Departamento De
Análisis Matemático

Retículos de Espacios Invariantes de Operadores de Composición

Memoria presentada por Manuel Ponce Escudero

para optar al Grado de Doctor en Matemáticas

por la Universidad de Sevilla

Fdo. Manuel Ponce Escudero

V$^{\text{O}}$B$^{\text{O}}$ del Director de la Memoria:

Dr. D. Alfonso Montes Rodríguez

Sevilla, Enero de 2008

Contents

Resumen en Castellano

Para hacer más accesibles sus contenidos, se incluye en esta memoria de investigación el presente capítulo en castellano. En este capítulo se da cuenta de los principales resultados, enunciándolos en el orden en que aparecen y respetando su numeración original.

Capítulo 1: Introducción

Los objetos principales que trata esta memoria son los operadores de composición. Dada una función φ holomorfa que transforma el disco unidad $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ en sí mismo, se define el *operador de composición* C_φ como aquél que a cada función f le asigna $C_\varphi f = f \circ \varphi$. Como consecuencia del Teorema del Grafo Cerrado, los operadores de composición son acotados cuando actúan en el espacio de las funciones holomorfas en el disco unidad $\mathscr{H}(\mathbb{D})$.

Puesto que la composición de funciones es una operación fundamental en Matemáticas, los orígenes de los operadores de composición se remontan, de forma implícita, a los orígenes del Análisis Complejo. En concreto, a finales del siglo XIX, en trabajos de Schröder [48] y Königs [29] se estudian soluciones de ecuaciones que definen los autovectores de operadores de composición. Ya en 1925, el Principio de Subordinación de Littlewood [30] implica la acotación de operadores de composición en un gran número de espacios de funciones analíticas. Sin embargo, no fueron estudiados desde el punto de vista de la Teoría de Operadores hasta la aparición de dos trabajos a finales de la década de los sesenta del siglo pasado. En la tesis de Schwartz [49] aparecen resultados fundamentales y teoremas acerca de la compacidad de operadores de composición. Por su parte, Nordgren [35] encontró los espectros de operadores de composición inducidos por automorfismos del disco unidad.

Probablemente, uno de los problemas abiertos más antiguos e interesantes de la Teoría de Operadores es el conocido como *Problema del Subespacio Invariante*:

> ¿Tiene todo operador acotado en un espacio de Hilbert separable
> de dimension infinita un espacio invariante no trivial?

El problema se remonta a la década de los 30 del siglo pasado cuando John von Neumann encuentra una prueba no publicada de que todo operador compacto actuando en un espacio de Banach tiene un subespacio invariante no trivial. Esto estímulo la búsqueda de un resultado que afirmase la existencia de espacios invariantes para operadores generales actuando en espacios de Banach. Desafortunadamente, P. Enflo [14] resolvió el problema negativamente construyendo un operador sin espacios invariantes.

Sin embargo, el problema sigue abierto en el ámbito de los espacios de Hilbert separables de dimensión infinita. Aunque hay una serie de resultados positivos bajo diferentes hipótesis, se cree que la razón por la que el problema sigue abierto es la falta de ejemplos de operadores cuyos retículos de espacios invariantes sean conocidos. De hecho, el número de operadores cuyo retículo de espacios invariantes conocidos es bastante escaso.

En lo que respecta a los operadores de composición, hasta la fecha no se ha caracterizado el retículo de espacios invariantes de ninguno de ellos. Éste es uno de los motivos por los que se iniciaron los trabajos de investigación que resultaron en la presente memoria. En esta memoria abordamos el estudio de los espacios invariantes de ciertos operadores de composición inducidos por aplicaciones bilineales, caracterizando el retículo de espacios invariantes en un par de casos. Los resultados presentados en la memoria contribuirán a una mejor comprensión de los retículos de espacios invariantes de operadores en general y de operadores de composición en particular. Se espera que estos resultados abran el camino para resultados acerca de espacios invariantes para operadores de composición más generales tanto en el espacio de Hardy como en otros espacios de Hilbert de funciones analíticas.

En un principio puede parecer un objetivo poco ambicioso, pero hay que recordar que debido al gran número de aplicaciones holomorfas φ tales que $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ es inabordable una caracterización general del retículo de espacios invariantes de un operador de composición. Más aún, un resultado de Nordgren, Rosenthal y Wintrobe [37] implica que conocer los espacios invariantes de los operadores de composición inducidos por ciertas transformaciones bilineales implica resolver el Problema del Subespacio Invariante. En concreto,

> Sea $\varphi(z) = \dfrac{2z+1}{z+2}$. Entonces caracterizar $\mathrm{Lat}_{\mathscr{H}^2} C_\varphi$ implica resolver el Problema del Subespacio Invariante.

Capítulo 2: Preliminares

En este capítulo se introducen los objetos principales sobre los que versa esta memoria así como las propiedades básicas de éstos que se necesitarán más adelante. En concreto se comienza recordando la definición y propiedades del *espacio de Hardy* del disco unidad $\mathscr{H}^2$ y de los operadores de composición C_φ.

Más adelante, se estudian las aplicaciones bilineales definidas en el plano complejo como

$$\sigma(z) = \frac{az+b}{cz+d}.$$

Aquellas que transforman el disco unidad en sí mismo, que son las que inducen operadores de composición en el espacio de Hardy, se clasifican en cuatro grupos. El siguiente resultado aporta información sobre las aplicaciones que pertenecen a cada uno de estos grupos.

Teorema 2.3.2 (Clasificación de aplicaciones bilineales $\sigma(\mathbb{D}) \subseteq \mathbb{D}$). *Supongamos que σ es una aplicación bilineal tal que $\sigma(\mathbb{D}) \subseteq \mathbb{D}$. Entonces:*

- *Si σ es parabólica, entonces tiene un punto fijo en $\partial\mathbb{D}$. Equivalentemente, σ es parabólica si es conjugada a una traslación en el semiplano superior.*
- *Si σ es hiperbólica, tiene un punto fijo atractivo en $\overline{\mathbb{D}}$ y el otro punto fijo fuera de $\mathbb{D}$. Ambos puntos fijos están en $\partial\mathbb{D}$ si y sólo si σ es un automorfismo de $\mathbb{D}$.*

- *Si σ loxodrómica o elíptica, un punto fijo está en $\mathbb{D}$ y el otro punto fijo está fuera de $\overline{\mathbb{D}}$. Las aplicaciones elípticas son aquellas que son automorfismos de $\mathbb{D}$ con esta configuración de puntos fijos.*

Posteriormente se introducen los conceptos básicos relacionados con la teoría de espacios invariantes. Por subespacio de un espacio de Hilbert se entiende un subespacio vectorial cerrado bajo la topología inducida en el espacio de Hilbert por su producto escalar. Un subespacio $\mathcal{M}$ es invariante por el operador T siempre y cuando $T\mathcal{M} \subseteq \mathcal{M}$.

Se concluye el capítulo recordando los elementos más destacados de la literatura en los que se estudian espacios invariantes de operadores de composición. El número de referencias es escaso y hemos de destacar que en ninguna de ellas se caracteriza completamente el retículo de espacios invariantes de un operador de composición.

Capítulo 3: El Operador de Composición Elíptico

En este capítulo se caracterizan los retículos de espacios invariantes de los operadores de composición inducidos por una aplicación bilineal elíptica. Toda transformación bilineal elíptica es conjugada a una rotación del plano complejo centrada en 0. Por tanto, para todo operador de composición C_φ inducido por una aplicación bilineal elíptica existe $0 < \theta < 2\pi$ tal que C_φ es similar a $C_{e^{i\theta}z}$. Puesto que los retículos de espacios invariantes de dos operadores similares tienen la misma estructura, el trabajo se centra en el estudio de los operadores $C_{e^{i\theta}z}$.

Los operadores $C_{e^{i\theta}z}$ se dividen en dos grupos en función del valor de $e^{i\theta}$. El primer grupo está formado por aquellos tales que $e^{i\theta}$ es una raíz de la unidad o, equivalentemente, aquellos tales que θ es un múltiplo irracional de 2π. El segundo grupo lo forman los operadores para los que $e^{i\theta}$ es una raíz de la unidad, en cuyo caso es una raíz primitiva p-ésima de la unidad para cierto número natural p; o, equivalentemente, aquellos operadores para los que θ es un múltiplo racional de 2π. Esto genera comportamientos y propiedades radicalmente distintos para cada grupo. Mientras que en el primer grupo todos los operadores son cíclicos, en el segundo son fuertemente no cíclicos ya que el número de elementos en la órbita de toda función bajo el operador es finito.

En primer lugar se caracterizan los autovalores de estos operadores. La caracterización refleja los distintos comportamientos mencionados anteriormente.

Proposición 3.1.1. *Si $e^{i\theta}$ no es una raíz de la unidad, entonces los autovalores de $C_{e^{i\theta}z}$ son $e^{in\theta}$. Cada $e^{in\theta}$ tiene como único autovector a z^n, para todo $n = 0, 1, 2, \ldots$*

Proposición 3.1.2. *Si $e^{i\theta}$ es una raíz primitiva p-ésima de la unidad, entonces el número complejo λ es un autovalor de $C_{e^{i\theta}z}$ si y sólo si $\lambda = e^{ik\theta}$ para algún $k \in \{0, 1, \ldots, p-1\}$. Además, f es un autovector correspondiente a $e^{ik\theta}$ si y sólo si*

$$f = \sum_{n=0}^{\infty} a_n z^{np+k}, \qquad \text{para algún } \{a_n\}_{n\geq 0} \in \ell^2 \backslash \{0\}.$$

Una vez caracterizados los autovectores, se prueba que en ambos casos el espacio invariante generado por la órbita de cualquier función de $\mathscr{H}^2$ bajo el operador $C_{e^{i\theta}z}$ está generado por autofunciones del operador. Así se obtienen sendas caracterizaciones de los subespacios invariantes de $C_{e^{i\theta}z}$ en función de si $e^{i\theta}$ es o no raíz de la unidad.

Teorema 3.2.2. *Si $e^{i\theta}$ no es raíz de la unidad, entonces*

$$\operatorname{Lat} C_{e^{i\theta}z} = \{\overline{\operatorname{span}}\{z^n : n \in N\} : N \in \mathcal{P}(\mathbb{N})\}.$$

Teorema 3.2.4. *Si $e^{i\theta} \neq 1$ es raíz de la unidad, entonces*

$$\operatorname{Lat} C_{e^{i\theta}z} = \{\overline{\operatorname{span}}\{f : f \in N\} : N \text{ es un conjunto de autovalores}\}.$$

Como consecuencia, se obtiene la siguiente descripción del retículo un operador de composición inducido por una transformación bilineal elíptica, debido a que cada uno de estos operadores es similar a uno de los operadores anteriores.

Corolario 3.2.5. *Sea φ una transformación bilineal elíptica. Entonces* $\operatorname{Lat} C_\varphi$ *está formado por todos los subespacios de $\mathscr{H}^2$ generados por autovectores de C_φ.*

Capítulo 4: El Parabólico No Automorfismo

En este capítulo se caracteriza completamente el retículo de espacios invariantes de un operador de composición inducido por una transformación bilineal parabólica que no es automorfismo del disco unidad. El espectro de estos operadores fue caracterizado por C. Cowen.

Teorema 4.2.2 (Cowen, 1983). *Sea φ_a un parabólico no automorfismo que transforma el disco unidad en sí mismo. Entonces*

$$\sigma(C_{\varphi_a}) = \{e^{-at} \colon t \in [0, +\infty)\} \cup \{0\}.$$

Las autofunciones del operador C_{φ_a} son una familia de funciones bien conocidas:

$$C_{\varphi_a} e_t = e^{-at} e_t, \text{ siendo } e_t(z) = \exp\left(t \frac{z+1}{z-1}\right) \text{ para cada } t \geq 0.$$

Entre las propiedades de las autofunciones de C_{φ_a} cabe destacar que son un conjunto generador del espacio de Hardy.

Proposición 4.2.4. *El conjunto de autofunciones de C_{φ_a} genera el espacio $\mathscr{H}^2$. Es decir,*

$$\overline{\operatorname{span}} \{e_t \colon t \geq 0\} = \mathscr{H}^2.$$

Gracias a un resultado de Halmos [19, Problema 85] sabemos que cuando las autofunciones de un operador generan el espacio en el que actúa el operador, el adjunto de éste es similar a un operador de multiplicación en un cierto espacio de Hilbert funcional. El problema radica en que encontrar ese espacio de Hilbert funcional no siempre es una tarea sencilla.

En este caso, se identifica este espacio como un espacio de Sobolev. Recordamos que el espacio de Sobolev $W^{1,2}[0,\infty)$ está formado por todas aquellas funciones f en $L^2[0,\infty)$ absolutamente continuas en cada subintervalo de $[0,\infty)$ cuyas derivadas también pertenecen a $L^2[0,\infty)$. El espacio $W^{1,2}[0,\infty)$ es un espacio de Hilbert cuando se le dota del producto escalar

$$\langle f,g\rangle_{1,2} = \frac{1}{2}\int_0^\infty \left(f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\right) dt.$$

El espacio de Sobolev $W^{1,2}(\mathbb{R})$ se define de forma análoga. Consideremos el operador definido en $L^2(\mathbb{T})$ como

$$(\Psi f)(t) = \langle f, e_t\rangle_{L^2(\mathbb{T})}, \qquad t \in \mathbb{R}.$$

Usando el Teorema de Plancherel se prueba el siguiente

Teorema 4.3.1. *El operador Ψ es un isomorfismo isométrico de $L^2(\mathbb{T})$ en $W^{1,2}(\mathbb{R})$.*

Como consecuencia, tenemos el isomorfismo deseado entre el espacio de Hardy y el espacio de Sobolev $W^{1,2}[0,\infty)$. Se define Φ como aquel que a cada función de $\mathscr{H}^2$ le asigna

$$(\Phi f)(t) = \langle f, e_t\rangle_{\mathscr{H}^2}, \qquad t \geq 0.$$

Corolario 4.3.2. *El operador Φ define un isomorfismo de $\mathscr{H}^2$ con valores en $W^{1,2}[0,\infty)$. De hecho, $\|\Phi f\|_{1,2}^2 = \|f\|_{\mathscr{H}^2}^2 - |f(0)|^2/2$.*

De la definición de Φ se obtiene inmediatamente la siguiente

Proposición 4.3.4. *Sea φ_a, con $\Re a \geq 0$, un no automorfismo del disco unidad parabólico. Entonces el adjunto de C_{φ_a} actuando en $\mathscr{H}^2$ es similar bajo el isomorfismo Φ al operador de multiplicación M_ψ, siendo $\psi(t) = e^{-\bar{a}t}$, actuando en $W^{1,2}[0,\infty)$.*

Así pues, gracias a esta última proposición somos capaces de cambiar la naturaleza de nuestro problema. Los espacios invariantes de C_{φ_a} se corresponden con los de su adjunto $C_{\varphi_a}^*$ y éstos, a su vez, se corresponden con los del operador de multiplicación M_ψ actuando en el espacio de Sobolev. Para encontrar los espacios invariantes de este operador de multiplicación recurrimos a la siguiente proposición.

Proposición 4.1.2. *Sea $\mathscr{A}$ una álgebra de Banach. Entonces los subespacios invariantes de un operador de multiplicación por un elemento cíclico son justamente los ideales cerrados de $\mathscr{A}$.*

Recuérdese que un operador es cíclico cuando existe un elemento f del espacio tal que el subespacio generado por todas las potencias del operador aplicadas a f es denso en el espacio total. En ese caso, se dice que f es un vector cíclico del operador. Se tiene que M_ψ es cíclico.

Proposición 4.3.5. *El operador M_ψ, donde $\psi(t) = e^{-\bar{a}t}$ y $\Re a > 0$, actuando en $W^{1,2}[0,\infty)$ es cíclico con vector cíclico igual a ψ.*

El hecho de que el espacio $W^{1,2}[0,\infty)$ es un algebra de Banach es algo conocido por los expertos. En esta memoria se incluye una prueba de esto. La novedad radica en que en la prueba que sólo se hace uso del isomorfismo Φ construido y que, al parecer, no era conocido previamente. Por tanto M_ψ verifica las condiciones de la Proposición 4.1.2 y así sus espacios invariantes coinciden con los ideales cerrados del álgebra de Banach $W^{1,2}[0,\infty)$.

Para caracterizar los ideales cerrados de $W^{1,2}[0,\infty)$ hemos de recurrir a ciertos resultados clásicos de la teoría de álgebras de Banach. En el caso del espacio de Sobolev resulta que cada ideal cerrado es intersección de ideales maximales. Así, tras caracterizar los funcionales lineales multiplicativos de $W^{1,2}[0,\infty)$ se obtiene que cada ideal maximal está formado por el conjunto de funciones que se anulan en un determinado punto. De esto se infiere el siguiente resultado donde $\varkappa$ denota un funcional lineal multiplicativo de $W^{1,2}[0,\infty)$ y $\mathbb{F}[0,\infty)$ denota el conjunto de subconjuntos cerrados de $[0,\infty)$.

Proposición 4.4.5. *El álgebra de Banach $W^{1,2}[0,\infty)$ es semisimple y regular. Además, la correspondencia $F \to \bigcap_{\varkappa\in F} \ker \varkappa$ es una biyección entre $\mathbb{F}[0,\infty)$ y el conjunto de ideales cerrados de $W^{1,2}[0,\infty)$.*

Así cada ideal cerrado está formado por todas las funciones que se anulan en un determinado subconjunto cerrado de $[0,\infty)$. Los subespacios invariantes de M_ψ quedan caracterizados como sigue.

Corolario 4.5.3. *Sea $M_{e^{-\bar{a}t}}$ el operador de multiplicación por $e^{-\bar{a}t}$ actuando sobre el espacio de Sobolev $W^{1,2}[0,\infty)$. Entonces*

$$\operatorname{Lat} M_{e^{-\bar{a}t}} = \Big\{\{f \in W^{1,2}[0,\infty) \colon f \text{ se anula en } F\} \colon F \in \mathbb{F}[0,\infty)\Big\}.$$

Volviendo atrás al espacio de Hardy con el isomorfismo Φ se obtienen los espacios invariantes del operador adjunto $C^\star_{\varphi_a}$.

Corolario 4.5.2. *Sea φ un no automorfismo del disco unidad parabólico. Entonces*

$$\operatorname{Lat} C^\star_\varphi = \Big\{\{f \in \mathscr{H}^2 \colon \langle f, e_t\rangle_{\mathscr{H}^2} = 0 \text{ para } t \in F\} \colon F \in \mathbb{F}[0,\infty)\Big\}.$$

Es bien conocido que un subespacio es invariante por un operador sí y sólo sí el ortogonal de ese espacio es invariante por el adjunto de ese operador. Así, del corolario anterior se deduce el teorema principal de este capítulo.

Teorema 4.1.1. *Sea φ un no automorfismo del disco unidad parabólico. Entonces*

$$\operatorname{Lat} C_\varphi = \Big\{\overline{\operatorname{span}}\,\{e_t : t \in F\} \colon F \in \mathbb{F}[0,\infty)\Big\}.$$

Tres resultados que aportan información del operador C_{φ_a} y de su adjunto se siguen de la caracterización de sus espacios invariantes.

Corolario 4.5.1. *Todos los operadores de composición inducidos por no automorfismos del disco unidad parabólicos poseen el mismo retículo de espacios invariantes y comparten sus vectores cíclicos.*

Teorema 4.5.4. *Sea φ un no automorfismo del disco unidad parabólico. Entonces C_φ no posee ningún espacio reductor no trivial.*

Corolario 4.5.5. *Sea φ un no automorfismo del disco unidad parabólico. Entonces una función f en $\mathscr{H}^2$ es un vector cíclico de $C^\star_{\varphi_a}$ si y sólo si*

$$\langle f, e_t\rangle_{\mathscr{H}^2} \neq 0 \quad \textit{para todo } t \geq 0.$$

Capitulo 5: El Retículo en Otros Espacios

En este capítulo se estudian los espacios invariantes del operador de composición inducido por un no automorfismo parabólico del disco unidad en otros espacios de funciones analíticas distintos al espacio de Hardy. Puesto que en el espacio de Hardy la caracterización del retículo del operador dependía fuertemente de su espectro y del hecho de que las autofunciones generan el espacio, cabe preguntarse qué sucede en un espacio en el que el espectro del operador siga siendo el mismo pero que no contenga a las autofunciones e_t. Un espacio que reúne estas características es el espacio de Dirichlet $\mathscr{D}$.

Resulta que en el espacio de Dirichlet módulo las constantes $\mathscr{D}_0$ todos los subespacios invariantes son reductores, lo cual supone un claro contraste con lo que sucedía en el espacio de Hardy donde ningún subespacio invariante era reductor.

Corolario 5.1.4. *Sea φ un no automorfismo del disco unidad parabólico. Entonces* $\mathrm{Lat}_{\mathscr{D}_0} \widetilde{C}_\varphi$ *es la imagen inversa bajo $\mathcal{F}C_\sigma$ de*

$$\big\{\{f \in L^2(\mathbb{R}^+, tdt) : f \text{ se anula en } A\} : A \in \mathbb{A}(0,\infty)\big\}.$$

Siendo la aplicación $\mathcal{F}C_\sigma$ la isometría entre $\mathscr{D}_0$ y $L^2((0,\infty), tdt)$ definida para toda función f en $\mathscr{D}_0$ como

$$\mathcal{F}C_\sigma f(t) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) \frac{e^{t\frac{x+1}{x-1}}}{(1-x)^2}\, dx.$$

La caracterización de los espacios invariantes en $\mathscr{D}_0$ junto con un estudio de las órbitas del operador permite caracterizar los espacios invariantes del operador en el espacio de Dirichlet. Si denotamos por $\widetilde{C}_\varphi$ a la compresión del operador C_φ al subespacio $\mathscr{D}_0$, se tiene el siguiente

Teorema 5.1.6. *Sea φ un no automorfismo del disco unidad parabólico.*

Considerando C_φ actuando en el espacio de Dirichlet $\mathcal{D}$, se tiene que

$$\operatorname{Lat} C_\varphi = \{0\} \cup \big\{[1] \oplus \mathcal{M} \colon \mathcal{M} \in \operatorname{Lat} \widetilde{C}_\varphi\big\}.$$

Una vez caracterizados los espacios invariante en el espacio de Dirichlet y obtenido, como era de esperar, que no son los mismos que en el espacio de Hardy, cabe plantearse otra pregunta natural. Puesto que la densidad de las combinaciones lineales de las autofunciones e_t era una pieza fundamental en la construcción del isomorfismo Φ que llevó a la caracterización de los espacios invariantes en el espacio de Hardy, ¿qué ocurre en otro espacio de funciones analíticas que sí contenga las autofunciones e_t y que a su vez éstas generen el espacio?. Esta es la situación en los espacios de Bergman con pesos $\mathcal{A}^2_\alpha$ para todo $\alpha > -1$. En este caso, al igual que en el espacio de Hardy, cabría esperar que el retículo de subespacios invariantes estuviese compuesto únicamente por subespacios generados por autofunciones. Pero, sorprendentemente, no es el caso. A medida que α crece, aparecen nuevos espacios invariantes que no se corresponden con subespacios invariantes en el espacio de Hardy y que por tanto no están generados por autofunciones.

Empezamos construyendo una aplicación análoga a Φ definida como

$$(\Phi_\alpha f)(t) = \langle f, e_t \rangle_{\mathcal{A}^2_\alpha}$$

para toda función f en $\mathcal{A}^2_\alpha$ y todo $\alpha > -1$. Para determinados valores de α podemos determinar el espacio de llegada como un espacio de Sobolev.

Teorema 5.3.2. *Para cada entero no negativo k, la aplicación Φ_k es un isomorfismo sobreyectivo entre $\mathcal{A}^2_k(\mathbb{D})$ y $W^{k+2,2}_{k+1}[0,\infty)$.*

Es más, gracias a la definición de Φ_α, el adjunto del operador de composición es similar a un operador de multiplicación.

Proposición 5.3.3. *Sea φ_a un no automorfismo del disco unidad parabólico. Entonces, para todo entero no negativo, el adjunto de C_{φ_a} actuando en $\mathcal{A}^2_k$ es similar a través de Φ_k al operador de multiplicación M_ψ, siendo $\psi(t) = e^{-\bar{a}t}$, actuando en $W^{k+2,2}_{k+1}[0,\infty)$.*

Cuando k es un número par se puede probar que el espacio de Sobolev es un álgebra de Banach y que las derivadas de sus funciones hasta cierto orden están acotadas.

Proposición 5.4.1. *Para cada entero par $k \geq 0$, el espacio $W_k^{k+1,2}[0,\infty)$ es un álgebra de Banach conmutativa sin elemento unidad. Además, la convergencia en este espacio implica la convergencia uniforme de las derivadas de orden $l < (k+1)/2$.*

Puesto que M_ψ es cíclico en cada uno de los nuevos espacios de Sobolev, la Proposición 4.1.2 implica que los subespacios invariantes del operador M_ψ son exactamente los ideales cerrados del álgebra $W_k^{k+1,2}[0,\infty)$. Aquí es donde radica la diferencia con el espacio de Hardy. La convergencia uniforme de la derivada en $W_k^{k+1,2}[0,\infty)$ hace que aparezcan nuevos ideales distintos al ideal formado por todas las funciones que se anulan en un punto. En concreto,

Proposición 5.4.4. *Para cada entero par $k \geq 2$, el conjunto*

$$I_F = \{f \in W_k^{k+1,2}[0,\infty) \colon f^{(l)}(t) = 0 \text{ para todo } t \in F,\ 0 \leq l \leq\}$$

es un ideal cerrado para cada $F \in \mathbb{F}[0,\infty)$ y cada $0 \leq j < (k+1)/2$. Más aún, en caso de que F tenga puntos aislados y $j \geq 1$,

$$I_F \neq \{f \in W_k^{k+1,2}[0,\infty) \colon f(t) = 0 \text{ para todo } t \in F\}.$$

Esto genera subespacios invariantes del operador de composición C_{φ_a} actuando en $\mathcal{A}_\alpha^2$ que son distintos a subespacios generados por sus autofunciones. En particular,

Teorema 5.4.5. *Para cada entero par $k \geq 2$, consideremos C_{φ_a} actuando en $\mathcal{A}_{k-1}^2$. Entonces, para cada $F \in \mathbb{F}[0,\infty)$ y cada $0 \leq j < (k+1)/2$, el subespacio*

$$\mathcal{M}_F = \overline{span}\left\{\left(\frac{z+1}{z-1}\right)^l e_t(z) \colon t \in F \text{ y } 0 \leq l \leq j\right\}$$

pertenece a $LatC_{\varphi_a}$. En particular, si F tiene puntos aislados, el subespacio $\mathcal{M}_F$ no está generado por autofunciones.

Capítulo 6: El Automorfismo Parabólico

El caso de un operador de composición inducido por un automorfismo parabólico es mucho más complejo. En primer lugar hemos de resaltar que en este caso el operador es hipercíclico, ver [6], por tanto aunque también sea conjugado a una traslación del semiplano superior su comportamiento es mucho más caótico que el de una aplicación bilineal parabólica no automorfismo. Comenzamos con un resultado que caracteriza los autovectores del operador.

Proposición 6.1.1. *Sea $a \neq 0$ tal que $\Re a = 0$ y $\lambda = e^{-at_0}$, donde $0 \leq t_0 < 2\pi/|a|$. Entonces ℓ^2 es isomorfo a $\ker(C_{\varphi_a} - \lambda I)$ a través del operador que asigna a cada sucesión $\{a_n\}$ la función $f = \sum_{n=0}^{\infty} a_n e_{t_0 + 2\pi n/|a|}$.*

Este resultado arroja cierta luz sobre el retículo de espacios invariantes de este operador. En principio, aparece una cantidad enorme de espacios invariantes que no son invariantes para el operador de composición inducido por un no automorfismo parabólico. Es el caso de los espacios vectoriales $\mathcal{M}_t = [e_t + e_{2\pi/|a|}]$ para todo número t no negativo.

Estos espacios invariantes $\mathcal{M}_t$ también muestran otra diferencia con el caso anterior. Los operadores de composición inducidos por automorfismos parabólicos no comparten sus retículos de espacios invariantes. Debido a la gran cantidad de autovectores nuevos que aparecen en este caso, cabría esperar que al menos compartiese con el caso no automorfismo parabólico el hecho de que todo espacio invariante está generado por autovectores. Desafortunadamente, no es el caso como muestra la siguiente proposición.

Proposición 6.2.1. *Sea φ_a un automorfismo parabólico del disco unidad. Entonces C_{φ_a} tiene un espacio invariante no trivial de dimensión infinita que contiene únicamente a la autofunción* 1.

Por tanto, aún queda mucho camino por recorrer para encontrar una caracterización del retículo de espacios invariantes de un operador de composición inducido por un automorfismo parabólico del disco unidad.

Chapter 1

Introduction

Probably one of the most natural algebraic operations that can be defined between functions is composition. Given two functions, in case the range of one of them is included in the domain of the other one we can always compose them. Given a collection S of analytic functions on some domain and a holomorphic map φ from that domain onto itself, we can define the *composition operator* C_φ on S as $C_\varphi f = f \circ \varphi$ for each f in S. In principle, there is no reason that $C_\varphi f$ should even belong to S.

In spite of being composition a natural operation when working with functions, it took a long time to study it as an operator between spaces of functions. The idea of studying the general properties of composition operators seems to go back to 1968 in Eric Nordgren's work [35], where the author characterized the spectra of composition operators induced by automorphism of the unit disk. Almost at the same time, H. J. Schwartz presented his doctoral dissertation [49] in 1969 containing fundamental results and theorems about compactness of composition operators.

Nevertheless, many basic aspects of composition operators implicity date back to the beginnings of complex analysis. For instance, in the late nineteenth century, two different works of Schröder [48] and Königs [29] were published studying solutions of equations that define the eigenvectors of composition operators. Another instance is Littlewood's Subordination Theorem of harmonic functions, see [30], appeared in 1925 that implies the boundedness of composition operators in a great number of spaces of analytic functions.

Since their introduction, composition operators have attracted much attention, possibly due to its naive definition. Many works have appeared relating operator theoretic properties of C_φ with geometrical or functional properties of its inducing symbol φ. Among others, boundedness, compactness and spectral properties have been studied in different spaces of analytic functions. Although most of the research on composition operators has been carried out in spaces of analytic functions, they can also be defined in other spaces such as Lebesgue spaces as done by Sarason in [46].

Probably the oldest and one of the most interesting open problems in Operator Theory is the so called *Invariant Subspace Problem*:

> Does each bounded operator acting on the separable infinite-dimensional Hilbert space have a nontrivial closed invariant subspace?

The history of this problem goes back at least to John von Neumann. In the early 1930's he found a proof of the existence of non-trivial invariant subspaces for compact operators in Hilbert spaces; the proof was never published. Later, von Neumann's proof was rediscovered by Aronszajn and extended to compact operators acting on Banach spaces in [4]. The Invariant Subspace Problem was solved in the negative by Enflo who announced his result in 1975 in [13]. However, its Acta Mathematica paper [14] solving the problem did not appear until 1987 since due to its difficulty it remained unrefereed for years. In the meanwhile, C. J. Read simplified Enflo's counterexample in [41] and was able to publish it 1984, before the publication of Enflo's paper. One year later Read [42] published another proof where he constructed an operator in the sequence space ℓ^1 without non-trivial invariant subspaces. See the survey paper [51] for an account of the history and the present state of the Invariant Subspace Problem.

However, for infinite-dimensional separable Hilbert space the question remains open. Nevertheless, there are a number of affirmative results under various hypothesis. It is believed that the Invariant Subspace Problem remains unsolved due to the lack of examples of operators whose lattice of invariant subspaces has been characterized.

Although intensively studied during last decades, the works researching invariant subspaces for composition operators are scarce. Specially interesting is [37] where the authors prove that the composition operator induced by a hyperbolic disk automorphism is universal. Thus every operator acting on a Hilbert space is similar to the restriction of that composition operator to one of its invariant subspaces. In particular, showing that all minimal invariant subspaces of the composition operator induced by a hyperbolic disk automorphism are one-dimensional implies answering the Invariant Subspace Problem in the affirmative. Although there are a number of similar results, this one is striking because of the simplicity of the operator and since it acts in the well understood Hardy space.

This is the starting point for this work. To provide characterization of lattices of invariant subspaces of composition operators. This will furnish new examples to the literature that will lead to a better understanding of the structure of lattices of invariant subspaces of general operators and in particular of composition operators. It is hoped that the results presented here will point the way toward results about more general composition operators, both on the Hardy space and on related Hilbert spaces of analytic functions.

The contents of this work are structured as follows. In the first chapter we introduce the main characters in which the work is done in the following chapters: Hardy spaces, composition operators and invariant subspaces. The basic properties of these objects that will be needed in the future are also introduced. In the second chapter, the lattices of invariant subspaces of composition operators induced by elliptic fractional maps are studied. Two different kinds of lattices appear, according wether the operator is similar to a rotation through a rational multiple of 2π or not.

In the next two chapters composition operators induced by parabolic non-automorphisms are studied. First we focus in studying their lattices of invariant subspaces when acting in the Hardy space. To achieve the characterization of their lattices, the key point is to establish an isomorphism between the Hardy space and a Sobolev space. Although it is known that all infinite dimensional separable Hilbert spaces are isomorphic, it is not always easy to give an explicit expression for the isomorphism. This isomorphism makes the adjoint of the composition operator similar to a multiplication operator acting on the Sobolev space $W^{1,2}[0,\infty)$. The invariant subspaces of that multiplication operator are identified using some elements of Gelfand Theory and Banach algebras techniques. Hence the lattice of invariant subspaces of the composition operator turns out to be formed exclusively by all its eigenspaces, that is, the subspaces spanned by its eigenvectors.

Latter on we continue the study in other spaces of analytic functions. Since the characterization of invariant subspaces in the Hardy space relies heavily in the spectra and eigenfunctions of these operators, first we explore what happens if the spaces does not contain the eigenfunctions but the operator still has the same spectrum that in the Hardy space: either the interval $[0,1]$ or a downward spiral that starts at the point 1 and converges to the origin winding infinitely many times around it. In the Dirichlet space the lattice of invariant subspaces of the operator are not eigenspaces anymore. It turns out that in the Dirichlet space modulo the constants, all the invariant subspaces are reducing in opposite contrast to the Hardy space case where no invariant subspace was reducing. Next we consider certain weighted Bergman spaces. These spaces contain the eigenfunctions and they span the whole space, thus the conditions are exactly the same as in the Hardy spaces. We are able to prove that still the adjoint of the operator is similar to a multiplication operator acting in certain Sobolev space, different to the one appearing in the Hardy space case. Now the structure of the ideals in this new Sobolev spaces is more complex than in the Hardy space case. Strikingly, new subspaces arise different to those spanned by eigenfunctions.

We end the work studying the composition operator induced by a parabolic automorphism. The characterization of the invariant subspaces of the composition operator induced by a parabolic disk automorphism in the Hardy space is far from being completed. It is shown that the situation is harder since the lattice is extremely rich. In this case a huge amount of new eigenspaces appear. In addition to the eigenspaces, it is shown that the lattice contains infinite-dimensional invariant subspaces with just one eigenfunction.

Chapter 2

Preliminaries

Throughout this work we will mainly deal with operators acting on complex Hilbert spaces of analytic functions. Recall that a complex Hilbert space is a vector space over the field of complex numbers which is endowed with an inner product compatible with the linear structure of the vector space and such that generates a metric that makes the vector space complete. By an *operator* we will always mean a linear map acting between Hilbert spaces. In other words, an operator will be a homomorphism of vector spaces, that is, a mapping that preserves the linear structure of the Hilbert space where it is acting. An operator acting on a Hilbert space will be said *bounded* whenever it is continuous with respect to the topology generated by the metric of the Hilbert space.

Although there is only one separable infinite dimensional Hilbert space, it can be represented in many different forms. Those forms of the Hilbert space we are going to work with are known as functional Hilbert spaces. Recall that a *functional Hilbert space* $\mathscr{H}$ is a non-trivial (i.e.: $\mathscr{H} \neq 0$) Hilbert space of complex valued functions defined on a set X such that for each $x \in X$ the point evaluation functional, $x \longmapsto f(x)$, is bounded.

Our results are set exclusively in Hilbert spaces of functions defined in the unit disk of the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We can define analogous of these spaces consisting of functions defined in a simply connected plane domain G. If σ is a univalent holomorphic mapping of $\mathbb{D}$ onto G, the main results in this work can be transferred to the space of functions defined on G with the aid of the composition operator C_σ.

Recall that all complex Hilbert spaces are particular cases of complex Banach spaces, that are complex vector spaces, where instead an inner product, each of them possesses a norm that induces a metric in the Banach space that makes it complete. Most of the objects and properties defined in this work can be carried out on the Banach space setting. However, we will stay into the friendly confines of the Hilbert space setting when defining them, since there is where the essential ideas occur.

2.1 Hardy Spaces

In this section we will introduce the Hardy space of the unit disk $\mathscr{H}^2$. It is a particular instance of a family of spaces: the Hardy spaces of the unit disk. The name of these spaces is in honor of G. H. Hardy, contributor to the fundamentals of the subject. For this reason these spaces are denoted by $\mathscr{H}^p(\mathbb{D})$ or simply by $\mathscr{H}^p$, where p satisfies $1 \leq p \leq \infty$. The historical starting point for the subject of the $\mathscr{H}^p$ spaces is Hardy's paper [20] appeared in 1915. Most of the work will be carried out in the Hardy space $\mathscr{H}^2$ which is the only one in this family of spaces that is a Hilbert space. All definitions and properties presented here are elementary facts in the theory of Hardy spaces that can be found, for instance, in [45, Chapter 17] or [11].

The Hardy space $\mathscr{H}^2$ is formed by all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic on the unit disk $\mathbb{D}$ such that the norm

$$\|f\|^2_{\mathscr{H}^2} = \sum_{n=0}^{\infty} |a_n|^2$$

is finite. The inner product that induces the norm above in $\mathscr{H}^2$ is

$$\langle f, g \rangle_{\mathscr{H}^2} = \sum_{n=0}^{\infty} a_n \overline{b_n},$$

for two arbitrary functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathscr{H}^2$.

Associated to each point α of the unit disk there is a function of particular interest called the *reproducing kernel* at α and defined as

$$k_\alpha = \sum_{n=0}^{\infty} (\overline{\alpha} z)^n = \frac{1}{1 - \overline{\alpha} z}.$$

The function belongs to $\mathscr{H}^2$ and its norm equals $1/\sqrt{1-|\alpha|^2}$. The name of the functions k_α comes from the property,

$$f(\alpha) = \langle f, k_\alpha \rangle_{\mathscr{H}^2}, \quad \text{for all } f \in \mathscr{H}^2,$$

that makes them to be closely related with the structure of $\mathscr{H}^2$. This property is a direct consequence of the definition of both k_α and inner product in $\mathscr{H}^2$. Therefore, all point evaluation functionals are bounded and thus $\mathscr{H}^2$ is a functional Hilbert space.

An equivalent norm can be defined in this space. All functions f in the Hardy space have radial limits

$$f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$$

for almost every $e^{i\theta}$ in $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. The boundary function f^* is the limit in $L^2(\mathbb{T}) = L^2(\mathbb{T}, d\theta/2\pi)$ of the dilated functions $f(re^{i\theta})$ as $r \to 1^-$. Therefore f^* belongs to the Lebesgue space $L^2(\mathbb{T})$ and if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then its boundary function is

$$f^*(z) = \sum_{n=0}^{\infty} a_n e^{in\theta}.$$

Thus, the orthogonality of $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ implies that the norm in the Hardy space previously defined coincides with the norm in $L^2(\mathbb{T})$ of the limit function, that is,

$$\|f\|^2_{\mathscr{H}^2} = \frac{1}{2\pi} \int_0^{2\pi} |f^*(e^{i\theta})|^2 \, d\theta.$$

Thus the mapping $f \to f^*$ is an isometry from $\mathscr{H}^2$ onto the subspace of $L^2(\mathbb{T})$ generated by the functions $\{e^{in\theta}\}_{n\geq 0}$. We will keep in mind this identification between $\mathscr{H}^2$ and the latter subspace of $L^2(\mathbb{T})$ and from now on we will drop the notation f^* for the boundary function and simply write f. According to the context, it will be clear wether we are dealing with a holomorphic function on the unit disk or with a measurable function in the unit circle.

We have also an equivalent formulation for the inner product in $\mathscr{H}^2$,

$$\langle f, g \rangle_{\mathscr{H}^2} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta$$

for any two functions f and g in $\mathscr{H}^2$.

The analyticity of functions in the Hardy space leads to a uniqueness theorem for their boundary functions.

Theorem 2.1.1 ([45, Theorem 17.18]). *If function f in $\mathscr{H}^2$ is not identically 0, then $f(e^{i\theta}) \neq 0$ almost everywhere in $\mathbb{T}$.*

It is well known that the zeroes of an analytic function cannot cluster inside its domain of analyticity. If the function belongs to the Hardy space, it can be said more about its zeroes.

Theorem 2.1.2 ([11, Theorem 2.3]). *If $\{z_n\}_{n\geq 0}$ is the set of zeros of a function in $\mathscr{H}^2$ repeated according to their multiplicity, then*

$$\sum_{n=0}^{\infty}(1 - |z_n|) < \infty. \tag{2.1}$$

A sequence satisfying condition (2.1) is called a *Blaschke sequence.*

One more property of functions belonging to Hardy spaces has to be highlighted. The finiteness of their Hardy norms implies that the growth of the functions is controlled.

Theorem 2.1.3 ([11, p. 36]). *If f belongs to $\mathscr{H}^2$, then*

$$|f(z)|^2 \leq 2\frac{\|f\|_{\mathscr{H}^2}}{1-r},$$

for every point z in the circle $\{r = |z|\}$.

In particular, convergence in the Hardy spaces implies uniform converge in compacta of the unit disk.

2.1.1 Hardy Space of the Upper Half-Plane

Let Π denote the upper half-plane of the complex plane, that is,

$$\Pi = \{x + iy \in \mathbb{C} \colon y > 0\}.$$

We define the *Hardy space of the upper half-plane* $\mathscr{H}^2(\Pi)$ as those functions F holomorphic in Π for which the norm

$$\|F\|^2_{\mathscr{H}^2(\Pi)} = \sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dx$$

is finite. As well as in the Hardy space of the unit disk, the boundary function

$$F^*(x) = \lim_{y \to 0} F(x + iy)$$

exists for almost every x in $\mathbb{R}$. Once again, the norm in $\mathscr{H}^2(\Pi)$ can be computed in the boundary,

$$\|F\|^2_{\mathscr{H}^2(\Pi)} = \int_{-\infty}^{+\infty} |F^*(x)|^2 \, dx,$$

and $\mathscr{H}^2(\Pi)$ becomes a Hilbert space if we define an inner product in the obvious way. We will denote the boundary function simply by F, identifying a function in $\mathscr{H}^2(\Pi)$ with its boundary values.

This shows that the Hardy space of the upper half-plane can be identified with a subspace of the Lebesgue space $L^2(\mathbb{R})$ formed by all measurable functions square-integrable over the real line. This subspace is completely characterized via the Fourier transform by a classical theorem of Paley and Wiener [38]. The *Fourier transform* of a function f in $L^1(\mathbb{R})$, the space of Lebesgue measurable functions with integrable modulus over $\mathbb{R}$, is defined as

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} f(x) e^{-itx} \, dx, \qquad t \in \mathbb{R}. \tag{2.2}$$

The Fourier transform is an intensively studied operator that possesses many useful properties. Among them is Plancherel Theorem, see [45, Theorem 9.13] or [7, Theorem X.6.16] for instance.

Theorem 2.1.4 (Plancherel Theorem). *The Fourier transform can be extended from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ in such a way that*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R})} = 2\pi \|f\|_{L^2(\mathbb{R})}, \quad \textit{for all } f \in L^2(\mathbb{R})$$

It has to be remarked at this point that the Fourier transform is sometimes defined multiplying the integral in (2.2) by the factor $1/\sqrt{2\pi}$. In that case Plancherel Theorem asserts that the Fourier transform can be extended to an isometric isomorphism on $L^2(\mathbb{R})$.

Fourier transform can be used to give a description of those functions in $L^2(\mathbb{R})$ that belong to $\mathscr{H}^2(\Pi)$. Recall that $\mathscr{H}^2$ can be identified with the subspace of those functions in $L^2(\mathbb{T})$ such that its negative Fourier coefficients vanish. Paley-Wiener theorem provides a similar characterization for the Hardy space of the upper half-plane: $\mathscr{H}^2(\Pi)$ is the subspace of $L^2(\mathbb{T})$ form by all functions whose Fourier transforms vanish on $(-\infty, 0]$, see [45, Chapter 19] or [18, p. 88].

Theorem 2.1.5 (Paley-Wiener, 1934). *To each function $F \in \mathscr{H}^2(\Pi)$ corresponds a function $f \in L^2(0,\infty)$ such that*

$$F(z) = \int_0^\infty f(t)e^{itz}\,dt, \quad z \in \Pi$$

and

$$\|F\|^2_{\mathscr{H}^2(\Pi)} = 2\pi \int_0^\infty |f(t)|^2\,dt.$$

Moreover, Plancherel's theorem asserts that $(\mathcal{F}F)(t) = f(t)$ for almost every t in $\mathbb{R}$. In particular, $(\mathcal{F}F)(t)$ vanishes for almost every $t < 0$.

In other words, this theorem shows that $\mathcal{F}(\mathscr{H}^2(\Pi)) = L^2(0,+\infty)$ and that $\mathcal{F}$ is an isomorphism between $\mathscr{H}^2(\Pi)$ and $L^2(0,+\infty)$. This is a very useful result as it enables one pass to the Fourier transform of a function in the Hardy space and perform calculations in the easily understood space $L^2(0,+\infty)$.

To end this section we will show up an isometric isomorphism between the Hardy space of the unit disk and the Hardy space of the upper half-plane. The isomorphism is, see [26, p. 106],

$$\begin{aligned} T:\ \mathscr{H}^2 &\longrightarrow \mathscr{H}^2(\Pi) \\ f &\longmapsto Tf(w) = \frac{1}{\sqrt{\pi}} f\left(\frac{w-i}{w+i}\right)\frac{1}{w+i} \end{aligned}$$

Observe that the *Cayley transform* $\sigma(w) = (w-i)/(w+i)$ maps conformally the upper half-plane onto the unit disk and in particular, it maps $\mathbb{R} \cup \{\infty\}$ onto $\mathbb{T}$. Thus, an obvious change of variables shows that

$$\begin{aligned} \|Tf\|^2_{\mathscr{H}^2(\Pi)} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left|f\left(\frac{w-i}{w+i}\right)\right|^2 \frac{1}{1+w^2}\,dw \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} |f(e^{i\theta})|^2\,d\theta \\ &= \|f\|^2_{\mathscr{H}^2}. \end{aligned}$$

2.2 Composition Operators

Given a holomorphic self-map of the unit disk into itself, $\varphi : \mathbb{D} \to \mathbb{D}$, we can define the *composition operator* C_φ as

$$C_\varphi f = f \circ \varphi$$

for every holomorphic function f. The class of composition operators has been intensively studied in the second half of last century. Part of the interest in studying composition operators comes from the fact that they are a transversal class inside the class of bounded operators acting on a Hilbert space. That is, given a concrete operator theoretic property, not all composition operators satisfy it. This fact, together with its naive definition, makes the study of composition operators as a testing set to generalize results to all bounded operators. The books [10] and [47] are well known introductions to the subject.

Note that composition preserves linearity and composition of holomorphic maps is a holomorphic map as well. Hence, it is a consequence of the Closed Graph Theorem that for every holomorphic self-map φ of $\mathbb{D}$, the operator C_φ is bounded on $\mathscr{H}(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$ endowed with the topology of uniform convergence on compacta. If we study the composition operator acting on spaces smaller than $\mathscr{H}(\mathbb{D})$, it is not at all obvious that the composition operator is still bounded. In fact, the boundedness, together with compactness, in different spaces of analytic functions have attracted most of the research devoted to composition operators. As for the Hardy space, the boundedness can be deduced from a classical theorem of J. E. Littlewood [30]. We state here a weaker version enough for our aims.

Theorem 2.2.1 (Littlewood's Subordination Theorem, 1925). *Let φ be a holomorphic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$. Then for each $f \in \mathscr{H}^2$, the function $f \circ \varphi$ belongs to $\mathscr{H}^2$ and $\|f \circ \varphi\|_{\mathscr{H}^2} \leq \|f\|_{\mathscr{H}^2}$.*

Littlewood's Theorem implies the boundedness of composition operators induced by holomorphic self-maps of $\mathbb{D}$ fixing the origin. For general holomorphic self-maps of $\mathbb{D}$, the boundedness is a consequence of Littlewood's Theorem and two basic properties of composition operators.

Proposition 2.2.2. *Let φ and ψ be holomorphic self-maps of the unit disk. Then $C_\varphi C_\psi$ is also a composition operator $C_\varphi C_\psi = C_{\psi\circ\varphi}$.*

Proposition 2.2.3 ([10, Theorem 1.6]). *Let φ be a holomorphic self-map of the unit disk. The composition operator C_φ acting on the Hardy space is invertible if and only if φ is a conformal mapping of $\mathbb{D}$ onto itself. In this case, $C_\varphi^{-1} = C_{\varphi^{-1}}$.*

Therefore all composition operators are bounded in the Hardy space and their norms can be estimated as follows.

Theorem 2.2.4 ([10, Corollary 3.7]). *Let φ be a holomorphic self-map of the unit disk. Then the composition operator C_φ is bounded in $\mathscr{H}^2$ and*

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/2} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}.$$

2.3 Linear Fractional Transformations

Due to the vast diversity of behaviors found among the class of composition operators, we have to restrict our study to a particular subclass of those operators. We will focus on the composition operators induced by linear fractional transformations taking the unit disk into itself.

Recall that a *linear fractional transformation*, also known as *Möbius transformation*, is a mapping of the form

$$\sigma(z) = \frac{az+b}{cz+d}, \tag{2.3}$$

where a, b, c and d are complex numbers such that $ad - bc \neq 0$, condition which is necessary and sufficient to guarantee that σ is not constant. With the usual conventions for the arithmetic with ∞, every linear fractional transformation is a bijective conformal self-map of the Riemann Sphere, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and they form a group under composition. In fact, they are the automorphism group of the Riemann Sphere. We will denote the set of all linear fractional transformations as $\mathrm{Aut}(\widehat{\mathbb{C}})$. Each of the linear fractional maps transforms a circle in $\widehat{\mathbb{C}}$, that is a circle or a line in the complex plane, to another circle in $\widehat{\mathbb{C}}$.

Each linear fractional map can be represented as a square two dimensional matrix with complex coefficients. To the linear fractional transformation described in (2.3) corresponds the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that the restriction $ad - bc \neq 0$ is equivalent to the condition that the determinant of the above matrix be nonzero. Observe that this representation is not unique, since matrices that differ by a non-zero scalar multiple represent the same linear fractional map. However, we can define a mapping

$$\begin{aligned} \pi : \quad \mathrm{GL}(2, \mathbb{C}) &\longrightarrow \mathrm{Aut}(\widehat{\mathbb{C}}), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \frac{az + b}{cz + d}, \end{aligned}$$

where $\mathrm{GL}(2, \mathbb{C})$ is the general linear group of degree 2, that is, the set of two dimensional invertible matrices with complex coefficients together with the operation of ordinary matrix multiplication. The utility of the matrix representation relies on the fact that the map π is a surjective group homomorphism. Thus the composition of two linear fractional transformations corresponds precisely to matrix multiplication of the corresponding matrices:

$$\pi(A) \circ \pi(B) = \pi(AB).$$

The mapping π is not an isomorphism since $\pi(A) = \pi(\lambda A)$ for every matrix A in $\mathrm{GL}(2, \mathbb{C})$ and every non-zero complex number λ.

Two linear fractional transformations, σ and τ, are said *conjugate* if there exists a third linear fractional map ϕ such that

$$\sigma = \phi \circ \tau \circ \phi^{-1}.$$

Thus, because of the homomorphism π, conjugate linear fractional transformations correspond to similar matrices. This concept of conjugation leads to a classification of linear fractional maps according to an invariant under conjugation, the trace.

The *trace* of a linear fractional transformation $\chi(\sigma)$ is the square of the trace, that is, the sum of the elements in the main diagonal, of a matrix A such that $\det A = 1$ and $\pi(A) = \sigma$. That is, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has $\det A = 1$ and $\pi(A) = \sigma$, then $\chi(\sigma) = (a+d)^2$. Note that in the definition the square of the trace of the matrix A is used in such a way that A and $-A$ have both the determinant equal to one but $\mathrm{tr}(-A) = -\mathrm{tr}(A)$. Four different types of linear fractional transformations arise: parabolic, elliptic, hyperbolic and loxodromic.

Theorem 2.3.1 (Classification by trace). *Let σ be a linear fractional transformation that is not the identity. Then σ is loxodromic if and only if its trace $\chi(\sigma)$ is not a positive number. In case $\chi(\sigma)$ is a positive number, then σ is:*

- *hyperbolic if and only if* $\chi(\sigma) > 4$,
- *parabolic if and only if* $\chi(\sigma) = 4$,
- *elliptic if and only if* $0 < \chi(\sigma) < 4$.

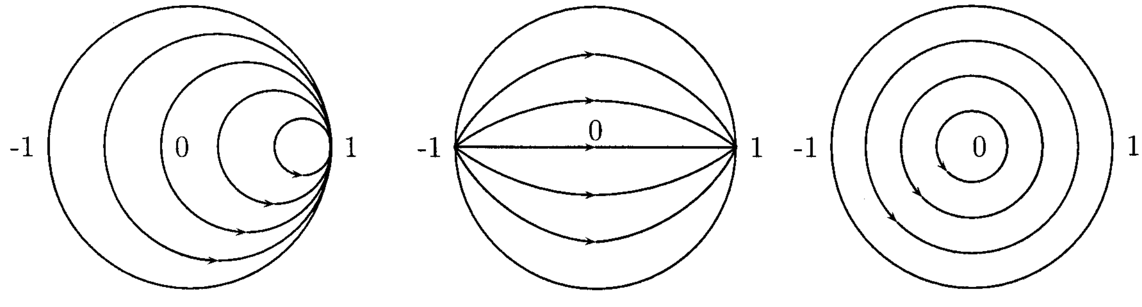
It turns out that the concept of trace of a linear fractional transformation σ is intimately related with the fixed points that σ has in $\widehat{\mathbb{C}}$. It can be shown that every linear fractional transformation, except the identity, has one or two fixed points in the Riemann sphere. The number of fixed points is invariant under conjugation. Therefore, each linear fractional transformation is conjugate to a normal form σ that has a single point at ∞ or two fixed points at 0 and ∞. The parabolic maps are the ones with just one fixed point, the others have two fixed points. In the former case σ is a translation, $\sigma(z) = z + a$ for some complex number a, and in the latter case it is a complex dilation $\sigma(z) = \lambda z$ for some complex number λ. According to the values of λ, the transformation is elliptic, if $|\lambda| = 1$, hyperbolic, if $\lambda > 0$, or loxodromic for the rest of values of λ. Thus, parabolic maps conjugate to translations, elliptic maps to rotations, hyperbolic ones to positive dilations and loxodromic maps to complex dilations. In fact, the hyperbolic maps can be seen as a particular case of loxodromic maps.

Since we will work with composition operators induced by linear fractional maps, our interest will be focused in the subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}})$ consisting of self-maps of the unit disk. The previous classification for linear fractional transformations leads to a classification for linear fractional self-maps of $\mathbb{D}$.

Theorem 2.3.2 (Classification of linear fractional self-maps of $\mathbb{D}$). *Suppose that σ is a linear fractional self-map of $\mathbb{D}$. Then:*

- *If σ is parabolic, then it fixed point is on $\partial\mathbb{D}$. Equivalently, σ is parabolic if it is conjugated to a translation in the upper half-plane.*
- *If σ is hyperbolic, it has an attractive fixed point in $\overline{\mathbb{D}}$ and the other fixed point outside of $\mathbb{D}$. Both fixed points are on $\partial\mathbb{D}$ if and only if σ is an automorphism of $\mathbb{D}$.*
- *If σ is loxodromic or elliptic, one fixed point is in $\mathbb{D}$ and the other fixed point is outside of $\overline{\mathbb{D}}$. The elliptic ones are precisely the automorphism of $\mathbb{D}$ with this fixed point configuration.*

We close this section with a drawing reflecting the dynamics of the different linear fractional self-maps of the unit disk.



Parabolic Hyperbolic Elliptic

2.4 Invariant Subspaces

Once we have introduced Hardy spaces and composition operators, it remains to introduce the third main character in this work, invariant subspaces. All definitions and properties are taken from Radjavi and Rosenthal's book [39].

By a *subspace* of a Hilbert space, we will always mean a subset of the space that is closed in the norm topology and which is also closed under the vector space operations. Given S a nonempty set of a Hilbert space, the *span of* S is the linear subspace formed by all finite linear combinations of elements of S. We will denote it by

$$\operatorname{span} S.$$

The span of the empty set is the subspace $\{0\}$. The closure of the span of S is a subspace that will be denoted by $\overline{\operatorname{span}}\, S$. Note that $\overline{\operatorname{span}}\, S$ coincides with the intersection of all linear subspaces containing S.

If T is a bounded operator acting on a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ a subspace of $\mathcal{H}$, we say that $\mathcal{M}$ is an *invariant subspace* of T –or $\mathcal{M}$ is invariant under T– if $T\mathcal{M} \subseteq \mathcal{M}$. The trivial subspaces, $\{0\}$ and $\mathscr{H}$, are always invariant under every operator bounded in $\mathscr{H}$. If $\mathcal{M}$ is invariant under T, then it makes sense to consider the *restriction of the operator* T to the subspace $\mathcal{M}$. We will denote this restriction by $T|_M$.

The set of all invariant subspaces of an operator $T : \mathscr{H} \to \mathscr{H}$ is a lattice that will be denoted by $\mathrm{Lat}_{\mathscr{H}} T$. When there is no risk of confusion, we will drop the subindex $\mathscr{H}$ and simply write $\mathrm{Lat}\, T$. Recall that a *lattice* is a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound. A lattice is said *complete* if every nonempty subset of the lattice has a least upper bound and a greatest lower bound.

It can be proved that the collection of invariant subspaces of a bounded operator T is a complete lattice under inclusion as an order operation. The least upper bound of a subset of $\mathrm{Lat}\, T$ is the closure of the span of the union of the elements of the subset. The greatest lower bound of a subset of $\mathrm{Lat}\, T$ is the intersection of its elements.

The lattice of invariant subspaces of an operator is isomorphic to the lattice of invariant subspaces of its adjoint, as the following proposition illustrates.

Proposition 2.4.1 ([39, Proposition 0.1]). *Let $\mathscr{H}$ be a Hilbert space and T be a bounded operator acting on $\mathscr{H}$. Then $\mathcal{M} \in \mathrm{Lat}\, T$ if and only if $\mathcal{M}^{\perp} \in \mathrm{Lat}\, T^{\star}$.*

Proof. The proposition follows from the definition of adjoint, since for $f \in \mathcal{M}$ and $g \in \mathcal{M}^{\perp}$, then

$$\langle Tf, g\rangle_{\mathscr{H}} = \langle f, T^{\star}g\rangle_{\mathscr{H}}.$$

The above identity shows that if M is invariant under T, then $T^{\star}g \in \mathcal{M}^{\perp}$ for each $g \in \mathcal{M}^{\perp}$ and vice versa. □

Given a subspace $\mathcal{M}$ of a Hilbert space $\mathscr{H}$, every element f of $\mathscr{H}$ can be written in a unique way in the form $f = g + h$, where $g \in \mathcal{M}$ and $h \in \mathcal{M}^{\perp}$. If $\mathcal{M}$ is a subspace of a Hilbert space, the *projection* onto $\mathcal{M}$ is the operator defined by $P_{\mathcal{M}}f = g$, where $f = g + h$ with $g \in \mathcal{M}$ and $h \in \mathcal{M}^{\perp}$. From basic properties of the Hilbert space, it can be deduced that every projection is a bounded operator with norm equal one, unless the projection onto the trivial subspace $\{0\}$. Even more, every projection is self-adjoint, $P_{\mathcal{M}} = P^{\star}_{\mathcal{M}}$, and idempotent, $P^2_{\mathcal{M}} = P_{\mathcal{M}}$.

Theorem 2.4.2 ([39, Theorem 0.1]). *Let $\mathscr{H}$ be a Hilbert space. If T is a bounded operator and $P_{\mathcal{M}}$ is the projection onto $\mathcal{M}$, then $\mathcal{M} \in Lat\ T$ if and only if $TP_{\mathcal{M}} = P_{\mathcal{M}}TP_{\mathcal{M}}$.*

Proof. If $\mathcal{M}$ is invariant under T, then for each $x \in \mathscr{H}$ we have $TP_{\mathcal{M}}x \in \mathcal{M}$. Thus $P_{\mathcal{M}}TP_{\mathcal{M}}x = TP_{\mathcal{M}}x$. Conversely, if $TP_{\mathcal{M}} = P_{\mathcal{M}}TP_{\mathcal{M}}$ then for all $x \in \mathcal{M}$, $Tx = P_{\mathcal{M}}Tx$ since $P_{\mathcal{M}}x = x$. Therefore Tx belongs to $\mathcal{M}$, since $P_{\mathcal{M}}\mathscr{H} = \mathcal{M}$, and $\mathcal{M} \in \text{Lat}\ T$. □

The decomposition $\mathcal{M} \oplus \mathcal{M}^{\perp}$ of $\mathscr{H}$ induces a block matrix representation of every bounded operator T acting on $\mathscr{H}$ as

$$T = \begin{pmatrix} P_{\mathcal{M}}TP_{\mathcal{M}} & P_{\mathcal{M}}TP_{\mathcal{M}^{\perp}} \\ P_{\mathcal{M}^{\perp}}TP_{\mathcal{M}} & P_{\mathcal{M}^{\perp}}TP_{\mathcal{M}^{\perp}} \end{pmatrix}.$$

If the subspace $\mathcal{M}$ belongs to $\text{Lat}\,T$, then $P_{\mathcal{M}^{\perp}}TP_{\mathcal{M}} = 0$ and we can write the operator as

$$T = \begin{pmatrix} P_{\mathcal{M}}TP_{\mathcal{M}} & P_{\mathcal{M}}TP_{\mathcal{M}^{\perp}} \\ 0 & P_{\mathcal{M}^{\perp}}TP_{\mathcal{M}^{\perp}} \end{pmatrix}.$$

There is a particularly relevant class of invariant subspaces. The subspace $\mathcal{M}$ is a *reducing subspace* if $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are invariant under T.

Theorem 2.4.3 ([39, Theorem 0.2]). *Let T be a bounded operator acting on the Hilbert space $\mathscr{H}$ and $\mathcal{M}$ a subspace of $\mathscr{H}$. Then the following assertions are equivalent:*

1. *The subspace $\mathcal{M}$ is reducing for T.*

2. *The projection $P_{\mathcal{M}}$ commutes with T, that is, $P_{\mathcal{M}}T = TP_{\mathcal{M}}$.*

3. *The subspace $\mathcal{M}$ is invariant under both T and $T^{\star}$.*

Proof. For the equivalence between assertions *1* and *2*, let $\mathcal{M}$ be a reducing subspace for T. It is easily checked that $P_{\mathcal{M}^{\perp}} = I - P_{\mathcal{M}}$, being I the identity operator. Since $\mathcal{M}^{\perp}$ is invariant under T, Theorem 2.4.2 implies that

$$(I - P_{\mathcal{M}})T = (I - P_{\mathcal{M}})T(I - P_{\mathcal{M}}).$$

The above equation is equivalent to $P_{\mathcal{M}}TP_{\mathcal{M}} = P_{\mathcal{M}}T$. Since $\mathcal{M}$ is also invariant under T, Theorem 2.4.2 implies that $TP_{\mathcal{M}} = P_{\mathcal{M}}TP_{\mathcal{M}}$. Therefore, $TP_{\mathcal{M}} = P_{\mathcal{M}}T$.

For the converse, assume $TP_{\mathcal{M}} = P_{\mathcal{M}}T$. For each function $f \in \mathcal{M}$, we have $P_{\mathcal{M}}f = f$ and thus $Tf = P_{\mathcal{M}}Tf$ what is equivalent to say that f is in $\mathcal{M}$. Thus $\mathcal{M}$ is invariant under T. Note that $TP_{\mathcal{M}} = P_{\mathcal{M}}T$ implies that T also commutes with $P_{\mathcal{M}^{\perp}}$, since $T(I - P_{\mathcal{M}}) = (I - P_{\mathcal{M}})T$. An analogous argument shows that $\mathcal{M}^{\perp}$ is invariant under T.

The equivalence of assertions *1* and *3* is a direct consequence of Proposition 2.4.1. □

From equivalence 2 in above theorem, it can be seen that the subspace $\mathcal{M}$ reduces T if and only if the decomposition of T with respect to $\mathcal{M} \oplus \mathcal{M}^{\perp}$ has the following diagonal form

$$T = \begin{pmatrix} P_{\mathcal{M}}TP_{\mathcal{M}} & 0 \\ 0 & P_{\mathcal{M}^{\perp}}TP_{\mathcal{M}^{\perp}} \end{pmatrix}.$$

To end this section, we introduce a notion closely related to that of invariant subspace. An operator T acting on a Hilbert space $\mathscr{H}$ is said to be *cyclic* if there is a vector x in the space whose *orbit*,

$$\mathrm{Orb}\,(T, x) = \{T^n x \colon n = 0, 1, 2, \ldots\},$$

has dense linear span. In this case, x is called a *cyclic vector* for T. The connection with invariant subspaces comes from the following fact. For each vector x in $\mathscr{H}$, the closed linear span of $\operatorname{Orb}(T,x)$ is the smallest invariant subspace that contains x.

In addition, we can characterize the cyclic vectors of an operator in term of its invariant subspaces. Note that in case x is not cyclic for T, then $\overline{\operatorname{span}}\operatorname{Orb}(T,x)$ is an invariant subspace of T. Conversely, if x belongs to a non-trivial subspace $\mathcal{M}$ and $\mathcal{M}$ belongs to $\operatorname{Lat} T$, then $\operatorname{Orb}(T,x)$ lies completely in $\mathcal{M}$ and thus x cannot be cyclic. Thus, a vector x is cyclic for T if and only if it does not belong to $\bigcup_{\mathcal{M}\in\operatorname{Lat} T} \mathcal{M}$.

Finally, a *cyclic subspace* $\mathcal{M}$ of the Hilbert space $\mathscr{H}$ is a subspace that contains at least one vector whose orbit has dense linear span in $\mathcal{M}$.

2.5 Invariant Subspaces and Composition Operators so Far

In next chapters we will study invariant subspaces of composition operators induced by linear fractional self-maps of the unit disk. This may seems a narrow goal since the set of linear fractional maps is insignificant when it is related to the class of holomorphic self-maps of $\mathbb{D}$. However, in spite of their apparent simplicity, composition operators induced by linear fractional self-maps of $\mathbb{D}$ exhibit a wide diversity of behaviours. This diversity can be seen in their spectra. See [9] where some examples show that linear fractional transformations give rise to most of the major spectral types.

Other instance of this diversity appears in the study of cyclic properties of composition operators. At first glance, the study of cyclic phenomena its simpler than characterizing the lattice of invariant subspaces. This is due to the fact that once characterized the lattice of an operator, its cyclic vectors are those that do not belong to the union of invariant subspaces. Among the papers studying cyclic composition operators acting on the Hardy space, stands out the thorough study done in [6]. The authors study composition operators induced by linear fractional self-maps of $\mathbb{D}$ characterizing its cyclicity and hypercyclicity, a stronger form of cyclicity where the orbit itself of a vector is dense without the aid of its linear span. This work was improved and generalized in [17] where cyclicity and hypercyclicity where studied in different weighted Hardy spaces. In this work the authors restrict their study to linear fractional self-maps of $\mathbb{D}$ as well. They also covered the study of the concept of supercyclicity, a notion half way between cyclicity and hypercyclicity. Next are collected the cyclic properties of composition operators induced by linear fractional maps acting on the Hardy space.

TYPE OF φ	Cyclicity of C_φ	Example
ELLIPTIC RATIONAL ROTATION	Non-cyclic	$e^{2\pi i/3}z$
ELLIPTIC IRRATIONAL ROTATION	Cyclic	$e^{2i/3}z$
PARABOLIC NON-AUTOMORPHISM	Cyclic	$\dfrac{1}{2-z}$
PARABOLIC AUTOMORPHISM	Hypercyclic	$\dfrac{(1+i)z-1}{z+i-1}$
HYPERBOLIC NON-AUTOMORPHISM	Hypercyclic	$\dfrac{1+z}{2}$
HYPERBOLIC AUTOMORPHISM	Hypercyclic	$\dfrac{2z+1}{z+2}$

This seems to confirm the need of restrict our study to composition operators induced by linear fractional self-maps of $\mathbb{D}$ in order to obtain concrete results. As far as we know, no complete characterization of the lattice of invariant subspaces of any of such operators has been given. In fact, the number of works exploring invariant subspaces of composition operators is short. Probably the first attempt to describe the properties of lattices of invariant subspaces of composition operators is [31]. In this work, the author studies the consequences of the lattice of one composition operator being contained into another. Also, some results concerning the structure of an invariant subspace shared by two composition operators are given.

A particularly interesting result about invariant subspaces of composition operators is included in [37]. In this work it is carried out a deep study of the invariant subspaces of algebras generated by invertible compositions operators. One of the most striking results included in the work is a corollary that relates the invariant subspace problem with the lattice of invariant subspaces induced by a hyperbolic automorphism. The result states that in case φ is a hyperbolic automorphism of $\mathbb{D}$ acting on the Hardy space, then solving the invariant subspace problem is equivalent to showing that each minimal invariant subspace for C_φ is one-dimensional.

> Set $\varphi(z) = \dfrac{2z+1}{z+2}$. Then characterizing $\mathrm{Lat}_{\mathscr{H}^2} C_\varphi$ solves the Invariant Subspace Problem.

This striking result shows how chaotic can be the behaviour of C_φ when φ is the apparently innocent linear fractional transformation $(2z+1)/(z+2)$. Characterizing its invariant subspace lattice is a problem as difficult as the invariant subspace problem. Consequently, some authors paid attention to the study of minimal invariant subspaces of the composition operator induced by a hyperbolic automorphism.

Some works have appeared exploring the nature of the minimal invariant subspaces of the hyperbolic automorphism case. This is the case of R. Mortini's paper [34] or the works of V. Matache [32] and [33]. The authors study the eigenfunctions of the operator and certain cyclic subspaces with the aim of characterizing its minimal invariant subspaces. However, the understanding of the structure of the invariant subspaces of this operator is far to be achieved.

Chapter 3

An Appetizer: The Elliptic Composition Operator

Recall from the classification of linear fractional transformations in Section 2.3 that every elliptic transformation φ_a is an automorphism of $\mathbb{D}$ with a fixed point a in the unit disk. Every elliptic transformation is conjugated to a rotation (centered at 0) of the complex plane by an angle θ. Let

$$\psi_a(z) = \frac{a-z}{1-\overline{a}z}$$

be the automorphism of $\mathbb{D}$ that interchanges the origin and the point a. Since ψ_a is involutive, we have

$$\varphi_a = \psi_a \circ \sigma \circ \psi_a, \tag{3.1}$$

where $\sigma(z) = e^{i\theta}z$ for all $z \in \mathbb{D}$.

Equation (3.1) induces a similarity between the composition operator induced by φ_a and the one induced by σ. Recall that given two Hilbert spaces, $\mathscr{H}_1$ and $\mathscr{H}_2$, and two bounded operators, $T_1 : \mathscr{H}_1 \to \mathscr{H}_1$ and $T_2 : \mathscr{H}_2 \to \mathscr{H}_2$, the operators T_1 and T_2 are said *similar* if there exists an invertible operator $S : \mathscr{H}_1 \to \mathscr{H}_2$ such that $T_1 = S^{-1}T_2S$. Thus, from Propositions 2.2.2 and 2.2.3, we obtain

$$C_{\varphi_a} = C_{\psi_a} C_\sigma C_{\psi_a}. \tag{3.2}$$

Most of the "good" operator theoretic properties remain unchanged under similarity. In particular, similar operators share the same spectrum. In addition, if one characterizes the invariant subspaces of an operator, then also characterizes the invariant subspaces of every operator in its similarity orbit. For this reasons, we will focus our study on the composition operators $C_{e^{i\theta}z}$.

3.1 Spectra and Eigenvectors

Given an operator T acting on a Hilbert space $\mathscr{H}$, its *spectrum* is a subset of the complex plane defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not invertible in } \mathscr{H}\},$$

where, as usual, I denotes the identity operator in $\mathscr{H}$. As said before, similar operators have the same spectrum. This can be proved as follows. Suppose $T = V^{-1}SV$ and fix a number $\lambda \in \mathbb{C}$. Then, $T - \lambda I$ is invertible if and only if $V^{-1}(T - \lambda I)V$ is invertible, what is equivalent to say that $S - \lambda I$ is invertible. Therefore, $\sigma(T) = \sigma(S)$. Thus the spectra of elliptic composition operators will be totally determined by the spectra of composition operators induced by rotations centered at 0.

It is noteworthy to distinguish a subset of special points in the spectrum, the *eigenvalues*. A complex number λ in $\sigma(T)$ is said an eigenvalue of the operator T if there exists a vector $f \in \mathscr{H}$ different from 0 such that $Tf = \lambda f$. In that case, f is said to be an *eigenvector* of λ. When the elements of the Hilbert space are functions, eigenvectors are also referred to as *eigenfunctions*. Since all elements of the Hilbert or Banach spaces considered in this work are functions, both terms, eigenvector and eigenfunction, will be used indistinctively.

The spectra of the operators $C_{e^{i\theta}z}$ were characterized by E. A. Nordgren in his seminal work [35] as

$$\sigma(C_{e^{i\theta}z}) = \overline{\{e^{ik\theta} \colon k = 0, 1, 2, \ldots\}}.$$

Thus we have two different spectral pictures for an operator $C_{e^{i\theta}z}$. Recall that a point $e^{i\theta}$ in $\mathbb{T}$ is called a *p-th root of unity* if there exists a natural number p different from 0 such that $e^{ip\theta} = 1$. A p-th root of unity is called *primitive* if $e^{ik\theta} \neq 1$ for $k = 1, 2, \ldots, p-1$. Observe that for an arbitrary $e^{i\theta}$ in $\mathbb{T}$ there are two options: either the number is not a root of unity, in that case $e^{ik\theta} \neq 1$ for all $k = 1, 2, \ldots$, or the number is a root of unity, in that case there is a number p such that $e^{i\theta}$ is a primitive p-th root of unity. Note also that it can be decided wether $e^{i\theta}$ is a root of unity or not just by looking at θ. The number $e^{i\theta}$ is a root of unity if and only if θ is a rational multiple of 2π. Conversely, $e^{i\theta}$ is not a root of unity if and only if θ is an irrational multiple of 2π.

Hence the spectra of these operators is as follows. If $e^{i\theta}$ is not a root of unity, then the powers of $e^{i\theta}$ are dense in the unit circle and therefore the spectrum of $C_{e^{i\theta}z}$ equals $\mathbb{T}$. On the contrary, if $e^{i\theta}$ is a primitive p-th root of unity, then the spectrum of $C_{e^{i\theta}z}$ is formed by p isolated points, all the p-th roots of unity. This difference of spectra will be reflected on the structure of the lattices of invariant subspaces of the operators.

This dichotomy can also be seen in the matrix representation of the operator $C_{e^{i\theta}z}$. The standard orthonormal basis of $\mathscr{H}^2$ is the set $\{z^n\}_{n\geq 0}$. Every element of this set is an eigenvector of $C_{e^{i\theta}z}$ since

$$C_{e^{i\theta}z} z^n = e^{in\theta} z^n, \qquad \text{for all } n \in \mathbb{N}.$$

This shows that the infinite matrix associated to $C_{e^{i\theta}z}$ with respect to the basis $\{z^n\}_{n\geq 0}$ is diagonal,

$$C_{e^{i\theta}z} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & e^{i\theta} & 0 & \ldots & 0 & \ldots \\ 0 & 0 & e^{i2\theta} & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & e^{in\theta} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Therefore in case $e^{i\theta}$ is not a root of unity then all the entries in the main diagonal are different. In case $e^{i\theta}$ is a primitive p-th root of unity, then the diagonal has only p different entries. This representation of $C_{e^{i\theta}z}$ as a diagonal operator will be in the backstage of the characterization of its eigenvectors.

Let's start the study of the case when $e^{i\theta}$ is not a root of unity. Suppose that the complex number λ is an eigenvalue of $C_{e^{i\theta}z}$, then there exist a function $f = \sum_{n=0}^{\infty} a_n z^n$ in $\mathscr{H}^2$ different from 0 and such that $C_{e^{i\theta}z} f = \lambda f$, that is,

$$\sum_{n=0}^{\infty} a_n e^{in\theta} z^n = \sum_{n=0}^{\infty} \lambda a_n z^n.$$

Since f is analytic, it turns out that

$$e^{in\theta} a_n = \lambda a_n \text{ for all } n = 0, 1, \ldots.$$

Since $e^{i\theta}$ is not a root of unity, all $e^{in\theta}$ are different and the above display implies that all the Taylor coefficients a_n are zero, but one. Therefore,

$$a_n = 0, \text{for all } n \neq k \qquad \text{and} \qquad \lambda = e^{ik\theta}.$$

We have proved

Proposition 3.1.1. *If $e^{i\theta}$ is not a root of unity, then all the eigenvalues of $C_{e^{i\theta}z}$ are $e^{in\theta}$ with corresponding eigenvector z^n for all $n = 0, 1, 2, \ldots$*

Now suppose that $e^{i\theta}$ is a primitive p-th root of unity. Let λ be an eigenvector of $C_{e^{i\theta}z}$ with corresponding eigenvector $f = \sum_{n=0}^{\infty} a_n z^n$ in $\mathscr{H}^2$. Then equation $C_{e^{i\theta}z} f = \lambda f$ becomes

$$\sum_{n=0}^{\infty} a_n e^{in\theta} z^n = \sum_{n=0}^{\infty} \lambda a_n z^n.$$

The above equality is equivalent to

$$(1a_0, e^{i\theta} a_1, \ldots, e^{i(p-1)\theta} a_{p-1}, 1a_p, e^{i\theta} a_{p+1}, \ldots) = (\lambda a_0, \lambda a_1, \lambda a_2, \ldots).$$

From where we obtain p equalities:

$$e^{ik\theta}(a_k, a_{p+k}, \ldots, a_{np+k}, \ldots) = \lambda(a_k, a_{p+k}, \ldots, a_{np+k}, \ldots), \quad k = 0, 1, \ldots, p-1.$$

This can only be possible if there exists a natural number $k_0 \in \{0, 1, \ldots, p-1\}$ such that

$$a_n = 0, \qquad \text{for all } n \neq k_0 \pmod{p}.$$

In this case, $\lambda = e^{ik_0\theta}$ and the function f is

$$f = \sum_{n=0}^{\infty} a_{np+k_0} z^{np+k_0}$$

and since f belongs to $\mathscr{H}^2$, the sequence $\{a_n\}$ must be square-summable. Since each function as the one in the above display is obviously an eigenvector for $C_{e^{i\theta}z}$, we have proved

Proposition 3.1.2. *If $e^{i\theta}$ is a primitive p-th root of unity, then the complex number λ is an eigenvalue of $C_{e^{i\theta}z}$ if and only if $\lambda = e^{ik\theta}$ for some $k \in \{0, 1, \dots, p-1\}$. Furthermore, f is an eigenvector corresponding to $e^{ik\theta}$ if and only if*

$$f = \sum_{n=0}^{\infty} a_n z^{np+k}, \qquad \text{for some } \{a_n\}_{n\geq 0} \in \ell^2 \backslash \{0\}.$$

Now the similarity (3.2) allows us to identify all the eigenfunctions of an arbitrary elliptic composition operator.

Corollary 3.1.3. *Let φ_a be an elliptic fractional transformation that fixes a point a in $\mathbb{D}$ conjugated to $\sigma(z) = e^{i\theta} z$ via $\psi_a = \dfrac{a-z}{1-\bar{a}z}$. Then*

1. *If $e^{i\theta}$ is not a root of unity, then $e^{in\theta}$ is an eigenvalue with corresponding eigenvector ψ_a^n for all $n = 0, 1, \dots$*
2. *If $e^{i\theta}$ is a primitive p-th root of unity, then $e^{ik\theta}$ is an eigenvalue of infinite multiplicity for all $k = 0, 1, \dots, p-1$. A function f in $\mathscr{H}^2$ is an eigenvector with eigenvalue $e^{ik\theta}$ if and only if*

$$f = \sum_{n=0}^{\infty} a_n \psi_a^{np+k}, \qquad \text{for some } \{a_n\}_{n\geq 0} \in \ell^2.$$

3.2 The Lattice of Invariant Subspaces

As mentioned, given two similar operators their lattices of invariant subspaces are isomorphic. If T_1 and T_2 are two operators similar under the invertible operator S, then S induces a lattice isomorphism between $\operatorname{Lat} T_1$ and $\operatorname{Lat} T_2$. This is due to the fact that a subspace $\mathcal{M}$ belongs to $\operatorname{Lat} T_1$ if and only if $S\mathcal{M}$ belongs to $\operatorname{Lat} T_2$. Also, since S is invertible, it preserves the inclusions and the least upper bound and the greatest lower bound.

We will therefore study the invariant subspaces of $C_{e^{i\theta}z}$. To characterize the lattice we have to distinguish once again wether $e^{i\theta}$ is a root of unit or not.

However, in both cases every invariant subspace is generated by eigenvectors, that is, every invariant subspace is an *eigenspace.*

Let us suppose that $e^{i\theta}$ is not a root of unity. We will show

Proposition 3.2.1. *If* $f = \sum_{n=0}^{\infty} a_n z^n$ *is in* $\mathscr{H}^2$ *and* $N_f = \{n \in \mathbb{N}\colon a_n \neq 0\}$, *then*

$$\overline{\operatorname{span}}\,\{C^k_{e^{i\theta}z} f\colon k = 0, 1, 2, \ldots\} = \overline{\operatorname{span}}\{z^n\colon n \in N_f\}.$$

In particular, f *is cyclic for* $C_{e^{i\theta}z}$ *if and only if all its Taylor coefficients are different from zero.*

Proof. Set $M_f = \overline{\operatorname{span}}\{z^n\colon n \in N_f\}$. Since $C^k_{e^{i\theta}z} f = \sum_{n=0}^{\infty} a_n e^{ikn\theta} z^n$, the span of orbit of f under $C_{e^{i\theta}z}$ is included in M_f. To prove its density in M_f, let g be in M_f and suppose that

$$\langle C^k_{e^{i\theta}z} f, g\rangle_{\mathscr{H}^2} = 0, \qquad \text{for } k = 0, 1, 2, \ldots$$

The proof will be finished once we have shown that $g = 0$. If the Taylor series of g is $\sum_{n\in N_f} b_n z^n$, then above display becomes

$$0 = \sum_{n\in N_f} a_n e^{ikn\theta} \overline{b_n} = \sum_{n\in N_f} a_n \overline{b_n} (e^{ik\theta})^n, \qquad \text{for } k = 0, 1, 2, \ldots \tag{3.3}$$

Now we define the function $h = \sum_{n\in N_f} a_n \overline{b_n} z^n$. Observe that its Taylor coefficients verify

$$\sum_{n\in N_f} |a_n \overline{b_n}| \leq \left(\sum_{n\in N_f} |a_n|^2\right)^{1/2} \left(\sum_{n\in N_f} |b_n|^2\right)^{1/2} = \|f\|_{\mathscr{H}^2} \|g\|_{\mathscr{H}^2},$$

where Cauchy-Schwarz inequality has been used in the inequality above. Thus the function h belongs to $\mathcal{H}^\infty$ and it is continuous in $\overline{\mathbb{D}}$ [11, Theorem 6.1]. Hence, (3.3) implies that h is identically zero on $\mathbb{T}$ and therefore, by Theorem 2.1.1, h itself has to be the zero function. Therefore, $a_n \overline{b_n} = 0$ for every $n \in N_f$. This implies that $b_n = 0$ and therefore $g = 0$. The proof is finished. □

Since each monomial z^n is an eigenvector of $C_{e^{i\theta}z}$, then every subspace spanned by monomials belongs to $\operatorname{Lat} C_{e^{i\theta}z}$. It turns out that these are all the invariant subspaces. If we denote by $\mathcal{P}(\mathbb{N})$ the power set of $\mathbb{N}$, we can prove

Theorem 3.2.2. *If $e^{i\theta}$ is not a root of unity, then*

$$\operatorname{Lat} C_{e^{i\theta}z} = \{\overline{\operatorname{span}}\{z^n : n \in N\} : N \in \mathcal{P}(\mathbb{N})\}.$$

Proof. Let $\mathcal{M}$ be a subspace in $\operatorname{Lat} C_{e^{i\theta}z}$. We denote by $[z^n]$ the one-dimensional subspace of $\mathscr{H}^2$ spanned by the function z^n. Observe that the projection onto this subspace, $P_{[z^n]}$, is given by $P_{[z^n]}f = \langle f, z^n\rangle_{\mathscr{H}^2} z^n$ for every f in $\mathscr{H}^2$. If we define

$$N = \{n \in \mathbb{N} : P_{[z^n]}\mathcal{M} \neq \{0\}\},$$

then $\mathcal{M} \subseteq \overline{\operatorname{span}}\{z^n : n \in N\}$. The former inclusion is in fact an equality. Indeed, for all n in N, there is a function f in $\mathcal{M}$ such that its n-th Taylor coefficient is different from zero. By Proposition 3.2.1, we have that

$$z^n \in \overline{\operatorname{span}}\,\{C^k_{e^{i\theta}z}f : k = 0, 1, 2, \ldots\} \subseteq \mathcal{M},$$

where the inclusion holds for being $\mathcal{M}$ an invariant subspace of $C_{e^{i\theta}z}$. Hence, $\mathcal{M} \subseteq \overline{\operatorname{span}}\{z^n : n \in N\}$ and the proof is finished. □

Next corollary follows immediately from above proof and Proposition 3.2.1.

Corollary 3.2.3. *If $e^{i\theta}$ is not a root of unity then all its invariant subspaces are cyclic and reducing.*

Now suppose that $e^{i\theta}$ is a primitive p-th root of unity. In this case the situation is different since the operator $C_{e^{i\theta}z}$ is not cyclic. In fact, the orbit of each function under $C_{e^{i\theta}z}$ is finite dimensional as it has at most p elements. For this reason the lattice is richer in this case. Of course if $e^{i\theta} = 1$, then the invariant subspaces of $C_{e^{i\theta}z}$ consists of all subspaces of $\mathscr{H}^2$.

Theorem 3.2.4. *If $e^{i\theta} \neq 1$ is a root of unity, then*

$$\operatorname{Lat} C_{e^{i\theta}z} = \{\overline{\operatorname{span}}\{f : f \in N\} : N \textit{ is a set of eigenvalues}\}.$$

Proof. By definition of eigenvector, every eigenspace belongs to $\operatorname{Lat} C_{e^{i\theta}z}$. To prove the converse, let $\mathcal{M}$ be a subspace of $\mathscr{H}^2$ invariant under $C_{e^{i\theta}z}$. Assume that $e^{i\theta}$ is a p primitive root of the unit. We can decompose any function $f = \sum_{n=0}^{\infty} a_n z^n$ in $\mathscr{H}^2$ as

$$f = f_0 + f_1 + \ldots + f_{p-1},$$

where $f_j = \sum_{n=0}^{\infty} a_{np+j} z^{np+j}$ is an eigenvector for every $j = 0, 1, \ldots, p-1$. The iterates of $C_{e^{i\theta}z}$ acting on f are

$$C^k_{e^{i\theta}z} f = 1 f_0 + e^{ik\theta} f_1 + \ldots + e^{ik(p-1)\theta} f_{p-1}, \qquad k = 0, 1, \ldots, p-1.$$

We claim that

$$\operatorname{span}\{C^k_{e^{i\theta}z} f \colon k = 0, 1, \ldots, p-1\} = \operatorname{span}\{f_j \colon j = 0, 1, \ldots, p-1\}.$$

Observe that above display is equivalent to say that the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & e^{i\theta} & e^{i2\theta} & \ldots & e^{ip\theta} \\ 1 & e^{i2\theta} & e^{i4\theta} & \ldots & e^{i2p\theta} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & e^{i(p-1)\theta} & e^{i(p-1)2\theta} & \ldots & e^{i(p-1)^2\theta} \end{pmatrix}$$

can be triangularized with all its entries in the main diagonal different from zero. This can always be done. Indeed, observe that M is an example of what is known as Vandermonde Matrix [27, p. 29]. Then its determinant is given by formula

$$\det(M) = \prod_{0 \le j < k \le p-1} (e^{ik\theta} - e^{ij\theta}) \neq 0.$$

Since its determinant is not zero, all the eigenvalues of M are different from zero. Schur decomposition, see [27, p. 79], states that M can be triangularized via a unitary matrix as $M = U^*TU$ where T is a upper triangular matrix whose entries in the main diagonal are the eigenvalues of M. This finishes the proof. □

Hence we have completed the characterization of invariant subspaces of an arbitrary composition operator induced by an elliptic transformation.

Corollary 3.2.5. *Let φ be an elliptic fractional transformation. Then* Lat C_φ *consists in all the subspaces of $\mathscr{H}^2$ spanned by eigenvectors of C_φ.*

Chapter 4

The Parabolic Non-Automorphism

This chapter will be devoted to the study of composition operators induced by mappings φ that are parabolic non-automorphisms taking the unit disk into itself. Recall that in Section 2.3 we had characterized this mappings as those that are conjugated with a translation in the upper half plane. Set

$$\sigma(z) = i\frac{1+z}{1-z},$$

the conformal automorphism that takes the unit disk onto the upper half plane, $\Pi = \{x + iy \in \mathbb{C} \colon y > 0\}$, and whose inverse mapping,

$$\sigma^{-1}(z) = \frac{z-i}{z+i},$$

is known as the *Cayley transform*. Therefore,

$$\sigma \circ \varphi \circ \sigma^{-1} = \tau,$$

where $\tau(z) = z + ia$ is translation by ia in Π. Thus from the above display we obtain

$$\varphi = \sigma^{-1} \circ \tau \circ \sigma = \sigma^{-1}(\sigma + ia), \tag{4.1}$$

and after some simple calculations we can express φ as

$$\varphi(z) = \varphi_a(z) = \frac{(2-a)z + a}{-az + 2 + a}, \tag{4.2}$$

for certain complex number a such that $\Re a > 0$. Let us see where does the condition $\Re a > 0$ comes from. Since $z + ia$ is a translation that maps Π into itself, then $\Re a \geq 0$. But indeed, $\Re a > 0$ since φ is not an automorphism and the translation $z + ia$ is an automorphism of Π if and only if $\Re a = 0$. Note that the above formula is valid for a parabolic automorphism as well, but in this case a is a complex number such that $\Re a = 0$.

The goal of this chapter will be to characterize the lattice of invariant subspaces of the composition operator C_{φ_a}, where φ_a is a parabolic non-automorphism that takes the unit disk into itself. Let $\{e_t\}_{t\geq 0}$ be the family of functions defined as

$$e_t(z) = \exp\left(t\frac{z+1}{z-1}\right)$$

and let $\mathbb{F}[0, \infty)$ denote the family of closed subsets of $[0, \infty)$. We will prove the following

Theorem 4.1.1. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then*

$$\operatorname{Lat} C_\varphi = \left\{ \overline{\operatorname{span}} \{e_t : t \in F\} \colon F \in \mathbb{F}[0, \infty) \right\}.$$

To prove this theorem we will need some tools from Gelfand Theory in Banach algebras. These tools will be introduced in next section, where will also study the structure of closed ideals in some particular Banach algebras and the invariant subspaces of the operator of multiplication by a cyclic element in a Banach algebra. After that we will study the spectra of the operator C_{φ_a}, since it will be the key point to establish an isomorphism between $\mathscr{H}^2$ and a Sobolev space that will transform the adjoint of C_{φ_a} into a multiplication operator.

4.1 Banach Algebras Techniques

Recall that an *algebra* over the complex numbers is a vector space $\mathscr{A}$ with a binary operation defined on it, $(a, b) \mapsto ab$, that makes $\mathscr{A}$ into a ring and such that if $\lambda \in \mathbb{C}$ and $a, b \in \mathscr{A}$, then

$$\lambda(ab) = (\lambda a)b = a(\lambda b).$$

The binary operation defined on the algebra is usually referred to as *multiplication*. A *Banach algebra* is a complex vector space $\mathscr{A}$ endowed with a norm $\|\cdot\|$ under which $(\mathscr{A}, \|\cdot\|)$ is a Banach space and with a binary operation, $(a, b) \to ab$, which turns $\mathscr{A}$ into an algebra over the complex numbers. Both structures are linked by the inequality

$$\|ab\| \leq c\|a\|\|b\|, \qquad \text{for all } a, b \in \mathscr{A},$$

for some positive constant c. Although it is not required that $c = 1$, this can always be achieved by replacing the initial norm of $\mathscr{A}$ by an equivalent one, see [7] or [22] for instance. Observe that it is not required that the algebra $\mathscr{A}$ has identity. If there is an element $1 \in \mathscr{A}$ such that $\|1\| = 1$ and $1a = a1 = a$ for all $a \in \mathscr{A}$, then the Banach algebra $\mathscr{A}$ is said to have an *identity*. The Banach algebra $\mathscr{A}$ is said *commutative* whenever $ab = ba$ for all $a, b \in \mathscr{A}$. We would like to stress here that all Banach algebras we are going to deal with will be commutative, hence in what follows whenever we talk about a Banach algebra we will mean a commutative Banach algebra.

4.1.1 Banach Algebras with a Cyclic Element

Recall that an ideal $\mathcal{I}$ of a Banach algebra $\mathscr{A}$ is a subalgebra of $\mathscr{A}$ such that

$$ab \in \mathcal{I} \qquad \text{whenever } a \in \mathscr{A} \text{ and } b \in \mathcal{I}.$$

An element a in $\mathscr{A}$ is called *cyclic* if the subalgebra generated by a,

$$\operatorname{span}\{a^n \colon n \geq 1\},$$

is dense in $\mathscr{A}$. Note that if the Banach algebra $\mathscr{A}$ contains a cyclic element, then it is clearly separable and commutative. If $a \in \mathscr{A}$, we define the *operator of multiplication* by a acting on $\mathscr{A}$ as

$$M_a x = ax, \qquad x \in \mathscr{A}.$$

The continuity of the multiplication in the Banach algebra $\mathscr{A}$ implies that this operator is bounded. The proof of Theorem 4.1.1 will rely heavily on the following

Proposition 4.1.2. *Let $\mathscr{A}$ be a Banach algebra. Then the invariant subspaces of multiplication by a cyclic element are exactly the closed ideals of $\mathscr{A}$.*

Proof. First, since $\mathscr{A}$ has a cyclic element, it is commutative. Let a be a cyclic element of $\mathscr{A}$ and let $\mathcal{L}$ be an invariant subspace of M_a. Clearly,

$$\mathcal{M}_{\mathcal{L}} = \{b \in \mathscr{A} : bx \in \mathcal{L} \text{ for all } x \in \mathcal{L}\}$$

is a closed subalgebra of $\mathscr{A}$. Since $\mathcal{L}$ is an invariant subspace of M_a, we find that $a \in \mathcal{M}_{\mathcal{L}}$ and, therefore, $\mathcal{M}_{\mathcal{L}}$ contains the subalgebra generated by a and, being $\mathcal{M}_{\mathcal{L}}$ closed and a cyclic, it follows that $\mathcal{M}_{\mathcal{L}} = \mathscr{A}$. Hence, $\mathcal{L}$ is an ideal of $\mathscr{A}$. On the other hand, each ideal of $\mathscr{A}$ is invariant with respect to M_a, which finishes the proof. □

4.1.2 The Maximal Ideal Space and the Gelfand Transform

As said before, the proposition just proved will be a cornerstone in the proof of the main theorem of this chapter. As one may suppose, the lattice of invariant subspaces of the composition operator C_{φ_a} will be related to that of a multiplication operator in certain Banach algebra. For this reason we need to go deeper in the theory of Banach algebras and develop certain machinery that will lead to the identification of the closed ideals of certain Banach algebras.

A *multiplicative linear functional* on a Banach algebra $\mathscr{A}$ is a non-trivial linear functional $\varkappa : \mathscr{A} \to \mathbb{C}$ such that such that

$$\varkappa(ab) = \varkappa(a)\varkappa(b) \qquad \text{for each } a, b \in \mathscr{A}.$$

Equivalently, $\varkappa$ is an algebra homomorphism of $\mathscr{A}$ onto the complex numbers.

Lemma 4.1.3 ([28, p. 201]). *Every multiplicative linear functional on a Banach algebra is continuous and has norm bounded by 1.*

The *spectrum* of $\mathscr{A}$ will be denoted as $\Omega(\mathscr{A})$ and is the set of multiplicative linear functionals of $\mathscr{A}$. Note that the spectrum of $\mathscr{A}$ can be turn into a topological space if we equip it with the weak-star topology.

Theorem 4.1.4 ([28, p. 206]). *The spectrum of a Banach algebra with the weak-star topology is a Hausdorff locally compact topological space. If the algebra possesses an identity then its spectrum is compact.*

An ideal $\mathcal{I}$ of a Banach algebra $\mathscr{A}$ is called regular when the quotient algebra $\mathscr{A}/\mathcal{I}$ has identity. Let $\varkappa$ be a multiplicative linear functional, then its kernel $\ker\varkappa$ is an ideal of $\mathscr{A}$. Even more, since $\varkappa \not\equiv 0$ then $\varkappa(\mathscr{A}) = \mathbb{C}$. Thus, the First Isomorphism Theorem implies that the quotient algebra $\mathscr{A}/\ker\varkappa$ is isomorphic to $\mathbb{C}$, that is,

$$\mathscr{A}/\ker\varkappa \cong \mathbb{C}.$$

Therefore, the quotient algebra $\mathscr{A}/\ker\varkappa$ is a field. This implies that $\ker\varkappa$ is a maximal ideal that in fact is regular since $\mathbb{C}$ has an identity. Thus the kernel of every multiplicative linear functional of $\mathscr{A}$ is a maximal regular ideal. If we denote by $\mathfrak{M}$ the set of maximal regulars of the Banach algebra $\mathscr{A}$, we have just constructed a mapping

$$\varkappa \in \Omega(\mathscr{A}) \longmapsto \ker\varkappa \in \mathfrak{M}.$$

Theorem 4.1.5 ([28, p. 202]). *The mapping* $\varkappa \mapsto \ker\varkappa$ *defines a one-to-one correspondence between the spectrum of* $\mathscr{A}$ *and the set of its maximal regular ideals.*

Given $x \in \mathscr{A}$ we can define a mapping on $\mathfrak{M}$ such that for every $\mathcal{M} \in \mathfrak{M}$,

$$\widehat{x}(\mathcal{M}) = x \bmod \mathcal{M}.$$

That is, $\widehat{x}(\mathcal{M})$ is the image of x under the multiplicative linear functional corresponding to the regular maximal ideal $\mathcal{M}$. The mapping $x \mapsto \widehat{x}$ is usually referred to as *Gelfand map* and the function $\widehat{x}$ is called the *Gelfand transform* of x. Since $\mathfrak{M}$ is endowed with the weak-star topology, $\widehat{x}$ is continuous in $\mathfrak{M}$ for each $x \in \mathscr{A}$. In case $\mathscr{A}$ does not have an identity, it can be proved that the set

$$\{M \in \mathfrak{M} \colon |\widehat{x}(M)| \geq \varepsilon\}$$

is compact for every $\varepsilon > 0$ and all $x \in \mathscr{A}$.

Proposition 4.1.6 ([28, p. 207]). *Let $\mathscr{A}$ be a Banach algebra without an identity. The mapping $x \mapsto \widehat{x}$ is a homomorphism from $\mathscr{A}$ into $C_0(\mathfrak{M})$.*

The *Jacobson radical* of a Banach algebra is the intersection of all its regular maximal ideals. A Banach algebra is said *semisimple* if its Jacobson radical is zero. Thus a commutative Banach algebra $\mathscr{A}$ is semisimple if and only if the elements of $\Omega(\mathscr{A})$ separate points of $\mathscr{A}$. Therefore the Jacobson radical is the kernel of the Gelfand transform and we have the following equivalence.

Proposition 4.1.7 ([28, p. 207]). *The Gelfand transform is one-to-one if and only if $\mathscr{A}$ is semisimple.*

To end this subsection we introduce the concept of spectrum of an element of a Banach algebra. If $\mathscr{A}$ is a Banach algebra with an identity 1, then the *spectrum* of $a \in \mathscr{A}$ is the set defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} \colon x - \lambda 1 \text{ is not invertible in } \mathscr{A}\}.$$

For every $a \in \mathscr{A}$, its spectrum is a non-empty compact set of the complex plane. The Gelfand map provides an effective procedure to compute the spectrum of elements of Banach algebras.

Theorem 4.1.8 ([5, p. 16]). *Let $\mathscr{A}$ be a commutative Banach algebra with an identity. Then, for every $a \in \mathscr{A}$, we have*

$$\sigma(a) = \{\widehat{a}(\varkappa) \colon \varkappa \in \Omega(\mathscr{A})\}.$$

The last theorem will be used in next section to compute the spectrum of C_{φ_a}.

4.1.3 Ideals of Semisimple Regular Algebras

Let $\mathscr{A}$ be a Banach algebra and $\mathcal{I}$ be an ideal of $\mathscr{A}$. The *hull* of $\mathcal{I}$, denoted by $h(\mathcal{I})$, is the set of all maximal regular ideals $\mathcal{M}$ such that $\mathcal{I}$ is contained in $\mathcal{M}$. Equivalently, $h(\mathcal{I})$ is the set of all $\mathcal{M} \in \mathfrak{M}$ such that $\widehat{x}(\mathcal{M}) = 0$ for all $x \in \mathcal{I}$.

Let E be a subset of the maximal ideal space $\mathfrak{M}$. The *kernel* of E, denoted by $k(E)$, is the ideal $\bigcap_{\mathcal{M}\in E}\mathcal{M}$. Equivalently, $k(E)$ is the set of all $x \in \mathscr{A}$ such that $\widehat{x}$ equals zero on E.

Thus, if we denote by $\mathcal{P}(\mathfrak{M})$ the power set of $\mathfrak{M}$, we have constructed two mappings; the hull,

$$\begin{aligned} h: \ \{\text{Ideals of } \mathscr{A}\} &\longrightarrow \mathcal{P}(\mathfrak{M}) \\ \mathcal{I} &\longmapsto h(\mathcal{I}) = \{\mathcal{M} \in \mathfrak{M} \colon \mathcal{I} \subseteq \mathcal{M}\} \end{aligned}$$

and the kernel,

$$\begin{aligned} k: \ \mathcal{P}(\mathfrak{M}) &\longrightarrow \{\text{Ideals of } \mathscr{A}\} \\ E &\longmapsto k(E) = \bigcap_{\mathcal{M}\in E} \mathcal{M}. \end{aligned}$$

Recall also that a Banach algebra $\mathscr{A}$ is said to be a *regular Banach algebra* when each point in $\mathscr{A}$ has a neighborhood U such that $k(U)$ is a regular ideal.

For a closed set F in $\mathfrak{M}$, let $J(F,\infty)$ be the union of all ideals $k(U)$, where U is an open set containing F and having compact complement. Since $J(F,\infty)$ is the smallest ideal with hull equal to F, see [43, p. 91], for each closed ideal $\mathcal{I}$ the following holds

$$J(h(\mathcal{I}),\infty) \subset \mathcal{I} \subset k(h(\mathcal{I})).$$

If $\mathscr{A}$ is a semisimple regular algebra, then the closed sets of $\mathfrak{M}$ are exactly the hulls of closed ideals and a closed ideal is an intersection of maximal regular ideals if and only if it is equal to the kernel of its hull. Therefore, we have

Lemma 4.1.9 ([43, p. 92]). *Let $\mathscr{A}$ be a semisimple regular Banach algebra. Then every closed ideal $\mathcal{I}$ of $\mathscr{A}$ is equal to an intersection of maximal regular ideals if and only if* $\overline{J(h(\mathcal{I}),\infty)} = k(h(\mathcal{I}))$.

Using the definition of $J(h(\mathcal{I}),\infty)$, the equality in the preceding lemma is equivalent to the fact that for each closed ideal $\mathcal{I}$ and each $x \in k(h(\mathcal{I}))$, there exist open sets $U_n \supset h(\mathcal{I})$ with compact complement and $x_n \in h(U_n)$ such that $x_n \to x$. If we define $h(x) = \{\mathcal{M} \in \mathfrak{M} : x \in \mathcal{M}\}$, then it is easy to see that $h(x)$ equals to the hull of the ideal generated by x. Thus the equality in Lemma 4.1.9 is also equivalent to the fact that for each x

in $k(h(\mathcal{I}))$, there is a sequence $\{x_n\}$ such that $x_n \to x$ in $\mathscr{A}$ and $\widehat{x}_n$ equals zero in a neighborhood U_n of $h(x)$ with compact complement. Next corollary follows immediately from Proposition 4.1.2 and Lemma 4.1.9.

Corollary 4.1.10. *Let $\mathscr{A}$ be a semisimple regular commutative Banach algebra such that a is a cyclic element of $\mathscr{A}$. Then*

$$\text{Lat}\, M_a = \left\{ \bigcap_{\varkappa \in F} \ker \varkappa : \ F \text{ is closed in } \Omega(\mathscr{A}) \right\}$$

if and only if for each $x \in \mathscr{A}$, there exists a sequence $\{x_n\}$ tending to x in $\mathscr{A}$ and $\widehat{x}_n$ vanishes on a neighborhood U_n of $h(x)$ with compact complement.

4.2 The Spectrum

4.2.1 The Eigenfunctions of C_{φ_a}

We will introduce at this point a family of functions that will play a prominent role in the present chapter. The family in question is formed by the functions

$$e_t(z) = \exp\left(t\frac{z+1}{z-1}\right), \qquad \text{for each } t \geq 0. \tag{4.3}$$

Each of these functions is an *inner function*, that is, a holomorphic function in $\mathbb{D}$ having radial limit of modulus 1 for almost every point in $\mathbb{T}$. In fact, they are a family of atomic singular inner functions (see [11, p. 24]). Thus for every non-negative number t, the function e_t is analytic in $\mathbb{D}$ and bounded by 1. Therefore, it belongs to $\mathscr{H}^2$.

We have introduced this family because they are the eigenvectors of C_{φ_a}. Indeed, note that if we set $\sigma(z) = i(1+z)/(1-z)$ the conformal mapping that takes $\mathbb{D}$ onto Π, we can write e_t as

$$e_t(z) = \exp\left(t\frac{z+1}{z-1}\right) = \exp(it\sigma).$$

Thus,

$$(C_{\varphi_a}e_t)(z) = e_t(\varphi_a(z)) = \exp(it(\sigma \circ \varphi_a)). \tag{4.4}$$

Now since φ_a is conjugated to a translation, equation (4.1) implies that $\sigma \circ \varphi_a = \sigma + ia$ and thus

$$(C_{\varphi_a} e_t)(z) = \exp\left(it(\sigma + ia)\right) = e^{-at} e_t(z). \tag{4.5}$$

Hence, e_t is an eigenvector with corresponding eigenvalue e^{-at}. We have proved

Proposition 4.2.1. *Let φ_a be a parabolic non-automorphism that takes the unit disk into itself. For each number $t \geq 0$, e_t is an eigenvector of C_{φ_a} with corresponding eigenvalue e^{-at}.*

4.2.2 The Spectrum of C_{φ_a}

Note that the spectrum of an operator T acting in a Hilbert space $\mathscr{H}$ coincides with its spectrum in the Banach algebra of all bounded operators in $\mathscr{H}$. The computation of the spectrum of a composition operator induced by a parabolic non-automorphism that takes $\mathbb{D}$ into itself was first made by Carl Cowen in [8]. In fact, he did it for a wider class of composition operators. We will reproduce here its proof with some minors simplifications. To prove Cowen's result, we will need some of the concepts and results of Banach algebras theory introduced in last section.

Theorem 4.2.2 (Cowen, 1983). *Let φ_a be a parabolic non-automorphism that takes the unit disk into itself. Then*

$$\sigma(C_{\varphi_a}) = \{e^{-at} \colon t \in [0, +\infty)\} \cup \{0\}.$$

Proof. We consider the set $\{C_{\varphi_a} \colon \Re a > 0\}$. We start showing that this set is a holomorphic semigroup of operators, that is, it is a semigroup under composition and the mapping

$$a \longmapsto C_{\varphi_a}$$

is continuous and holomorphic in the norm topology for a in $\{z \in \mathbb{C} \colon \Re z > 0\}$. Proving that $\{C_{\varphi_a} \colon \Re a > 0\}$ is a semigroup is immediate if we have in mind that each C_{φ_a} is conjugated to the operator of translation by a. To prove the holomorphy of the mapping $a \longmapsto C_{\varphi_a}$, it is enough to prove that for each f in $\mathscr{H}^2$ and each α in $\mathbb{D}$, the mapping

$$a \longmapsto \langle C_{\varphi_a} f, k_\alpha \rangle_{\mathscr{H}^2}$$

is holomorphic in $\{z \in \mathbb{C}\colon \Re z > 0\}$. The last assertion is a standard reasoning in semi-group theory that follows from the uniform boundedness principle (see for instance Theorem 3.10.1 in the classical monograph [25]). Indeed, from formula (4.2) for the parabolic non-automorphism, the above display equals to

$$a \longmapsto f\left(\frac{(2-a)\alpha + a}{-a\alpha + 2 + a}\right),$$

which is a holomorphic function on $\{z \in \mathbb{C}\colon \Re z > 0\}$, since the inequality $\Re a > 0$ implies that $-a\alpha + 2 + a$ does not vanish.

Now let $\mathscr{A}$ be the norm closed algebra of operators generated by the set

$$\{I\} \cup \{C_{\varphi_a}\colon \Re a > 0\},$$

where I denotes the identity operator on $\mathscr{H}^2$. Since $\{C_{\varphi_a}\colon \Re a > 0\}$ is a semigroup and all operators commute with I, the algebra $\mathscr{A}$ is a commutative Banach algebra with identity. Then we can apply Theorem 4.1.8 and the spectrum of C_{φ_a} as an element of $\mathscr{A}$ is the set

$$\sigma_{\mathscr{A}}(C_{\varphi_a}) = \{\varkappa(C_{\varphi_a})\colon \varkappa \in \Omega(\mathscr{A})\}.$$

Let us identify all the elements in this set. For each multiplicative linear functional $\varkappa$ on $\mathscr{A}$, we will write

$$\kappa(a) = \varkappa(C_{\varphi_a})$$

for every complex number a such that $\Re a > 0$. Since the norm of $\varkappa$ is bounded by 1 (Lemma 4.1.3) and we already know that $\{C_{\varphi_a}\colon \Re a > 0\}$ is a norm holomorphic semigroup, the mapping $\kappa(a)$ is holomorphic in the right half plane. In addition,

$$\begin{aligned}\kappa(a_1 + a_2) &= \varkappa(C_{\varphi_{a_1+a_2}}) = \varkappa(C_{\varphi_{a_1}} C_{\varphi_{a_2}}) = \varkappa(C_{\varphi_{a_1}})\varkappa(C_{\varphi_{a_2}}) \\ &= \kappa(a_1)\kappa(a_2).\end{aligned}$$

This means that either $\kappa(a) \equiv 0$ or $\kappa(a) = e^{-\lambda a}$ for some complex number λ. In fact, λ has to be a non-negative real number since the norm continuity of $\varkappa$ implies

$$|e^{-\lambda a}| = \lim_{n\to\infty} |e^{-\lambda n a}|^{1/n} = \lim_{n\to\infty} |\varkappa(C^n_{\varphi_a})|^{1/n} \le \lim_{n\to\infty} \|C^n_{\varphi_a}\|^{1/n} = 1,$$

where last equality follows from Theorem 2.2.4 and formula (4.2) for φ_a. Since $|e^{-\lambda a}| = e^{-\Re(\lambda a)}$, above display implies that $\Re(\lambda a) \ge 0$ for every complex number such that $\Re a > 0$. Therefore, $\lambda \ge 0$ and we obtain that

$$\sigma_{\mathscr{A}}(C_{\varphi_a}) \subseteq \{e^{-at} : t \in [0, +\infty)\} \cup \{0\}. \tag{4.6}$$

Now observe that if $C_{\varphi_a} - \lambda I$ does not have an inverse in the algebra of bounded operators on $\mathscr{H}^2$, then the same is true in the smaller algebra $\mathscr{A}$. Hence

$$\sigma(C_{\varphi_a}) \subseteq \sigma_{\mathscr{A}}(C_{\varphi_a}).$$

Therefore, (4.6) implies that

$$\sigma(C_{\varphi_a}) \subseteq \{e^{-at} : t \in [0, +\infty)\} \cup \{0\}.$$

Since we already knew from Proposition 4.2.1 that

$$\{e^{-at} : t \in [0, +\infty)\} \subseteq \sigma(C_{\varphi_a})$$

and the spectrum is a closed subset of $\mathbb{C}$, then

$$\sigma(C_{\varphi_a}) = \{e^{-at} : t \in [0, +\infty)\} \cup \{0\}.$$

□

The family of eigenfunctions of C_{φ_a} possesses many interesting properties. Here we will list two of them.

Proposition 4.2.3. *The set $\{e_t\}_{t\ge 0}$ is a semigroup with respect to pointwise multiplication.*
In addition, the mapping

$$\begin{aligned} \Psi : \quad [0, +\infty) &\longrightarrow \{e_t\}_{t\ge 0} \\ t &\longmapsto e_t(z) = \exp\left(t\frac{z+1}{z-1}\right) \end{aligned}$$

is a homeomorphism if we endow $\{e_t\}_{t\ge 0}$ with the relative topology.

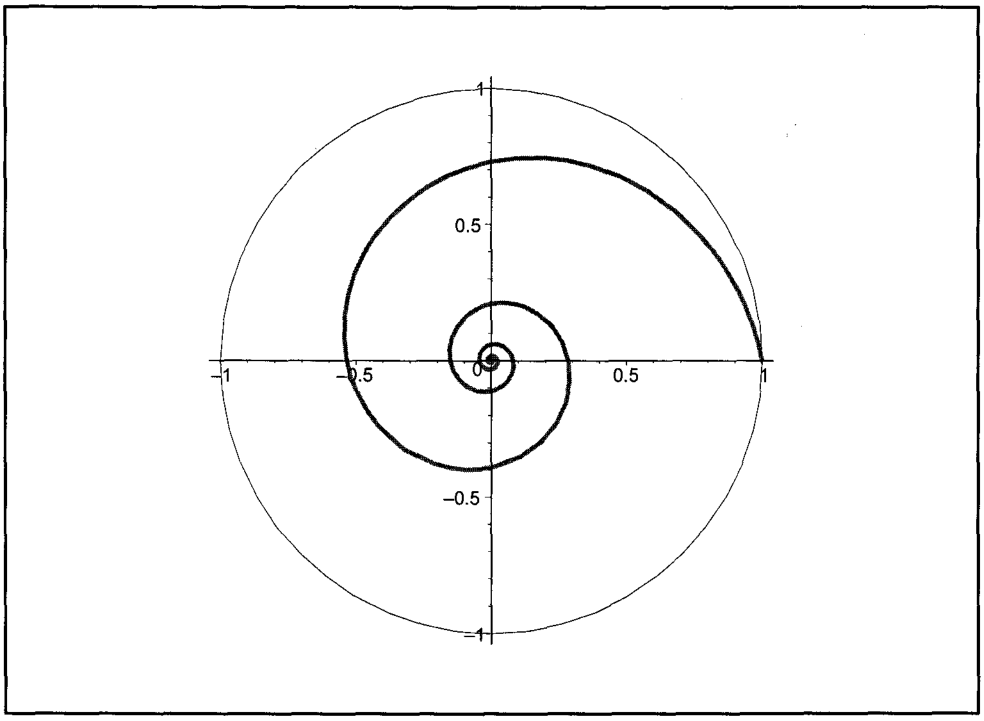


Figure 4.1: The spectrum of C_φ

Proof. The semigroup structure is inherited from the semigroup structure of $[0, \infty)$ since $e_t(z)e_s(z) = e_{t+s}(z)$.

To prove the second assertion, let $\{t_n\}_{n\geq 0}$ be a sequence of non-negative numbers and $t \geq 0$ fixed. Observe that

$$\|e_{t_n} - e_t\|^2_{\mathscr{H}^2} = \|e_t\|^2_{\mathscr{H}^2} - 2\Re(\langle e_{t_n}, e_t\rangle_{\mathscr{H}^2}) + \|e_{t_n}\|^2_{\mathscr{H}^2} = 2 - 2e^{-|t_n - t|}.$$

Therefore, $e_{t_n} \to e_t$ if and only if $2 - 2e^{-|t_n-t|} \to 0$ if and only if $e^{-|t_n-t|} \to 1$, what is equivalent to say that $t_n \to t$, since $\{t_n\}$ is a real sequence. □

Another interesting property is that the set of all finite linear combinations of the $e_t's$ is dense in $\mathscr{H}^2$.

Proposition 4.2.4. *The set of eigenfunctions of C_{φ_a} is a spanning set of $\mathscr{H}^2$. That is,*

$$\overline{\text{span}}\,\{e_t \colon t \geq 0\} = \mathscr{H}^2.$$

This result is well known for specialists. Because of his simplicity, we reproduce here the proof appearing in [16].

Proof. Under the standard isometry of $T : \mathscr{H}^2 \to \mathscr{H}^2(\Pi)$, an eigenfunction e_t is mapped to

$$Te_t(w) = \frac{1}{w+i}\, e_t\left(\frac{w-i}{w+i}\right) = \frac{e^{itw}}{w+i}.$$

The latter function is mapped under the Fourier transform to

$$\left(\mathcal{F}\frac{e^{itw}}{w+i}\right)(s) = \int_{-\infty}^{\infty} \frac{e^{itx}}{x+i}\, e^{-isx}\, dx = \left(\mathcal{F}\frac{1}{w+i}\right)(s-t).$$

Using the Residue Theorem, Fourier transform in the left hand side of above display is computed as

$$\left(\mathcal{F}\frac{1}{w+i}\right)(s) = -2\pi i e^{-s}\chi_{(0,\infty)}(s),$$

being $\chi_{(0,\infty)}$ the characteristic function of the interval $(0,\infty)$. Thus the set of eigenfunctions is mapped by the isomorphism $\mathcal{F}T : \mathscr{H}^2 \to L^2(0,+\infty)$ onto the set

$$\left\{-2\pi i e^{-s+t}\chi_{(t,\infty)}(s) \colon t \geq 0\right\}.$$

This set of functions is easily seen to span $L^2(0,\infty)$ and thus $\{e_t \colon t \geq 0\}$ spans $\mathscr{H}^2$. □

The density in the Hardy space of the eigenfunctions of C_{φ_a} will be essential in proving that its adjoint $C_{\varphi_a}^\star$ is similar to a multiplication operator. This can be done thanks to the following theorem of Halmos.

Theorem 4.2.5 ([19, Problem 85]). *A necessary and sufficient condition that an operator T on a Hilbert space $\mathscr{H}$ be representable as a multiplication on a functional Hilbert space is that the eigenvectors of $T^\star$ span $\mathscr{H}$.*

The way of proving this theorem is to construct an isomorphism between $\mathscr{H}$ and a functional Hilbert space. The idea to construct the isomorphism is the following. Let X be an index set such that for each $x \in X$ there is an eigenvector K_x of $T^\star$ with corresponding eigenvalue $\overline{\varphi(x)}$ and such that the K_x's span $\mathscr{H}$. Then for each f in $\mathscr{H}$, we define a mapping Φ on $\mathscr{H}$ as

$$(\Phi f)(x) = \langle f, K_x \rangle_{\mathscr{H}}.$$

The mapping Φ is a bounded operator on $\mathscr{H}$ and

$$\begin{aligned}(\Phi T f)(x) &= \langle Tf, K_x \rangle_{\mathscr{H}} \\ &= \langle f, T^\star K_x \rangle_{\mathscr{H}} \\ &= \langle f, \overline{\varphi(x)} K_x \rangle_{\mathscr{H}} \\ &= \varphi(x) \langle f, K_x \rangle_{\mathscr{H}} \\ &= \varphi(x)(\Phi f)(x).\end{aligned}$$

Thus if we endow the image space with the inner product

$$\langle \Phi f, \Phi g \rangle = \langle f, g \rangle_{\mathscr{H}},$$

then Φ becomes an isometric isomorphism from $\mathscr{H}$ onto $\Phi(\mathscr{H})$ and the operator T is similar, via Φ, to the operator of multiplication by $\varphi(x)$ in $\Phi(\mathscr{H})$, that can be proved to be a functional Hilbert space.

Since we already know from Proposition 4.2.4 that the eigenfunctions of C_{φ_a} span the whole space $\mathscr{H}^2$, following the procedure described above we can construct an isomorphism such that $C_{\varphi_a}^\star$ is isomorphic to a multiplication operator on certain functional Hilbert space. At the moment, there is only one thing we can say about this functional Hilbert space: Proposition 4.2.3 implies that all functions in the space are continuous.

The main issue will be to identify the target space $\Phi(\mathscr{H})$ with a Sobolev type space. This will be done in the next section.

This idea was previously used in [17, Chaps. IV and V]. However, the norm on the space $\Phi(\mathscr{H}^2)$ is defined as $\|\Phi(f)\| = \|f\|_{\mathscr{H}^2}$. Since the space $\Phi(\mathscr{H}^2)$ is not identified, it is more difficult to handle.

4.3 The Sobolev Space $W^{1,2}[0, \infty)$

The *Sobolev space* $W^{1,2}[0, \infty)$ consists of those functions f in $L^2[0, \infty)$ absolutely continuous on each bounded subinterval of $[0, \infty)$ and whose derivative belong to $L^2[0,\infty)$. It is well-known, see [1], and easy to check that the space $W^{1,2}[0,\infty)$ becomes a Hilbert space endowed with the inner product

$$\langle f,g\rangle_{1,2} = \frac{1}{2}\int_0^\infty \left(f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\right) dt.$$

The corresponding norm will be denoted by $\|\cdot\|_{1,2}$. The Sobolev space $W^{1,2}(\mathbb{R})$ can be defined in a similar way just replacing $[0,\infty)$ by $\mathbb{R}$. Sobolev spaces are named after the Soviet mathematician S. L. Sobolev who made major contributions to the subject in the late 1930's.

4.3.1 An Isomorphism from $\mathscr{H}^2$ onto $W^{1,2}[0,\infty)$

We will show up an isomorphism between the Hardy space $\mathscr{H}^2$ and the Sobolev space $W^{1,2}[0,\infty)$ that will be crucial to prove Theorem 4.1.1. The inner functions $e_t(z)$, with $t \geq 0$, allow us to consider a complex valued function for each f in $\mathscr{H}^2$ defined by

$$(\Phi f)(t) = \langle f, e_t\rangle_{\mathscr{H}^2}, \qquad t \geq 0.$$

The key point to prove that Φ is an isomorphism from $\mathscr{H}^2$ onto $W^{1,2}[0,\infty)$ is to consider the operator Ψ that for each f in $L^2(\mathbb{T})$, defined as

$$(\Psi f)(t) = \langle f, e_t\rangle_{L^2(\mathbb{T})}, \qquad t \in \mathbb{R}.$$

Let $W_0^{1,2}[0,\infty)$ denote the subspace of functions in $W^{1,2}(\mathbb{R})$ that vanish on $(-\infty,0]$. The space $W_0^{1,2}(-\infty,0]$ is defined similarly. We have

Theorem 4.3.1. *The operator Ψ is an isometric isomorphism from $L^2(\mathbb{T})$ onto $W^{1,2}(\mathbb{R})$. In addition, $\Psi(z\mathscr{H}^2) = W_0^{1,2}[0,\infty)$ and $\Psi(\overline{z\mathscr{H}^2}) = W_0^{1,2}(-\infty,0]$.*

Proof. For each f in $L^2(\mathbb{T})$, we have

$$(\Psi f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \exp\left(t\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) d\theta, \qquad t \in \mathbb{R}.$$

The change of variables $x = i(1+e^{i\theta})/(1-e^{i\theta})$ yields

$$(\Psi f)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{1+x^2}dx, \qquad t \in \mathbb{R}. \tag{4.7}$$

Therefore, $\Psi = \mathcal{F}MT$, where $\mathcal{F}$ denotes the Fourier transform defined as

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} f(x)e^{-itx}\,dx,$$

and M and T are the bounded operators defined as

$$(Mg)(y) = \frac{1}{\sqrt{\pi}}\frac{g(y)}{\sqrt{1+y^2}} \quad \text{and} \quad (Tf)(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+x^2}}f\left(\frac{x-i}{x+i}\right).$$

Recall that the linear fractional map known as the Cayley transform, $\tau(z) = (z-i)/(z+i)$, maps conformally Π onto $\mathbb{D}$. Thus it takes $\mathbb{R}$ onto $\mathbb{T}$ and the obvious change of variables shows that T is an isometric isomorphism from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$. The first statement of the proposition will be proved once we have shown that $\mathcal{F}M$ is an isometric isomorphism from $L^2(\mathbb{R})$ onto $W^{1,2}(\mathbb{R})$. For each function f in $L^2(\mathbb{R})$, we have

$$\|\mathcal{F}Mf\|_{1,2}^2 = \frac{1}{2\pi}\left\|\mathcal{F}\left(\frac{f(x)}{\sqrt{1+x^2}}\right)\right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2\pi}\left\|\left(\mathcal{F}\left(\frac{f(x)}{\sqrt{1+x^2}}\right)\right)'\right\|_{L^2(\mathbb{R})}^2.$$

The derivative of the Fourier transform of a function g is $(\mathcal{F}g)' = \mathcal{F}(-ixg(x))$, see [45, p. 179]. Hence, the above display becomes

$$\|\mathcal{F}Mf\|_{1,2}^2 = \frac{1}{2\pi}\left\|\mathcal{F}\left(\frac{f(x)}{\sqrt{1+x^2}}\right)\right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2\pi}\left\|\mathcal{F}\left(\frac{-ixf(x)}{\sqrt{1+x^2}}\right)\right\|_{L^2(\mathbb{R})}^2.$$

Upon applying Plancherel Theorem, Theorem 2.1.4, we obtain that

$$\begin{aligned}\|\mathcal{F}Mf\|_{1,2}^2 &= \left\|\frac{f(x)}{\sqrt{1+x^2}}\right\|_{L^2(\mathbb{R})}^2 + \left\|\frac{-ixf(x)}{\sqrt{1+x^2}}\right\|_{L^2(\mathbb{R})}^2 \\ &= \|f\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Therefore the first part of the theorem is proved.

Now, let f be in $z\mathscr{H}^2$, that is, $f(z) = zg(z)$ with g in $\mathscr{H}^2$. Using (4.7), we obtain

$$(\Psi f)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{(x+i)^2}\,dx, \qquad \text{for each } t \in \mathbb{R}.$$

Since the map

$$h \to \frac{1}{\sqrt{\pi}(x+i)}h\left(\frac{x-i}{x+i}\right),$$

is an isometric isomorphism from $\mathscr{H}^2$ onto $\mathscr{H}^2(\Pi)$, and multiplication by $(w+i)^{-1}$ is bounded on $\mathscr{H}^2(\Pi)$, we find that Ψf is the Fourier transform of a function of $\mathscr{H}^2(\Pi)$. Thus, the Paley-Wiener Theorem shows that Ψf, which is continuous, must vanish on $(-\infty, 0]$ and, therefore, $\Psi(z\mathscr{H}^2) \subset W_0^{1,2}[0,\infty)$. A similar argument shows that $\Psi(\overline{z\mathscr{H}^2}) \subset W_0^{1,2}(-\infty,0]$. The fact that $\Psi(z\mathscr{H}^2) = W_0^{1,2}[0,\infty)$ and $\Psi(\overline{z\mathscr{H}^2}) = W_0^{1,2}(-\infty,0]$ follows immediately from the orthogonal decomposition

$$W^{1,2}(\mathbb{R}) = W_0^{1,2}(-\infty,0] \oplus [e^{-|t|}] \oplus W_0^{1,2}[0,\infty),$$

which in turns follows, being Ψ an isometric isomorphism, from the orthogonal decomposition

$$L^2(\mathbb{T}) = \overline{z\mathscr{H}^2} \oplus [1] \oplus z\mathscr{H}^2$$

and the fact that $\Psi 1 = e^{-|t|}$, where $[f]$ denotes the one-dimensional linear space spanned by the vector f. The proof is complete. □

Corollary 4.3.2. *The operator Φ defines an isomorphism from $\mathscr{H}^2$ onto $W^{1,2}[0,\infty)$. Indeed, $\|\Phi f\|_{1,2}^2 = \|f\|_{\mathscr{H}^2}^2 - |f(0)|^2/2$.*

Proof. Upon applying Theorem 4.3.1, we find that Φ and Ψ coincide on $z\mathscr{H}^2$. Therefore, the map Φ defines an isometric isomorphism from $z\mathscr{H}^2$ onto $W_0^{1,2}[0,\infty)$. Since $e^{-|t|}$ is orthogonal to $W_0^{1,2}[0,\infty)$, so is $e^{-t}\chi_{[0,\infty)}$ and, therefore,

$$W^{1,2}[0,\infty) = [e^{-t}\chi_{[0,\infty)}] \oplus W_0^{1,2}[0,\infty) = (\Phi 1) \oplus \Phi(z\mathscr{H}^2) = \Phi(\mathscr{H}^2),$$

which proves that Φ is an isomorphism. The formula for the norm comes from the fact that $\|e^{-t}\chi_{[0,\infty)}\|_{1,2}^2 = 1/2$. The proof is complete. □

An interesting consequence of Corollary 4.3.2 is a summability theorem for the Laguerre polynomials. Set $u_n(z) = z^n$. Then

$$\widetilde{u}_n(t) = (\Phi u_n)(t) = L_n^{(-1)}(2t)e^{-t}, \quad t \geq 0$$

where $L_n^{(-1)}(t)$ is the Laguerre polynomial of degree n and of index -1. Indeed, $\widetilde{u}_n = \langle z^n, e_t(z)\rangle_{\mathscr{H}^2}$ is the complex conjugate of the n-th coefficient of the Taylor series of $e_t(z)$. By definition of the Laguerre polynomials see [50, p. 97], we have

$$e_t(z) = e^{-t}\exp\left(-\frac{2tz}{1-z}\right) = \sum_{n=0}^{\infty} e^{-t}L_n^{(-1)}(2t)z^n. \tag{4.8}$$

Since Laguerre polynomials are real valued, next corollary follows immediately.

Corollary 4.3.3. *Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers. Then the series $\widetilde{f}(t) = \sum_{n=0}^{\infty} a_n L_n^{(-1)}(2t)e^{-t}$ converges in $W^{1,2}[0,\infty)$ if and only if $\{a_n\}$ is in the sequence space ℓ^2. Indeed,*

$$\|\widetilde{f}\|_{1,2}^2 = -\frac{|a_0|^2}{2} + \|\{a_n\}_{n\geq 0}\|_2^2.$$

4.3.2 Similarity with a Multiplication Operator

Now, we can apply the argument described at the end of last section to prove that the adjoint of a composition operator induced by a parabolic non-automorphism can be seen as a multiplication operator on $W^{1,2}[0,\infty)$.

Proposition 4.3.4. *Let φ_a, with $\Re a \geq 0$, be as in (4.2). Then the adjoint of C_{φ_a} acting on $\mathscr{H}^2$ is similar under Φ to the multiplication operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$, acting on $W^{1,2}[0,\infty)$.*

Proof. Using the eigenvalue equation (4.5), for each $f \in \mathscr{H}^2$, we have

$$\begin{aligned}(\Phi C_{\varphi_a}^{\star} f)(t) &= \langle C_{\varphi_a}^{\star} f, e_t\rangle_{\mathscr{H}^2} \\ &= \langle f, C_{\varphi_a} e_t\rangle_{\mathscr{H}^2} \\ &= e^{-\bar{a}t}\langle f, e_t\rangle_{\mathscr{H}^2} \\ &= e^{-\bar{a}t}(\Phi f)(t),\end{aligned}$$

for each $t \geq 0$. Thus $M_\psi = \Phi C_{\varphi_a}^{\star}\Phi^{-1}$. The result is proved. □

Proposition 4.3.5. *The operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$ and $\Re a > 0$, acting on $W^{1,2}[0,\infty)$ is cyclic with cyclic vector ψ.*

Proof. Given a point w in $\mathbb{D}$, recall the notation k_w for the reproducing kernel for $\mathscr{H}^2$ at the point w. That is, $k_w(z) = (1 - \overline{w}z)^{-1}$ and $\langle f, k_w\rangle_{\mathscr{H}^2} = f(w)$. Observe also that $\sigma(z) = (\overline{z}-1)/(\overline{z}+1)$ maps bijectively the right half plane onto the unit disc. Therefore, given a with $\Re a > 0$ we set $\alpha = \sigma(a) \in \mathbb{D}$. An easy computation shows that $\Phi k_\alpha(t) = \langle k_\alpha, e_t\rangle_{\mathscr{H}^2} = e^{-\overline{a}t}$. Thus, by Proposition 4.3.4, it is enough to show that the function $k_\alpha(z)$ is cyclic for $C^\star_{\varphi_a}$.

Suppose that f in $\mathscr{H}^2$ is orthogonal to the orbit of k_α under $C^\star_{\varphi_a}$. Then, for each $n \geq 0$, we have

$$\begin{aligned} 0 &= \langle C^{\star n}_{\varphi_a} k_\alpha, f\rangle_{\mathscr{H}^2} \\ &= \langle k_\alpha, C^n_{\varphi_a} f\rangle_{\mathscr{H}^2} \\ &= \langle k_\alpha, C_{\varphi_{na}} f\rangle_{\mathscr{H}^2} \\ &= \langle k_\alpha, f \circ \varphi_{na}\rangle_{\mathscr{H}^2} \\ &= f(\varphi_{na}(\alpha)). \end{aligned}$$

Since $\{\varphi_{na}(\alpha)\}$ is not a Blaschke sequence, we find that f is the null function and the result follows. □

Observe that the same argument can be used to prove that each reproducing kernel is a cyclic vector for the adjoint operator $C^\star_{\varphi_a}$. We will devote here a few lines to the automorphism case. Suppose that C_{φ_a} is induced by a parabolic automorphism that takes the unit disk into itself. Since $\Re a = 0$, Proposition 4.3.4 is valid in this case and the adjoint operator $C^\star_{\varphi_a}$ is similar to $M_{e^{-\overline{a}t}}$. In this case we can reproduce the argument made in the above proof to show that $C^\star_{\varphi_a}$ is cyclic as well, being each reproducing kernel a cyclic vector for $C^\star_{\varphi_a}$. Thus, since cyclicity is preserved under similarity, $M_{e^{-\overline{a}t}}$ is also cyclic. The key issue here is that $e^{-\overline{a}t}$ is not a cyclic vector for $M_{e^{-\overline{a}t}}$ and we cannot apply Proposition 4.1.2. Indeed, the vector $e^{-\overline{a}t}$ is not cyclic for $M_{e^{-\overline{a}t}}$ because condition $\Re a = 0$ implies that $e^{-\overline{a}t}$ is not bounded and thus, as next proposition will show, $e^{-\overline{a}t}$ does not belong to $W^{1,2}[0, \infty)$.

4.4 The Lattice of the Operator C_{φ_a}

In this section, we will prove Theorem 4.1.1 stated at the beginning of this chapter. To this end, we have to show that $W^{1,2}[0,\infty)$ is a semisimple regular Banach algebra with respect to the pointwise multiplication and identify its closed ideals. First, we need to state some basic properties of $W^{1,2}[0,\infty)$.

4.4.1 The Sobolev Space $W^{1,2}[0,\infty)$ as a Banach Algebra

The content of the following two propositions is already known for specialists in the Sobolev space, see [1, Chapter V] for instance. However, we will include here a proof for each of them. Both proofs are interesting by their own, since they just make use of the isomorphism between $\mathscr{H}^2$ and $W^{1,2}[0,\infty)$ constructed in the previous section.

Proposition 4.4.1. *Each f in $W^{1,2}[0,\infty)$ satisfies $\|f\|_\infty \leq \sqrt{2}\|f\|_{1,2}$ and vanishes at ∞. In particular, each f in $W^{1,2}[0,\infty)$ is uniformly continuous and norm convergence in $W^{1,2}[0,\infty)$ implies uniform convergence.*

Proof. By Corollary 4.3.3, we can write $f(t) = \sum_{n=0}^{\infty} a_n L_n^{(-1)}(2t)e^{-t}$, where $\{a_n\}$ is in ℓ^2. The Cauchy-Schwarz inequality and Corollary 4.3.3, for each $t \geq 0$, yields

$$|f(t)| = \left|\sum_{n=0}^{\infty} a_n L_n^{(-1)}(2t)e^{-t}\right| \leq \|f\|_{1,2}\left(2e^{-2t} + \sum_{n=1}^{\infty}(L_n^{(-1)}(2t))^2 e^{-2t}\right)^{1/2}.$$

Since $\|e_t\|_{\mathscr{H}^2} = 1$, using (4.8), one easily checks that the quantity into the brackets above equals to $1 + e^{-2t} \leq 2$ and, therefore, $\|f\|_\infty \leq \sqrt{2}\|f\|_{1,2}$.

To show that f vanishes at ∞, for each positive integer m, we observe that

$$|f(t)| \leq \left|\sum_{n=0}^{m} a_n L_n^{(-1)}(2t)e^{-t}\right| + \left|\sum_{n=m+1}^{\infty} a_n L_n^{(-1)}(2t)e^{-t}\right|.$$

The second term in the right-hand side above is bounded by $\sqrt{2}\|\{a_n\}_{n\geq m+1}\|_2$ and, thus, we can take large enough m so that this term is small enough for each $t \geq 0$. For this m and large enough t, the first term in the right-hand side above is clearly as small as desired. The proof is complete. □

As a consequence of Proposition 4.4.1, we find that $W^{1,2}[0,\infty)$ is a Banach algebra.

Proposition 4.4.2. *The space* $W^{1,2}[0,\infty)$ *with the pointwise multiplication is a Banach algebra without unity.*

Proof. Let f and g be in $W^{1,2}[0,\infty)$. Upon applying Proposition 4.4.1, we see that

$$\|fg\|_2 \leq \|f\|_2\|g\|_\infty \leq 2\|f\|_{1,2}\|g\|_{1,2}$$

and

$$\|(fg)'\|_2 = \|f'g + fg'\|_2 \leq \|f'\|_2\|g\|_\infty + \|g'\|_2\|f\|_\infty \leq 4\|f\|_{1,2}\|g\|_{1,2},$$

which show that the statement holds. □

We will need a special dense subspace of $W^{1,2}[0,\infty)$. Let $C_c^\infty[0,\infty)$ denote the space of infinitely differentiable complex functions on $[0,\infty)$ that have compact support. The content of the next proposition is known, we include a proof for the sake of completeness.

Proposition 4.4.3. *The space* $C_c^\infty[0,\infty)$ *is dense in* $W^{1,2}[0,\infty)$.

Proof. Suppose that f in $W^{1,2}[0,\infty)$ satisfies

$$\int_0^\infty f(t)\overline{g(t)}\,dt + \int_0^\infty f'(t)\overline{g'(t)}\,dt = 0, \qquad \text{for each } g \in C_c^\infty[0,\infty).$$

Since g has compact support, integrating by parts and putting everything under the same integral sign, we find that

$$\int_0^\infty \left[f'(x) - \left(\int_0^x f(t)\,dt\right)\right]\overline{g'(x)}\,dx = 0, \qquad \text{for each } g \in C_c^\infty[0,\infty).$$

Observe that since g' has compact support, the second integral above is always over a finite interval. Let $a > 0$ be fixed. Since the set of functions g' with g in $C_c^\infty[0,a)$ is dense in $L^2[0,a]$, we have

$$f'(x) - \int_0^x f(t)\,dt = 0, \qquad \text{for each } 0 \leq x \leq a.$$

Therefore, it follows that $f(x) = c_1e^x + c_2e^{-x}$, for $0 \leq x \leq a$, where c_i, $i = 1, 2$, is constant. Since a was arbitrary, it follows that $f(x) = c_1e^x + c_2e^{-x}$ for $0 \leq x < \infty$. But $c_1 = 0$ because f is in $W^{1,2}[0,\infty)$ and $c_2 = 0$ because $f'(0) = 0$. Thus f is the zero function and the result follows. □

For each $t \geq 0$, let δ_t denote the reproducing kernel for $W^{1,2}[0,\infty)$ at the point t, that is, $f(t) = \langle f, \delta_t\rangle_{1,2} = \langle \Phi^{-1} f, e_t\rangle_{\mathscr{H}^2}$ for each $f \in W^{1,2}[0,\infty)$ and where Φ is the transform defined in Section 4.3. Recall that the spectrum $\Omega(W^{1,2}[0,\infty))$ is the space of multiplicative linear functional endowed with the weak-star topology that, since $W^{1,2}[0,\infty)$ is a Hilbert space, coincides with the weak topology.

Proposition 4.4.4. *The spectrum of the Banach algebra $W^{1,2}[0,\infty)$ is*

$$\Omega(W^{1,2}[0,\infty)) = \{\delta_t : t \geq 0\}.$$

Furthermore, the mapping that to each t assigns δ_t is a homeomorphism from $[0,\infty)$ onto $\Omega(W^{1,2}[0,\infty))$.

Proof. Clearly, for each $t \geq 0$, the functional δ_t is a multiplicative linear functional on $W^{1,2}[0,\infty)$, that is, δ_t is in $\Omega = \Omega(W^{1,2}[0,\infty))$. To prove that each multiplicative linear functional on $W^{1,2}[0,\infty)$ is one of the δ_t's, we begin by considering the Banach algebra

$$C^1[0,1] = \{f : [0,1] \to \mathbb{C}\colon f \text{ is differentiable and } f' \text{ is continuous}\}$$

with pointwise multiplication, endowed with the norm

$$\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}.$$

Consider also its Banach subalgebra $\mathscr{A}_0 = \{f \in C^1[0,1]\colon f(1) = 0\}$. A straightforward computation shows that $(Tf)(x) = f(x/(1+x))$ defines a bounded operator from $\mathscr{A}_0$ into $W^{1,2}[0,\infty)$, which is also an algebra homomorphism.

Now, given a multiplicative linear functional $\varkappa$ of $W^{1,2}[0,\infty)$, we can construct a functional $\widetilde{\varkappa}$ on $C^1[0,1]$ defined by $\widetilde{\varkappa}(f) = \varkappa(T(f - f(1))) + f(1)$. It is easily checked that $\widetilde{\varkappa}$ is also a multiplicative linear functional. Since the multiplicative linear functionals of $C^1[0,1]$ are exactly the point evaluations $f \to f(s)$, with $0 \leq s \leq 1$, see [28, p. 204], then there exists $0 \leq s \leq 1$ such that $\widetilde{\varkappa}(f) = f(s)$ for each f in $C^1[0,1]$.

If $s = 1$, it follows immediately that $\varkappa(Tf) = 0$ for each f in $\mathscr{A}_0$. Hence $\varkappa$ vanishes on the range of T, which is dense because it contains $C_c^\infty[0,\infty)$, see Proposition 4.4.3. Therefore, $\varkappa$ is the zero functional.

If $s \neq 1$, then set $t = s/(1-s) \geq 0$ and observe that for each $f \in \mathscr{A}_0$,

$$\varkappa(Tf) = \widetilde{\varkappa}(f) = f(s) = (Tf)(t).$$

Hence $\varkappa$ and and the point evaluation δ_t coincide on a dense set, which implies that $\varkappa = \delta_t$. Thus we have shown that $\Omega = \{\delta_t : t \geq 0\}$.

Next, since each f in $W^{1,2}[0,\infty)$ is continuous, so is the mapping $t \to \delta_t$ from $[0,\infty)$ onto Ω. Since $\|\delta_t\|_{1,2} \leq \|\Phi^{-1}\|\|e_t\|_{\mathscr{H}^2} = \|\Phi^{-1}\|$, we find that Ω is norm bounded on the dual space. Since the weak topology of a separable Hilbert space is metrizable on bounded sets, we may conclude that Ω is metrizable. Thus, to prove that $t \to \delta_t$ is a homeomorphism, it suffices to show that $t_n \to t_0$ whenever $\delta_{t_n} \to \delta_{t_0}$. Suppose that this is not the case, then there is $\varepsilon > 0$ such that $|t_n - t_0| > \varepsilon$ for each positive integer n. Consider the $W^{1,2}[0,\infty)$-function defined for $t \geq 0$ by

$$f(t) = \begin{cases} \varepsilon - |t_0 - t|, & \text{if } |t_0 - t| \leq \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\delta_{t_n}(f) = 0$ and $\delta_{t_0}(f) = \varepsilon$, we find that δ_{t_n} cannot converge to δ_{t_0}. Therefore, the mapping $t \to \delta_t$ is a homeomorphism. The result is proved. □

Proposition 4.4.5. *The Banach algebra $W^{1,2}[0,\infty)$ is semisimple and regular and the mapping $F \to \bigcap_{\varkappa\in F} \ker \varkappa$ is one-to-one from $\mathbb{F}[0,\infty)$ onto the set of closed ideals of $W^{1,2}[0,\infty)$.*

Proof. Since the multiplicative linear functionals δ_t's separate points, the Banach algebra $W^{1,2}[0,\infty)$ is semisimple. To prove that $W^{1,2}[0,\infty)$ is also regular, consider a maximal regular ideal $\mathcal{M}_0$ corresponding to the reproducing kernel δ_{t_0}. Suppose that $t_0 \in (a,b) \subset [0,\infty)$ and let U be the image of (a,b) under the homeomorphism furnished by Proposition 4.4.4. Then U is an open neighborhood of $\mathcal{M}_0$ and

$$k(U) = \{f \in W^{1,2}[0,\infty) : \; f \equiv 0 \text{ on } (a,b)\}$$

is a regular ideal. Indeed, if we take $g \in \mathcal{C}_c^\infty[0,\infty)$ such that $g \equiv 1$ on (a,b), then, for each $f \in W^{1,2}[0,\infty)$, we have $f - gf \in k(U)$ and therefore g is the unit in $W^{1,2}[0,\infty)/k(U)$.

It remains to show that the hypotheses of Lemma 4.1.9 are fulfilled. Indeed, the Gelfand transform of a function in $W^{1,2}[0,\infty)$ vanishes on a set if and only if the function vanishes on its preimage under the homeomorphism furnished by Proposition 4.4.4. Clearly, for each f in $W^{1,2}[0,\infty)$ there is a sequence $\{f_n\}$ in $\mathcal{C}_c^\infty[0,\infty)$ converging to f and such that the zero set of each f_n contains an open neighborhood U_n of the zero set of f. Then, by Lemma 4.1.9, each closed ideal of $W^{1,2}[0,\infty)$ is of the form $\bigcap_{\varkappa\in F}\ker\varkappa$ for some F in $\mathbb{F}[0,\infty)$. Thus the mapping $F\to\bigcap_{\varkappa\in F}\ker\varkappa$ is onto and, since

$$\bigcap_{\varkappa\in F}\ker\varkappa\neq\bigcap_{\varkappa\in G}\ker\varkappa$$

whenever $F\neq G$, it is also one-to-one. The result is proved. □

4.4.2 Proof of the Main Theorem

Now, we have all the tools at hand to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. By Proposition 4.3.5, the symbol ψ is a cyclic element of the semisimple regular Banach algebra $W^{1,2}[0,\infty)$. Thus, using Corollary 4.1.10 we obtain that $F\to\bigcap_{\varkappa\in F}\ker\varkappa$ is a one-to-one correspondence from the set of closed subsets of $\Omega(\mathscr{A})$ and $\operatorname{Lat} M_\psi$. By Proposition 4.4.4, we see that the map

$$F\to I_F=\{f\in W^{1,2}[0,\infty)\colon f\text{ vanishes on }F\}$$

is one-to-one from $\mathbb{F}[0,\infty)$ onto $\operatorname{Lat} M_\psi$. Since $M_\psi=\Phi C_\varphi^\star\Phi^{-1}$, it follows that the map

$$F\to J_F=\{f\in\mathscr{H}^2\colon\langle f,e_t\rangle_{\mathscr{H}^2}=0\text{ for }t\in F\}$$

is one-to-one from $\mathbb{F}[0,\infty)$ onto $\operatorname{Lat} C_\varphi^\star$. Since $\operatorname{Lat} C_\varphi$ consists of the orthogonal complements of $\operatorname{Lat} C_\varphi^\star$, we find that the map

$$F\to J_F^\perp$$

is one-to-one from $\mathbb{F}[0,\infty)$ onto $\operatorname{Lat} C_\varphi$. It remains to notice that $J_F^\perp=\overline{\operatorname{span}}\,\{e_t:t\in F\}$ for each F in $\mathbb{F}[0,\infty)$. The proof is complete. □

4.5 Consequences

Next we will show some direct consequences of the characterization of the lattice of invariant subspaces of C_{φ_a}. First, observe that in the proof of Theorem 4.1.1, parameter a did not play any role.

Corollary 4.5.1. *All composition operators induced by parabolic non-automorphisms that take the unit disk into itself share their lattices and their cyclic vectors.*

We can obtain two more corollaries from the proof of Theorem 4.1.1 that describe the lattices of the adjoint operator $C_\varphi^\star$ acting on the Hardy space and the multiplication operator $M_{e^{-\bar{a}t}}$ acting on the Sobolev space $W^{1,2}[0,\infty)$.

Corollary 4.5.2. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then*

$$\operatorname{Lat} C_\varphi^\star = \Big\{\{f \in \mathscr{H}^2 \colon \langle f, e_t\rangle_{\mathscr{H}^2} = 0 \text{ for } t \in F\} \colon F \in \mathbb{F}[0,\infty)\Big\}.$$

Corollary 4.5.3. *Let $M_{e^{-\bar{a}t}}$ be the operator of multiplication by $e^{-\bar{a}t}$ acting on the Sobolev space $W^{1,2}[0,\infty)$. Then*

$$\operatorname{Lat} M_{e^{-\bar{a}t}} = \Big\{\{f \in W^{1,2}[0,\infty) \colon f \text{ vanishes on } F\} \colon F \in \mathbb{F}[0,\infty)\Big\}.$$

Theorem 4.5.4. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then C_φ has no non-trivial reducing subspace.*

We will exhibit three different proofs based on the definition and the two different characterizations of reducing subspaces furnished by Theorem 2.4.3

First Proof. Let F be in $\mathbb{F}[0,+\infty)$ for which $N_F = \overline{\operatorname{span}}\,\{e_t : t \in F\}$ is non-trivial. We must show that its orthogonal complement $N_F^\perp$ is not invariant under C_φ. To this end we need to evaluate the inner product of two different eigenfunctions. Let $s > t \geq 0$, since e_t is an inner function then $|e_t(e^{i\theta})| = 1$ almost everywhere. Hence

$$\begin{aligned}
\langle e_t, e_s\rangle_{\mathscr{H}^2} &= \int_0^{2\pi} e_t(e^{i\theta})\overline{e_t(e^{i\theta})}\,\overline{e_{s-t}(e^{i\theta})}\,d\theta \\
&= \int_0^{2\pi} 1\,\overline{e_{s-t}(e^{i\theta})}\,d\theta \\
&= \overline{e_{s-t}(0)} \\
&= e^{-(s-t)}.
\end{aligned}$$

Interchanging the roles of s and t in case $t > s \geq 0$, we obtain that

$$\langle e_t, e_s \rangle_{\mathscr{H}^2} = e^{-|t-s|}, \qquad \text{for each } t, s \geq 0. \tag{4.9}$$

Now we are ready to prove the theorem.

First assume that 0 is not in F. Set $t_0 = \min F$. One easily checks that $f_{t_0} = 1 - e^{-t_0} e_{t_0}$ is orthogonal to e_t for each $t \geq t_0$, which means that f_{t_0} is in $N_F^\perp$. If $N_F^\perp$ is invariant under C_φ, then $f_{t_0} - C_\varphi f_{t_0}$ is in $N_F^\perp$. But $f_{t_0} - C_\varphi f_{t_0} = e^{-t_0}(1 - e^{-at_0}) e_{t_0}$ is also in N_F, which means that $f_{t_0} - C_\varphi f_{t_0} = 0$. Hence, $f_{t_0} \equiv 1$, a contradiction.

Assume now that 0 is in F. Let $s > 0$ be fixed and consider the operator M_{e_s} of multiplication by e_s. We have

$$M_{e_s}(N_F) = e_s \overline{\operatorname{span}} \{e_t : t \in F\} = \overline{\operatorname{span}} \{e_{s+t} : t \in F\} = N_{s+F}. \tag{4.10}$$

Clearly, M_{e_s} is a Hilbert space isometry preserving inner products. Therefore,

$$M_{e_s}\left(N_F^\perp\right) = \left(M_{e_s}(N_F)\right)^\perp. \tag{4.11}$$

Proceeding by contradiction, assume that $N_F^\perp$ is also invariant under C_φ. Then

$$M_{e_s}(C_\varphi(N_F^\perp)) \subseteq M_{e_s}(N_F^\perp). \tag{4.12}$$

Since, for each f in $\mathscr{H}^2$, we have

$$C_\varphi(M_{e_s} f) = C_\varphi(e_s f) = e^{-as} e_s C_\varphi f = e^{-as} M_{e_s}(C_\varphi f),$$

from (4.12), it follows that $C_\varphi(M_{e_s}(N_F^\perp))$ is included in $M_{e_s}(N_F^\perp)$. Therefore, from (4.10) and (4.11), we immediately see that

$$C_\varphi(N_{s+F}^\perp) \subseteq N_{s+F}^\perp.$$

This is a contradiction because 0 is not in $s + F$. The proof is complete. □

The next proof was indicated by El Hassan Zerouali.

Second Proof. According to assertion *2* in Theorem 2.4.3, $\mathcal{M}$ is reducing for C_{φ_a} if and only if $P_{\mathcal{M}}$ commutes with C_{φ_a}. Suppose T is in the commutant of C_{φ_a}, that is, $C_{\varphi_a} T = T C_{\varphi_a}$. Then, for all $t \geq 0$ we have

$$C_{\varphi_a}(Te_t) = T(C_{\varphi_a} e_t) = T(e^{-at} e_t) = e^{-at}(Te_t).$$

Thus Te_t is an eigenvector of C_{φ_a} and since all eigenvectors in $\sigma(C_{\varphi_a})$ have multiplicity one, then there exits a complex number $\lambda(t)$ such that $Te_t = \lambda(t)e_t$. Thus we have constructed a mapping $\lambda : [0, +\infty) \to \mathbb{C}$ such that

$$Te_t = \lambda(t)e_t \qquad \text{for all } t \geq 0.$$

Note that since T is bounded and e_{t_n} converges to e_t if and only if the sequence $\{t_n\}$ converges to t, the mapping λ is continuous.

Now suppose that T is a projection that commutes with C_{φ_a}. Since projections are idempotent operators, $T^2 = T$ and therefore $\lambda^2(t) = \lambda(t)$ for all $t \geq 0$. Therefore the range of λ is included in $\{0, 1\}$. Since we already know that λ is a continuous mapping, then either $\lambda \equiv 0$ or $\lambda \equiv 1$. Thus the only reducing subspaces are the trivial ones, $\{0\}$ and $\mathscr{H}^2$. □

Third Proof. According to assertion *3* in Theorem 2.4.3, a subspace $\mathcal{M}$ is reducing for C_{φ_a} if and only if it is invariant under both C_{φ_a} and $C^\star_{\varphi_a}$. Since $\mathcal{M} \in \text{Lat } C_{\varphi_a}$,

$$\mathcal{M} = \text{span}\{e_t \colon t \in F\}$$

for certain closed set $F \subseteq [0, +\infty)$. On the other side, $\mathcal{M} \in \text{Lat } C^\star_{\varphi_a}$. Corollary 4.5.2 shows that there exists F', a closed subset of $[0, +\infty)$, such that

$$\mathcal{M} = \left\{ f \in \mathscr{H}^2 \colon \langle f, e_t \rangle_{\mathscr{H}^2} = 0 \text{ for all } t \in F' \right\}.$$

Therefore $\mathcal{M} = \{0\}$, since $\langle e_t, e_s \rangle_{\mathscr{H}^2} = e^{-|t-s|} \neq 0$ for each non-negative numbers t and s. □

Recall that the cyclic vectors of an operator can be characterized as those that do not lie in any subspace invariant for this operator. This observation allows us to prove immediately the following corollary

Corollary 4.5.5. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then a function f in $\mathscr{H}^2$ is a cyclic vector of $C^\star_{\varphi_a}$ if and only if*

$$\langle f, e_t \rangle_{\mathscr{H}^2} \neq 0 \quad \textit{for all } t \geq 0.$$

Chapter 5

The Lattice in Other Spaces

In this chapter we characterize the lattice of invariant subspaces of the composition operator induced by a parabolic non-automorphism acting in a wide variety of spaces of analytic functions. First we start with some weighted Bergman spaces. Then we use that characterization to obtain the lattice of invariant subspaces in the rest of the Hardy spaces and in the remaining weighted Bergman spaces.

5.1 Dirichlet Space

Until now we have characterized the lattice of invariant subspaces of the composition operator induced by a parabolic non automorphism of the Hardy space. The characterization of the invariant subspaces relied heavily on the spectrum of the operator: any of its invariant subspaces is spanned by a set of eigenfunctions of the operator. An immediate question arises: Is still the lattice the same in a smaller space that does not contain the eigenfunctions but where the operator preserves the same spiral-like spectrum?.

The Dirichlet space fulfills the desired properties, it does not contain the eigenfunctions e_t's but the spectrum is still the same that the spectrum in the Hardy space [24]; nevertheless, there is a relevant difference: C_{φ_a} is normal in $\mathscr{D}_0$. Evenmore, the operator is completely normal. Recall that an operator is *normal* whenever it commutes with it adjoint. An operator is said *completely normal* in case it is normal and all its invariant subspaces are reducing, see

[39, p.22].

Let $A(z)$ stand for the normalized Lebesgue area of the unit disk. The Dirichlet space $\mathscr{D}$ is the space of functions f holomorphic on $\mathbb{D}$ for which the norm

$$\|f(z)\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

is finite. It is easy to see that the norm above comes from an inner product and $\mathscr{D}$ is indeed a Hilbert space. Let $\mathscr{D}_0$ denote the subspace of $\mathscr{D}$ that consists of the functions that vanish at 0 and let [1] the one dimensional subspace $\mathscr{D}$ formed by the constant functions. We clearly have the orthogonal decomposition $\mathscr{D} = [1] \oplus \mathscr{D}_0$. Let P denote the projection from $\mathscr{D}$ onto $\mathscr{D}_0$. Consider the compression $\widetilde{C}_\varphi = PC_\varphi P$ of C_φ to $\mathscr{D}_0$. Now, if $f = f(0) + g$, where g is in $\mathscr{D}_0$, then

$$C_\varphi f = C_\varphi f(0) + C_\varphi g = (f(0) + g(\varphi(0))) + \widetilde{C}_\varphi g = f(\varphi(0)) + \widetilde{C}_\varphi g.$$

When there is no risk of confusion, we will denote $\widetilde{C}_\varphi$ just by C_φ.

Proposition 5.1.1. *Let φ_a be a parabolic non-automorphism that takes the unit disk into itself. Then C_{φ_a} acting on $\mathscr{D}_0$ is completely normal.*

Proof. In [15], the adjoint of $C_{\varphi_a} : \mathscr{D}_0 \to \mathscr{D}_0$ is calculated. It turns out that $C^\star_{\varphi_a}$ is a composition operator as well, $C^\star_{\varphi_a} = C_{\psi_a}$, being its inducing symbol

$$\psi_a(z) = \frac{(2-\overline{a})z + \overline{a}}{-\overline{a}z + 2 + \overline{a}}.$$

Recall that any Möbius transform can be represented as a square two dimensional matrix and that the composition of two Möbius transformations corresponds precisely to matrix multiplication of the corresponding matrices. Thus $\varphi_a \circ \psi_a$ is represented as

$$\begin{pmatrix} 2-a & a \\ -a & 2+a \end{pmatrix} \begin{pmatrix} 2-\overline{a} & \overline{a} \\ -\overline{a} & 2+\overline{a} \end{pmatrix} = \begin{pmatrix} |2-a|^2 - |a|^2 & 2(a+\overline{a}) \\ -2(a+\overline{a}) & |2-a|^2 - |a|^2 \end{pmatrix}.$$

An the composition $\psi_a \circ \varphi_a$ corresponds to

$$\begin{pmatrix} 2-\overline{a} & \overline{a} \\ -\overline{a} & 2+\overline{a} \end{pmatrix} \begin{pmatrix} 2-a & a \\ -a & 2+a \end{pmatrix} = \begin{pmatrix} |2-a|^2 - |a|^2 & 2(a+\overline{a}) \\ -2(a+\overline{a}) & |2-a|^2 - |a|^2 \end{pmatrix}.$$

Therefore, $\varphi_a \circ \psi_a = \psi_a \circ \varphi_a$ and the composition operators mentioned above commute

$$C_{\psi_a} C_{\varphi_a} = C_{\varphi_a \circ \psi_a} = C_{\psi_a \circ \varphi_a} = C_{\varphi_a} C_{\psi_a}.$$

Since $C_{\psi_a} = C^{\star}_{\varphi_a}$, the operator C_{φ_a} is normal on $\mathscr{D}_0$.

To prove that the operator is completely normal, we make use of Theorem 1.23 of [39]. That theorem states that a normal operator whose spectrum is simply connected is completely normal. Since $\sigma(C_{\varphi_a})$ is a spiral, it is simply connected and therefore the operator is completely normal. ☐

5.1.1 An isomorphism between $\mathscr{D}_0$ and $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$

We will start defining an isometry between $\mathscr{D}_0$ and $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$. This will be accomplished in three steps. First we will move to the analog of the space $\mathscr{D}_0$ in the upper half-plane. Let Π denote the upper half plane. The Dirichlet space of the upper half plane $\mathscr{D}_\pi$ consists of those functions holomorphic on the upper half plane Π vanishing at i and for which

$$\|F\|^2_{\mathcal{D}_\pi} = \frac{1}{\pi} \int_\Pi \int_{-\infty}^{\infty} \int_0^{\infty} |f'(x+iy)|^2 \, dx dy$$

is finite.

Now, the change of variables

$$w = \sigma(z) = i\frac{1+z}{1-z}$$

that takes $\mathbb{D}$ into Π, shows that $F \in \mathscr{D}_\pi$ if and only if $F \circ \sigma \in \mathscr{D}_0$. Hence, the composition operator

$$C_\sigma : \mathscr{D}_\pi \to \mathscr{D}_0$$

defines an isometric isomorphism between $\mathscr{D}_\Pi$ and $\mathscr{D}_0$. Note that the isometry is well defined since $\sigma(0) = i$.

Second step will be to move to a Lebesgue space via the Fourier transform, $\mathcal{F}$. Following [24, Theorem 2.1], we know that $\mathcal{F} : \mathscr{D}_\Pi \to L^2(\mathbb{R}^+, t\, dt)$ is an isometry.

Finally, doing a change of variables we arrive to $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$ with the aid of the isometry defined as

$$\begin{array}{rccl} K_a : & L^2(\mathbb{R}^+, t\, dt) & \longrightarrow & L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right) \\ & f & \longmapsto & f\left(\frac{-\log t}{a}\right). \end{array}$$

Thus, we have constructed for any complex number with $\Re a > 0$ an isometry $\Psi_a = K_a \mathcal{F} C_\tau : \mathscr{D}_0 \to L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$. Let us see how the composition operator is transformed under this isometry.

Start with the composition operator C_{φ_a} acting on $\mathscr{D}_0$. It is similar to the operator $C_\tau C_{\varphi_a} C_\tau^{-1}$ acting on $\mathscr{D}_\Pi$. A straightforward computation shows up that $C_\tau C_{\varphi_a} C_\tau^{-1}$ is nothing but T_{ai}, the operator of translation by ai defined as $T_{ai} f(z) = f(z + ai)$. Thus, C_{φ_a} is similar to T_{ai} on $\mathscr{D}_\Pi$. At this point we recall one of the most basic properties of Fourier transform: it transforms translation by ai into multiplication by e^{-at}. Thus, via the Fourier transform, the operator T_{ai} is similar to $M_{e^{-at}}$ acting on $L^2(\mathbb{R}^+, t\,dt)$. Finally, the change of variables K_a transforms the operator $M_{e^{-at}}$ into another operator of multiplication, M_x, acting on $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$.

Theorem 5.1.2. *The operator defined as*

$$\Psi_a f(t) = K_a \mathcal{F} C_\tau f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f\left(\frac{x-i}{x+i}\right) t^{ix/a}\, dx$$

is an isometry from $\mathscr{D}_0$ onto $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$.

Furthermore, under this isometry the composition operator C_{φ_a} acting on $\mathscr{D}_0$ is similar to the multiplication operator M_x acting on $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$.

5.1.2 Invariant subspaces on $\mathscr{D}_0$

Now we are ready to study the invariant subspaces of C_{φ_a} and the isomorphism described in Theorem 5.1.2 will be the key tool for that together with

Proposition 5.1.3 ([44, Example 1]). *If X is a compact set of the complex plane and μ is a regular non-atomic Borel measure on X, then the invariant subspaces of $M_x : L^2(X, d\mu) \to L^2(X, d\mu)$ are*

$$\{f \in L^2(X, d\mu) \colon f = 0 \text{ a.e. on } E\}$$

for measurable sets E of (X, μ).

We will say that two Lebesgue measurable subsets A and B in $(0, \infty)$ are equivalent if $A \setminus B \cup B \setminus A$ has null measure. Let $\mathbb{A}(0, \infty)$ denote the set of the equivalent classes under the latter relation. We have,

Corollary 5.1.4. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then* $\operatorname{Lat} \widetilde{C}_\varphi$ *is the inverse image under $\mathcal{F}C_\sigma$ of*

$$\big\{\{f \in L^2(\mathbb{R}^+, tdt) : f \text{ vanish on } A\} : A \in \mathbb{A}(0,\infty)\big\}.$$

Proof. Theorem 5.1.2 asserts that C_{φ_a} is similar to multiplication by the independent variable on $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$. Applying Proposition 5.1.3 we obtain that any subspace of $L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$ invariant under M_x is of the type

$$\left\{f \in L^2\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right) : f = 0 \text{ a.e. on } E\right\}.$$

Now we can go back to $L^2(\mathbb{R}^+, t\,dt)$ using the isometry K_a. This isometry establishes a bijection between the invariant subspaces of $M_{e^{-at}}$ and those of M_x. Thus, a subspace $N \subseteq L^2(\mathbb{R}^+, t\,dt)$ is invariant under $M_{e^{-at}}$ if and only if

$$\mathcal{N} = \{f \in L^2(\mathbb{R}^+, t\,dt) \colon K_a f = 0 \text{ a.e. on } E_{\mathcal{N}}\}$$

for certain measurable set $E_{\mathcal{N}}$ of $\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$. Note that the mapping $\frac{-\log t}{a}$ transforms bijectively the σ-algebra of measurable sets of $(\mathbb{R}^+, t\,dt)$ onto that of measurable sets of $\left(\sigma(C_{\varphi_a}), \frac{-\log t}{a^2 t}\right)$. Thus,

$$\mathcal{N} = \{f \in L^2(\mathbb{R}^+, t\,dt) \colon f = 0 \text{ a.e. on } F\}$$

for some measurable set F of $(\mathbb{R}^+, t\,dt)$. The theorem is already proved since a subspace $\mathcal{M} \subseteq \mathscr{D}_0$ is invariant under C_{φ_a} if and only if $\mathcal{M}$ is mapped by $\mathcal{F}C_\tau$ onto one of the invariant subspaces of $M_{e^{-at}}$ characterized above. □

We can obtain a more explicit expression for the invariant subspaces of C_{φ_a}. Composing the isometries,

$$\mathcal{F}C_\tau f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f\left(\frac{x-i}{x+i}\right) e^{-ixt}\, dx.$$

Performing a change of variables in the above display, a subspace $\mathcal{M} \subset \mathscr{D}_0$ is invariant under C_{φ_a} if and only if there exists a measurable set F of $(\mathbb{R}^+, t\,dt)$ such that

$$\mathcal{M} = \left\{f \in \mathscr{D}_0 \colon \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) \frac{e^{t\frac{x+1}{x-1}}}{(1-x)^2}\, dx = 0 \text{ for a.e. t in } F\right\}.$$

5.1.3 Invariant subspaces on $\mathscr{D}$

At this point we wonder what can be said about the invariant subspaces of C_{φ_a} on $\mathscr{D}$. We have decomposed the Dirichlet space as the orthogonal sum of two of its subspaces

$$\mathscr{D} = [1] \oplus \mathscr{D}_0.$$

We recover the notation $\widetilde{C}_{\varphi_a}$ for the operator $PC_{\varphi_a}P$ acting on $\mathscr{D}_0$ and C_{φ_a} will denote the composition operator itself acting on the whole space $\mathscr{D}$. Since the constant function 1 is an eigenvector of C_{φ_a}, the subspace [1] is invariant under C_{φ_a} and the operator has the following matrix representation

$$C_{\varphi_a} = \begin{pmatrix} 1 & B \\ 0 & \widetilde{C}_{\varphi_a} \end{pmatrix},$$

where 1 represents the identity operator, relative to the orthogonal decomposition of $\mathscr{D}$ mentioned above. Thus, the composition operator acting on the Dirichlet space can be decomposed as the sum of two operators,

$$C_{\varphi_a} = 1 \oplus \widetilde{C}_{\varphi_a} + S = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{C}_{\varphi_a} \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \qquad (5.1)$$

where S is a rank one operator and C can be identified with the operator $C_{\varphi_a} : \mathscr{D}_0 \to \mathscr{D}_0$. Note that B is not zero since $\mathscr{D}_0$ is not invariant under C_{φ_a}. Even more, the following lemma asserts that there is no non-trivial invariant subspace of c_φ included in the subspace $\mathscr{D}_0$.

Lemma 5.1.5. *Let $\mathcal{M} \in$ Lat C_{φ_a} be different from the null space. If $P_{[1]} : \mathscr{D} \to \mathbb{C}$ denotes the orthogonal projection onto the constants, then $P_{[1]}M = \mathbb{C}$.*

Proof. Let $\mathcal{M} \in$ Lat C_{φ_a} be different from $\{0\}$ and suppose that $\mathcal{M} \subseteq \mathscr{D}_0$. Take a function f in $\mathcal{M}$. Since $\mathcal{M}$ is invariant under C_{φ_a}, the orbit $\{C^n_{\varphi_a} f\}_{n\geq 0}$ is included in M and therefore $C^n_{\varphi_a} f(0) = 0$ for any $n \geq 0$. Recall that φ_a is conjugated to a translation un the upper half-plane, thus $C^n_{\varphi_a} = C_{\varphi_{na}}$ and therefore $0 = C_{\varphi_{na}} f(0) = f(\varphi_{na}(0))$ for any $n \geq 0$. Hence any function in the subspace $\mathcal{M}$ must vanish at the sequence $\{\varphi_{na}(0) = na/(2+na)\}_{n\geq 0}$. But this sequence is not a Blaschke sequence, so the only function vanishing at this sequence is the zero function. This contradicts the assumption and proves the result. □

Now we are in position to prove the main theorem.

Theorem 5.1.6. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then for C_φ acting on the Dirichlet space $\mathcal{D}$, we have*

$$\operatorname{Lat} C_\varphi = \{0\} \cup \{[1] \oplus \mathcal{M} \text{ where } \mathcal{M} \in \operatorname{Lat} \widetilde{C}_\varphi\}.$$

Proof. First, suppose that $f \in [1] \oplus \mathcal{M}$, where $\mathcal{M}$ is an invariant subspace under $\widetilde{C}_\varphi$ acting on $\mathcal{D}_0$. Thus $f = f(0) + g$, where $g \in \mathcal{M}$. Therefore,

$$C_\varphi f = C_\varphi f(0) + C_\varphi g = (f(0) + g(\varphi(0))) + \widetilde{C}_\varphi g = f(\varphi(0)) + \widetilde{C}_\varphi g,$$

which belongs to $[1] \oplus \mathcal{M}$, since $\mathcal{M}$ is invariant under $\widetilde{C}_\varphi$. Thus $[1] \oplus \mathcal{M}$ is invariant for C_φ.

Conversely, suppose that $\mathcal{N}$ is invariant under C_φ. We may suppose that $\mathcal{N} \neq [1]$, since in such a case there is nothing to prove. In addition, Lemma 5.1.5 implies that $\mathcal{N}$ is not contained in $\mathcal{D}_0$. Let $\mathcal{M}$ denote the image of $\mathcal{N}$ under the orthogonal projection from $\mathcal{D}$ onto $\mathcal{D}_0$. Clearly, $\mathcal{M}$ is different from $\{0\}$ and is invariant under $\widetilde{C}_\varphi$. Indeed, choose an arbitrary function $g \in \mathcal{M}$. Then $g(z) = P_{\mathscr{D}_0} f(z)$ for certain f in $\mathcal{N}$. Therefore,

$$\widetilde{C}_\varphi g = P_{\mathscr{D}_0} C_\varphi P_{\mathscr{D}_0} g = P_{\mathscr{D}_0} C_\varphi g = P_{\mathscr{D}_0}(f(0) + C_\varphi g) = P_{\mathscr{D}_0} C_\varphi f,$$

where we have used in the second equality above we have used that $P_{\mathscr{D}_0}$ is idempotent and in the third one we used that $P_{\mathscr{D}_0} f(0) = 0$. Thus, since $\mathcal{N}$ is invariant under C_φ then $\widetilde{C}_\varphi g$ belongs to $\mathcal{M}$ and actually $\mathcal{M}$ is invariant under $\widetilde{C}_\varphi$. We will show that the constant function 1 belongs to $\mathcal{N}$, from which the result follows immediately.

Since $\mathcal{M} \neq \{0\}$, using the description of the invariant subspaces of $\widetilde{C}_\varphi$, we may choose a non-zero function $g \in \mathcal{M}$ such that $\mathcal{F} C_\sigma g$ has compact support. By a Paley-Wiener theorem, see [45, p. 375], $\mathcal{F}^{-1}(\mathcal{F} C_\sigma g)$ is an entire function. Therefore, g is analytic on the closed unit disk $\overline{\mathbb{D}}$. Thus the corresponding function $f = f(0) + g$ in $\mathcal{N}$ is also analytic on $\overline{\mathbb{D}}$. In particular,

$$f(z) = (1 - z)^k h(z)$$

for some non-negative integer k and h analytic on $\overline{\mathbb{D}}$ with $h(1) \neq 0$. We will show that

$$\frac{1}{(1-\varphi_n(0))^k} C_\varphi^n f = \frac{(1-\varphi_n(z))^k}{(1-\varphi_n(0))^k} h(\varphi_n(z))$$

tends, in the norm of $\mathcal{D}$, to $h(1)$. To this end, it is enough to show that

$$\frac{(2+na)^k}{2^k} \int_{\mathbb{D}} |(f(\varphi_n(z)))'| \, dA(z)$$

tends to zero in $\mathcal{D}$. The change of variables $w = \varphi_n(z)$ in the integral above yields

$$\frac{k(2+na)^k}{2^k} \int_{\varphi_n(\mathbb{D})} |(1-w)^{k-1} h(z)| \, dA(z) + \frac{(2+na)^k}{2^k} \int_{\varphi_n(\mathbb{D})} |(1-w)^k h'(z)| dA(z),$$

where the first term above does not appear in case $k = 0$. The above display is less than or equal to

$$\frac{(2+na)^k}{2^k} A(\varphi_n(\mathbb{D})) \left(k \max_{\varphi_n(\mathbb{D})} \{|1-w|^{k-1} |h(w)|\} + \max_{\varphi_n(\mathbb{D})} \{|1-w|^k |h(w)|\} \right).$$

Set $M' = \max_{\mathbb{D}} |h'(w)|$ and $M = \max_{\mathbb{D}} |h(w)|$. Then the above display is less than or equal to

$$\frac{(2+na)^k}{2^k} A(\varphi_n(\mathbb{D})) \left(\left| 1 - \frac{n\Re a - 1}{n\Re a + 1} \right|^{k-1} kM' + \left| 1 - \frac{n\Re a - 1}{n\Re a + 1} \right|^k M \right)$$

Since

$$A(\varphi_n(\mathbb{D})) = \frac{1}{2^2} \left(1 - \frac{n\Re a - 1}{n\Re a + 1} \right)^2,$$

the result follows. Since $h(1) \neq 0$, it follows that 1 is in $\mathcal{M}$. The proof is complete. □

5.2 Bergman Spaces

We have already characterized the lattice of invariant subspaces in the Dirichlet space, a space of analytic functions that does not contains the eigenfunctions of the composition operator but where the spectrum is the same. As expected, the lattice of invariant subspaces was totally different to the lattice of the operator acting on the Hardy space. Recall that the density of the span of the eigenfunctions was a key point to characterize the invariant subspaces of C_φ.

Now we turn our attention to weighted Bergman spaces of the unit disk. The Hardy space is densely contained in each of the weighted Bergman spaces. Therefore the eigenfunctions e_t are included in these spaces and they span it. Hence one would expect that as in the Hardy space case, each invariant subspace is spanned by eigenfunctions. As we will see, this is not the case in weighted Bergman spaces.

The *Bergman space* of the unit disk, denoted by $\mathcal{A}^2$, is defined as the set of analytic functions with square modulus integrable in $\mathbb{D}$. That is,

$$\mathcal{A}^2 = \left\{ f \in \mathscr{H}(\mathbb{D}) \colon \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \right\},$$

where $dA(z)$ denotes the normalized area measure on $\mathbb{D}$. It is a Hilbert space when endowed with the inner product

$$\langle f, g \rangle_{\mathcal{A}^2} = \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z),$$

for any two functions $f, g \in \mathcal{A}^2$.

The Bergman space is a particular instance of a wider class of spaces known as weighted Bergman spaces. For any real number $\alpha > -1$ we define the measure

$$dA_\alpha = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z).$$

For each $\alpha > -1$, the *weighted Bergman space* of the unit disk $\mathcal{A}^2_\alpha$ is the space formed by all analytic functions in $L^2(\mathbb{D}, dA_\alpha)$. The space $\mathcal{A}^2_\alpha$ is endowed with the inner product that inherits from $L^2(\mathbb{D}, dA_\alpha)$ that will be denoted as

$$\langle f, g \rangle_{\mathcal{A}^2_\alpha} = \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA_\alpha(z).$$

Equipped with the above inner products, the weighted Bergman spaces are Hilbert spaces. Bergman spaces are named after S. Bergman. Most of the theory of Bergman spaces has been developed during last four decades, although the major breakthroughs were made during the 1990's. Two books, appeared recently, are good introductions to the subject [12] and [21].

An orthonormal basis for $\mathcal{A}^2_\alpha$ is formed by the functions

$$\sqrt{\frac{\Gamma(n+2+\alpha)}{n!\,\Gamma(2+\alpha)}}\, z^n, \quad z \in \mathbb{D},$$

for $n = 0, 1, 2, \ldots$, see [21, p. 4]. Here Γ represents the usual Gamma function that generalizes the notion of factorial function to complex numbers.

Once we know about the existence of such an exceptional basis for $\mathcal{A}^2_\alpha$, we can rewrite its norm an inner product. Let

$$f = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g = \sum_{n=0}^{\infty} b_n z^n$$

be two functions in $\mathcal{A}^2_\alpha$, then

$$\|f\|^2_{\mathcal{A}^2_\alpha} = \sum_{n=0}^{\infty} \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} |a_n|^2 \tag{5.2}$$

and

$$\langle f, g \rangle_{\mathcal{A}^2_\alpha} = \sum_{n=0}^{\infty} \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}\, a_n \overline{b_n}. \tag{5.3}$$

From now on, we will use these equivalent norm and inner product in $\mathcal{A}^2_\alpha$. As an immediate consequence of the Stirling's formula for the Gamma function, the weight appearing in the inner product (5.3) can be asymptotically estimated as

$$\frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} \approx \Gamma(2+\alpha)(n+1)^{-1-\alpha} \approx (n+1)^{-1-\alpha}, \tag{5.4}$$

where the notation $f_n \approx g_n$ indicates that the quotient sequence $\{f_n/g_n\}$ is bounded from above and below by two positive constants as n tends to infinity. Thus we can define an equivalent inner product in $\mathcal{A}^2_\alpha$ as

$$\langle f, g \rangle_\alpha = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{(n+1)^{1+\alpha}}. \tag{5.5}$$

With this equivalent inner product, the Hardy space is easily seen to be a vector subspace of each $\mathcal{A}^2_\alpha$. In fact, the Hardy space can be regarded as the limit case $\alpha = -1$. An orthonormal basis for this new inner product is

$$(n+1)^{(1+\alpha)/2} z^n, \quad n = 0, 1, 2, \ldots \tag{5.6}$$

From the formula of the new equivalent inner product it is immediate to see that in case $p = 2$, the weighted Bergman spaces $\mathcal{A}^2_\alpha$ are totally ordered with respect to inclusion. In particular, if $\alpha_1 < \alpha_2$, then $\mathcal{A}^2_{\alpha_1} \subsetneq \mathcal{A}^2_{\alpha_2}$.

We close the section mentioning that Littlewood's Subordination Theorem implies that all composition operators are bounded in each weighted Bergman space $\mathcal{A}^2_\alpha$ for $\alpha > -1$, see [10, Theorem 3.1]. Thus it makes sense to study the lattice of invariant subspaces in each weighted Bergman space.

5.3 An Isomorphism Between $\mathcal{A}^2_k$ and $W^{k+2,2}_{k+1}$

In order to extend the characterization of $\mathrm{Lat}_{\mathscr{H}^2} C_{\varphi_a}$ to the weighted Bergman spaces $\mathcal{A}^2_\alpha$, the first attempt could be the following. Since we have develop certain machinery in the Sobolev space $W^{1,2}[0,\infty)$ and characterized its closed ideals, the first step would be to find an isomorphism between $\mathcal{A}^2_\alpha$ and $W^{1,2}[0,\infty)$. This can be done easily for each $\alpha > -1$.

Proposition 5.3.1. *Fix $\alpha > -1$ and set $\beta = (-1+\alpha)/2$. Then the mapping*

$$\begin{array}{rccl} \Phi_\alpha : & \mathcal{A}^2_\alpha & \longrightarrow & W^{1,2}[0,\infty) \\ & f & \longmapsto & (\Phi_\alpha f)(t) = \langle f, e_t \rangle_{\mathcal{A}^2_\beta} \end{array}$$

is an isomorphism of Hilbert spaces.
In addition, $\|\Phi_\alpha f\|^2_{1,2} = \|f\|^2_{\mathcal{A}^2_\alpha} - \dfrac{|f(0)|^2}{2}$.

Proof. Due to the particular choice of β, the image of the basis (5.6) for $\mathcal{A}^2_\alpha$ under the mapping Φ_α is

$$(\Phi_\alpha (n+1)^{(1+\alpha)/2} z^n)(t) = \langle (n+1)^{(1+\alpha)/2} z^n, e_t \rangle_{\mathcal{A}^2_\beta} = \langle z^n, e_t \rangle_{\mathscr{H}^2} = (\Phi z^n)(t).$$

Hence the image of the basis coincides with the image under Φ of the standard basis for the Hardy space. Since we already proved in Corollary 4.3.2 that Φ is an isomorphism, so is Φ_α.

Therefore Φ_α is well defined since it can be extended continuously to the whole space $\mathcal{A}^2_\alpha$. The formula for the norm is a consequence of the formula for the norm appearing in Corollary 4.3.2. ∎

Therefore we have already obtained an isomorphism between each $\mathcal{A}^2_\alpha$ and the Sobolev space $W^{1,2}[0,\infty)$. Next step would be to check wether $C^\star_{\varphi_a}$ is similar under the isomorphism Φ_α to the multiplication operator $M_{e^{-\overline{a}t}}$. Fix $\alpha > -1$ and, as above, let $\beta = (-1+\alpha)/2$. Then for each function f in $\mathcal{A}^2_\alpha$ we can write

$$(\Phi_\alpha C^\star_{\varphi_a} f)(t) = \langle C^\star_{\varphi_a} f, e_t\rangle_{\mathcal{A}^2_\beta} = \langle f, C_{\varphi_a} e_t\rangle_{\mathcal{A}^2_\beta} = e^{-\overline{a}t}\langle f, e_t\rangle_{\mathcal{A}^2_\beta} = e^{-\overline{a}t}(\Phi_\alpha f)(t).$$

The above chain of equalities equals to say that $\Phi_\alpha C^\star_{\varphi_a} = M_{e^{-\overline{a}t}}\Phi_\alpha$. However, there is an error in second equality in above display. The operator $C^\star_{\varphi_a}$ denotes the adjoint of the composition operator C_{φ_a} in the Bergman space $\mathcal{A}^2_\alpha$, therefore it is defined through the equality $\langle C^\star_{\varphi_a} f, e_t\rangle_\alpha = \langle f, C_{\varphi_a} e_t\rangle_{\mathcal{A}^2_\alpha}$. Hence for the Bergman space $\mathcal{A}^2_\beta$ we have that

$$\langle C^\star_{\varphi_a} f, e_t\rangle_{\mathcal{A}^2_\beta} \neq \langle f, C_{\varphi_a} e_t\rangle_{\mathcal{A}^2_\beta}.$$

since $C^\star_{\varphi_a}$ is not the adjoint of C_{φ_a} acting on $\mathcal{A}^2_\beta$. Thus, if we want the adjoint operator $C^\star_{\varphi_a}$ to be similar to the multiplication operator $M_{e^{-\overline{a}t}}$, there is no choice that redefine the isomorphism Φ_α as

$$(\Phi_\alpha f)(t) = \langle f, e_t\rangle_{\mathcal{A}^2_\alpha}, \quad t \geq 0, \tag{5.7}$$

for each function $f \in \mathcal{A}^2_\alpha$ and identify the arriving space as certain Sobolev space.

For each non-negative integer k, consider the Hilbert space $L^2_k[0,\infty)$ of complex Lebesgue measurable functions defined on $[0,\infty)$ endowed with the inner product

$$\langle f, g\rangle_k = \int_0^\infty f(t)\overline{g(t)}t^k\,dt, \qquad f, g \in L^2_k[0,\infty).$$

For each non-negative integer k, the Sobolev space $W^{k+1,2}_k[0,\infty)$ consists of those functions f in $L^2_k[0,\infty)$ which have derivatives of order j, $1 \leq j \leq k$, and $f^{(k)}$ is absolutely continuous on each bounded subinterval of $[0,\infty)$ and $f^{(k+1)}$ belongs to $L^2_k[0,\infty)$. It is simple to check that the space $W^{k+1,2}_k[0,\infty)$ becomes a Hilbert space endowed with the inner product

$$\langle f, g\rangle_{W^{k+1,2}_k} = \sum_{j=0}^{k+1}\binom{k+1}{j}\int_0^\infty f^{(j)}(t)\overline{g^{(j)}(t)}t^k dt, \qquad f, g \in W^{k+1,2}_k,$$

where, as usual, $f^{(0)}$ denotes the function itself.

Set the notation $u_n(z) = (n+1)^{(1+\alpha)/2} z^n$ for the members of the orthonormal basis of $\mathcal{A}^2_\alpha$. Recall from (4.8) that the Taylor coefficients of e_t are Laguerre polynomials evaluated at $2t$ multiplied by e^{-t}. Therefore the orthonormal basis of $\mathcal{A}^2_\alpha$ is mapped under Φ_α to

$$(\Phi_\alpha u_n)(t) = \langle u_n, e_t \rangle_{\mathcal{A}^2_\alpha} = (n+1)^{-(1+\alpha)/2} e^{-t} L_n^{(-1)}(2t). \tag{5.8}$$

Before proceeding further, we recall some properties of the Laguerre polynomials that will be needed in what follows. For each $\alpha \geq -1$, the Laguerre polynomial of index α and degree n arises from the generating function

$$\frac{1}{(1-z)^{1+\alpha}} \exp\left(\frac{-zt}{1-z}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(t) z^n.$$

The characteristic feature of the Laguerre polynomials is that the sequence $\{e^{-t/2} L_n^{(\alpha)}(t)\}_{n\geq 0}$ forms a complete orthogonal system for the space $L^2_\alpha[0,\infty)$. Indeed, if δ_{nm} denotes Kronecker's delta, then

$$\int_0^\infty t^\alpha e^{-t} L_n^{(\alpha)}(t) L_m^{(\alpha)}(t)\, dt = \frac{\Gamma(1+\alpha+n)}{n!} \delta_{nm}. \tag{5.9}$$

It is easy to see that for $\alpha \geq 1$, the Laguerre polynomials satisfy

$$L_n^{(\alpha)}(t) = L_n^{(\alpha+1)}(t) - L_{n-1}^{(\alpha+1)}(t)$$

and

$$D L_n^{\alpha}(t) = -L_{n-1}^{(\alpha+1)}(t),$$

where D denotes the derivative operator, see [40, p. 203]. From these two properties, straightforward easy calculations show that the functions defined by

$$\mathcal{L}_n^{(\alpha)}(t) = e^{-t} L_n^{(\alpha)}(2t)$$

for each $n \geq 0$ and for each $\alpha \geq 1$, satisfy

$$\mathcal{L}_n^{(\alpha)} = \mathcal{L}_n^{(\alpha+1)} - \mathcal{L}_{n-1}^{(\alpha+1)} \tag{5.10}$$

and

$$D\mathcal{L}_n^{(\alpha)} = -\mathcal{L}_n^{(\alpha+1)} - \mathcal{L}_{n-1}^{(\alpha+1)}. \tag{5.11}$$

As above, in all that follows, $\mathcal{L}_n^{(\alpha)}$ is assumed to be 0 whenever $n < 0$. A change of variables shows that the fundamental relations in (5.9) written in terms of $\mathcal{L}_n^{(\alpha)}$ are

$$\langle \mathcal{L}_n^{(\alpha)}, \mathcal{L}_m^{(\alpha)} \rangle_{\mathcal{A}_\alpha^2} = \frac{1}{2^{1+\alpha}} \frac{\Gamma(1+\alpha+n)}{n!} \delta_{nm}. \tag{5.12}$$

We have,

Theorem 5.3.2. *For each non-negative integer k, the map Φ_k is an isomorphism from $\mathcal{A}_k^2(\mathbb{D})$ onto $W_{k+1}^{k+2,2}[0,\infty)$.*

Proof. We start by proving that $\{(\Phi_k u_n)(t)\}_{n\geq 0}$, is a complete orthogonal system of $W_{k+1}^{k+2,2}[0,\infty)$. To this end, observe that fixed a natural number n the functions

$$\Phi_k u_n(t) = (n+1)^{-(1+k)/2} \mathcal{L}_n^{(-1)}(t)$$

are essentially the same, they just differ by a constant multiple. Hence, showing that for each non-negative integer k the set $\{(\Phi_k u_n)(t)\}_{n\geq 0}$ is a complete orthogonal system of $W_{k+1}^{k+2,2}[0,\infty)$ equals proving that the system $\{\mathcal{L}_n^{(-1)}\}_{n\geq 0}$ is complete and orthogonal for each Sobolev space $W_{k+1}^{k+2,2}[0,\infty)$. To handle a simpler notation, we will prove that the latter system is a complete orthogonal system for $W_k^{k+1,2}[0,\infty)$ for each non-negative integer k.

In order to do this, we will show that the inner product in $W_k^{k+1,2}[0,\infty)$ of two functions in $\{\mathcal{L}_n^{(-1)}\}_{n\geq 0}$ is given by

$$\sum_{l=0}^{k+1} \binom{k+1}{l} \langle D^l \mathcal{L}_n^{(-1)}, D^l \mathcal{L}_m^{(-1)} \rangle = 2^{k+1} \sum_{l=0}^{k+1} \binom{k+1}{l} \langle \mathcal{L}_{n-l}^{(k)}, \mathcal{L}_{m-l}^{(k)} \rangle. \tag{5.13}$$

We stress here that identity above is just a consequence of the linearity of the inner product and properties (5.10) and (5.11) of the functions $\mathcal{L}_n^{(-1)}$ and not of cancelations arising from their orthogonality properties shown in (5.12). In fact, this is the reason why we can get rid of the index in the inner product above. In particular, we may and will use the identity in (5.13) for other functions different from $\mathcal{L}_n^{(-1)}$ as soon as they satisfy the properties in (5.10) and (5.11).

The identity in (5.13) will be proved by induction on k. For $k = 0$, the left hand side in (5.13) is equal to

$$\langle \mathcal{L}_n^{(-1)}, \mathcal{L}_m^{(-1)} \rangle + \langle (\mathcal{L}_n^{(-1)})', (\mathcal{L}_m^{(-1)})' \rangle.$$

Using (5.10) and (5.11), we see that the above display is equal to

$$\langle \mathcal{L}_n^{(0)} - \mathcal{L}_{n-1}^{(0)}, \mathcal{L}_m^{(0)} - \mathcal{L}_{m-1}^{(0)} \rangle + \langle \mathcal{L}_n^{(0)} + \mathcal{L}_{n-1}^{(0)}, \mathcal{L}_m^{(0)} + \mathcal{L}_{m-1}^{(0)} \rangle,$$

which, using the linearity of inner product, is equal to

$$2\langle \mathcal{L}_n^{(0)}, \mathcal{L}_m^{(0)} \rangle + 2\langle \mathcal{L}_{n-1}^{(0)}, \mathcal{L}_{m-1}^{(0)} \rangle.$$

Now, suppose that we have already proved the identity in (5.13) for $k-1$. Using the binomial relations

$$\binom{k}{l-1} + \binom{k}{l} = \binom{k+1}{l}, \tag{5.14}$$

we see that the left-hand side in (5.13) is equal to

$$\sum_{l=1}^{k+1} \binom{k}{l-1} \langle D^l \mathcal{L}_n^{(-1)}, D^l \mathcal{L}_m^{(-1)} \rangle + \sum_{l=0}^{k} \binom{k}{l} \langle D^l \mathcal{L}_n^{(-1)}, D^l \mathcal{L}_m^{(-1)} \rangle. \tag{5.15}$$

The induction hypothesis shows that the second term above is equal to

$$2^k \sum_{l=0}^{k} \binom{k}{l} \langle \mathcal{L}_{n-l}^{(k-1)}, \mathcal{L}_{m-l}^{(k-1)} \rangle. \tag{5.16}$$

On the other hand, a change of indexes shows that the first term in (5.15) is equal to

$$\sum_{l=0}^{k} \binom{k}{l} \langle D^{l+1} \mathcal{L}_n^{(-1)}, D^{l+1} \mathcal{L}_m^{(-1)} \rangle = 2^k \sum_{l=0}^{k} \binom{k}{l} \langle D\mathcal{L}_{n-l}^{(k-1)}, D\mathcal{L}_{m-l}^{(k-1)} \rangle \tag{5.17}$$

where we have applied the induction hypothesis again with $\mathcal{L}_n^{(\alpha)}$ replaced by $D\mathcal{L}_n^{(\alpha)}$, which also satisfy the properties in (5.10) and (5.11). Finally, using (5.10) and (5.11) in (5.16) and (5.17), respectively, we obtain that (5.15) is equal to

$$2^k \sum_{l=0}^{k} \binom{k}{l} \left(\langle \mathcal{L}_{n-l}^{(k)} + \mathcal{L}_{n-l-1}^{(k)}, \mathcal{L}_{m-l}^{(k)} + \mathcal{L}_{m-l-1}^{(k)} \rangle + \langle \mathcal{L}_{n-l}^{(k)} - \mathcal{L}_{n-l-1}^{(k)}, \mathcal{L}_{m-l}^{(k)} - \mathcal{L}_{m-l-1}^{(k)} \rangle \right)$$

Cancelations along with the binomial relation in (5.14) show that the above display is equal to

$$2^{k+1} \sum_{l=0}^{k+1} \binom{k+1}{l} \langle \mathcal{L}_{n-l}^{(k)}, \mathcal{L}_{m-l}^{(k)} \rangle$$

an the induction is complete. Now, the orthogonality of the system $\{\mathcal{L}_n^{(-1)}\}$ in $W_k^{k+1,2}[0,\infty)$ follows immediately from formulas in (5.12) and (5.13). The fact that the system $\{\mathcal{L}_n^{(-1)}\}$ is complete in $W_k^{k+1,2}$ is standard and a proof that can be done following the lines of the proof of Theorem 5.7.1 in [50, p. 107]. Thus we omit the proof here.

Finally, we show that Φ is indeed an isomorphism. Indeed, for each $n \geq 0$, we clearly have $\Phi_k u_n$ is different from 0, otherwise u_n would be orthogonal to each e_t and this is not possible since the eigenfunctions span the space $\mathcal{A}_k^2$. In addition, for $n \geq k$, we deduce from (5.13) and (5.12) that

$$\begin{aligned}
\|\Phi_k u_n\|^2_{W_{k+1}^{k+2,2}} &= \frac{1}{(n+1)^{k+1}} \|\mathcal{L}_n^{(-1)}\|^2_{W_{k+1}^{k+2,2}} \\
&= \frac{2^{k+2}}{(n+1)^{k+1}} \sum_{l=0}^{k+2} \binom{k+2}{l} \langle \mathcal{L}_{n-l}^{(k+1)}, \mathcal{L}_{n-l}^{(k+1)} \rangle_{k+1} \\
&= \frac{1}{(n+1)^{k+1}} \sum_{l=0}^{k+2} \binom{k+2}{l} \frac{(n-l+k+1)!}{(n-l)!}.
\end{aligned}$$

Upon making n tend to ∞, the last display tends to

$$\sum_{l=0}^{k+2} \binom{k+2}{l} = 2^{k+2}.$$

Therefore, $\{\Phi_k u_n\}$ is bounded above and bounded away from 0. Since $\{u_n\}$ and $\{\widetilde{u}_n\}$ are complete orthogonal systems, it immediately follows that Φ_k is an isomorphism between $\mathcal{A}_k^2(\mathbb{D})$ and $W_{k+1}^{k+2,2}[0,\infty)$. The proof is complete. □

Now we can obtain the desired similarity between the adjoint of the composition operator and a multiplication operator acting on the Sobolev space $W_{k+1}^{k+2,2}[0,\infty)$.

Proposition 5.3.3. *Let φ_a be a parabolic, either automorphism or not, self-map of the unit disk. Then the adjoint of C_{φ_a} acting on $\mathcal{A}_k^2$ is similar under Φ_k to the multiplication operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$, acting on $W_{k+1}^{k+2,2}[0,\infty)$.*

Proof. Using the eigenvalue equation (4.5), for each $f \in \mathcal{A}_k^2(t)$, we have

$$(\Phi_k C_{\varphi_a}^\star f)(t) = \langle C_{\varphi_a}^\star f, e_t \rangle_{\mathcal{A}_k^2} = \langle f, C_{\varphi_a} e_t \rangle_{\mathcal{A}_k^2} = e^{-\bar{a}t} \langle f, e_t \rangle_{\mathcal{A}_k^2} = e^{-\bar{a}t} (\Phi_k f)(t),$$

for each $t \geq 0$. Thus $M_\psi = \Phi_k C_{\varphi_a}^\star \Phi_k^{-1}$. The result is proved. □

5.4 The Lattice in $\mathcal{A}_\alpha^2$

Proposition 5.4.1. *For each even integer $k \geq 0$, the space $W_k^{k+1,2}[0, \infty)$ is a commutative Banach algebra without identity.*

Furthermore, $\|f^{(l)}\|_\infty \leq \|f\|_{W_k^{k+1,2}}$ for each $0 \leq l < (k+1)/2$. Hence, convergence in $W_k^{k+1,2}[0, \infty)$ implies uniform convergence of the derivatives up to order $(k+1)/2$.

Proof. First, we will show that for each positive integer $l < (k+1)/2$, the operator D^l from $W_k^{k+1,2}[0, \infty)$ into $L^\infty[0, \infty)$ is bounded. Each function f in $W_k^{k+1,2}[0, \infty)$ can be written as

$$f(t) = (\Phi_{k-1} h)(t) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^k} \mathcal{L}_n^{(-1)}(t),$$

where $h = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{A}_{k-1}^2$. Upon iterating (5.11), we have the second equality below,

$$\begin{aligned} |(D^l f)(t)| &= \left| \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^k} (D^l \mathcal{L}_n^{(-1)})(t) \right| \\ &= \left| \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^k} \sum_{j=0}^{l} \binom{l}{j} \mathcal{L}_{n-j}^{(l-1)}(t) \right| \\ &\leq \sum_{j=0}^{l} \binom{l}{j} \left| \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^k} \mathcal{L}_{n-j}^{(l-1)}(t) \right|. \end{aligned}$$

Now, recall that $\mathcal{L}_{n-j}^{(l-1)}$ is null for $n - j < 0$. Thus we have the equality below. Applying the Cauchy-Schwarz inequality in the first inequality below and the fact that Φ_{k-1} is an isomorphism in the second, we have

$$\begin{aligned}
\left|\sum_{n=0}^{\infty}\frac{a_n}{(n+1)^k}\mathcal{L}_{n-j}^{(l-1)}(t)\right| &= \left|\sum_{n=0}^{\infty}\frac{a_{n+j}}{(n+j+1)^k}L_n^{(l-1)}(2t)e^{-t}\right| \\
&\leq \|h\|_{\mathcal{A}^2_{k-1}}\left(2e^{-2t}+\sum_{n=1}^{\infty}\frac{(L_n^{(l-1)}(2t))^2}{(n+1)^k}e^{-2t}\right)^{1/2} \\
&\leq C\|f\|_{W_k^{k+1,2}}\left(2e^{-2t}+\sum_{n=1}^{\infty}\frac{(L_n^{(l-1)}(2t))^2}{(n+1)^k}e^{-2t}\right)^{1/2} \\
&\leq C\|(1-z)^{-l}\|_{\mathcal{A}^2_{k-1}}\|f\|_{W_k^{k+1,2}},
\end{aligned}$$

where the last inequality follows from the identity

$$\sum_{n=0}^{\infty}e^{-t}L_n^{(l-1)}(2t)z^n = \frac{e^{-t}}{(1-z)^l}\exp\left(\frac{-2tz}{1-z}\right),$$

and the fact that $|e_t(z)|\leq 1$. The power series of the function $(1-z)^{-l}$ is

$$\frac{1}{(1-z)^l}=\sum_{n=0}^{\infty}\frac{\Gamma(n+l)}{n!\,\Gamma(l)}z^n.$$

Hence, we can apply Stirling's formula to estimate its coefficients as in (5.4) and obtain that $(1-z)^{-l}$ belongs $\mathcal{A}^2_{k-1}$ if and only if $l<(k+1)/2$. Putting everything together, there is a positive constant C, which does not depend on f, such that

$$\|D^l f(t)\|_\infty \leq C\|f\|_{W_k^{k+1,2}}, \text{ for } 0\leq l<k.$$

The above inequality is what in fact justifies taking the derivatives in the sum above.

Now, assume that f and g belong to $W_k^{k+1,2}[0,\infty)$. We claim that

$$\|f^{(l)}g^{(j-l)}\|_k \leq C\|f\|_{W_k^{k+1,2}}\|g\|_{W_k^{k+1,2}}, \qquad \text{for } 0\leq l\leq j\leq k+1. \tag{5.18}$$

Indeed, note that $(k+1)/2<k$ if and only if $k\geq 2$. Thus, for $0\leq l\leq (k+1)/2$, we have

$$\|f^{(l)}g^{(j-l)}\|_k \leq C\|f^{(l)}\|_\infty\|g^{(j-l)}\|_k \leq C\|f\|_{W_k^{k+1,2}}\|g\|_{W_k^{k+1,2}}.$$

For $l > (k+1)/2$, it must be $j - l \le (k+1)/2$ and again, we have

$$\|f^{(l)} g^{(j-l)}\|_k \le C\|f^{(l)}\|_k \|g^{(j-l)}\|_\infty \le C\|f\|_{W_k^{k+1,2}} \|g\|_{W_k^{k+1,2}}.$$

Therefore, (5.18) holds and we can use it in the second inequality below.

$$\begin{aligned}
\|fg\|^2_{W_k^{k+1,2}} =& \sum_{j=0}^{k+1} \binom{k+1}{j} \|(fg)^j\|_k^2 \\
=& \sum_{j=0}^{k+1} \binom{k+1}{j} \left\| \sum_{l=0}^{j} \binom{j}{l} f^{(j)} g^{(j-l)} \right\|_k^2 \\
\le& \sum_{j=0}^{k+1} \binom{k+1}{j} \sum_{l=0}^{j} \binom{j}{l} \|f^{(j)} g^{(j-l)}\|_k^2 \\
\le& C \sum_{j=0}^{k+1} \binom{k+1}{j} \sum_{l=0}^{j} \binom{j}{l} \|f\|^2_{W_k^{k+1,2}} \|g\|^2_{W_k^{k+1,2}} \\
=& C3^{k+1} \|f\|^2_{W_k^{k+1,2}} \|g\|^2_{W_k^{k+1,2}}.
\end{aligned}$$

The result is proved. □

Note that in case k is an odd integer, then one cannot proceed as in the proof of Proposition 5.4.1. Indeed, for each f in $W_1^{2,2}[0,\infty)$ there exists a function $h(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{A}^2$ such that

$$f(t) = (\Phi_0 h)(t) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \mathcal{L}_n^{(-1)}(t).$$

Its derivative is given by

$$f'(t) = -\sum_{n=0}^{\infty} \frac{a_n}{n+1} (\mathcal{L}_n^{(0)}(t) + \mathcal{L}_{n-1}^{(0)}(t)).$$

The derivative f' does not need to be bounded. Indeed, $\mathcal{L}_n^{(0)}(0) = 1$ for every n and there are plenty of functions in $\mathcal{A}^2$ for which the series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

diverges.

We continue with the study of the lattice of invariant subspaces in $\mathcal{A}_k^2$ for k an even non-negative integer. Given a point w in $\mathbb{D}$, the reproducing kernel for the weighted Bergman space $\mathcal{A}_\alpha^2$ at w is the function

$$k_{w,\alpha}(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}},$$

see [21, p. 6].

Proposition 5.4.2. *Let $\psi(t) = e^{-\overline{a}t}$ and $\Re a > 0$. Then the operator M_ψ acting on $W_k^{k+1,2}[0,\infty)$ is cyclic with cyclic vector ψ for each non-negative integer k.*

Proof. Recall that $\sigma(z) = (\overline{z}-1)/(\overline{z}+1)$ maps bijectively the right half plane onto the unit disc. Therefore, for any a with $\Re a > 0$ we set $w = \sigma(a) \in \mathbb{D}$. Then $(w+1)/(w-1) = a$ and hence

$$\Phi_k k_{w,k}(t) = \langle k_{w,k}, e_t\rangle_{\mathcal{A}_k^2} = \overline{e^{-t\frac{w+1}{w-1}}} = e^{-\overline{a}t}.$$

Since cyclicity is preserved under similarities, proving the statement is equivalent to prove that the function $k_\alpha(z)$ is cyclic for $C_{\varphi_a}^\star$ on the space $\mathcal{A}_k^2$.

Suppose that f in $\mathcal{A}_k^2$ is orthogonal to the orbit of k_α under $C_{\varphi_a}^\star$. Then, for each $n \geq 0$, we have

$$\begin{aligned} 0 &= \langle C_{\varphi_a}^{\star n} k_\alpha, f\rangle_{\mathcal{A}_k^2} \\ &= \langle k_\alpha, C_{\varphi_a}^n f\rangle_{\mathcal{A}_k^2} \\ &= \langle k_\alpha, C_{\varphi_{na}} f\rangle_{\mathcal{A}_k^2} \\ &= \langle k_\alpha, f \circ \varphi_{na}\rangle_{\mathcal{A}_k^2} \\ &= f(\varphi_{na}(\alpha)). \end{aligned}$$

The sequence $\{\varphi_{na}(\alpha)\}$ is not a Blaschke sequence, but this does not necessarily imply that the Bergman function f is the null function, since not all zero sequences of Bergman functions are Blaschke sequences. Thus, in order to ensure that f equals zero, we must ask an extra property to the sequence $\{\varphi_{na}(\alpha)\}$. This extra property is furnished by Lemma 4.3 in [17, p. 51] which states that if a function g in $\mathcal{A}_k^2$ vanishes in a sequence $\{w_n\}$ and this sequence is included in $D(1/2, 1/2)$, an Euclidean disk with center at $1/2$ and radius 1/2, then $\{w_n\}$ is a Blaschke sequence. Since $\varphi_{na}(\mathbb{D}) \subseteq D(1/2, 1/2)$ for a sufficiently large n, then Lemma 4.3 of [17] implies that f is the null function and the proposition is proved. $\square$

Observe that, once again, every reproducing kernel is a cyclic vector for the adjoint operator $C^\star_{\varphi_a}$.

Now that we already know that $W^{k+2,2}_{k+1}[0,\infty)$ is a Banach algebra and that $M_{e^{-\overline{a}t}}$ is cyclic with cyclic vector $e^{-\overline{a}t}$, Proposition 4.1.2 implies that the lattice of invariant subspaces of $M_{e^{-\overline{a}t}}$ is formed by all closed ideals of $W^{k+2,2}_{k+1}[0,\infty)$. To identify the ideals in that Sobolev space, we will make use of the characterization of the closed ideals in $W^{1,2}[0,\infty)$. To this end, we start proving that the closure in $W^{1,2}[0,\infty)$ of a closed ideal of $W^{k+1,2}_k[0,\infty)$ is again a closed ideal in $W^{1,2}[0,\infty)$.

Proposition 5.4.3. *Let I be a closed ideal of $W^{k+1,2}_k[0,\infty)$. Then $\overline{I}^{W^{1,2}[0,\infty)}$, its closure in $W^{1,2}[0,\infty)$, is a closed ideal in $W^{1,2}[0,\infty)$.*

Proof. Let I be a closed ideal of $W^{k+1,2}_k[0,\infty)$. We start showing that $\overline{I}^{W^{1,2}}$ is a closed ideal of $W^{1,2}[0,\infty)$. Indeed, let f be a function in $\overline{I}^{W^{1,2}}$. Then there exists a sequence $\{f_n\}_{n\geq 0} \subseteq I$ converging to f in $W^{1,2}[0,\infty)$. Now let, g be an arbitrary function in $W^{1,2}[0,\infty)$. Since $W^{k+2,2}_{k+1}[0,\infty)$ is dense in the space $W^{1,2}[0,\infty)$, then there exists a sequence $\{g_n\} \subseteq W^{k+2,2}_{k+1}[0,\infty)$ such that g_n converges to g in $W^{1,2}[0,\infty)$. Since $W^{1,2}[0,\infty)$ is a Banach algebra, we have

$$\begin{aligned}\|g_nf_n - gf\|_{1,2} &\leq \|g_nf_n - g_nf\|_{1,2} + \|g_nf - gf\|_{1,2}\\ &\leq C\|g_n\|_{1,2}\|f_n - f\|_{1,2} + C\|g_n - g\|_{1,2}\|f\|_{1,2},\end{aligned}$$

for any natural number n. Thus the sequence $\{g_nf_n\}$ converges to gf. Since I is an ideal of $W^{k+2,2}_{k+1}[0,\infty)$ and $\{f_n\} \subseteq I$, then $g_nf_n \in I$ for all $n \geq 0$. Therefore $gf \in \overline{I}^{W^{1,2}}$ and, since g was arbitrary, $\overline{I}^{W^{1,2}}$ is an ideal of $W^{1,2}[0,\infty)$. $\square$

However, last proposition does not imply that all ideals in $W^{k+1,2}_k[0,\infty)$ are of the form $\{f \in W^{k+1,2}_k[0,\infty) \colon f(t) = 0 \text{ para todo } t \in F\}$. As an immediate consequence of 5.4.1 we have the following

Proposition 5.4.4. *For each even integer $k \geq 2$, the set*

$$I_F = \{f \in W_k^{k+1,2}[0,\infty) \colon f^{(l)}(t) = 0 \text{ for all } t \in F,\ 0 \leq l \leq j\}$$

is a closed ideal of $W_k^{k+1,2}[0,\infty)$ for each $F \in \mathbb{F}[0,\infty)$ and each $0 \leq j < (k+1)/2$.

Furthermore, in case F has isolated points and $j \geq 1$,

$$I_F \neq \{f \in W_k^{k+1,2}[0,\infty) \colon f(t) = 0 \text{ for all } t \in F\}.$$

Proof. The first assertion follows immediately from the uniform convergence of the derivatives

For the second, set t_0 be an isolated point of F and (a,b) an interval such that $(a,b) \cap F = \{t\}$. Let f be a C^∞ function that equals 0 outside (a,b) and does not vanish on (a,b). Then $(t_0 - t)f(t)$ is in $\{f \in W_k^{k+1,2}[0,\infty) \colon f(t) = 0$ for all $t \in F\}$ but it does not belong to I_F since $f'(t_0) = f(t_0) \neq 0$. □

Thus new invariant subspaces arise for the operator M_ψ acting on the space $W_k^{k+1,2}[0,\infty)$. These new subspaces correspond to new invariant subspaces that do not lie in the lattice of invariant subspaces of C_{φ_a} acting in $\mathscr{H}^2$. In particular, we have

Theorem 5.4.5. *For each even integer $k \geq 2$, consider C_{φ_a} acting on $\mathcal{A}^2_{k-1}$. Then, for each $F \in \mathbb{F}[0,\infty)$ and each $j < (k+1)/2$ the subspace*

$$\mathcal{M}_F = \overline{span}\left\{ \left(\frac{z+1}{z-1}\right)^l e_t(z) \colon t \in F \text{ and } 0 \leq l \leq j \right\}$$

belongs to $LatC_{\varphi_a}$.

In particular, in case F does not have isolated points, the invariant subspace $\mathcal{M}_F$ is not generated by eigenfunctions e_t.

Proof. It is an straightforward consequence of Proposition 5.4.4 and the fact that

$$(\Phi_{k-1}f)^{(l)}(t) = \sum_{j=0}^{l} \binom{l}{j} \left(\frac{z+1}{z-1}\right)^j e_t(z).$$

□

To close the chapter we show the following striking property of the eigenfunctions e_t. Suppose that the closed set $F \subseteq [0,\infty)$ does not have isolated points or, equivalently, F equals the closure of its interior. Then, in case a function f in $W_k^{k+1,2}$ vanishes on F also its derivatives vanish on F. Hence, for any $0 \le j < (k+1)/2$ we have

$$\left\{ f \in W_k^{k+1,2}[0,\infty)\colon f_{|F} = 0 \right\} = \left\{ f \in W_k^{k+1,2}[0,\infty)\colon f^{(j)}{}_{|F} = 0,\, 0 \le l \le j \right\}.$$

Carrying the above set to $\mathcal{A}_{k-1}$ with the isomorphism Φ_{k-1}, we have that

$$\{ f \in \mathcal{A}^2_{k-1}\colon \Phi_{k-1} f_{|F} = 0 \} = \left\{ f \in \mathcal{A}^2_{k-1}\colon (\Phi_{k-1} f)^{(j)}{}_{|F} = 0,\, 0 \le l \le j \right\}.$$

Since both subspaces above are equal, their respective orthogonal subspaces are equal and we obtain that

$$\overline{\text{span}}\,\{e_t\colon t \in F\} = \overline{\text{span}} \left\{ \left(\frac{z+1}{z-1}\right)^l e_t\colon t \in F,\, 0 \le l \le j \right\}.$$

Chapter 6

The Parabolic Automorphism

When $\Re a = 0$ in formula (4.2), then φ is a parabolic automorphism of $\mathbb{D}$ and still satisfies the eigenfunction equation (4.4) with the same eigenfunctions. But, instead of a spiral, the spectrum is the unit circle. Now, the lattice becomes much more complicated. The reason for this is that the eigenspaces asociated to each eigenvalue are infinite dimensional. If we fix t_0 with $0 \leq t_0 < 2\pi/|a|$, then it is clear that

$$\ker\left(C_{\varphi_a} - e^{-at_0}I\right) = \overline{\text{span}}\left\{e_{t_0+2\pi n/|a|} : n = 0, 1, \ldots\right\}.$$

We have

6.1 The Eigenvectors

Proposition 6.1.1. *Let $a \neq 0$ with $\Re a = 0$ and $\lambda = e^{-at_0}$, where $0 \leq t_0 < 2\pi/|a|$. Then ℓ^2 is isomorphic to $\ker(C_{\varphi_a} - \lambda I)$ under the operator that to each sequence $\{a_n\}$ assigns the function $f = \sum_{n=0}^{\infty} a_n e_{t_0+2\pi n/|a|}$.*

Proof. Suppose that $\Im a > 0$. If $\Im a < 0$, the proof runs analogously. Since the operators C_{φ_a}, with $\Im a > 0$, are similar to each other, we may assume that $a = i/(2\pi)$. Since multiplication by e^{-it_0} is an isometric isomorphism, we may also assume that $t_0 = 0$.

Recall from (4.9) that $\langle e_t, e_s\rangle_{\mathscr{H}^2} = e^{-|t-s|}$ for each $t, s \geq 0$. From that formula, one immediately checks that the functions defined as

$$f_n = e_n - e^{-1}e_{n+1}, \quad n \geq 0,$$

are pairwise orthogonal. Since $e_0 = \sum_{k=0}^{\infty} e^{-k} f_k$, the set $\{f_n\}$ forms a complete orthogonal system of $\ker(C_{\varphi_{i/(2\pi)}} - I)$. Thus, since $\|f_n\|_{\mathcal{H}^2}^2 = 1 - e^{-2}$, we need only to prove that the operator T defined by

$$Tf_n = e_n = \sum_{k=0}^{\infty} e^{-k} f_{n+k}$$

is bounded with bounded inverse. But observe that T can be written as

$$T = (I - e^{-1}S)^{-1},$$

where S is defined by $Sf_n = f_{n+1}$ is the operator known as unilateral shift. Clearly, $I - e^{-1}S$ is bounded and has bounded inverse because $\|e^{-1}S\| < 1$. The result is proved. □

Therefore a large number of new invariant subspaces appear. Let φ_a be a parabolic automorphism that takes the unit disk into itself. Then, for each $t \geq 0$ let $\mathcal{M}_t = [e_t + e_{t+2\pi/|a|}]$ be the one-dimensional subspace generated by the eigenfunction $e_t + e_{t+2\pi/|a|}$. Thus $\mathcal{M}_t$ is invariant under C_{φ_a}. It is clear that for any parabolic non-automorphism φ_b that takes the unit disk into itself, the subspace $\mathcal{M}_t$ is not invariant under C_{φ_b}. Indeed,

$$\operatorname{span} \operatorname{Orb}(C_{\varphi_b}, e_t + e_{t+2\pi/|a|}) = \operatorname{span}\{e_t, e_{t+2\pi/|a|}\}.$$

Therefore any eigenfunction of φ_a that it is not an eigenfunction of any of the parabolic non-automorphisms generates a new invariant subspace. The same is true for subspaces generated by an infinite number of eigenfunctions corresponding to different eigenvalues.

The above observation shows up another difference between the automorphism and the non-automorphism case. Recall from 4.5.1 that all composition operators induced by parabolic non-automorphisms that take the unit disk into itself share their lattices of invariant subspaces. This is not the case for parabolic automorphism. Let a_1 and a_2 be two different pure imaginary numbers. Then the subspace $\mathcal{M} = [e_0 + e_{2\pi/|a_1|}]$ is invariant under $C_{\pi_{a_1}}$ but it is not invariant under $C_{\pi_{a_2}}$ since Proposition 6.1.1 implies that $e_0 + e_{2\pi/|a_1|}$ is not an eigenvector of $C_{\pi_{a_2}}$.

6.2 Invariant Subspaces Without Eigenvectors

The following proposition shows that there are a lot of invariant subspaces which are not spanned by eigenfunctions.

Proposition 6.2.1. *Let φ_a be a parabolic automorphism of the unit disk. Then C_{φ_a} has a non-trivial infinite-dimensional invariant subspace with at most the eigenfunction* 1.

Proof. As in the proof of Proposition 6.1.1, it suffices to consider the case $a = i/2\pi$. Before constructing $\mathcal{M}$ we will study the image of the eigenfunctions of $C_{\varphi_{i/2\pi}}$ under Φ. Recall that $(\Phi e_s)(t) = e^{-|s-t|}$. Integrating by parts it is easy to obtain the relationship between Φe_s and the reproducing kernel δ_s. Indeed,

$$\Phi e_0 = \frac{1}{2}\delta_0 \qquad \text{and} \qquad \Phi e_s = \delta_s - \frac{e^{-s}}{2}\delta_0,$$

for any $s > 0$. Now, the image under Φ of an arbitrary eigenfunction of $C_{\varphi_{i/2\pi}}$ is

$$h_{t_0} = \Phi\left(\sum_{j=0}^{\infty} a_j e_{t_0+j}\right) = \sum_{j=0}^{\infty} a_j\left(\delta_{t_0+j} - \frac{e^{-(t_0+j)}}{2}\delta_0\right),$$

where $\{a_j\} \in \ell^2$ and $0 \le t_0 < 1$. Those are the eigenfunctions of $M_\psi^\star$.

By Theorem 4.3.1 we know that Φ is an isometric isomorphism between $z\mathcal{H}^2$ and $W_0^{1,2}[0,\infty)$. Therefore it preserves orthogonality between these spaces. Using the similarity provided by Proposition 4.3.4, in order to conclude the proof it is enough to prove that there is a subspace $\mathcal{M} \subseteq W_0^{1,2}[0,\infty)$ invariant under M_ψ such that its orthogonal complement $\mathcal{M}^\perp$ has no eigenfunction for the adjoint $M_\psi^\star$ but the eigenfunction δ_0.

Now we are ready to construct $\mathcal{M}$. Take f_0 in $W_0^{1,2}[0,\infty)$ such that $f_0(t) \neq 0$ for each $t > 0$ and

$$\int_0^{1/2} \ln|f_0(t)|\,dt = -\infty. \tag{6.1}$$

We also take f_1 in $W_0^{1,2}[0,\infty)$ such that $f_1(t) > 0$ for each $t > 1$ and vanishing on $[0,1]$. For each $n \ge 2$, set $x_n = n - 2 + 2^{-n+1}$ and take f_n in $W_0^{1,2}[0,\infty)$

such that $f_n(t) \neq 0$ for $t \in (x_n, x_{n+1})$ and $f_n(t) = 0$ otherwise. The required subspace is

$$\mathcal{M} = \overline{\text{span}}\,\{M_\psi^k f_n : k \in \mathbb{Z} \text{ and } n = 0, 1, 2, \ldots\}.$$

Clearly, $\mathcal{M}$ is invariant under M_ψ.

Now let

$$h_{t_0} = \sum_{j=0}^{\infty} a_j \left(\delta_{t_0+j} - \frac{e^{-(t_0+j)}}{2} \delta_0 \right),$$

where $0 \leq t_0 < 1$ and $\{a_j\} \in \ell^2$, be an arbitrary eigenfunction of $M_\psi^\star$. Assume that h_{t_0} is orthogonal to $\mathcal{M}$. Since $\mathcal{M} \subseteq W_0^{1,2}[0, \infty)$, this is equivalent to $\sum_{j=0}^{\infty} a_j \delta_{t_0+j}$ being orthogonal to $\mathcal{M}$.

First suppose that $0 < t_0 < 1$. Then the equality $t_0 + j = x_n$, $j \geq 0$ and $n \geq 2$, holds for at most just one $n \geq 2$. Suppose that $t_0 + j \neq x_n$ for every $j \geq 1$ and every $n \geq 1$. Then $t_0 + j$, for each $j \geq 1$, belongs to a unique (x_n, x_{n+1}). It follows that

$$0 = \langle f_n, h_{t_0} \rangle = \overline{a_j} f_n(t_0 + j)$$

for each $j \geq 1$, which implies that $a_j = 0$ for each $j \geq 1$. Then $h_{t_0} = a_0(\delta_{t_0} - e^{-t_0}\delta_0/2)$, but we have

$$0 = \langle f_0, h \rangle = \overline{a_0} f_0(t_0).$$

Thus $a_0 = 0$ and h_{t_0} is the zero function.

In case there is an n such that $t_0 + k = x_n$, then as above we deduce that $a_j = 0$ for every j different from k. In addition, since h_{t_0} is orthogonal to f_0 and f_1, we have

$$\overline{a_0} f_0(t_0) + \overline{a_k} f_0(t_0 + k) = 0$$

and

$$\overline{a_0} f_1(t_0) + \overline{a_k} f_1(t_0 + k) = 0.$$

Since f_1 vanishes only on $[0, 1]$, then $a_k = 0$. Thus $a_0 = 0$ and h_{t_0} is the zero function again.

In case $t_0 = 0$, then $j \neq x_n$ for each $j \geq 1$ and each $n \geq 2$. In this case any $j \geq 1$ belongs at most to one of the intervals (x_n, x_{n+1}). Hence, as in the previous case, $a_j = 0$ for each $j \geq 1$. Then $h_{t_0} = a_0\delta_0$, which in fact is orthogonal to $\mathcal{M}$.

Therefore, the unique eigenfunction of $M_\psi^\star$ orthogonal to $\mathcal{M}$ is δ_0.

Finally, we see that there are infinitely many functions other than δ_0 in $\mathcal{M}^\perp$. In fact, $\mathcal{M}^\perp$ is infinite-dimensional since $\mathcal{M}$ cannot span all functions in $W^{1,2}[0,1/2]$. Indeed, f_0 is the only function non vanishing on $[0,1/2]$ and by (6.1), Szegö's Theorem, see [23], implies that $\{e^{kat}f_0(t)\}_{k\in\mathbb{Z}}$ does not span $L^2[0,1/2]$ and therefore neither $W^{1,2}[0,1/2]$. The result is proved. □

Chapter 7

Further Developments

The work just presented here suggests multiple directions where the research can be continued. The first one would be to complete the characterization of invariant subspaces of a composition operator induced by a parabolic automorphism.

As for the composition operator induced by a hyperbolic transformation, all the work remains to be done. The eigenvectors of these composition operators has been already characterized. In case φ is a hyperbolic fractional map, wether automorphism or non-automorphism, the eigenvectors of C_φ are the functions

$$\left(\frac{1+z}{1-z}\right)^{s+it},$$

for $t \in \mathbb{R}$ and $-1/2 < s < 1/2$, see [10, Lemma 7.24]. Note that the power is defined in terms of the principal branch of the logarithm. Hence, to identify their invariant subspaces a similar approach to the one made for the parabolic non-automorphism can be made. First step would be to prove that the eigenfunctions span the Hardy space and after that the composition operator will be similar to a multiplication operator acting on a certain functional Hilbert space. After identifying that functional Hilbert space, some results could be obtained concerning invariant subspaces for these operators.

However, one cannot expect to complete the characterization of the lattice of invariant subspaces of the composition operator induced by a hyperbolic automorphism since that problem is as difficult as the Invariant Subspace Problem.

The remaining case, the characterization of the lattice of invariant subspaces of composition operators induced by loxodromic maps, is a quite difficult task as well. Even the study of its cyclic properties, which in principle is easier than the study of its invariant subspaces, is complex. In [6] the authors characterize the cyclicity and hypercyclicity of the elliptic, parabolic and hyperbolic cases, but the skip the study of the loxodromic case.

Bibliography

[1] R. A. Adams, *Sobolev spaces.* Pure and Applied Mathematics, Vol. 65. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[2] A. Aleman, H. Hedenmalm and S. Richter, *Recent progress and open problems in the Bergman space. Quadrature domains and their applications*, Oper. Theory Adv. Appl. **156** (2005), 27–59.

[3] A. Aleman, S. Richter and C. Sundberg, *Beurling's theorem for the Bergman space*, Acta Math. **177** (1996), 275–310.

[4] N. Aronszajn and K. T. Smith, *Invariant Subspaces of Completely Continuous Operators*, Annals of Math. **60** (1954), 345–340.

[5] W. Arveson, *A Short Course in Spectral Theory.* Graduate Texts in Mathematics, 209. Springer-Verlag, New York, 2002.

[6] P. S. Bourdon and J. H. Shapiro, *Cyclic phenomena for composition operators*, Mem. Amer. Math. Soc. **125** (1997), no. 596, x+105 pp.

[7] J. B. Conway, *A Course in Functional Analysis.* Second edition. Springer-Verlag, New York, 1990.

[8] C. C. Cowen, *Composition operators on* H^2, J. Operator Theory **9** (1983), 77–106.

[9] C. C. Cowen, *Linear fractional composition operators on* H^2, Integral Eqns. Op. Th. **11** (1988), 151–160.

[10] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, 1995.

[11] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38 Academic Press, New York-London, 1970.

[12] P. L. Duren and A. Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, 100. American Mathematical Society, Providence, RI, 2004.

[13] P. Enflo, *On the invariant subspace problem for Banach spaces*, Seminaire Maurey-Schwarz (1975–1978).

[14] P. Enflo, *On the invariant subspace problem for Banach spaces*, Acta Math. **158** (1987), 213–313.

[15] E. A. Gallardo-Gutiérrez, A. Montes-Rodríguez, *Adjoints of linear fractional composition operators on the Dirichlet space*, Math. Ann. **327**, (2003), 117-134.

[16] E. Gallardo-Gutiérrez and A. Montes-Rodríguez, *The role of the angle in supercyclic behavior*, J. Funct. Anal. **203** (2003), 27–43.

[17] E. Gallardo-Gutiérrez and A. Montes-Rodríguez, *The role of the spectrum in the cyclic behavior of composition operators*, Mem. Amer. Math. Soc., **167** (2004), no. 791, x+81 pp.

[18] Garnett, J. B., *Bounded Analytic Functions*, Academic Press, New York, 1981.

[19] P. R. Halmos, *A Hilbert space problem book*, Graduate Texts in Mathematics, vol. 19. Second edition. Springer-Verlag, New York-Berlin, 1982.

[20] G. H. Hardy, *The mean value theorem of the modulus of an analytic function*, Proc. London Math. Soc., **14** (1915) 269–277.

[21] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, vol. 199. Springer-Verlag, New York, 2000.

[22] A. Ya. Helemsky, *Banach and locally convex algebras,* Oxford University Press, New York, 1998.

[23] H. Helson, *Lectures on invariant subspaces*, Academic Press, New York, 1964.

[24] W. M. Higdon, *The spectra of composition operators from linear fractional maps acting upon the Dirichlet space*, J. Func. Anal., **220**, (2005), 55–75.

[25] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Rev. ed. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957.

[26] K. Hoffman, *Banach spaces of analytic functions*, Dover Publications Inc., New York, 1968.

[27] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1990.

[28] K. Y. Katznelson, *An introduction to harmonic analysis*, Dover Publications Inc., New York, 1976.

[29] G. Königs, *Recherches sur les intégrales de certaines équations fonctionelles*, Annales Scientifiques de l'École Normale Supérieure Sér. 3, **1** (1884), 3–41 (suplement).

[30] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23** (1925), 481–519.

[31] A. Mahvidi, *Invariant subspaces of composition operators*, J. Operator Theory **46** (2001) 453–476.

[32] V. Matache, *On the minimal invariant subspaces of the hyperbolic composition operator*, Proc. Amer. Math. Soc. **119** (1993), 837–841.

[33] V. Matache, *The eigenfunctions of a certain composition operator*, Contemp. Math., **213** (1998), 121–136.

[34] R. Mortini, *Cyclic subspaces and eigenvectors of the hyperbolic composition operator*, Travaux mathématiques, Fasc. VII, (1995), 69–79, Sém. Math. Luxembourg, Centre Univ. Luxembourg, Luxembourg.

[35] E. A. Nordgren, *Composition operators*, Canadian J. Math. 20 (1968), 442-449.

[36] E. A. Nordgren, P. Rosenthal and F. S. Wintrobe, *Composition operators and the invariant subspace problem*, C. R. Math. Rep. Acad. Sci. Canada **6** (1984), 279–283.

[37] E. A. Nordgren, P. Rosenthal and F. S. Wintrobe *Invertible composition operators on H^p*, J. Funct. Anal. **73** (1987), 324–344.

[38] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society, Providence, RI, 1934.

[39] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77. Springer-Verlag, New York-Heidelberg, 1973.

[40] E. D. Rainville, *Special functions*, Chelsea Publ. Company, New York, 1971.

[41] C. J. Read, *A solution to the invariant subspace problem*, Bull. London Math. Soc. **16** (1984), 337–401.

[42] C. J. Read, *A solution to the invariant subspace problem on the space ℓ^1*, Bull. London Math. Soc. **17** (1985), 305–317.

[43] C. E. Rickart, *General theory of Banach algebras*, The University Series in Higher Mathematics, D. van Nostrand Co., Inc., Princeton, New York, 1960.

[44] P. Rosenthal, *Invariant subspace lattices*, Duke Math. J., **37** (1970), 103–112.

[45] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1987.

[46] D. Sarason, *Composition operators as integral operators*, Analysis and partial differential equations, 545–565. Lecture Notes in Pure and Appl. Math., **122**, Dekker, New York, 1990.

[47] Shapiro, J. H., *Composition Operators and Classical Function Theory*, Springer Verlag, 1993.

[48] E. Schröder, *Über iterierte Funktionen*, Math. Ann. **3** (1871), 296–322.

[49] H. J. Schwartz, *Composition Operators on H^p*, dissertation, University of Toledo, 1969.

[50] G. Szegö, *Orthogonal polynomials* American Mathematical Society Colloquium Publications, vol. 23., New York, 1939.

[51] B. S. Yadav, *The Present State and Heritages of the Invariant Subspace Problem*, Milan J. Math. **73** (2005), 289–316.

Symbol Index

Subject Index