On a q-extension of the linear harmonic oscillator with the continuous orthogonality property on $\mathbb{R}$

R. Álvarez-Nodarse*, M. K. Atakishiyeva†, and N. M. Atakishiyev‡

* Departamento de Análisis Matemático, Universidad de Sevilla,
Apdo. 1160, E-41080 Sevilla, Spain and
Instituto Carlos I de Física Teórica y Computacional,
Universidad de Granada, E-18071 Granada, Spain
E-mail: ran@us.es

†Facultad de Ciencias, UAEM, Apartado Postal 396-3,
CP 62250, Cuernavaca, Morelos, México
E-mail: mesuma@servm.fc.uaem.mx

‡Instituto de Matemáticas, UNAM, Apartado Postal 273-3,
C.P. 62210 Cuernavaca, Morelos, México
E-mail: natig@matcuer.unam.mx

Abstract

We discuss a q-analogue of the linear harmonic oscillator in quantum mechanics, based on a q-extension of the classical Hermite polynomials $H_n(x)$, recently introduced by us in [1]. The wave functions in this q-model of the quantum harmonic oscillator possess the continuous orthogonality property on the whole real line $\mathbb{R}$ with respect to a positive weight function. A detailed description of the corresponding q-system is carried out.

1 Introduction

In [1] we introduced a q-extension of the classical Hermite polynomials $H_n(x)$, which satisfy the following requirements: They are polynomials in the variable x, which obey a three-term recurrence relation; They are orthogonal on the whole real line $\mathbb{R}$ with respect to a continuous positive weight function; In the limit as $q \to 1$ they coincide with the Hermite polynomials $H_n(x)$. Such a family enables one to build a q-deformed version of the linear harmonic oscillator in quantum mechanics, which is still defined on the whole real line $\mathbb{R}$ and enjoys the continuous orthogonality property on $\mathbb{R}$ with respect to a positive weight function. Let us point out here that there are several publications (see [2]–[10] and references therein) devoted to the study of explicit realizations, which represent q-extensions of the Hermite functions (or the wave functions of the linear harmonic oscillator) $H_n(x)\, e^{-x^2/2}$. But none of these realizations satisfies all of the aforementioned requirements: the continuous weight functions in [2, 4, 7] are supported on the finite intervals; the continuous weight functions in [3, 8] are not positive; the q-extensions in [2], [4]–[9] are not expressed in terms of polynomials in the independent variable; and, finally, the orthogonality relations in [5]–[7], [10] are discrete.

Our main goal in this paper has been to employ this q-extension of the Hermite polynomials, $\mathcal{H}_n(x;q)$, in order to built a q-analogue to the linear harmonic oscillator in quantum mechanics. Section 2 collects those known results from [1] about the polynomials $\mathcal{H}_n(x;q)$, which are needed in section 3 to derive an explicit form of the wave functions $\psi_n(x;q)$ in this q-model and their properties. Section 4 is devoted to explicit construction of the generators of the dynamical symmetry algebra $su_q(1,1)$ in terms of the lowering and raising q-difference operators $a(x;q)$ and $a^{\dagger}(x;q)$. Concluding section 5 contains a brief discussion of q-coherent states for this q-extension of the quantum harmonic oscillator.

2 Definition and properties of the polynomials $\mathcal{H}_n(x;q)$

In [1] the following family was introduced

$$\mathcal{H}_{2n}(x;q) := (-1)^n\,(q;q)_n\,L_n^{(-1/2)}(x^2;q)$$
$$= (-1)^n\,(q^{1/2};q)_n\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{1/2}\end{matrix}\,\middle|\, q;-q^{n+1/2}x^2\right) = (-1)^n\,{}_2\phi_1\left(\begin{matrix} q^{-n},-x^2 \\ 0\end{matrix}\,\middle|\, q;q^{n+1/2}\right),$$
$$\mathcal{H}_{2n+1}(x;q) := (-1)^n\,(q;q)_n\,x\,L_n^{(1/2)}(x^2;q)$$
$$= (-1)^n\,(q^{3/2};q)_n\,x\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{3/2}\end{matrix}\,\middle|\, q;-q^{n+3/2}x^2\right) = (-1)^n\,x\,{}_2\phi_1\left(\begin{matrix} q^{-n},-x^2 \\ 0\end{matrix}\,\middle|\, q;q^{n+3/2}\right), \tag{2.1}$$

where $L_n^{(\alpha)}(x;q)$ are q-Laguerre polynomials, ${}_1\phi_1$ and ${}_2\phi_1$ denote the basic hypergeometric polynomials and $(a;q)_n$ is the q-shifted factorial (we employ standard notations of q-analysis, see, for example, [11] or [12]). In (2.1) and throughout the sequel it is assumed that q is a fixed number such that $0 < q < 1$.

This family is generated by the three-term recurrence relation

$$x\,\mathcal{H}_n(x;q) = q^{-n/2}\,\mathcal{H}_{n+1}(x;q) - (1-q^{-n/2})\,\mathcal{H}_{n-1}(x;q)\,, \qquad n=0,1,2,\ldots, \tag{2.2}$$

with the initial condition $\mathcal{H}_0(x;q) \equiv 1$.

The polynomials (2.1) satisfy the continuous orthogonality relation

$$\int_{-\infty}^{\infty} \mathcal{H}_m(x;q)\,\mathcal{H}_n(x;q)\frac{dx}{E_q(x^2)} = \pi q^{-n/2}\,(q^{1/2};q^{1/2})_n\,(q^{1/2};q)_{1/2}\,\delta_{mn} \tag{2.3}$$

on the whole real line $\mathbb{R}$ with respect to the positive weight function $w(x) = 1/E_q(x^2) = 1/(-x^2;q)_\infty$ [1].

The polynomials $\mathcal{H}_n(x;q)$ constitute a q-extension of the classical Hermite polynomials $H_n(x)$ since these polynomials reduce to the latter in the limit as $q \to 1$, i.e.,

$$\lim_{q\to 1}\,(1-q)^{-n/2}\,\mathcal{H}_n(\sqrt{1-q}\,x;q) = 2^{-n}H_n(x)\,, \tag{2.4}$$

From the recurrence relation (2.2) it follows that the $\mathcal{H}_n(x;q)$ can be expressed in terms of the discrete q-Hermite polynomials $\tilde{h}(x;q)$ of type II as

$$\mathcal{H}_n(x;q^2) = q^{n(n-1)/2}\;\tilde{h}_n(x;q) := i^{-\,n}\,{}_2\phi_0\left(\begin{matrix} q^{-\,n},\,ix \\ -\end{matrix}\,\middle|\, q;-q^{\,n}\right)\,. \tag{2.5}$$

So from the known q-difference equation for the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ (see [13], (3.29.5), p.119) one deduces that

$$(1-q^{n/2})\,x^2\,\mathcal{H}_n(x;q) = (1+q^{1/2}+x^2)\,\mathcal{H}_n(x;q)$$
$$-\,(1+x^2)\,\mathcal{H}_n(q^{1/2}x;q) - \,q^{1/2}\,\mathcal{H}_n(q^{-1/2}x;q)\,. \tag{2.6}$$

Similarly, one readily verifies that the forward and backward shift operators for the polynomials $\mathcal{H}_n(x;q)$ are of the form

$$\left[q^{-\frac{1}{2}\,x\,\frac{d}{dx}} - 1\right]\,\mathcal{H}_n(x;q) = q^{-1/2}\,(1-q^{n/2})\,x\,\mathcal{H}_{n-1}(x;q),$$
$$\left[(1+x^2)\,q^{\frac{1}{2}\,x\,\frac{d}{dx}} - 1\right]\,\mathcal{H}_n(x;q) = x\,\mathcal{H}_{n+1}(x;q), \tag{2.7}$$

respectively, where $q^{\,a\,x\,\frac{d}{dx}}$ is the dilation operator, i.e., $q^{\,a\,x\,\frac{d}{dx}}\,f(x) = f(\,q^a\,x)$.

A Rodrigues-type difference formula for the polynomials $\mathcal{H}_n(x;q)$ can be written as

$$\mathcal{H}_n(x;q) = (-x)^{-n}\, E_q(x^2)\,(q^{\,\frac{1}{2}\,x\,\frac{d}{dx}}; q^{-1/2})_n\, E_q^{-1}(x^2)\,, \tag{2.8}$$

where we have slightly simplified the n-th power of the q-derivative operator $\mathcal{D}_q$ (*cf* (3.29.10) in [13], p.119) by representing it in the form

$$\mathcal{D}_q^n \;\equiv\; \frac{1}{(1-q)^n\,x^n}\,(q^{\,x\,\frac{d}{dx}}\,; q^{-1})_n\,, \qquad n = 0,1,2,\ldots . \tag{2.9}$$

It is not difficult to prove (2.9) by induction on the power n.

Finally, using the generation function for the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II [13], one finds that

$$\frac{(-xt;q^{1/2})_\infty}{(-t^2;q)_\infty} = \sum_{n=0}^{\infty} \frac{1}{(q^{1/2};q^{1/2})_n}\,\mathcal{H}_n(x;q)\,t^{\,n}\,. \tag{2.10}$$

3 Wave functions $\psi_n(x;q)$ and their properties

We wish to discuss a q-model of the linear harmonic oscillator, which is described by the wave functions of the form

$$\psi_n(x;q) := d_n^{-1}(q)\,\mathcal{H}_n(x;q)\,E_q^{-1/2}(x^2) \tag{3.1}$$

with the normalization constant $d_n(q) := q^{-n/4}\,\sqrt{\pi\,(q^{1/2};q)_{1/2}\,(q^{1/2};q^{1/2})_n}$. Then, by continuous orthogonality relation (2.3), these functions are orthonormal on $\mathbb{R}$, that is,

$$\int_{-\infty}^{\infty} \psi_m(x;q)\,\psi_n(x;q)\,dx = \delta_{mn}. \tag{3.2}$$

The wave functions $\psi_n(x;q)$ are defined by (3.1) in such a way that in the limit as $q \to 1$ they coincide with the orthonormalized Hermite functions (or the wave functions of the linear harmonic oscillator in non-relativistic quantum mechanics):

$$\lim_{q\to 1}\,\psi_n\left(\sqrt{1-q}\,\xi; q\right) = \frac{1}{\sqrt{\sqrt{\pi}\,2^n\,n!}}\,H_n(\xi)\,\exp\left(-\xi^2/2\right) =: \psi_n(\xi)\,. \tag{3.3}$$

This limit property of $\psi_n(x;q)$ follows immediately from (2.4) and the well-known fact

$$\lim_{q\to 1}\,E_q\,\left((1-q)\,z\right) = e^z \tag{3.4}$$

about the Jackson q-exponential function $E_q\,(z)$ (see [11] or [12]).

From (2.6) and (3.1) one obtains that the wave functions $\psi_n(x;q)$ are eigenfunctions of the q-Hamiltonian $H(x;q)$,

$$H(x;q)\,\psi_n(x;q) = E_n(q)\,\psi_n(x;q)\,, \quad E_n(q) := \frac{1-q^{n/2}}{1-q^{1/2}}\,. \tag{3.5}$$

By equation (2.6), the explicit form of this self-adjoint q-difference operator is

$$H(x;q) := \frac{1}{(1-q^{1/2})\,x^2}\,\left[(1+x^2+q^{1/2})\,I - \sqrt{1+x^2}\,q^{\,\frac{1}{2}\,x\,\frac{d}{dx}} - q^{\,\frac{1}{2}\,(1-x\,\frac{d}{dx})}\,\sqrt{1+x^2}\,\right], \tag{3.6}$$

where I is the identity operator. This expression for $H(x;q)$ in terms of the dilation operators $q^{\pm\frac{1}{2}\,x\,\frac{d}{dx}}$ may create an impression that the $H(x;q)$ contains singularity at $x=0$ due to the presence of the factor x^2 in the denominator. To remove this doubt one should take into account that, by definition (3.6),

$$H(x;q)\,\psi_n(x;q) = \frac{1}{(1-\sqrt{q}\,)\,x^2}\,\Big[(1+\sqrt{q}+x^2)\,\psi_n(x;q) - \sqrt{1+x^2}\;\psi_n(q^{1/2}\,x;q) - \sqrt{q+x^2}\;\psi_n(q^{-1/2}x;q)\Big] \tag{3.7}$$

for all $n=0,1,2,\ldots$. Besides, from (3.1) it is evident that the wave functions $\psi_n(x;q)$ have regular behavior around $x=0$. Now substituting the sum of first two terms $c_0+c_1\,x$ from the expansion of $\psi_n(x;q)$ around $x=0$ into expression in square brackets in (3.7) and keeping only constant and linear in x terms, one readily verifies that

$$(1+\sqrt{q}\,)\,(c_0+c_1\,x) - (c_0+c_1\,\sqrt{q}\,x) - \sqrt{q}\,\left(c_0+\frac{c_1}{\sqrt{q}}\,x\right) = 0\,.$$

Consequently, the total combination inside the square brackets in (3.7) behaves like x^2 in the $x\to 0$ limit and the right side of (3.7) therefore assumes a constant value at $x=0$. This confirms that there is no singularity at $x=0$.

We observe also that the eigenvalues $E_n(q)$ of $H(x;q)$ are bounded from above by the asymptotic value $E_\infty(q) = 1/(1-q^{1/2})$ and, since $E_{n+1}(q)-E_n(q) = q^{n/2}$, they are not equidistant.

From (2.2) it follows that the wave functions $\psi_n(x;q)$ satisfy the three-term recurrence relation

$$x\,\psi_n(x;q) = q^{-(2n+1)/4}\,\sqrt{1-q^{(n+1)/2}}\;\psi_{n+1}(x;q) + q^{(1-2n)/4}\,\sqrt{1-q^{n/2}}\;\psi_{n-1}(x;q) \tag{3.8}$$

with the initial condition that the ground state $\psi_0(x;q) = d_0^{-1}(q)\,E_q^{-1/2}(x^2)$.

Likewise, from the explicit form of the forward and backward shift operators (2.7) it follows that

$$a(x;q)\,\psi_n(x;q) = \sqrt{E_n(q)}\,\psi_{n-1}(x;q)\,, \quad a^\dagger(x;q)\,\psi_n(x;q) = \sqrt{E_{n+1}(q)}\,\psi_{n+1}(x;q)\,, \tag{3.9}$$

where the q-difference lowering and raising operators $a(x;q)$ and $a^\dagger(x;q)$ are given by

$$\begin{aligned} a(x;q) &= \frac{q^{1/4}}{\sqrt{1-q^{1/2}}\,x}\,\left(q^{-\frac{1}{2}\,x\,\frac{d}{dx}}\,\sqrt{1+x^2} - I\right)\,, \\ a^\dagger(x;q) &= \frac{q^{1/4}}{\sqrt{1-q^{1/2}}\,x}\,\left(\sqrt{1+x^2}\,q^{\frac{1}{2}\,x\,\frac{d}{dx}} - I\right)\,, \end{aligned} \tag{3.10}$$

respectively. We invite the reader to verify that these operators are indeed mutually adjoint in the Hilbert space $L^2(\mathbb{R},dx)$ of square integrable functions $f(x)$ with respect to dx.

Similar to the case of the quantum linear harmonic oscillator, the lowering and raising operators (3.10) factorize the Hamiltonian (3.6), that is,

$$H(x;q) = a^\dagger(x;q)\,a(x;q)\,. \tag{3.11}$$

Moreover, it is not difficult to verify, by using (3.10), that their another (i.e., when the operator $a(x;q)$ is right multiplied by its adjoint operator $a^\dagger(x;q)$) product $a(x;q)\,a^\dagger(x;q)$

is equal to $I + q^{1/2}\, H(x;q)$. This means that the operators $a(x;q)$ and $a^\dagger(x;q)$ satisfy the q-commutation relation of the form

$$a(x;q)\, a^\dagger(x;q) - q^{1/2}\, a^\dagger(x;q)\, a(x;q) \equiv \Big[\, a(x;q)\,,\, a^\dagger(x;q)\,\Big]_{q^{1/2}} = I\,. \tag{3.12}$$

It should be noted at this point that we have used above the known explicit form of the forward and backward shift operators (2.7) for the polynomials $\mathcal{H}_n(x;q)$ in order to find the lowering and raising operators $a(x;q)$ and $a^\dagger(x;q)$. But we could have started equivalently with the q-difference equation (3.5) itself and have directly factorized it in terms of the same operators $a(x;q)$ and $a^\dagger(x;q)$ (for a more detailed discussion of the factorization of difference equations, see, for example, [15, 16]).

So we have established that our q-model is governed by the Hamiltonian (3.6), which admits the factorization (3.11) in terms of the operators $a(x;q)$ and $a^\dagger(x;q)$, satisfying the q-commutation relation (3.12). This characteristic property of the Hamiltonian (3.6) is known to reflect the fact that the dynamical symmetry of this q-model is described by the quantum algebra $su_q(1,1)$ [14]. In the next section we construct explicitly the generators of this algebra in terms of the lowering and raising operators $a(x;q)$ and $a^\dagger(x;q)$.

4 Dynamical symmetry

In this section we remind the reader first how one constructs a dynamical symmetry algebra for the linear harmonic oscillator, which is governed in non-relativistic quantum mechanics by the well-known Hamiltonian

$$H(x) := \frac{\hbar\omega}{2}\left(\xi^2 - \frac{d^2}{d\xi^2}\right) \equiv \hbar\omega\left(N(x) + \frac{1}{2}\right), \tag{4.1}$$

where $\xi = \sqrt{mw/\hbar}\, x$ is a dimensionless coordinate, $N(x)$ is the *particle number operator*,

$$N(x) := a^\dagger(x)\, a(x)\,, \tag{4.2}$$

and the annihilation and creation operators are defined as usual:

$$\begin{gathered} a(x) = \tfrac{1}{\sqrt{2}}\left(\xi + \tfrac{d}{d\xi}\right), \qquad a^\dagger(x) = \tfrac{1}{\sqrt{2}}\left(\xi - \tfrac{d}{d\xi}\right), \\ \left[\, a(x), a^\dagger(x)\,\right] \equiv a(x)\, a^\dagger(x) - a^\dagger(x)\, a(x) = I\,. \end{gathered} \tag{4.3}$$

By using (4.1) and (4.3) one readily verifies that

$$[\, H(x), a(x)\,] = -\, a(x)\,, \qquad \Big[\, H(x), a^\dagger(x)\Big] = a^\dagger(x)\,. \tag{4.4}$$

Observe that in the case of the linear harmonic oscillator (4.1) there is no much difference between the Hamiltonian $H(x)$ and the particle number operator $N(x)$: the former operator, divided by the factor $\hbar\,\omega$, is equal to the latter one plus a constant term 1/2. So, the particle number operator $N(x)$ satisfies the same commutation relations (4.4) with the annihilation and creation operators $a(x)$ and $a^\dagger(x)$.

Having factorized the Hamiltonian $H(x)$ (or, equivalently, the particle number operator $N(x)$) in terms of the annihilation $a(x)$ and creation $a^\dagger(x)$ operators, one explicitly constructs the closed Lie algebra $su(1,1)$ with the three generators

$$K_0(x) := \frac{1}{2\hbar\omega}\, H(x) \equiv \frac{1}{2}\left(N(x) + \frac{1}{2}\right), \quad K_+(x) := \frac{1}{2}\left(a^\dagger(x)\right)^2, \quad K_-(x) := \frac{1}{2}\, a^2(x)\,. \tag{4.5}$$

Indeed, it is not difficult to verify that thus defined generators satisfy the standard commutation relations

$$[K_0(x), K_\pm(x)] = \pm\, K_\pm(x)\,, \qquad [K_-(x), K_+(x)] = 2\, K_0(x)\,, \tag{4.6}$$

of the algebra $su(1,1)$. Unitary irreducible representations of this algebra are known to be characterized by eigenvalues of the invariant (that is, commuting with all three generators (4.5)) Casimir operator

$$C := K_0(x)\,[\,K_0(x) - I\,] - K_+(x)\, K_-(x) = s(s-1)\, I\,. \tag{4.7}$$

A direct calculation of the Casimir operator (4.7) with the aid of (4.5) shows that the eigenvalue $s(s-1)$ in this particular case is equal to $-3/16$. This means that the parameter s may be equal to either $s_1 = 1/4$ or $s_2 = 3/4$. Each of these two values of s defines a unitary irreducible representation of the algebra $su(1,1)$: $D^+(1/4)$ consists of those eigenstates of the Hamiltonian $H(x)$, which correspond to the eigenvalues $s_1 + n = n + 1/4 = (2n + 1/2)/2$, $n = 0, 1, 2, ...$, of the generator $K_0(x) = H(x)/2\,\hbar\omega$; whereas $D^+(3/4)$ corresponds to the eigenvalues $s_2 + n = n + 3/4 = (2n + 1 + 1/2)/2$ of the same generator $K_0(x)$. So in this way one arrives at the correct spectrum $E_n = \hbar\omega\,(n + 1/2)$ of the Hamiltonian $H(x)$, without solving an eigenvalue problem for the appropriate Schrödinger equation. Thus eigenstates of $H(x)$ with the eigenvalues E_{2n} form the unitary irreducible representation $D^+(1/4)$ and those with E_{2n+1} form another one, $D^+(3/4)$.

The Fock space $\mathcal{H}_F$ of all eigenfunctions $\{\psi_n(x)\}$ of the Hamiltonian $H(x)$ splits into two $su(1,1)$-irreducible subspaces for $H(x)$ is symmetric with respect to the inversion $x \to -x$. Therefore the inversion operator P, $P\,x = -\,x$, commutes with all three generators (4.5) and $\mathcal{H}_F$ decomposes into two irreducible components,

$$\mathcal{H}_F \;=\; \mathcal{H}_0 \,\oplus\, \mathcal{H}_1\,, \tag{4.8}$$

consisting of the wave functions $\psi_n(x)$ with even and odd indices n, respectively. The irreducible subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are characterized by the eigenvalues $(-1)^\epsilon$ of the operator P with $\epsilon = 0$ in $\mathcal{H}_0$ and $\epsilon = 1$ in $\mathcal{H}_1$. It is clear that the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ correspond to the unitary irreducible representations $D^+(1/4)$ and $D^+(3/4)$, respectively.

Now we are in a position to discuss a dynamical symmetry algebra for the q-model (3.1). To construct it one needs to introduce first the operator [14]

$$N(x;q) := \frac{2}{\ln q}\, \ln\Big[1 - (1 - q^{1/2})\, H(x;q)\Big]\,. \tag{4.9}$$

Since the wave functions $\psi_n(x;q)$ are eigenfunctions of the q-Hamiltonian $H_n(x;q)$ with the eigenvalues $E_n = (1 - q^{n/2})/(1 - q^{1/2})$, from the definition (4.9) one deduces that

$$N(x;q)\, \psi_n(x;q) = n\, \psi_n(x;q)\,, \tag{4.10}$$

that is, $N(x;q)$ is the *particle number operator* and

$$[N(x;q),\, a(x;q)] = -\, a(x;q), \qquad \Big[N(x;q),\, a^\dagger(x;q)\Big] = a^\dagger(x;q)\,. \tag{4.11}$$

At the next step one defines a new set of the operators

$$b(x;q) := q^{-N(x;q)/8}\, a(x;q)\,, \qquad b^\dagger(x;q) := a^\dagger(x;q)\, q^{-N(x;q)/8}\,, \tag{4.12}$$

which satisfy, according to (4.11), the following commutation relation

$$b(x;q)\, b^\dagger(x;q) -\, q^{1/4}\, b^\dagger(x;q)\, b(x;q) = q^{-N(x;q)/4}\,. \tag{4.13}$$

This is readily verified with the aid of (4.11). The operators $b(x;q)$, $b^\dagger(x;q)$, and $N(x;q)$ directly lead to the dynamical algebra $su_{q^{1/2}}(1,1)$ with the generators

$$K_+(x;q) := \gamma\, \left(b^\dagger(x;q)\right)^2 , \quad K_-(x;q) := \gamma\, b^2(x;q), \quad K_0(x;q) := \tfrac{1}{2}\left(N(x;q) + \tfrac{1}{2}\right) ,$$
$$\gamma = [\,1/2\,]_{q^{1/2}} \; . \tag{4.14}$$

It is not difficult to check that thus defined generators (4.14) satisfy the standard commutation relations

$$[K_0(x;q),\, K_\pm(x;q)] = \pm\, K_\pm(x;q)\,, \qquad [K_-(x;q),\, K_+(x;q)] = [2K_0(x;q)]_{q^{1/2}} \,, \tag{4.15}$$

of the quantum algebra $su_{q^{1/2}}(1,1)$. The q-number $[\,A\,]_q$ in (4.14) is given by the common expression

$$[\,A\,]_q := \frac{q^A - q^{-A}}{q - q^{-1}} \,. \tag{4.16}$$

We are interested in the positive discrete series representations of the quantum algebra $su_q(1,1)$ with lowest weights. These irreducible representations of $su_q(1,1)$ are denoted by T_l^+, where l is the lowest weight, which can be any positive number (see, for example, [17]). It is the characteristic property of every T_l^+ that the generator $K_0(x;q)$ has the eigenvalues $l+n,\;\; n = 0,1,2,...,$ in T_l^+.

The invariant Casimir operator in the case under discussion is equal to

$$C(q) := [\,K_0(x;q) - 1/2\,]^2_{q^{1/2}} - K_+(x;q)\, K_-(x;q) - \frac{1}{4}\; I = \Big([\,1/4\,]^2_{q^{1/2}} - 1/4\Big)\; I. \tag{4.17}$$

This means that two possible values of the parameter s in this case are

$$s_1(q) = 1/2 \;-\; [\,1/4\,]_{q^{1/2}}\,, \qquad s_2(q) = 1/2 \;+\; [\,1/4\,]_{q^{1/2}} \; . \tag{4.18}$$

Since $[a]_q \to a$ in the limit as $q \to 1$ by definition of the q-number (4.16), the eigenvalue of $C(q)$ in (4.17) reduces in this limit to the eigenvalue for the Casimir operator in the case of the linear harmonic oscillator (4.7). Evidently, the same happens with the values of $s_1(q)$ and $s_2(q)$: they coincide in this limit with the corresponding values of the parameter s in (4.7), i.e.,

$$\lim_{q\to 1}\, s_1(q) = \frac{1}{4}\,, \qquad \lim_{q\to 1}\, s_2(q) = \frac{3}{4}\;. \tag{4.19}$$

From (4.17) it now follows that the lowest weights in our case are 1/4 and 3/4. Therefore by (4.14) the eigenvalues of the particle number operator $N(x;q) \equiv 2K_0(x;q) - 1/2$ are equal to $2n$ and $2n+1,\;\; n = 0,1,2,...,$ respectively. Taking into account interrelation (4.9) between the operators $N(x;q)$ and $H(x;q)$, one thus arrives at the correct spectrum (3.5) for the Hamiltonian $H(x;q)$, without solving an eigenvalue problem for $H(x;q)$.

So we conclude that the wave functions $\psi_n(x;q)$, defined in (3.1), form a representation of the quantum algebra $su_{q^{1/2}}(1,1)$ in the Fock space $\mathcal{H}_F$. This representation in the space $\mathcal{H}_F$ is reducible precisely for the same reason as in the case of the linear harmonic oscillator (4.1). Thus $\mathcal{H}_F$ splits into two $su_{q^{1/2}}(1,1)$-irreducible subspaces $\mathcal{H}_0 \equiv T^+_{1/4}$ and $\mathcal{H}_1 \equiv T^+_{3/4}$, consisting of the wave functions $\psi_n(x;q)$ with even and odd indices n, respectively.

5 q-coherent states

As in the case of the non-relativistic linear harmonic oscillator, one can construct q-coherent states for this model as eigenfunctions of the lowering operator $a(x;q)$, that is,

$$a(x;q)\,\varphi_\zeta(x;q) = \zeta\,\varphi_\zeta(x;q)\,, \tag{5.1}$$

where ζ is some arbitrary number. To find an explicit form of these states $\varphi_\zeta(x;q)$, we first note that by (3.8)

$$\psi_n(x;q) = c_n(q)\,\left[a^\dagger(x;q)\right]^n\,\psi_0(x;q)\,, \qquad c_n(q) := \sqrt{\frac{(1-q^{1/2})^n}{(q^{1/2};q^{1/2})_n}}\,. \tag{5.2}$$

Consequently, with the aid of (3.8) it is not difficult to verify that the states

$$\varphi_\zeta(x;q) := f_q(\zeta)\sum_{n=0}^{\infty} c_n(q)\,\zeta^n\,\psi_n(x;q)\,, \tag{5.3}$$

where $f_q(\zeta)$ is some normalization factor (see below), are indeed the eigenstates of the operator $a(x;q)$ with the eigenvalues ζ. They form an overcomplete system in the Hilbert space $\mathcal{H}_F$ and they are not orthogonal in this space. In fact, one can prove, by using expansion (5.3) and orthogonality relation (3.2), that

$$\int_{-\infty}^{\infty}\varphi_\zeta(x;q)\,\varphi_{\zeta'}(x;q)\,dx = f_q(\zeta)\,f_q(\zeta')\,e_{q^{1/2}}\left((1-q^{1/2})\zeta\,\zeta'\right)\,, \tag{5.4}$$

where

$$e_q(z) := \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}\,, \qquad |z|<1\,.$$

The normalization condition that the integral on the left of (5.4) is equal to 1 requires to choose $f_q(\zeta) = \sqrt{E_{q^{1/2}}\left(-(1-q^{1/2})\,\zeta^2\right)}$. Thus,

$$\varphi_\zeta(x;q) = \sqrt{E_{q^{1/2}}\left(-(1-q^{1/2})\,\zeta^2\right)}\,\sum_{n=0}^{\infty} c_n(q)\,\zeta^n\,\psi_n(x;q)\,, \tag{5.5}$$

Substitute now into expansion (5.5) explicit form of the coefficients $c_n(q)$ from (5.2) and the normalization constants $d_n(q)$ for the wave functions $\psi_n(x;q)$ from (3.1) and employ then the generating function (2.10) for the polynomials $\mathcal{H}_n(x;q)$. This yields the final form of the normalized q-coherent eigenfunctions of the lowering operator $a(x;q)$:

$$\varphi_\zeta(x;q) = \sqrt{\frac{E_{q^{1/2}}\left(-(1-q^{1/2})\,\zeta^2\right)}{\pi\,(q^{1/2};q)_{1/2}\,E_q(x^2)}}\;\frac{E_{q^{1/2}}\left(q^{1/4}\sqrt{1-q^{1/2}}\,x\zeta\right)}{E_q\left(q^{1/2}\,(1-q^{1/2})\,\zeta^2\right)}\,. \tag{5.6}$$

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References

[1] R. Álvarez-Nodarse, M. K. Atakishiyeva, and N. M. Atakishiyev. On a q-extension of the Hermite polynomials $H_n(x)$ with the continuous orthogonality property on $\mathbb{R}$. Boletín de la Sociedad Matemática Mexicana (3), **8**, No.2, pp.127–139, 2002.

[2] A. J. Macfarlane. On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$. J. Phys. A: Math. Gen., **22**, No.21, pp. 4581–4588, 1989.

[3] E. D. Kagramanov, R. M. Mir-Kasimov, and Sh. M. Nagiyev. The covariant linear oscillator and generalized realization of the dynamical $SU(1,1)$ symmetry algebra. J. Math. Phys., **31**, No.7, pp.1733–1738, 1990.

[4] N. M. Atakishiyev and S. K. Suslov. Difference analogs of the harmonic oscillator. Theor. Math. Phys., **85**, No.1, pp.1055–1062, 1991.

[5] N. M. Atakishiyev and S. K. Suslov. A realization of the q-harmonic oscillator. Theor. Math. Phys., **87**, No.1, pp.442–444, 1991.

[6] R. Askey and S. K. Suslov. The q-harmonic oscillator and an analogue of the Charlier polynomials. J. Phys. A: Math. Gen., **26**, No.15, pp.L693–L698, 1993.

[7] R. Askey and S. K. Suslov. The q-harmonic oscillator and the Al-Salam and Carlitz polynomials. Lett. Math. Phys., **29**, No.2, pp.123–132, 1993.

[8] N. M. Atakishiyev and Sh. M. Nagiyev. On the wave functions of a covariant linear oscillator. Theor. Math. Phys., **98**, No.2, pp.162–166, 1994.

[9] N. M. Atakishiyev, A. Frank, and K. B. Wolf. A simple difference realization of the Heisenberg q-algebra. J. Math. Phys., **35**, No.7, pp.3253–3260, 1994.

[10] R. Hinterding and J. Wess. q-deformed Hermite polynomials in q-quantum mechanics. Eur. Phys. J. C, **6**, No.1, pp.183–186, 1999.

[11] G. Gasper and M. Rahman. **Basic Hypergeometric Series**, Cambridge University Press, Cambridge, 1990.

[12] G. E. Andrews, R. Askey, and R. Roy. **Special Functions**, Cambridge University Press, Cambridge, 1999.

[13] R. Koekoek and R. F. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 98–17, Delft University of Technology, Delft, 1998.

[14] P. P. Kulish and E. V. Damaskinsky. On the q-oscillator and the quantum algebra $su_q(1,1)$. J. Phys.A: Math. Gen., **23**, No.9, pp.L415–L419, 1990.

[15] G. Bangerezako. Discrete Darboux transformation for discrete polynomials of hypergeometric type. J. Phys. A: Math. Gen., **31**, No.9, pp.2191–2196, 1998.

[16] R. Álvarez-Nodarse, N. M. Atakishiyev, and R. S. Costas-Santos. Factorization of the hypergeometic-type difference equation on non-uniform lattices: dynamical algebra. J. Phys. A: Math. Gen., **38**, No.1, pp.152–174, 2005.

[17] I. M. Burban and A. U. Klimyk. Representations of the quantum algebra $U_q(su_{1,1})$. J. Phys. A: Math. Gen., **26**, No.9, pp.2139–2151, 1993.