In this case $\Omega_1 = \Omega_2 = 0$ and since

$$\sum_{i=1}^{N-t} i^M \sim (N-t)^{M+1}, \quad N >> 1$$

Eq. (63) reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N^{M+1}. \quad (67)$$

To simplify further this expression, one examines Eq. (66) of M. It is easy to find three different subcases corresponding to $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$, $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$ and $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$, respectively. Let us study what happens for each subcase.

1. (a) $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$. Notice that

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{positive}} R' + \frac{m}{2}(k_0 - l_0)$$

Then the dominant term is obtained when $R' = m$ and $R = 0$, i.e. for the partition $(m, 0, 0, ..., 0)$. Therefore $M = (g_0 - h_0)m$ and expression (67) reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(m, 0, 0, ..., 0) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{m} N^{(g_0 - h_0)m+1}.$$

Since $F(m, 0, 0, ..., 0) = 1$ according to (9), it is clear that this relation is the expression (11) of Theorem 1.

(b) $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$. Then $M = \frac{m}{2}(k_0 - l_0)$ and (67) takes the form

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r_1', r_1, ..., r_j, r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} N^{\frac{m}{2}(k_0 - l_0)+1}.$$

This expression coincides with (12) given in Theorem 1.

(c) $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$. Notice that

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{negative}} R' + \frac{m}{2}(k_0 - l_0)$$

Then the dominant term is obtained when $2R = m$ and $R' = 0$,i.e., for the partition $(0, m, 0, ..., 0)$. Therefore $M = \frac{1}{2}(k_0 - l_0)$ and

$$\mu_m'^{(N)} \sim \sum_{(m)} F(0, m, 0, 0, ..., 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} N^{\frac{1}{2}(k_0 - l_0)+1},$$

which is the expression (13) given in Theorem 1, since $F(0, m, 0, 0, ..., 0) = 1$.

Case 2: $d_0 - e_0 \neq 0$ and/or $\frac{1}{2}(f_0 - s_0) \neq 0$.

Here one is obliged to perform the $i - summation$ of (63). One has

$$\sum_{i=1}^{N-t} i^M q^{i\Omega_1} = \frac{1}{(ln\, q)^M} \sum_{i=1}^{N-t} \frac{d^M}{d\Omega_1^M} q^{i\Omega_1}$$

$$= \frac{1}{(ln\, q)^M} \frac{d^M}{d\Omega_1^M} \sum_{i=1}^{N-t} q^{i\Omega_1}$$

$$= \frac{1}{(ln\, q)^M} \frac{d^M}{d\Omega_1^M} \left[\frac{q^{\Omega_1} - q^{\Omega_1(N-t+1)}}{1 - q^{\Omega_1}} \right].$$

Depending on whether q^{Ω_1} is smaller or bigger than unity, this summation has a N-behaviour or another, Indeed,

$$q^{\Omega_1} - q^{\Omega_1(N-t+1)} \sim \begin{cases} q^{\Omega_1} & \text{if } q^{\Omega_1} < 1 \\ -q^{\Omega_1(N-t+1)} & \text{if } q^{\Omega_1} > 1 \end{cases} \tag{68}$$

Then

$$\sum_{i=1}^{N-t} i^M q^{i\Omega_1} \sim \frac{1}{(ln\, q)^M} \frac{d^M}{d\Omega_1^M} \left[\frac{q^{\Omega_1}}{1 - q^{\Omega_1}} \right] \quad \text{if } q^{\Omega_1} < 1 \tag{69}$$

and

$$\sum_{i=1}^{N-t} i^M q^{i\Omega_1} \sim \left[\frac{q^{\Omega_1(N-t+1)}}{q^{\Omega_1} - 1} N^M \right] \quad \text{if } q^{\Omega_1} > 1 \tag{70}$$

Therefore, from (68) it is clear that to further reduce the expression (63) of the quantities $\mu_m'^{(N)}$ one has necessarily to distinguish the following two subcases: $q^{\Omega_1} < 1$ (i.e. $\Omega_1 < 0$) for all partitions of m and $q^{\Omega_1} > 1$ (i.e. $\Omega_1 > 0$) for at least one partition of m. Taking into account (65), these two subcases occur provided that

1. (a) $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$
   (b) $d_0 - e_0 = 0$ and $f_0 - s_0 < 0$
   (c) $d_0 - e_0 < 0$ and $f_0 - s_0 = 0$
2. $d_0 - e_0 > 0$ and/or $f_0 - s_0 > 0$

respectively. Let us see how the moments $\mu_m'^{(N)}$ given by (63) simplify in these two cases separately.

(a) Subcase (2a):

i. $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$ in such a way that $\Omega_1 \neq 0$. The replacement of *i-summation* given by (69) in (63) leads to

$$\mu_m'^{(N)} \sim \sum_{(m)} \frac{F(r_1', r_1, ..., r_{j+1}')}{q^{-\Omega_2}(ln\, q)^M} \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}} \right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}} \right]^{R} \frac{d^M}{d\Omega_1^M} \left(\frac{q^{\Omega_1}}{1 - q^{\Omega_1}} \right),$$

which is the expression (14) of Theorem 1.

ii. $d_0 - e_0 = 0$ and $f_0 - s_0 < 0$ and $g_0 - h_0 = k_0 - l_0 = 0$. Since

$$M = [(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)]R' + \frac{m}{2}(k_0 - l_0) = 0,$$

then

$$\sum_{i=1}^{N-t} i^M q^{i\Omega_1} = \sum_{i=1}^{N-t} q^{i\Omega_1} = q^{\Omega_1}\left[\frac{1 - q^{\Omega_1(N-t)}}{1 - q^{\Omega_1}}\right],$$

where $\Omega_1 = (f_0 - s_0)R$ (see (64)). For $N >> 1$ it is clear from the last expression that the *i-summation* is a decreasing and convex upward function, which has a maximum when $\Omega_1 = 0$, i.e. when $R = 0$ and $R' = m$ and it is equal to N. This corresponds to all partitions $(r'_1, 0, ..., 0, r'_{j+1})$. Notice that (see (64))

$$\Omega_2 = \underbrace{(d_0 - e_0)}_{=0}\sum_{k=1}^{j} k r'_{k+1} + 2(f_0 - s_0)\sum_{k=1}^{j-1} k\, \underbrace{r_{k+1}}_{=0} = 0.$$

Then (63) reduces as follows

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r'_1, 0, ..., 0, r'_{j+1})\left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} N.$$

which coincides with expression (15) of Theorem 1.

iii. $d_0 - e_0 < 0$ and $f_0 - s_0 = 0$ and $g_0 - h_0 = k_0 - l_0 = 0$. Here $\Omega_1 = (d_0 - e_0)R' \le 0$. Then, as in the previous case, we have the conditions

$$\Omega_1 = 0, \quad \Omega_2 = 0, \quad i - summation = N\,.$$

and (63) reduces as expression (16) of Theorem 1.

(b) Subcase(2b): $d_0 - e_0 > 0$ and/or $f_0 - s_0 > 0$. Here from (70) and (63) one gets

$$\mu_m'^{(N)} \sim \sum_{(m)} F(r'_1, r_1, ..., r_j, r'_{j+1})\left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'}\left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R}\frac{q^{\Omega_2 + (1-t)\Omega_1}}{q^{i\Omega_1} - 1} q^{i\Omega_1 N} N^M \tag{71}$$

To go further in the analysis of the $N-$dependence of $\mu_m'^{(N)}$ one has to analize the expression (65) which defines Ω_1. A simple study allows us to distinguish the following three situations

i. $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$
ii. $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$
iii. $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$

Now we shall examine the reduction of (71) in these situations.

i. $d_0 - e_0 > \frac{1}{2}(f_0 - s_0)$. From (65) and (71) one easily finds that the dominant term in the *(m)-summation* correspond to that for which $R' = m$, because

$$\Omega_1 = \underbrace{[(d_0 - e_0) - \frac{1}{2}(f_0 - s_0)]}_{\text{positive}} R' + \frac{m}{2}(f_0 - s_0).$$

Then $R = 0$, $\Omega_1 = m(d_0 - e_0)$, $M = (g_0 - h_0)$, the corresponding partition is $(m, 0, ..., 0)$ and then $\Omega_2 = 0$ an $t = 0$. Therefore

$$\mu_m'^{(N)} \sim \sum_{(m)} F(m,0,0,...,0) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} N^{(g_0-h_0)m}.$$

Since $F(m, 0, 0, ..., 0) = 1$ acording to (9) this relation is the expression (17) of Theorem 1.

ii. $d_0 - e_0 = \frac{1}{2}(f_0 - s_0)$. Here one has $\Omega_1 = \frac{m}{2}(f_0 - s_0) = (d_0 - e_0)m$, that is fixed number for all partitions of m. Then, in the expression (71) one is obliged to study the parameter M given by (66) to know the $N-$dominant term of the *(m)-summation*. The analysis of expression (66) leads to separate the following three possibilities:

A. $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$
B. $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$
C. $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$

For the case $g_0 - h_0 > \frac{1}{2}(k_0 - l_0)$ the dominant term is the one corresponding to the condition when N^M is maximum. It occurs when $R' = m$, $R = 0$ because

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{positive}} R' + \frac{m}{2}(k_0 - l_0)$$

It corresponds to the partition $(m, 0, ..., 0)$, for which $F(m, 0, 0, ..., 0) = 1$, $t = 0$, $\Omega_2 = 0$, $M = (g_0 - h_0)m$. Then, Eq. (71) reduces as

$$\mu_m'^{(N)} \sim \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \frac{q^{m(N+1)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} N^{(g_0-h_0)m},$$

which coincides with Eq. (18) of Theorem 1.

For the case $g_0 - h_0 = \frac{1}{2}(k_0 - l_0)$ it turns out that $M = (g_0 - h_0)m$, $\Omega_1 = (d_0 - e_0)$ and expression (71) easily transforms into (19) of Theorem 1.

For the case $g_0 - h_0 < \frac{1}{2}(k_0 - l_0)$ we have, as before, $\Omega_1 = (d_0 - e_0)m$ and the dominant term is the one corresponding to the partition $(0, m, 0, ..., 0)$. It is because

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{negative}} R' + \frac{m}{2}(k_0 - l_0).$$

Then, the maximum of N^M occurs for $R' = 0$, $R = \frac{m}{2}$. Therefore $t = 1$, $\Omega_2 = 0$, $M = \frac{1}{2}(k_0 - l_0)$ and (71) reduces as

$$\mu_m'^{(N)} \sim F(0,m,0,...,0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{(d_0-e_0)mN}}{q^{(d_0-e_0)m}-1} N^{\frac{1}{2}(k_0-l_0)m},$$

which is the expression (20) of Theorem 1 since $F(0, m, 0, ..., 0) = 1$.

iii. $d_0 - e_0 < \frac{1}{2}(f_0 - s_0)$. Since

$$M = \underbrace{\left[(g_0 - h_0) - \frac{1}{2}(k_0 - l_0)\right]}_{\text{negative}} R' + \frac{m}{2}(k_0 - l_0),$$

then the dominant term in the *(m)-summation* of the expression (71) is the one corresponding to the partition $(0, m, 0, ..., 0)$. Therefore $R' = 0$, $R = \frac{m}{2}$, $t = 1$, $\Omega_2 = 0$, $M = \frac{1}{2}(k_0 - l_0)$ and

$$\mu_m'^{(N)} \sim F(0, m, 0, ..., 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}} \frac{q^{\frac{1}{2}(f_0 - s_0)mN}}{q^{\frac{1}{2}(f_0 - s_0)m} - 1} N^{\frac{1}{2}(f_0 - s_0)m},$$

which coincides with (21) since $F(0, m, 0, ..., 0) = 1$.

This completely proves the Theorem 1. ■

As a conclusion to this section we provide the scheme with all different posibilities obtained in this section.

Scheme: The caracterization of general q-polynomials by its spectral properties.

$$1. \quad \begin{array}{c} d_0 - e_0 = \\ \frac{1}{2}(f_0 - s_0) \end{array} \left\{ \begin{array}{l} (a)\ g_0 - h_0 > \frac{1}{2}(k_0 - l_0) \\ (b)\ g_0 - h_0 = \frac{1}{2}(k_0 - l_0) \\ (c)\ g_0 - h_0 < \frac{1}{2}(k_0 - l_0) \end{array} \right.$$

$$2. \quad \begin{array}{c} d_0 - e_0 \neq 0 \\ f_0 - s_0 \neq 0 \end{array} \left\{ \begin{array}{ll} (a) \begin{array}{c} d_0 - e_0 \leq 0 \\ f_0 - s_0 \leq 0 \end{array} & \left\{ \begin{array}{ll} (i) \left\{ \begin{array}{l} d_0 - e_0 < 0 \\ f_0 - s_0 < 0 \end{array} \right. & \Omega_1 \neq 0 \\ (ii) \left\{ \begin{array}{l} d_0 - e_0 = 0 \\ f_0 - s_0 < 0 \end{array} \right. & g_0 - h_0 = k_0 - l_0 = 0 \\ (iii) \left\{ \begin{array}{l} d_0 - e_0 < 0 \\ f_0 - s_0 = 0 \end{array} \right. & g_0 - h_0 = k_0 - l_0 = 0 \end{array} \right. \\ (b) \begin{array}{c} d_0 - e_0 > 0 \\ and/or \\ f_0 - s_0 > 0 \end{array} & \left\{ \begin{array}{l} (i)\ d_0 - e_0 > \frac{1}{2}(f_0 - s_0) \\ (ii)\ d_0 - e_0 = \frac{1}{2}(f_0 - s_0) \left\{ \begin{array}{l} A)\ g_0 - h_0 > \frac{1}{2}(k_0 - l_0) \\ B)\ g_0 - h_0 = \frac{1}{2}(k_0 - l_0) \\ C)\ g_0 - h_0 < \frac{1}{2}(k_0 - l_0) \end{array} \right. \\ (iii)\ d_0 - e_0 < \frac{1}{2}(f_0 - s_0) \end{array} \right. \end{array} \right.$$

5 Searching for a normalized density of zeros.

In this Section the asymptotic distribution of zeros of the polynomial $P_N(x)_q$ defined by Eqs. (1)-(4) will be discussed. In particular Theorems 2-4 will be proved. The starting point will be Theorem 1.

From Theorem 1, one observes that the moments $\mu_m'^{(N)}$ of the (non-normalized) density of zeros $\rho_N(x)$ depends on N as follows:

$$\begin{array}{ll} N^{am+1} & \text{in case 1,} \\ Constant & \text{in subcase 2(a)i,} \\ N & \text{in subcases 2(a)ii-2(a)iii,} \\ N^{am} q^{bmN} & \text{in case 2b,} \end{array} \tag{72}$$

where the constants a and b are known and distinct for each case. Obviously we would like to have a normalized density of zeros $\rho_N^{norm}(x)$. The usual way to have it is to impose that the *moment of order zero* be equal unity, what permits to write

$$\rho_N^{norm}(x) = \frac{1}{N}\rho_N(x), \tag{73}$$

whose moments $\tilde{\mu'}_m^{(N)}$ will be related to those of $\rho_N(x)$ by

$$\tilde{\mu'}_m^{(N)} = \frac{1}{N}\mu_m'^{(N)}, \quad m \geq 0. \tag{74}$$

Then, from (72) and (74) it is clear that the N-dependence of the moments of the *normalized to unity* density of zeros is given by

$$\begin{array}{ll} N^{am} & \text{in case 1,} \\ N^{-1} & \text{in subcase 2(a)i,} \\ Constant & \text{in subcases 2(a)ii-2(a)iii,} \\ N^{am-1}q^{bmN} & \text{in case 2b,} \end{array} \tag{75}$$

As said before, we are interested in the asymptotic density of zeros. If this is defined by

$$\rho(x) = \lim_{N\to\infty} \rho_N(x), \tag{76}$$

then taking into account that $\mu_m'^{(N)}$ have a N-dependence of the form (72), its moments μ_m' given by

$$\mu_m' = \lim_{N\to\infty} \mu_m'^{(N)}$$

will be infinity in case 1, subcases 2(a)ii and 2(a)iii and in case 2b; and constant given by (14) in subcase 2(a)i. Therefore, the expressions (27), (33), (35) and (37) of theorems 2, 3 and 4, respectively, have been proved.

If one wants to have some information about the asymptotic distribution of zeros in case 1, subcases 2(a)ii and 2(a)iii and in case 2b, one needs to introduce a normalization factor and/or a scaling factor into the density $\rho_N(x)$ in the sense disscused in Eq. (55) and (56). Let us first think of a *scaled* density. For the case 1 there is no scaling factor D which leads to an asymptotic density of zeros whose moments have non-zero, finite values unless the scaling factor be of the form $D = N^{-a-\frac{1}{m}}$ but this is not useful since it would oblige to define a different *scaled* asymptotic density function for each moment. Contrary to this, for the case 2b one can consider scaling factor $D = N^{-a}q^{-bN}$ and define the discrete density of zeros given by

$$\rho_N^{**}(x) = \rho_N\left(\frac{x}{q^{bN}N^a}\right)$$

and the asymptotic density of zeros given by

$$\rho^{**}(x) = \lim_{N\to\infty} \rho_N\left(\frac{x}{q^{bN}N^a}\right) \tag{77}$$

whowe moments μ_m^{**} are acording to (56), as follows

$$\mu_m^{**} = \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{q^{mbN}N^{am}}. \tag{78}$$

From (72) and (78), it is clear that all the quantities μ_m^{**} have finite values. It is only missing to take the parameters a and b for the different subcases of 2b.

For the subcases 2(b)i, 2(b)iiA and 2(b)iiB it turns out that $a = g_0 - h_0$ and $b = d_0 - e_0$. Then, as in expression (77), one can define the asymptotic density function $\rho_1^{**}(x)$ in the form

$$\rho_1^{**}(x) = \lim_{N\to\infty} \rho_N\left(\frac{xq^{-(d_0-e_0)N}}{N^{(g_0-h_0)}}\right), \tag{79}$$

whose moments $\mu_m^{**}(1)$ given by

$$\mu_m^{**}(1) = \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N^{(g_0-h_0)m}q^{(d_0-e_0)mN}}, \tag{80}$$

have, according to (17) and (18), the values (for $m \geq 1$)

$$\mu_m^{**}(1) = \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m \frac{q^{m(d_0-e_0)}}{q^{m(d_0-e_0)}-1} \tag{81}$$

in the subcases 2(b)i and 2(b)iiA, and, according to (19), the values

$$\mu_m^{**}(1) = \sum_{(m)} F(r_1', r_1, ..., r_{j+1}') \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \frac{q^{\Omega_2+m(1-t)(d_0-e_0)}}{q^{m(d_0-e_0)}-1} \tag{82}$$

in the subcase 2(b)iiB. Remark that the expressions (81) and (82) are identical to (47) and (49) of Theorem 4, respectively. Similary, for the subcases 2(b)iiC it turns out that $a = \frac{1}{2}(k_0 - l_0)$ and $b = d_0 - e_0$. Then, as in expression (77), one defines the asymptotic density function $\rho_2^{**}(x)$ by (40), whose moments $\mu_m^{**}(2)$ given by (43) have, according to (20), the values given by (51). Finally, for the subcase 2(b)iii one has the density $\rho_3^{**}(x)$ defined by (40), whose moments $\mu_m^{**}(3)$, given by (43), have according to (21), the values given by (53). For the entire case 2b it happens that, according to (78) and since $\mu_0'^{(N)} = N$,

$$\mu_0^{**} = \mu_0^{**}(1) = \mu_0^{**}(2) = \mu_0^{**}(3) = \infty,$$

as in Theorem 4 is also pointed out.

Let us now searshed for a *normalized to unity* asymptotic density of zeros. The simplest way is to define it as

$$\rho_1(x) = \lim_{N\to\infty} \rho_N^{norm}(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N(x), \tag{83}$$

where the Eq. (73) has been used. Its moments given by

$$\mu_0'(1) = 1, \quad \mu_m'(1) = \lim_{N\to\infty} \frac{1}{N}\mu_m'^{(N)}, \quad m \geq 1, \tag{84}$$

have, taking into account (75), the following values

$$\mu'_0(1) = 1$$

$$\mu'_m(1) = \begin{cases} \infty \quad m \geq 1 & \text{in cases 1 and 2b} \\ 0 \quad m \geq 1 & \text{subcase 2(a)i} \\ \sum_{(m)} F(r'_1, 0, ..., 0, r'_{j+1}) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \quad m \geq 1 & \text{subcase 2(a)ii} \\ \sum_{(m)} F(0, r_1, 0, ..., r_j, 0) \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R} \quad m \geq 1 & \text{subcase 2(a)iii} \end{cases} \tag{85}$$

Then expressions (34)-(38) of Theorem 3 has been demonstrated. So, Theorem 3 has entirely proved.

For the case 1 and subcase 2b one would like to have information more useful than that expressed by (85), keeping the *normalization to unity* of the density $\rho_1(x)$ given by (83). Therefore one has to *compress* the spectrum of zeros by introducing a scaling factor. In the case 1 it is very easy to find that factor by looking at the expression (75): it is $D = N^{-a}$. Then one defines from (75) and (83) the density function

$$\rho^*(x) = \lim_{N\to\infty} \rho_N^{norm}\left(\frac{x}{N^a}\right) = \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^a}\right) \tag{86}$$

whose moments are according to (56) and (84) as

$$\mu_0^* = 1, \quad \mu_m^* = \lim_{N\to\infty} \frac{\mu_m'^{(N)}}{N^{am+1}}, \quad m \geq 1. \tag{87}$$

From (72) and (87) it is obvious that the quantities μ_m^* have finite values. One has only to take the values of a in the different subcases of the case 1. For the subcase 11, $a = g_0 - h_0$; then here it is convenient to define, according to (86), the following asymptotic density of zeros

$$\rho_1^*(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^{g_0-h_0}}\right),$$

whose moments are, according to (87) and (11), as follows

$$\mu_0^*(1) = 1; \quad \mu_m^*(1) = \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^m, \quad m \geq 1,$$

which is the expression (28) of Theorem 2.

For the subcases 12 and 13, it turns out that $a = \frac{1}{2}(k_0 - l_0)$, which defines the following asymptotic density of zeros

$$\rho_2^*(x) = \lim_{N\to\infty} \frac{1}{N}\rho_N\left(\frac{x}{N^{\frac{1}{2}(k_0-l_0)}}\right),$$

whose moments have, according to (87) and (12), the values ($\mu_0^*(2) = 1$)

$$\mu_m^*(2) = \sum_{(m)} F(r'_1, r_1, ..., r'_{j+1}) \left[\frac{\alpha_0^{(0)}}{\beta_0^{(0)}}\right]^{R'} \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{R}, \quad m \geq 1$$

for the subcase 12, and, according to (87) and (13), the values

$$\mu_0^*(2) = 1; \quad \mu_m^*(2) = \left[\frac{\theta_0^{(0)}}{\gamma_0^{(0)}}\right]^{\frac{m}{2}}, \quad m \geq 1,$$

for the subcase 13. Remarks that the last two expressions coincide with the expressions (29) and (30) of Theorem 2, respectively. Then this Theorem has been entirely proved.

For the subcase 2b the scaled *normalization to unity* asymptotic density function of the form (86) would also have all its moments of order other than zero equal to infinite. No other scaling factor would be able to make finite these moments unless $D = N^{-a+\frac{1}{m}} q^{mN}$, but this factor is of usefulness for reasons already discussed. Therefore one is obliged to change the normalization factor in this subcase. Here the discrete density of zeros $\rho_N^+(x)$ is normalized so that its moments are defined by

$$\mu_m^{+(N)} = \frac{q^m - 1}{q^{mN} - 1} \mu_m'^{(N)}, \quad m \geq 0,$$

i.e., that

$$\rho_N^+(x) = \frac{(m)_q}{(mN)_q} \rho_N(x) \tag{88}$$

when $(m)_q$ and $(mN)_q$ are q-numbers defined by Eq.(45). This normalization factor has the following relevant property: It tends to N^{-1} if $m \to 0$ and $q \to 1$. In particular, this implies that

$$\mu_0^{+(N)} = 1.$$

Furthermore, for the case 2b under consideration it turns out that the $N-$dependence of $\mu_m^{+(N)}$ is as $N^{am} q^{(b-1)mN}$. this dependence suggest to analize the asymptotic spectrum of zeros by means of the asymptotic density function defined by

$$\rho^{++}(x) = \lim_{N \to \infty} \rho_N \left(\frac{x}{N^a q^{(b-1)N}} \right), \tag{89}$$

whose moments μ_m^{++} are given by

$$\mu_m^{++} = \lim_{N \to \infty} \frac{\mu_m^{+(N)}}{N^{am} q^{(b-1)mN}} = \lim_{N \to \infty} \frac{(q^m - 1)\mu_m'^{(N)}}{(q^{mN} - 1) N^{am} q^{(b-1)mN}}. \tag{90}$$

Taking into account this expression together with the values (17)-(21) of $\mu_m'^{(N)}$ given in the Theorem 1, one observes that for the subcases 2(b)i, 2(b)iiA and 2(b)iiB the parameter a and b take the values

$$a = g_0 - h_0, \quad b = d_0 - e_0$$

and the appropiate asymptotic density of zeros is, according to (88)-(89), the function $\rho_1^{++}(x)$ given by (41) in Theorem 4.

For the subcase 2(b)iiC it turns out that

$$a = \frac{1}{2}(k_0 - l_0), \quad b = d_0 - e_0.$$

Then, the appropiate asymptotic density of zeros for this subcase is, according to (88)-(89), the function $\rho_2^{++}(x)$ given by (41) in Theorem 4.

Finally for the case 2(b)iii $a=\frac{1}{2}(k_0-l_0)$, $b=\frac{1}{2}(f_0-s_0)$ and the appropiate asymptotic density of zeros is, according to (88)-(89), the function $\rho_3^{++}(x)$ given by (41) in Theorem 4.

Now the Eq. (90) and the values (17)-(21) for $\mu_m'^{(N)}$ gives in a straightforward manner the moments $\mu_m^{++}(1)$, $\mu_m^{++}(2)$ and $\mu_m^{++}(3)$ ot the asymptotic density functions $\rho_1^{++}(x)$, $\rho_2^{++}(x)$ and $\rho_3^{++}(x)$. Indeed, the values of these quantities are given by the Eqs. (48) for the subcases 2(b)i, 2(b)iiA, (50) for the subcase 2(b)iiB, (52) for the subcase 2(b)iiC and (54) for the subcase 2(b)iii, respectively. This entirely proves Theorems 2-4. ■

6 Applications.

In this Section we will use the theorems obtained in the two previous sections to investigate the spectral properties of several known families of orthogonal q-polynomials. Let us make the observation that for a finite polynomial sequence (e.g. Hahn, Racah and Kravchuk polynomials), i.e., when the degree n of the polynomial is bounded by a fixed parameter N (not to be confused with the same letter previously used as generic degree of polynomials), it is assumed that N is sufficiently large and $1 << n \leq N$ so that Eq. (61) be fulfilled.

The q-Hahn polynomials $h_n^{\alpha,\beta}(q^{-x},N)$.

The q-Hahn polynomials $h_n^{\alpha,\beta}(q^{-x},N)$ play a fundamental role in the Representation Theory of the q-Algebras $SU_q(2)$ and $SU_q(1,1)$ (see [20], [22], [21]). They also appear in numerous physical applications since e.g. the Clebsh-Gordan Coefficients of the q-Algebras $SU_q(2)$ and $SU_q(1,1)$ are proportional to them. The theory and applications of the q-Hahn and classical Hahn polynomials have some close parallels. So, e.g. q-Hahn and classical Hahn polynomials appears in the analysis of functions on the lattice of subspaces of a finite vector space and the lattice of subsets of a finite set, respectively. These polynomials verify the recurrence relation [3] (page 59)

$$h_n^{\alpha,\beta}(q^{-x},N)=[q^{-x}-(1-A_{n-1}-C_{n-1})]h_{n-1}^{\alpha,\beta}(q^{-x},N)+B_{n-1}h_{n-2}^{\alpha,\beta}(q^{-x},N), \tag{91}$$

where $B_n=A_{n-1}C_n$, and the A and C parameters are

$$A_n=\frac{(1-\alpha\,q^{1+n})\,(1-\alpha\,\beta\,q^{1+n})\,\left(1-q^{-N+n}\right)}{(1-\alpha\,\beta\,q^{1+2\,n})\,(1-\alpha\,\beta\,q^{2+2\,n})},$$

$$C_n=-\frac{\alpha\,q^n\,(1-q^n)\,(1-\beta\,q^n)\,\left(q^{-N}-\alpha\,\beta\,q^{1+n}\right)}{(1-\alpha\,\beta\,q^{2\,n})\,(1-\alpha\,\beta\,q^{1+2\,n})}.$$

Let us also point out that

$$\kappa_{n+1}A_n=\kappa_n \tag{92}$$

where κ_n is the leading coefficient of the polynomial. The comparison of Eqs. (91) and (1) gives that

$$a_n^{num}=\alpha_0^{(0)}q^{3n}=\alpha^2\beta(1+\beta)q^{N+1}q^{3n},\quad a_n^{den}=\beta_0^{(0)}q^{4n}=\alpha^2\beta^2q^Nq^{4n}.$$

and

$$(b_n^{num})^2=\theta_0^{(0)}q^{7n}=\alpha^4\beta^3q^{-N}q^{7n},\quad (b_n^{den})^2=\gamma_0^{(0)}q^{8n}=\alpha^4\beta^4q^{8n}.$$

Then, $g_m=h_m=k_m=l_m=0$ for all $m=0,1,..N$ and

$$d_0=3,\ e_0=4,\ f_0=7,\ s_0=8.$$

This is the case $d_0-e_0<0$ and $f_0-s_0<0$, i.e., case 2(a)i. Therefore, Eqs. (33) and (34) of Theorem 3 give us the moments

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, ..., r_j, r'_{j+1}) q^{-\sum_{k=1}^{j} k r'_{k+1} - 2\sum_{k=1}^{j-1} k} \left[\frac{q(1+\beta)}{\beta}\right]^{R'} \left[\frac{\alpha}{q^N(q+q^{-1})}\right]^{R} \frac{1}{q^{\frac{m}{2}-1}} \quad (93)$$

for the asymptotic density of zeros $\rho(x)$ defined by Eq. (31), and

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geq 1 \end{cases} \quad (94)$$

for the corresponding asymptotic quantity $\rho_1(x)$ given by Eq. (31).

q-Kravchuk polynomials $k_n^p(q^{-x}, N)$.

The matrix elements of the representations T^l of the $U_q(sl_2)$ quantum algebra are proportional to the q-Kravchuk polynomials (see [21], Vol. III, page 64). According to [3] (page 76) and taking into account Eq. (92) the three term recurrence relation of these polynomials can be expressed as

$$k_n^p(q^{-x}, N) = [q^{-x} - (1 - A_{n-1} - C_{n-1})] k_{n-1}^p(q^{-x}, N) + B_{n-1} k_{n-2}^p(q^{-x}, N), \quad (95)$$

where $B_{n-1} = A_{n-1} C_{n-1}$, and

$$A_n = \frac{(1 + p\, q^n)\left(1 - q^{-K+n}\right)}{(1 + p\, q^{2\, n})\,(1 + p\, q^{1+2\, n})},$$

$$C_n = -\frac{p\, q^{-1-K+2\, n}\,(1 - q^n)\left(1 + p\, q^{K+n}\right)}{(1 + p\, q^{2\, n})\,(1 + p\, q^{-1+2\, n})}.$$

The comparison with (1) gives that

$$a_n^{num} = \alpha_0^{(0)} q^{3n} = pq(pq^N - 1)q^{3n}, \quad a_n^{den} = \beta_0^{(0)} q^{4n} = p^2 q^N q^{4n}.$$

and

$$(b_n^{num})^2 = \theta_0^{(0)} q^{6n} = p^3 q^{-2} q^{6n}, \quad (b_n^{den})^2 = \gamma_0^{(0)} q^{8n} = q^{-3} p^4 q^N q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, ..N$ and

$$d_0 = 3,\ e_0 = 4,\ f_0 = 6,\ s_0 = 8.$$

This is the case $d_0 - e_0 = -1 < 0$ and $f_0 - s_0 = -2 < 0$, i.e., case 2(a)i. Therefore, Eq. (34) of Theorem 3 gives us the values

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geq 1 \end{cases} \quad (96)$$

for the moments of the asymptotic density of zeros $\rho_1(x)$. Furthermore, since $\Omega_1 = \frac{1}{2}(f_0 - s_0) = -m$, $\Omega_2 = -(\sum_{k=1}^{j} k r'_{k+1} - 4\sum_{k=1}^{j-1} k r_{k+1})$ and $M = 0$, Eq. (33) of Theorem 3 gives us

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, ..., r'_{j+1}) q^{-\Omega_2} \left[\frac{q(pq^N - 1)}{pq^N}\right]^{R'} \left[\frac{q^{1-N}}{p}\right]^{R} \frac{1}{q^m - 1}, \quad m \geq 0, \quad (97)$$

for the moments of the spectral quantity $\rho(x)$ defined by Eq. (31).

q-Racah polynomials $R_n(\mu(x),\alpha,\beta,\gamma,\delta)$. $\mu(x)=q^{-x}+\gamma\delta q^{x+1}$.

It is well known the important role that the 6j symbols play in the quantum angular momentum theory (see [6]). It is known that the q-analog of the Racah coefficients (6j symbols) for the q-algebra $U_q(sl_2)$ are proportional to the q-Racah polynomials (see [21], Vol. III, page 70). From the three term recurrence relation of these polynomials [3] (page 53), as well as Eq. (92), we can rewrite [3] Eq.(3.15.3) in the form

$$\mu(x)R_{n-1}(\mu(x),\alpha,\beta,\gamma,\delta)=R_n(\mu(x),\alpha,\beta,\gamma,\delta)+ \\ +[1+\gamma\delta q-(1-A_{n-1}-C_{n-1})]R_{n-1}(\mu(x),\alpha,\beta,\gamma,\delta)+B_{n-1}R_{n-2}(\mu(x),\alpha,\beta,\gamma,\delta), \tag{98}$$

where $B_{n-1}=A_{n-1}C_{n-1}$, and

$$A_n=\frac{(1-\alpha\, q^{1+n})\ (1-\alpha\,\beta\, q^{1+n})\ (1-\beta\,\delta\, q^{1+n})\ (1-\gamma\, q^{1+n})}{(1-\alpha\,\beta\, q^{1+2\,n})\ (1-\alpha\,\beta\, q^{2+2\,n})},$$

$$C_n=\frac{q\ (1-q^n)\ (\delta-\alpha\, q^n)\ (1-\beta\, q^n)\ (\gamma-\alpha\,\beta\, q^n)}{(1-\alpha\,\beta\, q^{2\,n})\ (1-\alpha\,\beta\, q^{1+2\,n})}.$$

The comparison with (1) gives that

$$a_n^{num}= -\alpha_0^{(0)}q^{3n}=q\alpha\beta(\alpha+\gamma+\alpha\beta+\beta\delta+\alpha\beta\delta+\alpha\gamma+\delta\gamma+\beta\gamma\delta)q^{3n},$$

$$a_n^{den}= -\beta_0^{(0)}q^{4n}=\alpha^2\beta^2q^{4n}.$$

and

$$(b_n^{num})^2=\theta_0^{(0)}q^{8n}=q\alpha^4\beta^4\delta\gamma q^{8n},\quad (b_n^{den})^2=\gamma_0^{(0)}q^{8n}=\alpha^4\beta^4q^{8n}.$$

Then, $g_m=h_m=k_m=l_m=0$ for all $m=0,1,..N$ and

$$d_0=3,\ e_0=4,\ f_0=8,\ s_0=8.$$

This is the case $d_0-e_0=-1<0$ and $f_0-s_0=0$, i.e., case 2(a)iii. Therefore, Eq (16) of Theorem 3 yield the moments

$$\mu'_m(1)=\begin{cases} 1 & m=0 \\ \sum_{(m)} F(0,r_1,0,...,r_j,0)\,[q\delta\gamma]^R & m\geq 1 \end{cases} \tag{99}$$

for the asymptotic densities of zeros $\rho_1(x)$ defined by Eq. (31).

q-Askey & Wilson polynomials $p_n(x,a,b,c,d)$.

According to [3] (page 51) and Eq. (92), the three term recurrence relation for the q-Askey & Wilson polynomials can be rewritten as

$$xp_{n-1}(x,a,b,c,d)=p_n(x,a,b,c,d)+\tfrac{1}{2}[a+a^{-1}-(A_{n-1}+C_{n-1})]p_{n-1}(x,a,b,c,d)+ \\ +B_{n-1}p_{n-2}(x,a,b,c,d), \tag{100}$$

where $B_{n-1}=A_{n-1}C_{n-1}$, and

$$A_n=\frac{(1-a\,b\,c\,d\,q^{-1+n})\ (1-a\,b\,q^n)\ (1-a\,c\,q^n)\ (1-a\,d\,q^n)}{a\ (1-a\,b\,c\,d\,q^{2\,n})\ (1-a\,b\,c\,d\,q^{-1+2\,n})},$$

$$C_n = \frac{a\,(1 - b\,c\,q^{-1+n})\,(1 - b\,d\,q^{-1+n})\,(1 - c\,d\,q^{-1+n})\,(1 - q^n)}{(1 - a\,b\,c\,d\,q^{-2+2\,n})\,(1 - a\,b\,c\,d\,q^{-1+2\,n})}.$$

The comparison with (1) gives

$$a_n^{num} = -\alpha_0^{(0)} q^{3n} = qabcd(abc + abd + acd + bcd + q(a + b + c + d))q^{3n},$$

$$a_n^{den} = -\beta_0^{(0)} q^{4n} = 2a^2b^2c^2d^2q^{4n}.$$

and

$$(b_n^{num})^2 = \theta_0^{(0)} q^{8n} = a^4b^4c^4d^4q^{8n}, \quad (b_n^{den})^2 = \gamma_0^{(0)} q^{nn} = a^4b^4c^4d^4q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, ..N$ and

$$d_0 = 3,\ e_0 = 4,\ f_0 = 8,\ s_0 = 8.$$

This is the case $d_0 - e_0 = -1 < 0$ and $f_0 - s_0 = 0$, i.e., case 2(a)iii. Therefore, Eq. (38) of Theorem 3 gives us the moments

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ \sum_{(m)} F(0, r_1, 0, ..., r_j, 0) & m \geq 1. \end{cases} \tag{101}$$

for the asymptotic density of zeros $\rho_1(x)$ defined by Eq. (31).

Al Salam and Carlitz polynomials $u_n^\mu(x)$ and $v_n^\mu(x)$.

In dealing with the q-harmonic oscillator, Askey and Suslov [16] have introduced the q-polynomials

$$u_n^\mu(x) = \mu^{-n} q^{-\frac{n(n-1)}{2}} U_n^{(-\mu)}(x).$$

where $U_n^{-\mu}(x)$ are the so called Al Salam and Carlitz polynomials. These polynomials satisfy the recurrence relation [16]

$$xu_{n-1}^\mu(x) = u_n^\mu(x) + (1 - \mu)q^{n-1}u_{n-1}^\mu(x) + \mu q^{n-2}(1 - q^{n-1})u_{n-2}^\mu(x), \tag{102}$$

which is of the type (1) with the coefficients

$$a_n^{num} = -\alpha_0^{(0)} q^n = (1 - \mu)q^{-1}q^n, \quad a_n^{den} = 1,$$

and

$$(b_n^{num})^2 = \theta_0^{(0)} q^{2n} = \mu q^{-1} q^{2n}, \quad (b_n^{den})^2 = \gamma_0^{(0)} q^{s_0 n} = 1.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, ..N$ and

$$d_0 = 1,\ e_0 = 0,\ f_0 = 2,\ s_0 = 0.$$

This is the case $d_0 - e_0 = 1$ and $f_0 - s_0 = 2$, i.e., case 2(b)iiB. Therefore, Eqs. (49) and (50) of Theorem 4 give us the moments

$$\mu_m^{**}(1) = \begin{cases} \infty & m = 0 \\ \sum_{(m)} F(r'_1, r_1, ..., r'_{j+1})\,[1 - \mu]^{R'}\,\mu^R \dfrac{q^{\Omega_2}}{q^m - 1} & m \geq 1 \end{cases} \tag{103}$$

and

$$\mu_m^{++}(1) = \begin{cases} 1 & m = 0 \\ \sum_{(m)} F(r'_1, r_1, ..., r'_{j+1}) \, [1-\mu]^{R'} \, \mu^R q^{\Omega_2} & m \geq 1 \end{cases} \tag{104}$$

(where $\Omega_2 = \sum_{k=1}^{j} k r'_{k+1} + 4 \sum_{k=1}^{j-1} k r_{k+1} - mt$) corresponding to the asymptotic quantities $\rho_1^{**}(x)$ and $\rho_1^{++}(x)$, respectively.

It has been encountered [15] that another class of Al Salam and Carlitz polinomials, to be denoted by $v_n^{\mu}(x)$, is related also to the q-oscillator. So, it seems natural to search for its distribution of zeros. These polynomials satisfy the relation [15]

$$x v_{n-1}^{\mu}(x) = v_n^{\mu}(x) + (q+\mu) q^{-n-2} v_{n-1}^{\mu}(x) + \mu q^{-n-3} (q^{-n-1} - 1) v_{n-2}^{\mu}(x). \tag{105}$$

Therefore, $d_0 = -1, \ e_0 = 0, \ f_0 = -1, \ s_0 = 0$. This corresponds to the case 2(a)i. Then, Eq. (34) of Theorem 3 gives us the moments

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geq 1. \end{cases} \tag{106}$$

for the asymptotic density $\rho(x)$. Furthermore, since $\Omega_1 = -\frac{1}{2}(R' + m)$, $\Omega_2 = -(\sum_{k=1}^{j} k r'_{k+1} - 2\sum_{k=1}^{j-1} k r_{k+1})$ and $M = 0$, Eq. (33) of Theorem 3 gives

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, ..., r'_{j+1}) q^{-\Omega_2} \left[q^{-1}(q+\mu) \right]^{R'} \mu^R \frac{q^{-m}}{q^{\frac{1}{2}(R'+m)} - 1} \tag{107}$$

for the moments of the normalized -to-$\frac{1}{N}$ spectral quantity $\rho_1(x)$ defined by Eq. (31).

The little q-Jacobi polynomials $p_n(x, a, b)$.

The little q-Jacobi polynomials $p_n(x, a, b)$ play a fundamental role (see e.g. [20]) in the Representation Theory of the q-Algebra $U_q(sl_2)$ because they are the matrix elements of the representations T^l (see [21], Vol. III, page 51). They satisfy the three term recurrence relation [3] (page 59)

$$p_n(x, a, b) = [x + A_{n-1} + C_{n-1}] p_{n-1}(x, a, b) + B_{n-1} p_{n-2}(x, a, b), \tag{108}$$

where A and C parameters are given by

$$A_n = \frac{q^n \, (1 - a\, q^{1+n}) \, (1 - a\, b\, q^{1+n})}{(1 - a\, b\, q^{1+2\,n}) \, (1 - a\, b\, q^{2+2\,n})},$$

$$C_n = \frac{a\, q^n \, (1 - q^n) \, (1 - b\, q^n)}{(1 - a\, b\, q^{2\,n}) \, (1 - a\, b\, q^{1+2\,n})},$$

and $B_n = A_{n-1} C_n$. This relation is of the type (1) with the coefficients

$$a_n^{num} = \alpha_0^{(0)} q^{3n} = -ab(1+a) q^{3n}, \quad a_n^{den} = \beta_0^{(0)} q^{4n} = a^4 b^4 q^{4n}.$$

and

$$(b_n^{num})^2 = \theta_0^{(0)} q^{6n} = a^3 b^2 q^{6n}, \quad (b_n^{den})^2 = \gamma_0^{(0)} q^{8n} = q a^4 b^4 q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, ..N$ and

$$d_0 = 3,\ e_0 = 4,\ f_0 = 6,\ s_0 = 8.$$

This is the case $d_0 - e_0 = -1 < 0$ and $f_0 - s_0 = -2 < 0$, i.e., case 2(a)i. Therefore, Eq. (34) of Theorem 3 gives us the moments

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geq 1 \end{cases} \tag{109}$$

for the asymptotic density of zeros $\rho(x)$. Furthermore, since $\Omega_1 = \frac{1}{2}(f_0 - s_0) = -m$, $\Omega_2 = -(\sum_{k=1}^{j} kr'_{k+1} - 4\sum_{k=1}^{j-1} kr_{k+1})$ and $M = 0$, Eq. (33) of Theorem 3 gives us

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, ..., r'_{j+1}) q^{-\Omega_2} \left[\frac{1+a}{a}\right]^{R'} \left[\frac{-1}{aq}\right]^{R} \frac{1}{(q^m - 1)b^m}. \tag{110}$$

which are the moments of asymptotic spectral quantity $\rho(x)$.

The big q-Jacobi polynomials $P_n(x, a, b, c)$.

The big q-Jacobi polynomials $P_n(x, a, b, c)$ are defined in [3] (page 57). By use of the parameters

$$A_n = \frac{(1 - a\,q^{1+n})\ (1 - a\,b\,q^{1+n})\ (1 - c\,q^{1+n})}{(1 - a\,b\,q^{1+2\,n})\ (1 - a\,b\,q^{2+2\,n})},$$
$$C_n = -\frac{a\,c\,q^{1+n}\ (1 - q^n)\ (1 - b\,q^n)\ \left(1 - \frac{a\,b\,q^n}{c}\right)}{(1 - a\,b\,q^{2\,n})\ (1 - a\,b\,q^{1+2\,n})},$$

we can rewrite its three term recurrence relation [3] (page 59) in the form

$$P_n(x, a, b, c) = [x + 1 - A_{n-1} - C_{n-1}]P_{n-1}(x, a, b, c) + B_{n-1}P_{n-2}(x, a, b, c), \tag{111}$$

where $B_n = A_{n-1}C_n$. The comparison with (1) gives

$$a_n^{num} = \alpha_0^{(0)} q^{3n} = -qab(b+1)(a+c)q^{3n}, \quad a_n^{den} = \beta_0^{(0)} q^{4n} = -a^2b^2q^{4n}$$

and

$$(b_n^{num})^2 = \theta_0^{(0)} q^{7n} = a^4b^3cqq^{7n}, \quad (b_n^{den})^2 = \gamma_0^{(0)} q^{6n} = a^4b^4q^{-1}q^{8n}.$$

Then, $g_m = h_m = k_m = l_m = 0$ for all $m = 0, 1, ..N$ and

$$d_0 = 3,\ e_0 = 4,\ f_0 = 7,\ s_0 = 8.$$

This is the case $d_0 - e_0 < 0$ and $f_0 - s_0 < 0$, i.e., case 2(a)i. Therefore, Eqs. (33) and (34) of Theorem 3 give us the moments

$$\mu'_m = \sum_{(m)} F(r'_1, r_1, ..., r_j, r'_{j+1}) q^{-\sum_{k=1}^{j} kr'_{k+1} - 2\sum_{k=1}^{j-1} k} \times \\ \times \left[\frac{(b+1)(a+c)}{ab}\right]^{R'} \left[\frac{c}{b}\right]^{R} \frac{1}{q^{-R} - q^{-m}}, \tag{112}$$

for the asymptotic density of zeros $\rho(x)$ defined by Eq. (31), and

$$\mu'_m(1) = \begin{cases} 1 & m = 0 \\ 0 & m \geq 1 \end{cases} \tag{113}$$

for the corresponding asymptotic quantity $\rho_1(x)$ given by Eq. (31).

The q-Dual Hahn polynomials in the lattice $x(s) = [s]_q[s+1]_q$.

In this section we provide the asymptotic behavior of the moments of zeros of the q-Dual Hahn polynomials $W_n^{(c)}(x(s), a, b)_q$. This polynomials are connected with the Clebsh-Gordan of the q-Algebras $SU_q(2)$ and $SU_q(1,1)$ [24]. Using the above formulas we find the following asymptotic values of the moments $\mu_m'^{(N)}$ $(m \geq 1)$

$$\mu_m'^{(N)} \sim \sum_{(m)} m \prod_{i=1}^{j+1} \frac{(r_{i-1}+r_i'+r_i-1)!}{(r_{i-1}-1)!\,r_i!\,r_i'!} \frac{q^{5m-3R}}{[q^{-2c}-1]^{m-2R}} \frac{q^{2(c+a-b)m}}{(q-q^{-1})^{2m}} \frac{q^{8m(N-t)}}{q^{4m}-1},$$

Using the normalized density of zeros

$$\rho_1^{++}(x) = \lim_{N\to\infty} \frac{q^m-1}{q^{mN}-1} \rho_N\left(xq^{-7N}\right),$$

then, the corresponding moments are given by the expression for $m \geq 1$

$$\mu_0^{++}(1) = 1$$

$$\mu_m^{++}(1) = \sum_{(m)} m \prod_{i=1}^{j+1} \frac{(r_{i-1}+r_i'+r_i-1)!}{(r_{i-1}-1)!\,r_i!\,r_i'!} \frac{q^{5m-3R}}{[q^{-2c}-1]^{m-2R}} \frac{q^{2(c+a-b)m}}{(q-q^{-1})^{2m}} \frac{q^m-1}{q^{4m}-1}.$$

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