Pullback attractors to analyze the effect of random disturbances in the chemostat model

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joint work with T. Caraballo, M. J. Garrido-Atienza (PhD advisors) and A. Rapaport (UMR INRA-SupAgro 0729 MISTEA, Montpellier, France)

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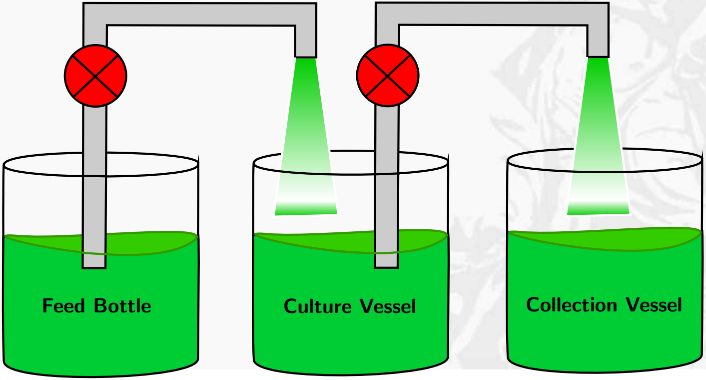
What is a chemostat?

- **Laboratory device** used for growing microorganisms in a cultured environment
- **Standard assumptions** for simple chemostats:
  - **Availability** of nutrient and its supply rate are both fixed
  - **Wall growth** is not taken into account



- However...

VERY STRONG RESTRICTIONS!

since

  - real world is non-autonomous and STOCHASTIC
  - more realistic situation: WALL GROWTH

- It encourages us to study

STOCHASTIC CHEMOSTATS WITH WALL GROWTH

Importance of chemostat models

1. They play an important role in *ecological studies*
2. They are used as models of *wastewater treatment processes*
3. They can be considered as starting point for other several models:
   - Problems of genetically altered organisms
   - Antibiotic production models
   - Fermentation models: WINE, BEER...!!!!!!

The simplest chemostat model

$$\frac{dS}{dt} = (S^0 - S)D - \frac{mSx}{a+S},$$
$$\frac{dx}{dt} = -Dx + \frac{mSx}{a+S},$$

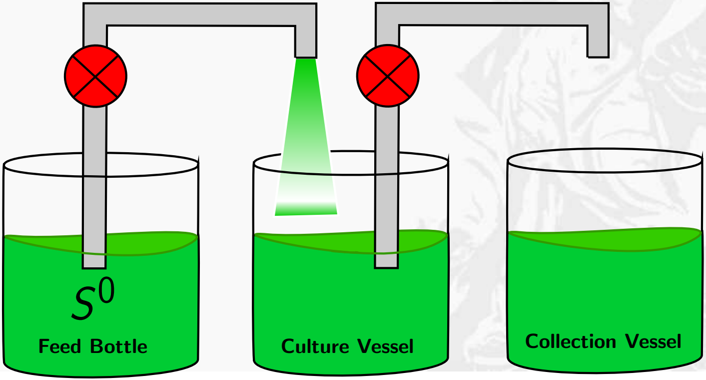
where

- $S(t)$: concentration of the nutrient
- $x(t)$: concentration of the microbial biomass
- $S^0 > 0$: input concentration
- $a > 0$: half-saturation constant
- $D > 0$: dilution rate
- $m > 0$: maximal consumption rate of the nutrient and maximal specific growth rate of microorganisms

Setting up the simplest chemostat model

$$\frac{dS}{dt} = (S^0 - S)D - \frac{mSx}{a+S}$$ **Equation for the nutrient**

$$\frac{dx}{dt} = -Dx + \frac{mSx}{a+S}$$ **Equation for the specie**



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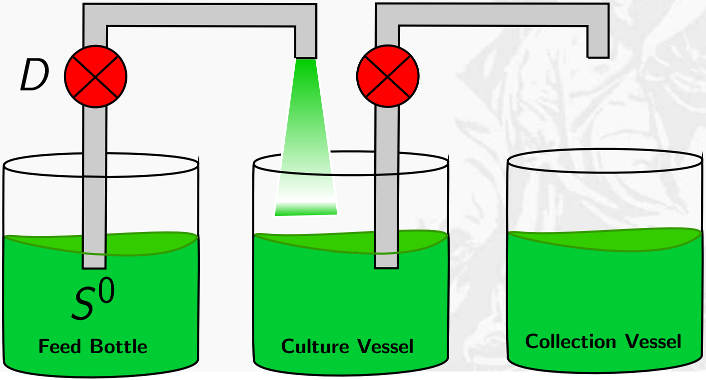

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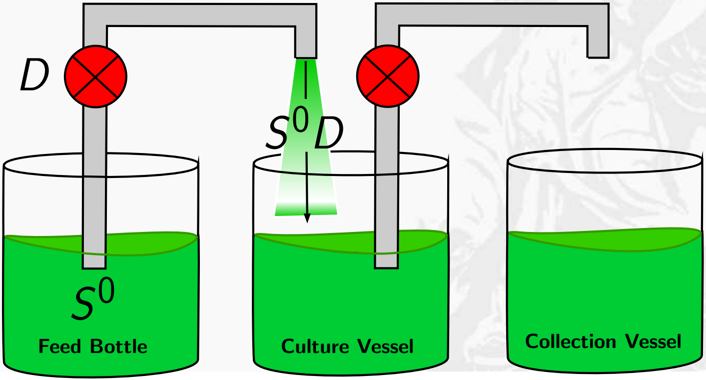

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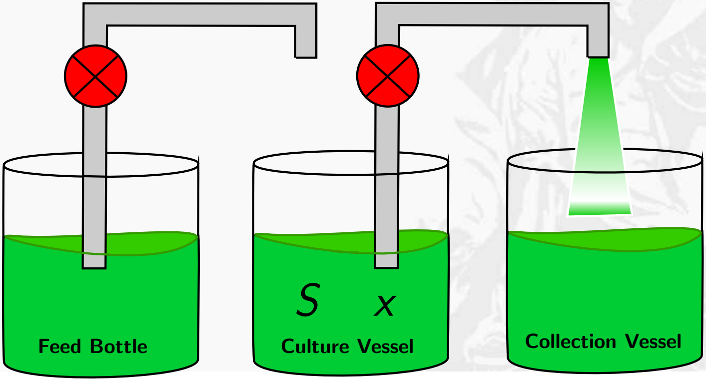

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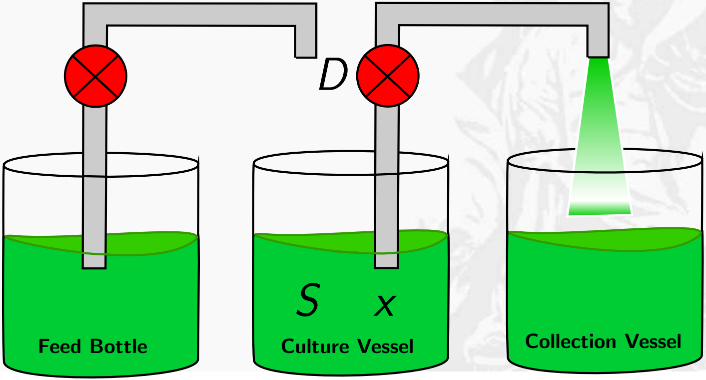

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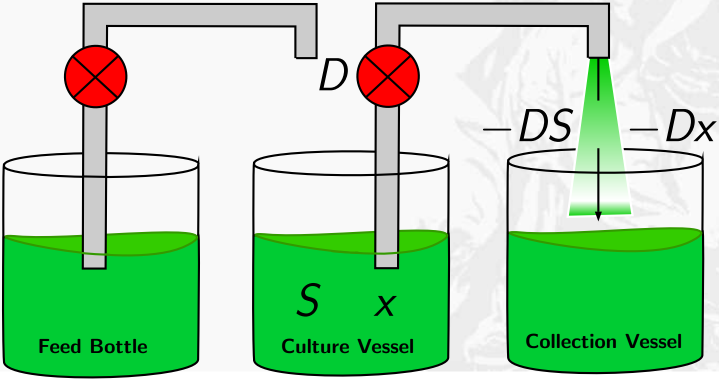

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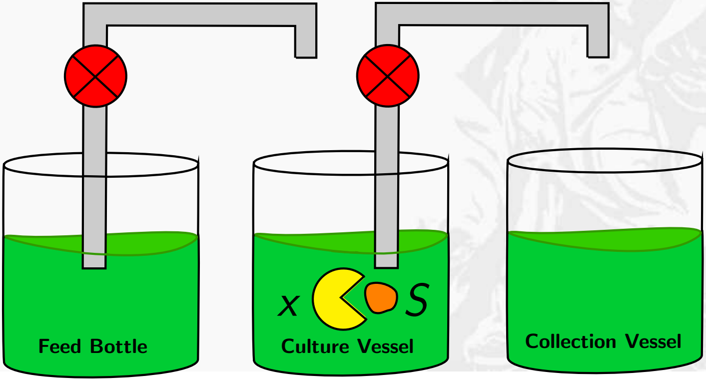

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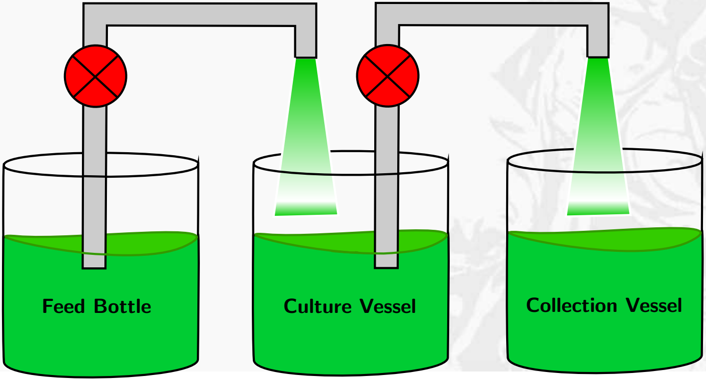


Chemostat with wall growth

$$\begin{aligned}
\frac{dS}{dt} &= D(S^0 - S) - \frac{mS}{a+S}x_1 - \frac{mS}{a+S}x_2 + b\nu x_1, \\
\frac{dx_1}{dt} &= -(\nu + D)x_1 + c\frac{S}{a+S}x_1 - r_1 x_1 + r_2 x_2, \\
\frac{dx_2}{dt} &= -\nu x_2 + c\frac{S}{a+S}x_2 + r_1 x_1 - r_2 x_2
\end{aligned}$$

- $S(t), x_1(t), x_2(t)$: concentrations of the nutrient and the two different microorganisms
- $b \in (0, 1)$: fraction of dead biomass which is recycled
- $\nu > 0$: collective death rate coefficient
- $r_1, r_2 > 0$: rates at which the species stick on to and shear off the walls, respectively
- $0 < c \leq m$: growth rate coefficient of the consumer species

The Wiener process

Definition

A *standard Wiener process* is a family of random variables $W(t)(\cdot) : \omega \in \Omega \mapsto W(t)(\omega) \in \mathbb{R}$, $t \geq 0$, s.t. $\mathbb{P}-$almost surely

- W(0)=0
- continuous (but NOT bounded variation) paths: $t \in \mathbb{R}^+ \mapsto W(t)(\omega) \in \mathbb{R}$
- independent increments: for $0 < t_1 < \cdots < t_n$, $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent random variables
- stationarity: joint distribution of $\{W(t_1 + t), \ldots, W(t_k + t)\}$ does NOT depend on t
- $W(t) - W(s)$, $0 \leq s \leq t$, is a Gaussian variable with mean 0 and variance $t - s$

The Wiener process

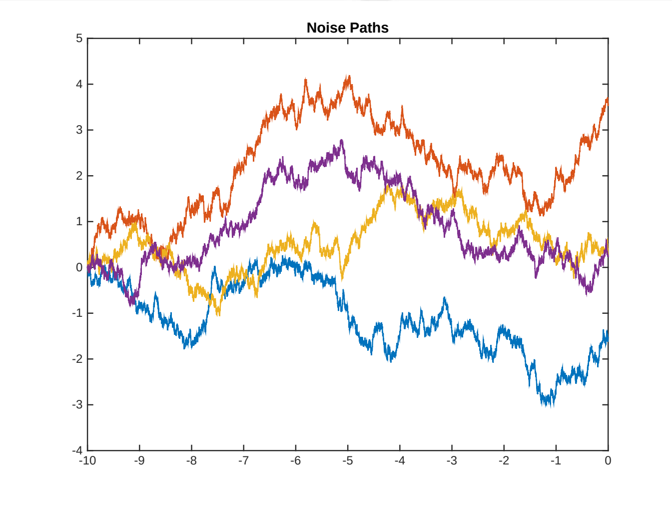


Figure: Realizations of the standard Wiener process

The Wiener process

- W: Wiener process
- *Kolmogorov's theorem* ensures that W has a continuous version, ω, whose canonical interpretation is:
  - $\Omega := \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$
  - $\mathcal{F}$: Borel $\sigma-$algebra on Ω
  - $\mathbb{P}$: Wiener measure on $\mathcal{F}$
- We consider the Wiener shift flow

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a *metric dynamical system*

The Ornstein-Uhlenbeck (OU) process

- The OU process on $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is defined as

$$z^*(\theta_t\omega) = -\int_{-\infty}^{0} e^s \theta_t\omega(s)ds, t \in \mathbb{R}, \omega \in \Omega,$$

which solves the Langevin equation

$$dz + zdt = d\omega(t), t \in \mathbb{R} \tag{1}$$

T. Caraballo, P. E. Kloeden and B. Schmalfuß,
Exponentially stable stationary solutions for stochastic evolution equations and their perturbation,
Applied Mathematics & Optimization, vol. **50**, no. 3, (2004) 183–207.

The Ornstein-Uhlenbeck (OU) process

Proposition *[T. Caraballo, P. E. Kloeden and B. Schmalfuß, 2004]*

There exists a θ_t-invariant set $\widetilde{\Omega} \in \mathcal{F}$ of Ω of full $\mathbb{P}$ measure such that for $\omega \in \widetilde{\Omega}$, we have

(i) the random variable $|z^*(\omega)|$ is tempered.

(ii) the mapping $(t, \omega) \to z^*(\theta_t \omega) = -\int_{-\infty}^{0} e^s \omega(t+s) \mathrm{d}s + \omega(t)$ is a stationary solution of (1) with continuous trajectories;

(iii) in addition, for any $\omega \in \tilde{\Omega}$:

$$\begin{aligned} \lim_{t \to \pm\infty} \frac{|z^*(\theta_t \omega)|}{t} &= 0; \\ \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t z^*(\theta_s \omega) ds &= 0; \\ \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t |z^*(\theta_s \omega)| ds &= \mathbb{E}[z^*] < \infty. \end{aligned}$$

Random dynamical systems (RDSs)

$(X, \|\cdot\|_X)$ separable Banach space

Definition

An *RDS* on X consists of two ingredients: (a) a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and the family of mappings $\theta_t : \Omega \to \Omega$ satisfies

(1) $\theta_0 = \mathrm{Id}_\Omega$,

(2) $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$,

(3) the mapping $(t, \omega) \mapsto \theta_t\omega$ is measurable,

(4) the probability measure $\mathbb{P}$ is preserved by θ_t, i.e., $\theta_t\mathbb{P} = \mathbb{P}$

and (b) a mapping $\varphi : [0, \infty) \times \Omega \times X \to X$ which is $(\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$- measurable, such that for each $\omega \in \Omega$,

(i) mapping $\varphi(t, \omega) : X \to X$, $x \mapsto \varphi(t, \omega)x$ is cont. for every $t \geq 0$,

(ii) $\varphi(0, \omega)$ is the identity operator on X,

(iii) (cocycle property) $\varphi(t + s, \omega) = \varphi(t, \theta_s\omega)\varphi(s, \omega)$ for all $s, t \geq 0$.

Random dynamical systems (RDSs)

Definition

A *random set* K is a measurable subset of $X \times \Omega$ w.r.t. $\sigma-$algebra $\mathcal{B}(X) \times \mathcal{F}$. Moreover K will be said a closed (compact) random set if $K(\omega) = \{x : (x, \omega) \in K\}$, $\omega \in \Omega$, is closed (compact) for $\mathbb{P}-$a.e. $\omega \in \Omega$.

Definition

A bounded random set $K(\omega) \subset X$ is said to be *tempered with respect to* $\{\theta_t\}_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{x\in K(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be *tempered with respect to* $\{\theta_t\}_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{t\in\mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0.$$

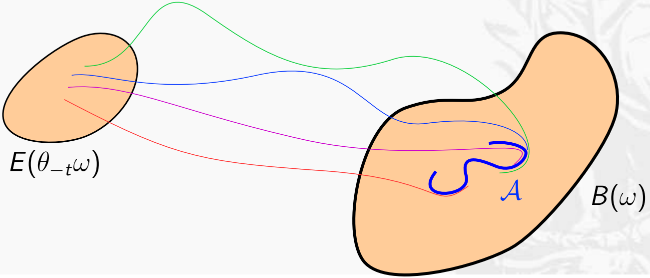
Random dynamical systems (RDSs)

$\mathcal{E}(X)$: set of all tempered random sets of X

Definition

A random set $B(\omega) \subset X$ is called a *random absorbing set* in $\mathcal{E}(X)$ if for any $E \in \mathcal{E}(X)$ and a.e. $\omega \in \Omega$, there exists $T_E(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega)E(\theta_{-t}\omega) \subset B(\omega), \quad \text{for all } t \geq T_E(\omega).$$



Random dynamical systems (RDSs)

Definition

Let $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ be an RDS over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ with state space X and let $A(\omega)(\subset X)$ be a random set. Then $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega}$ is called a *global random $\mathcal{E}-$attractor (or pullback $\mathcal{E}-$attractor)* for $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ if

(i) (compactness) $A(\omega)$ is a compact set of X for any $\omega \in \Omega$;

(ii) (invariance) for any $\omega \in \Omega$ and all $t \geq 0$, it holds

$$\varphi(t,\omega)A(\omega) = A(\theta_t\omega);$$

(iii) (attracting property) for any $E \in \mathcal{E}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} \mathrm{dist}_X(\varphi(t,\theta_{-t}\omega)E(\theta_{-t}\omega), A(\omega)) = 0,$$

where $\mathrm{dist}_X(G,H) = \sup_{g\in G} \inf_{h\in H} \|g-h\|_X$ is the Hausdorff semi-metric for $G, H \subseteq X$.

Random dynamical systems (RDSs)

Proposition

Let $B \in \mathcal{E}(X)$ be a closed absorbing set for the continuous RDS $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ that satisfies the asymptotic compactness condition for $a.e.$ $\omega \in \Omega$, i.e., each sequence $x_n \in \varphi(t_n, \theta_{-t_n}\omega)B(\theta_{-t_n}\omega)$ has a convergent subsequence in X when $t_n \to \infty$. Then φ has a unique global random attractor $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega}$ with component subsets

$$A(\omega) = \bigcap_{\tau \geq T_B(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)}.$$

Remark

The asymptotic compactness follows trivially if $X = \mathbb{R}^d$ as in the current work

Random dynamical systems (RDSs)

Lemma

Let φ_u be an RDS on X. Suppose that the mapping $\mathcal{T} : \Omega \times X \to X$ possesses the following properties: for fixed $\omega \in \Omega$, $\mathcal{T}(\omega, \cdot)$ is a homeomorphism on X, and for $x \in X$, the mappings $\mathcal{T}(\cdot, x)$, $\mathcal{T}^{-1}(\cdot, x)$ are measurable. Then the following mapping defines a (conjugated) RDS:

$$(t, \omega, x) \to \varphi_v(t, \omega)x := \mathcal{T}^{-1}(\theta_t\omega, \varphi_u(t, \omega)\mathcal{T}(\omega, x))$$

L. Arnold,
Random Dynamical Systems,
Springer-Verlag, Berlin (1998).

T. Caraballo and X. Han,
Applied Nonautonomous and Random Dynamical Systems, Applied Dynamical Systems,
Springer, 2016.

READY TO DEAL WITH STOCHASTIC CHEMOSTATS!

Modeling stochastic perturbations

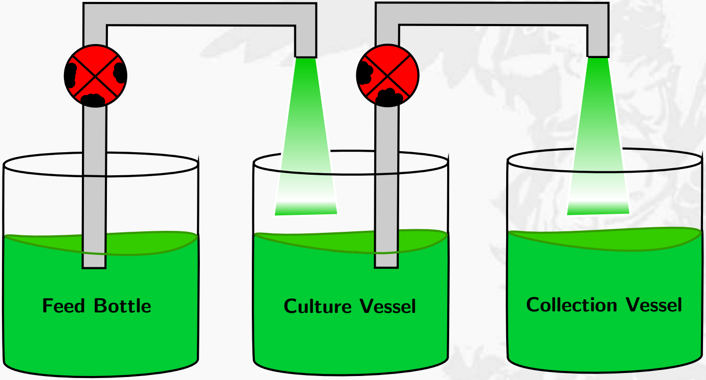
- Many different ways of modeling **randomness** and **stochasticity**

Some questions:

- What kind of stochastic perturbation can we introduce?
- How can we do it?
- Is it **realistic** from the biological point of view?
- And... Is it **tractable** from the mathematical point of view?

Perturbing the input flows

- **Biological process:** particles of dirt inside the pumps



Biologists claim that in the real life

- The flow passing through the pumps **IS NOT CONSTANT**
- The best approach: **STOCHASTIC/RANDOM FLUCTUATIONS**

How can we perturb the input flows?

We will analyze two different ways of modeling perturbed input flows in the chemostat model

- **Classical way:** by using the standard Wiener process

$$D \rightsquigarrow D + \alpha\dot{\omega}(t)$$

SEVERAL INCONNVENIENTS!!

- **New idea:** by using DIRECTLY the Ornstein-Uhlenbeck process

$$D \rightsquigarrow D + \alpha z^*(\theta_t\omega)$$

EXCELLENT APPROACH TO THE REAL LIFE!!

Classical way: the Wiener process

We consider the chemostat model

$$\begin{aligned}\frac{dS}{dt} &= (S^0 - S)D - \frac{mSx}{a+S},\\ \frac{dx}{dt} &= -Dx + \frac{mSx}{a+S},\end{aligned}$$

and introduce the following perturbation

$$D \rightsquigarrow D + \alpha\dot{\omega}(t)$$

then we obtain the following stochastic model understood in Itô's sense

$$\begin{aligned}dS &= \left[(S^0 - S)D - \frac{mSx}{a+S}\right]dt + \alpha(S^0 - S)d\omega(t),\\ dx &= \left[-Dx + \frac{mSx}{a+S}\right]dt - \alpha x d\omega(t).\end{aligned}$$

Classical way: the Wiener process

Itô vs Stratonovich conversion

$$
\begin{aligned}
dX_t &= a(t, X_t)dt + b(t, X_t)dW_t, \\
dX_t &= \left(a(t, X_t) - \frac{1}{2} b(t, X_t) b'(t, X_t) \right) dt + b(t, X_t) \circ dW_t
\end{aligned}
$$

Then, we obtain the following stochastic chemostat model

$$
dS = \left[(S^0 - S)\bar{D} - \frac{mSx}{a + S} \right] dt + \alpha (S^0 - S) \circ d\omega(t), \tag{2}
$$

$$
dx = \left[-\bar{D}x + \frac{mSx}{a + S} \right] dt - \alpha x \circ d\omega(t), \tag{3}
$$

in Stratonovich sense, where

$$
\bar{D} := D + \frac{\alpha^2}{2}. \tag{4}
$$

Classical way: the Wiener process

We define the following variable change

$$\sigma(t) = (S(t) - S^0)e^{\alpha z^*(\theta_t\omega)}, \quad (5)$$

$$\kappa(t) = x(t)e^{\alpha z^*(\theta_t\omega)}. \quad (6)$$

Then, by differentiation, we have the following **RANDOM** chemostat model given by

$$\frac{d\sigma}{dt} = -(\bar{D} + \alpha z^*)\sigma - \frac{m(S^0 + \sigma e^{-\alpha z^*(\theta_t\omega)}}{a + S^0 + \sigma e^{-\alpha z^*(\theta_t\omega)}}, \quad (7)$$

$$\frac{d\kappa}{dt} = -(\bar{D} + \alpha z^*)\kappa + \frac{m(S^0 + \sigma e^{-\alpha z^*(\theta_t\omega)}}{a + S^0 + \sigma e^{-\alpha z^*(\theta_t\omega)}}. \quad (8)$$

Classical way: the Wiener process

Now, we define $\mathcal{X} := \{(x,y) \in \mathbb{R}^2 \,:\, x \in \mathbb{R}, y \geq 0\}$

Theorem (Caraballo *et al*, 2017)

For any $\omega \in \Omega$ and any initial value $u_0 := (\sigma_0, \kappa_0) \in \mathcal{X}$, where $\sigma_0 := \sigma(0)$ and $\kappa_0 := \kappa(0)$, system (7)-(8) possesses a unique global solution $u(\cdot; 0, \omega, u_0) := (\sigma(\cdot; 0, \omega, u_0), \kappa(\cdot; 0, \omega, u_0)) \in \mathcal{C}^1([0, +\infty), \mathcal{X})$ with $u(0; 0, \omega, u_0) = u_0$. Moreover, the solution mapping generates a RDS $\varphi_u : \mathbb{R}^+ \times \Omega \times \mathcal{X} \to \mathcal{X}$ defined as

$$\varphi_u(t, \omega)u_0 := u(t; 0, \omega, u_0), \qquad \text{for all } t \in \mathbb{R}^+, \, u_0 \in \mathcal{X}, \, \omega \in \Omega,$$

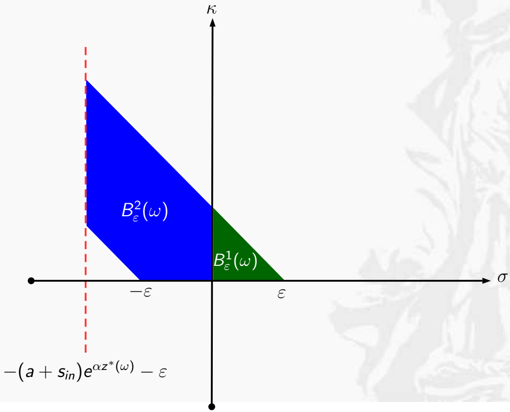
the value at time t of the solution of system (7)-(8) with initial value u_0 at time zero.

- **Key for the proof:** classical results from the theory of ODEs.
- **REMARK:** some state variable can take negative values.

Classical way: the Wiener process

Theorem (T. Caraballo *et al*, 2017)

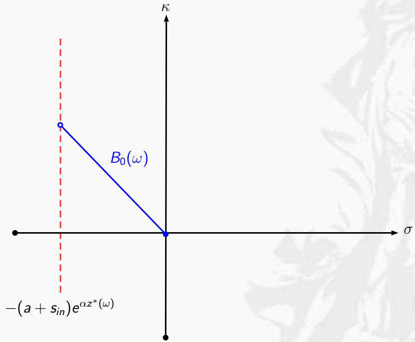
There exists a tempered compact random absorbing set for the RDS $\{\varphi_u(t,\omega)\}_{t\geq 0,\omega\in\Omega}$.



- Thanks to Proposition 25, we deduce that the RDS generated by the system (7)-(8) possesses a unique random pullback attractor given by

$$\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \subset B_\varepsilon(\omega), \quad \text{for any } \varepsilon > 0.$$

- Thus,



$$\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \subset B_0(\omega).$$

Classical way: the Wiener process

- Analysis of the internal structure of the random pullback attractor

Proposition (T. Caraballo *et al*, 2017)

The pullback random attractor of system (7)-(8) consists of a singleton components given by

$$\mathcal{A}(\omega) = \{(0,0)\}$$

as long as

$$\bar{D} > \mu(S^0) \tag{9}$$

EXTINCTION OF THE MICROORGANISMS!!

Classical way: the Wiener process

- Define now a mapping $\mathcal{T} : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ as

$$\mathcal{T}(\omega, \zeta) = \left((\zeta_1 - S^0) e^{\alpha z^*(\omega)}, \zeta_2 e^{\alpha z^*(\omega)} \right)$$

$$\mathcal{T}^{-1}(\omega, \zeta) = \left(S^0 + \zeta_1 e^{-\alpha z^*(\omega)}, \zeta_2 e^{-\alpha z^*(\omega)} \right)$$

- Denoting $v(t) = (S(t), x(t))$ and $u(t) = (\sigma(t), \kappa(t))$, since $\mathcal{T}$ is a homeomorphism, thanks to Lemma 26 we obtain a conjugated RDS given by

$$\varphi_v(t, \omega) v_0 \quad := \quad v(t; 0, \omega, v_0)$$

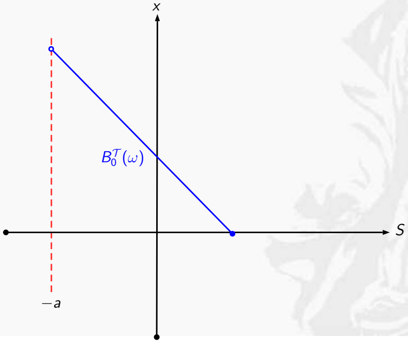
- Hence, φ_v is an RDS for our original stochastic system (2)-(3)

Moreover,

$$\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \subset B_0(\omega) \quad \Longrightarrow \quad \mathcal{A}^{\mathcal{T}} = \{A^{\mathcal{T}}(\omega)\}_{\omega\in\Omega} \subset B_0^{\mathcal{T}}(\omega),$$

where

$$B_0^{\mathcal{T}}(\omega) := \left\{(S,x) \in \mathcal{X} : S + x = S^0, s > -a\right\}$$



Recovering the stochastic chemostat

- Similarly to random case, the internal structure of the attractor consists of singleton subsets

$$A^{\mathcal{T}}(\omega) = \left(S^0, 0\right)$$

as long as $\bar{D} > \mu(S^0)$

- It is not possible to ensure the persistence of the microorganism otherwise even though our simulations show that we can get the persistence for several values of the parameters

Classical way: the Wiener process

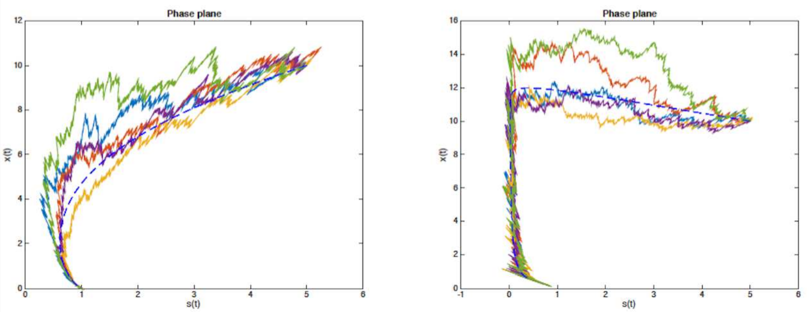


Figure: Value of parameters: $D = 3.5$, $S^0 = 1$, $a = 0.8$, $m = 1.5$ and $\alpha = 0.5$ (left), $D = 2$, $S^0 = 1$, $a = 0.6$, $m = 5$ and $\alpha = 0.5$ (right)

- Some state variables can take negative values
- The Wiener process is not a very good choice in our case

New idea: the Ornstein-Uhlenbeck process

We consider the chemostat model

$$\begin{aligned} \frac{dS}{dt} &= (S^0 - S)D - \frac{mSx}{a+S}, \\ \frac{dx}{dt} &= -Dx + \frac{mSx}{a+S}, \end{aligned}$$

and introduce the following perturbation

$$D \rightsquigarrow D + \alpha z^*(\theta_{-t}\omega)$$

then we obtain the following RANDOM model

$$\frac{dS}{dt} = -\left[D + \alpha z^*(\theta_t\omega)\right] S - \frac{mSx}{a+S} + S^0 D + \alpha S^0 z^*(\theta_t\omega), \quad (10)$$

$$\frac{dx}{dt} = -\left[D + \alpha z^*(\theta_t\omega)\right] x + \frac{mSx}{a+S}. \quad (11)$$

New idea: the Ornstein-Uhlenbeck process

Remark

Some parameters will be considered when defining the OU process

- **Langevin equation:**

$$dz = -\beta z dt + \nu d\omega(t), \quad t \in \mathbb{R}.$$

- **OU process:**

$$z^*_{\beta,\nu}(\theta_t \omega) := -\beta\nu \int_{-\infty}^{0} e^{\beta s} \theta_t \omega(s) ds, \quad t \in \mathbb{R}, \omega \in \Omega, \beta, \nu > 0.$$

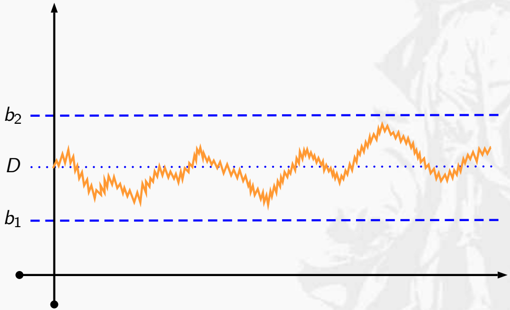
These parameters will allow us to control the noise!

New idea: the Ornstein-Uhlenbeck process

Proposition (T. Caraballo *et al*, 2017)

For any $\omega \in \Omega$,

$$\lim_{\beta\to\infty} z^*_{\beta,\nu}(\theta_t\omega) = 0, \quad \text{for all } t \in \mathbb{R}.$$



It is possible to take β large enough such that the OU is bounded

New idea: the OU process

A new framework is required

- **The new set of events:**

$$\Omega_\beta := \left\{\omega \in \Omega \ : \ b_1 \leq D + z^*_{\beta,\nu}(\theta_t\omega) \leq b_2, \text{ for all } t \in \mathbb{R}\right\},$$

- **The new $\sigma-$algebra $\mathcal{F}_\beta$:**

$$\mathcal{F}_\beta := \{A \cap \Omega_\beta, \ A \in \mathcal{F}\}$$

- **The new probability measure $\mathbb{P}_\beta : \mathcal{F}_\beta \longrightarrow [0,1]$:**

$$\mathbb{P}_\beta(F_\beta) := \frac{\mathbb{P}(F_\beta)}{\mathbb{P}(\Omega_\beta)}, \qquad \text{for all } F_\beta \in \mathcal{F}_\beta,$$

We can make use of the theory of RDS since
$(\Omega_\beta, \mathcal{F}_\beta, \mathbb{P}_\beta, \{\theta_t\}_{t\in\mathbb{R}}) \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$

New idea: the OU process

Theorem (T. Caraballo *et al*, 2017)

For any $\omega \in \Omega_\beta$ and any initial value $v_0 := (S_0, x_0) \in \mathcal{X}$, where $S_0 := S(0)$ and $x_0 := x(0)$, system (10)-(11) possesses a unique global solution $v(\cdot; 0, \omega, v_0) := (S(\cdot; 0, \omega, v_0), x(\cdot; 0, \omega, v_0)) \in \mathcal{C}^1([0, +\infty), \mathcal{X})$ with $v(0; 0, \omega, v_0) = v_0$. Moreover the solution mapping generates an RDS $\varphi_v : \mathbb{R}^+ \times \Omega_\beta \times \mathcal{X} \to \mathcal{X}$ defined by

$$\varphi_v(t, \omega)v_0 := v(t; 0, \omega, v_0), \quad \text{for all } t \in \mathbb{R}^+,\ v_0 \in \mathcal{X},\ \omega \in \Omega_\beta.$$

Theorem (T. Caraballo *et al*, 2017)

There exists a tempered compact random absorbing set $B_0(\omega) \in \mathcal{E}(\mathcal{X})$ of the RDS $\{\varphi_v(t, \omega)\}_{t \geq 0,\, \omega \in \Omega}$.

- The absorbing set:

$$B_0(\omega) := \{(S, x) \in \mathcal{X} \ : \ S + x = S^0\}.$$

New idea: the OU process

Proposition (T. Caraballo *et al*, 2017)

Assuming that the following condition $D > \mu(S^0)$ holds, the pullback random attractor of the chemostat model (10)-(11) is reduced to a singleton component which is given by $\mathcal{A}(\omega) = \{(S^0, 0)\}$.

In this case the microorganisms become EXTINCT!

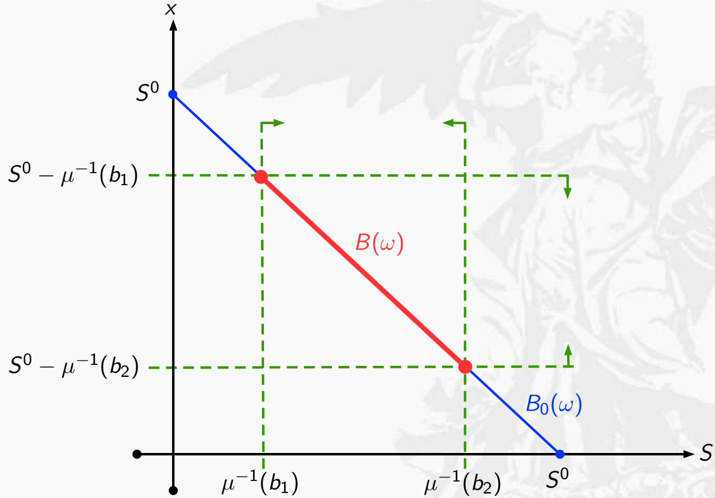
Theorem (T. Caraballo *et al*, 2017)

Assume

$$\mu^{-1}(b_2) < S^0$$

holds true. Then, there exists a **strictly positive** tempered absorbing set $B(\omega) \in \mathcal{E}(\mathcal{X})$ of the RDS $\{\varphi_v(t,\omega)\}_{t\geq 0, \omega\in\Omega}$.

New idea: the Ornstein-Uhlenbeck process



New idea: the Ornstein-Uhlenbeck process

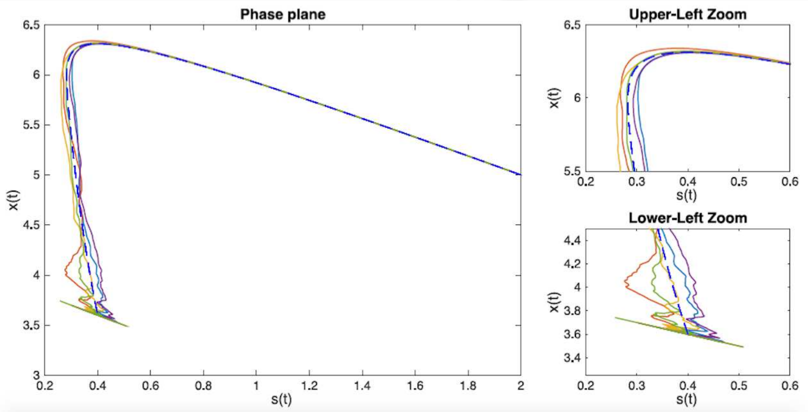


Figure: Value of parameters: $D = 2$, $S^0 = 4$, $a = 0.6$, $m = 5$, $\alpha = 0.5$, $\beta = 1$, $\nu = 0.7$, $S(0) = 2$ and $x(0) = 5$

New idea: the Ornstein-Uhlenbeck process

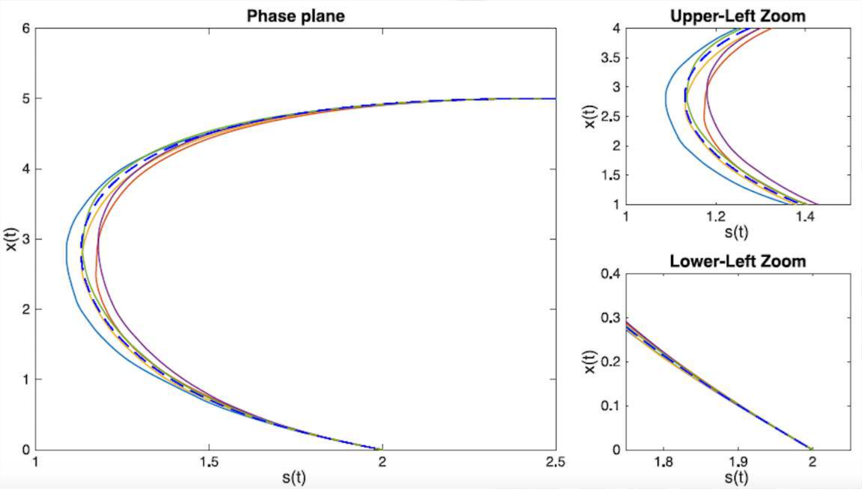


Figure: Value of parameters: $D = 3.5$, $S^0 = 2$, $a = 0.8$, $m = 0.5$, $\alpha = 0.5$, $\beta = 1$, $\nu = 0.7$, $S(0) = 2.5$ and $x(0) = 5$

Comparison: Wiener vs OU

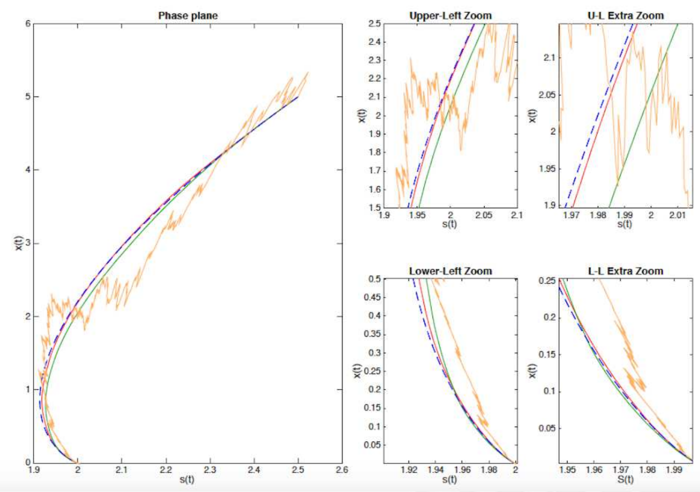


Figure: Value of parameters: $S^0 = 2$, $D = 3.5$, $a = 0.8$, $m = 0.5$, $\alpha = 0.8$, $\sigma = 0.8$, $x(0) = 5$, $S(0) = 2.5$, $\beta = 2$ (red) and $\beta = 0.5$ (green)

Comparison: Wiener vs OU

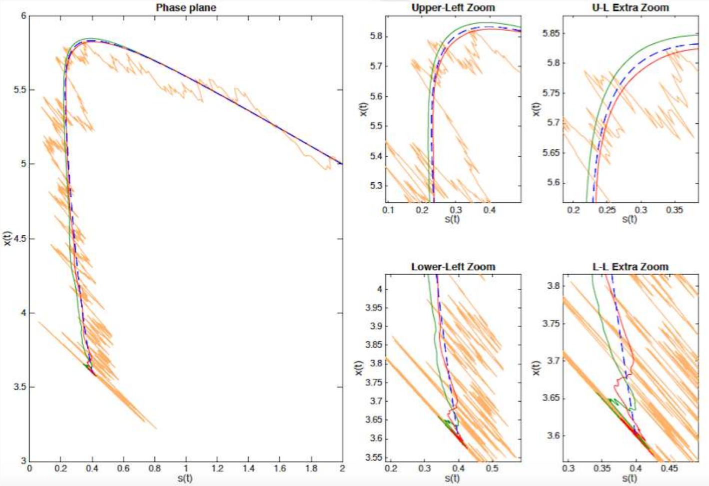


Figure: Value of parameters: $S^0 = 4$, $D = 2$, $a = 0.6$, $m = 5$, $\alpha = 0.15$, $\sigma = 0.8$, $x(0) = 5$, $S(0) = 2$, $\beta = 2$ (red) and $\beta = 0.5$ (green)

Conclusions and future works

- Classical way: $D \rightsquigarrow D + \alpha\dot{\omega}(t)$
  - Some state variables can take negative values...
  - Perturbations $D + \alpha\dot{\omega}$ are not realistic
- New way: $D \rightsquigarrow D + \alpha z^*_{\beta,\nu}(\theta_t\omega)$
  - Perturbations can be controlled by taking a suitable O-U
  - Random pullback attractor strictly contained in the positive cone
  - Hence...

PERSISTENCE OF THE MICROORGANISM CAN BE GUARANTEED!!!!

Future works

- Chemostat model with wall growth.
- Generalize the new idea to perturb other models.

Muito obrigado
pela sua atenção!!!!

Thank you very much
for your attention!!!!