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LINEAR AND ALGEBRAIC STRUCTURES IN FUNCTION SEQUENCE SPACES

Memoria presentada por Pablo José Gerlach Mena

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Mª del Carmen Calderón Moreno
Prof. Titular Universidad
Dpto. Análisis Matemático
Universidad de Sevilla

Vº Bº de la Directora

José Antonio Prado Bassas
Prof. Titular Universidad
Dpto. Análisis Matemático
Universidad de Sevilla

Vº Bº del Director

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Resumen

Históricamente han sido muchos los matemáticos de todas las épocas que se han sentido atraídos y fascinados por la existencia de grandes estructuras algebraicas que satisfacen ciertas propiedades que, *a priori*, pueden contradecir a la intuición matemática.

El objetivo de la presente Memoria es el estudio de la lineabilidad de diversas familias de sucesiones de funciones con propiedades muy específicas.

La Memoria se divide en 6 capítulos, donde los Capítulos 1, 2 y 3 se centran en introducir la notación básica y la terminología principal de la teoría de la Lineabilidad y de los modos de convergencia que usaremos a lo largo de esta Memoria.

En el Capítulo 4 comenzamos el estudio del tamaño algebraico de dos familias de sucesiones de funciones con distintos modos de convergencia en el intervalo unidad cerrado $[0, 1]$: convergencia en medida pero no puntual en casi todo y convergencia puntual pero no uniforme.

En el Capítulo 5 centramos nuestra atención en el marco de las funciones integrables (Lebesgue). Comenzamos con sucesiones de funciones integrables y distintos modos de convergencia en comparación con la convergencia en norma L^1, y finalizamos el capítulo con el tamaño algebraico de las familias de funciones no acotadas, continuas e integrables en $[0, +\infty)$, y las sucesiones de ellas.

Finalmente, en el Capítulo 6 trabajamos en el ámbito de las series de funciones, obteniendo resultados positivos sobre el tamaño lineal y algebraico de la familia de sucesiones de funciones cuya serie asociada converge uniformemente pero no verifica las hipótesis del Criterio M de Weierstrass.

Abstract

Historically, many mathematicians of all ages have been attracted and fascinated by the existence of large algebraic structures that satisfy certain properties that, *a priori*, contradict the mathematical intuition.

The aim of the present Dissertation is the study of the lineability of certain families of sequences of functions with very specific properties.

The Dissertation is divided in 6 chapters, where Chapters 1, 2 and 3 focus on introducing the basic notation and main terminology of the theory of Lineability and modes of convergence that will be used along this Dissertation.

In Chapter 4 we begin with the study of the algebraic size of two families of sequences of functions with different modes of convergence in the closed unit interval $[0, 1]$: convergence in measure but pointwise almost everywhere and pointwise but not uniform convergence.

In Chapter 5 we focus our attention on the setting of (Lebesgue) integrable functions. We start with sequences of integrable functions with different modes of convergence in comparison to the L^1-convergence, and finish the chapter with the algebraic size of the family of unbounded, continuous and integrable functions on $[0, +\infty)$ and sequences of them.

Finally, in Chapter 6 we turn into the setting of series of functions, obtaining positive results about the linear and algebraic size of the family of sequences of functions whose series converges absolutely and uniformly but does not verify the hypothesis of the Weierstrass M-test.

Introduction

In the field of Functional Analysis, one of the branches that has experienced a wider development in the past few decades is the theory of Lineability. In this branch we can find the study of the existence of vector spaces inside spaces with a strong non-linear setting, as well as the existence of linear algebras. This is precisely the framework in which the present Dissertation is set.

The theory of Lineability started in 1966 with a paper of the Russian mathematician Vladimir I. Gurariy [30], where he proved that the family of continuous functions in the closed unit interval $[0, 1]$ that are nowhere differentiable contains, except for the null function, an infinite dimensional vector space. The concept of *lineability* (existence of an infinite dimensional vector space) was coined by Gurariy himself. He later partnered with R.M. Aron, F.J. García-Pacheco, D. Pérez-García, J.B. Seoane-Sepúlveda [6] and L. Bernal [15], taking into account the topological structure of the space, its maximal dimension, and the existence of linear algebras, introduced the concepts of *dense lineability, spaceability, maximal lineability* and *algebrability* (see Chapter 2 for rigorous definitions).

In this sense, in 1999, V. Fonf, Gurariy and V. Kadets [28] proved that the previously defined family is spaceable in the space of continuous functions on $[0, 1]$. Later, in 2013, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández and Seoane-Sepúlveda [35], provided the first constructive proof of the $\mathfrak{c}$-lineability, that is, the existence of a $\mathfrak{c}$-dimensional vector space of this family, where $\mathfrak{c}$ is the dimension of continuum.

Finally, F. Bayart and L. Quarta [14] showed in 2007 the dense-algebrability of the family of continuous and nowhere Hölder functions on $[0,1]$, which is a more restrictive family than the initially considered by Gurariy.

Focusing our attention on the space of continuous functions with additional properties, the list of names that have dedicated their time to study the size of these families under the point of view of Lineability increases considerably. In 2015, Bernal, M.C. Calderón-Moreno and J.A. Prado-Bassas [17] proved that the family of continuous functions $f : [0,1] \to \mathbb{R}^2$ whose image has non-empty interior and a zero-(Lebesgue) measure boundary is maximal lineable and strongly algebrable, as well as the maximal dense-lineability of the family of continuous functions $f : [0,1] \to \mathbb{R}^2$ whose image has positive (Lebesgue) measure. Some related results can be found in [2] and [19].

But not everything is positive, as B. Levine and D. Milman [42] already showed in 1940. In particular, the family of continuous bounded variation functions on $[0,1]$ does not contain a closed infinite dimensional vector space when we consider this family as a subspace of the space of continuous functions endowed with the topology of the uniform convergence. On the other hand, in 2004, Gurariy and Quarta [32] proved that in some cases, even a large linear structure cannot be found. In particular, they proved that the family of continuous functions on $[0,1]$ which admit one and only one absolute maximum is 1-lineable but not 2-lineable. Moreover, if the closed unit interval $[0,1]$ is replaced by the whole real line $\mathbb{R}$, they showed that the previous family turns out to be 2-lineable, while the family of the corresponding subset of continuous functions vanishing at infinity is not 3-lineable.

In the setting of sequences of functions and modes of convergence of them, in 2017, G. Araújo *et al.* [4] showed the $\mathfrak{c}$-lineability of the family of sequences of (Lebesgue) measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ such that $(f_n)_n$ converges pointwise to zero on $\mathbb{R}$ and $f_n(I) = \mathbb{R}$ for any non-degenerate interval $I \subset \mathbb{R}$ and every $n \in \mathbb{N}$. Moreover, by considering the family of all sequences of (Lebesgue) measureable functions $f_n : [0,1] \to \mathbb{R}$ $(n \in \mathbb{N})$ such that $(f_n)_n$ converges in measure to zero but not pointwise almost everywhere on $[0,1]$, they were able to prove the maximal dense-lineability of this family in the vector space of sequences of (Lebesgue) measurable functions on $[0,1]$.

In the same year, A. Conejero *et al.* [27] focused their attention on the probability theory setting, and they studied the lineability and algebrability of diverse problems, with both positive and negative results about the size within martingales, random variables, and certain stochastic processes.

Continuing within the framework of integrable functions, Muñoz-Fernández *et al.* [44] proved, in 2008, the $\mathfrak{c}$-lineability of both the spaces $L^p([0,1]) \setminus L^q([0,1])$ for $1 \leq p < q$, and of $L^p(I) \setminus L^q(I)$ for $1 \leq q < p$ and any unbounded interval $I \subset \mathbb{R}$. Some generalizations and extensions of these results can be found in [6], [15], [18], [19], [21], [22] and [39].

Regarding the existence of linear algebras, in 2009, García-Pacheco *et al.* [29] proved the existence of an infinite generated closed algebra contained in the family of almost everywhere continuous functions $f : I \to \mathbb{R}$ (being $I \subset \mathbb{R}$ an arbitrary unbounded interval) that are not Riemann integrable. Moreover, they proved that the family of continuous and bounded functions $f : I \to \mathbb{R}$ that are not Riemann integrable on an unbounded interval $I \subset \mathbb{R}$ is spaceable. In addition, for the families of functions that are Riemann integrable but not Lebesgue integrable, and the ones that are Lebesgue integrable but not Riemann integable, they proved its lineability and spaceability, respectively. In this direction, Bernal and Ordóñez-Cabrera [19] showed in 2014 the maximal dense-lineability of the family of continuous and Riemann integrable functions on $[0,+\infty)$ which do not belong to $L^p([0,+\infty))$ for any $0 < p < +\infty$.

Going back to continuous functions, in the setting of sequences of functions there are, up to our knowledge, only few and recent results concerning its lineability. In fact, the first result is ascribed to Bernal and M. Ordóñez-Cabrera [19] in 2014, when they considered the family of sequences of continuous bounded and integrable functions $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ such that $\|f_n\|_\infty \to +\infty$ $(n \to \infty)$, $\sup \left\{ \|f_n\|_1 \, : \, n \in \mathbb{N} \right\} < +\infty$ but $\|f_n\|_1 \not\to 0$ $(n \to \infty)$, and proved its maximal lineability.

In the setting of series of functions, we can find a wide plethora of results concerning the study of the divergence of the series and its algebraic size. Its origin goes back to E. du Bois-Reymond (1873, see [40]), who was the first one to exhibit an example of a continuous function f on the unit circle $\mathbb{T}$ whose Fourier series diverges at a point. This was later improved in 1966 by J.P. Kahane and V. Katznelson (see [36],[38]), extending the divergence to a set $E \subset \mathbb{T}$ of (Lebesgue) measure zero. If we drop away the continuity of the functions, in 1926 A. Kolmogorov [41]) was able to find a function $f \in L^1(\mathbb{T})$ such that its Fourier series diverges everywhere on $\mathbb{T}$.

In 2005 Bayart [13] showed that the set of continuous functions on $\mathbb{T}$ whose Fourier series diverges on a set $E \subset \mathbb{T}$ of measure zero is dense-lineable. A year later, Aron *et al.* stated its dense-algebrability. When E is countable, results on lineability of divergent Fourier series with additional properties are obtained in [16] or [43].

Again Bayart [12], [13] showed that the set of Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ that are bounded in the right half-plane and diverge everywhere in the imaginary axis is lineable and spaceable. Together with Quarta [14], they were also able to establish the algebrability.

In the setting of Banach spaces, A. Aizpuru, C. Pérez-Eslava and Seoane-Sepúlveda in 2006 [1], asserted that the set of unconditionally convergent but not absolutely convergent series of any infinite dimensional Banach space is $\mathfrak{c}$-lineable. Moreover, the set of sequences such that its partial sums are bounded but the series diverges is also $\mathfrak{c}$-lineable. If we focus on the complex plane, in 2011 A. Bartoszewicz, S. Glab and T. Poreda proved in [11] that the set of non-absolutely convergent complex series and the set of divergent complex series with bounded partial sums are $\mathfrak{c}$-algebrable.

Following with sequences of real numbers, in 2013 Bartoszewicz and Glab [10] showed that $c_0 \setminus \bigcup_{p\geq 1} \ell_p$ is densely strongly $\mathfrak{c}$-algebrable, where c_0 denotes the set of all real sequences converging to zero, and ℓ_p is the set of all p-sumable real sequences.

Later, in 2017, Araújo *el al.* [4] studied the linear structure of sequences of real numbers such that its series fails the root and ratio test. Concretely, they show that the set of sequences in ℓ_1 that generate series for which the ratio or the root tests fail and the set of sequences in ω (the vector space of all real sequences endowed with the product topology) that generate divergent series for which the ratio and the root tests fail are $\mathfrak{c}$-dense-lineable in ℓ_1 or ω.

In the present Dissertation we generalize and extend some of the previous results. In Chapter 1 we establish the main notation and basic definitions that are needed to the correct development of this Dissertation.

In Chapter 2, definitions of the theory of Lineability are introduced, as well as different relations among them, with some general results about the existence of these linear structures.

In Chapter 3 we recall different modes of convergence in the setting of function sequence spaces. We show the different relations among them, including counter-examples when the reciprocals are not true.

Chapter 4 brings modes of convergence into the setting of Lineability. We focus our attention on sequences of functions with different modes of convergence, and study the size of these families under the point of view of Lineability, providing positive results about the existence of both linear and algebraic structures. In particular, we show the spaceability and strong $\mathfrak{c}$-algebrability of the family of sequences of (Lebesgue) measurable functions $f_n : [0,1] \to \mathbb{R}$ ($n \in \mathbb{N}$) such that they converge to zero in measure but not pointwise almost everywhere on $[0,1]$, extending the work in [4], and the maximal dense-lineability, spacebility and strong $\mathfrak{c}$-algebrability of the family of sequences pointwise but not uniformly convergent to zero on $[0,1]$, satisfying that for any $\varepsilon > 0$ there is a measurable set $E \subset [0,1]$ with $m(E) < \varepsilon$ such that $\operatorname{ess\,sup}_{[0,1]\setminus E} |f_n| \to 0$ $(n \to \infty)$.

In Chapter 5 we focus our attention on the setting of (Lebesgue) integrable functions. First, we recall the convergence in L^1-norm, and complete the results of Chapter 3 with the existence of sequences of functions $f_n : [0, +\infty) \to \mathbb{R}$ converging to zero in measure but not in L^1-norm, converging to zero in L^1-norm but not uniformly on $[0, +\infty)$, and converging to zero uniformly on $[0, +\infty)$ but not in L^1-norm. In fact, we study the algebraic size of this last family, obtaining its maximal dense-lineability, spaceability and strong $\mathfrak{c}$-algebrability. Later, we consider unbounded, continuous and integrable functions and sequences of them, and show that they are $\aleph_0$-algebrable and maximal dense-lineable. We finish this chapter with some final remarks about the maximal possible convergence of the sequences of functions, its growth and smoothness.

Finally, in Chapter 6 we deal with series of functions, inspired in the previous existing results about convergence of the series. In particular, we study the existence of vector spaces and linear algebras inside the family of uniformly and absolutely convergent series on a closed interval $[a, b] \subset \mathbb{R}$ that do not fulfill all the hypothesis of the Weierstrass M-test.

Most of all the results exposed in Chapters 3, 4, 5 and 6 are original. The results from Chapters 3, 4 and 5 are published in [24] and [25]. The ones from Chapter 6 are collected in [26].

Chapter 1

Preliminaries

1.1 Basic concepts and definitions

In order for this Dissertation to be as self-contained as possible, in this first Chapter we include basic concepts and definitions with which we deal in this work, as well as many results that will become useful while developing the upcomings chapters.

By $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the set of all natural numbers, the set of rational numbers, the real line and the complex plane, respectively. By $\mathbb{K}$ we will denote indistinctly the set of real or complex numbers, that is, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The cardinality of $\mathbb{N}$ will be $\aleph_0$, and the continuum $\mathfrak{c}$ will be the cardinality of the real line $\mathbb{R}$. The different concepts that we introduce next can be briefly consulted in [3], [49], [53].

Let $X \neq \varnothing$ be a set. An application $d : X \times X \to [0, +\infty)$ is a *metric* over X if for any $x, y, z \in X$ we have $d(x, y) = 0 \Leftrightarrow x = 0$; $d(x, y) = d(y, x)$; and $d(x, y) \leq d(x, z) + d(z, y)$. The pair (X, d) is called *metric space*. A topologic space (X, τ) is *metrizable* if there exists a metric d over X which induces the topology τ, that is, the family of open balls

$$B(x,\varepsilon) := \{y \in X \,:\, d(x,y) < \varepsilon\}, \qquad (x \in X, \varepsilon > 0)$$

is a basis of τ. When X is a vector space and τ is a topology on X where the singletons are closed and compatible with the linear structure, that is, the applications of addition $+ : (x,y) \in X \times X \mapsto x+y \in X$ and product by scalars $\cdot : (\lambda, x) \in \mathbb{K} \times X \mapsto \lambda \cdot x \in X$ are continuous, we say that (X,τ) is a *topological vector space*. A metric d defined on a vector space X is *translation-invariant* if $d(x+z, y+z) = d(x,y)$ for every $x,y,z \in X$. A metric space (X,d) is *complete* if every Cauchy sequence $(x_n)_n \subset X$ is convergent on X, that is, there exists $x \in X$ such that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for every $n \geq N$. We denote by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$ the convergence of a sequence $(x_n)_n$ to an element x.

A topological vector space X which is metrizable by a translation-invariant metric and complete is called an *F-space*. If, in addition, the F-space X is locally convex, we say that X is a *Fréchet space*. Recall that a topological vector space is *locally convex* if it is Hausdorff and the zero element has a neighbourhood basis formed by convex sets.

For any Fréchet space (X,d), the application $\|\cdot\| : X \to [0,+\infty)$ defined as $\|x\| := d(x,0)$ is a *seminorm*, that is, $\|x+y\| \leq \|x\| + \|y\|$ and $\|\lambda x\| = |\lambda| \cdot \|x\|$. If, in addition, $\|x\| = 0$ if and only if $x = 0$, we have that $\|\cdot\|$ is a norm and $(X, \|\cdot\|)$ is a complete normed space, that is $(X, \|\cdot\|)$ is a *Banach space*.

Let (X,τ) be a topological space. Given $A \subset X$ we denote by $\operatorname{int}(A)$ and $\overline{A}$ the interior and, respectively, the closure of A. We will use the following category concepts to study the topological size of some sets. A set $A \subset X$ is said to be:

(1) *dense* in X if $\overline{A} = X$;

(2) *nowhere dense* in X if $\operatorname{int}\big(\overline{A}\big) = \varnothing$;

(3) of *first category* if $A = \bigcup_{n=1}^{\infty} F_n$, where each F_n is nowhere dense in X;

(4) of *second category* if A is not of first category; and

(5) *residual* if its complement $A^c := X \setminus A$ is of first category.

A topological space (X, τ) is a *Baire space* if any countable intersection of dense open sets is also dense in X.

Theorem 1.1 (Baire's Theorem). *Every complete metric space (X, d) is a Baire space.*

In particular, we have the following result.

Corollary 1.2. *Let (X, d) be a complete metric space and $A \subset X$ be a subset. If A is of first category, then $X \setminus A$ is dense in X.*

1.2 Some classical spaces

In this Section we recall some well-known spaces which we will use in this work.

1.2.1 Continuous functions

Let $[a, b] \subset \mathbb{R}$ be a closed interval of the real line, where in $\mathbb{R}$ we consider always the euclidean norm. We define $C([a, b])$ as the space of all functions $f : [a, b] \to \mathbb{K}$ that are continuous on $[a, b]$. This space becomes a separable Banach space when endowed with the supremum norm $\| \cdot \|_\infty$ given by

$$\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|, \qquad \text{for every } f \in C([a, b]).$$

The space of all sequences of continuous functions on $[a, b]$ will be denoted by $C([a, b])^{\mathbb{N}}$, and it becomes a separable Banach vector space when endowed with the norm $\|(f_n)_n\| := \sup_{n \in \mathbb{N}} \|f_n\|_\infty$.

For the interval $[0, +\infty)$, we can consider the space $C([0, +\infty))$ of all functions $f : [0, +\infty) \to \mathbb{K}$ that are continuous on $[0, +\infty)$. Although it is not a Banach space, it does become a Fréchet space when endowed with the compact-open topology. Recall that for this topology we can consider the metric

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f - g\|_{\infty,n}}{1 + \|f - g\|_{\infty,n}},$$

where $\|h\|_{\infty,n} := \sup \big\{ |h(x)| \, : \, x \in [0, n] \big\}$ for any $h \in C([0, +\infty))$ and $n \in \mathbb{N}$. We have that $f_n \to f$ $(n \to \infty)$ in the metric d if and only if $f_n \to f$ $(n \to \infty)$ uniformly on compacta if and only if $\|f_n - f\|_{\infty,n} \to 0$ $(n \to \infty)$.

1.2.2 Sequence spaces

The ℓ^p sequence spaces for $1 \leq p < +\infty$

Let $p \in [1, +\infty)$. Given a sequence $(x_n)_n \subset \mathbb{K}$ we define its ℓ^p-norm or simply p-norm as

$$\|x_n\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

The sequence space ℓ^p is defined as those sequences $(x_n)_n \subset \mathbb{K}$ for which its ℓ^p-norm is finite, that is,

$$\ell^p := \big\{ (x_n)_n \subset \mathbb{K} \, : \, \|x_n\|_p < +\infty \big\},$$

and its becomes a (separable) Banach space when endowed with the $\| \cdot \|_p$ norm for every $1 \leq p < +\infty$.

The ℓ^∞ sequence space

For the case $p = +\infty$, the ℓ^∞-norm is taken slightly different. We define the ℓ^∞-norm, ∞-norm or supremum norm as

$$\|x_n\|_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

The sequence space ℓ^∞ is defined as those sequences $(x_n)_n \subset \mathbb{K}$ for which its ∞-norm is finite, that is,

$$\ell^\infty := \big\{(x_n)_n \subset \mathbb{K} : \|x_n\|_\infty < +\infty\big\}.$$

The pair $(\ell^\infty, \|\cdot\|_\infty)$ becomes a Banach space, whose elements are precisely those sequences $(x_n)_n \subset \mathbb{K}$ that are bounded. Moreover, the space $(\ell^\infty, \|\cdot\|_\infty)$ is not separable.

The c_0 and c_{00} sequence spaces

The family of all sequences $(x_n)_n \subset \mathbb{K}$ converging to zero is denoted by c_0, that is,

$$c_0 := \big\{(x_n)_n \subset \mathbb{K} : x_n \to 0 \qquad (n \to \infty)\big\}.$$

Observe that c_0 is a subspace of ℓ^∞, since convergence of a sequence implies its boundedness. Furthermore, if we endow c_0 with the supremum norm of ℓ^∞, we obtain that $(c_0, \|\cdot\|_\infty)$ is a closed subspace of $(\ell^\infty, \|\cdot\|_\infty)$, and hence it is also a Banach space.

If we consider the family c_{00} of all vanishing sequences, that is,

$$c_{00} := \{(x_n)_n \subset \mathbb{K} : \text{there exists } N \in \mathbb{N} \text{ such that } x_n = 0 \text{ for every } n \geq N\},$$

then c_{00} is a dense subspace of $(c_0, \|\cdot\|_\infty)$. Indeed, for any $x = (x_n)_n \in c_0$, the sequences $\omega^k = (\omega_n^k)_n = (x_1, x_2, \ldots, x_k, 0, 0, \ldots) \in c_{00}$ verify

$$\|\omega^k - x\|_\infty = \sup_{n\in\mathbb{N}} |w_n^k - x_n| = \sup_{n\geq k} |x_n|.$$

But $x \in c_0$, so given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n| < \varepsilon$ for every $n \geq N$. Hence, $\|\omega^k - x\|_\infty < \varepsilon$ for any $k \geq N$.

In fact, the set

$$\{(r_n)_n \subset \mathbb{Q} \,:\, \text{there exits } N \in \mathbb{N} \text{ such that } r_n = 0 \text{ for } n \geq N\}$$

is also dense in c_0, and hence, $(c_0, \|\cdot\|_\infty)$ is separable.

1.2.3 The space $\mathcal{L}^0(X, \mu)$

Let $X \neq \varnothing$ be a non-empty set. A family $\mathcal{M}$ of subsets of X is a *σ-algebra* on X if $\varnothing, X \in \mathcal{M}$; $X \setminus A \in \mathcal{M}$ whenever $A \in \mathcal{M}$; and every countable union of sets of $\mathcal{M}$ is again a member of $\mathcal{M}$.

An application $\mu : \mathcal{M} \to [0, +\infty)$ is a *measure* on the σ-algebra $\mathcal{M}$ if $\mu(\varnothing) = 0$; and $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ for every collection $(A_n)_n \subset \mathcal{M}$ of pairwise disjoints elements. The triplet $(X, \mathcal{M}, \mu)$ is called *measure space* over X.

A function $f : X \to \mathbb{K}$ is *measurable* (with respect to the measure μ) on X if for every open subset $B \subset \mathbb{K}$ we have that $f^{-1}(B) \in \mathcal{M}$. We denote by $\mathcal{L}^0(X, \mu)$ the set of all measurable functions from X into $\mathbb{K}$.

Given two functions $f, g \in \mathcal{L}^0(X, \mu)$ we say that $f = g$ almost everywhere, and we denote it by $f \sim g$, if there exists a measurable set $A \in \mathcal{M}$ such that $\mu(A) = 0$ and $f = g$ on $X \setminus A$. This is a relation of equivalence, and we denote by $[f] := \{g \in \mathcal{L}^0(X, \mu) \,:\, f \sim g\}$ the equivalence class of a measurable function f, and by $L^0(X, \mu)$ the quotient space under this relation, that is,

$$L^0(X, \mu) := \mathcal{L}^0(X, \mu) / \sim .$$

It is known that for any pair f and g of measurable functions such that $f \sim g$, the Lebesgue integrals for f and g in any $A \in \mathcal{M}$ with respect to μ are equals. So, the application $\|\cdot\|_1 : L^0(X,\mu) \to [0,+\infty]$ given by

$$\|f\|_1 := \int_X |f| \, d\mu,$$

is well-defined.

We define the space $L^1(X,\mu)$ as all the classes of measurable functions that are integrable over X, that is,

$$L^1(X,\mu) := \left\{ f \in L^0(X,\mu) \ : \ \|f\|_1 < +\infty \right\}.$$

In $L^1(X,\mu)$, the application $\|\cdot\|_1$ defines a norm. In fact, there is no problem if we consider powers of the functions.

1.2.4 $L^p(X,\mu)$ spaces of functions for $1 \le p < +\infty$

Let $f : X \to \mathbb{K}$ be a measurable function on X. For $1 \le p < +\infty$ we define the L^p-norm or simply p-norm as

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

The space $L^p(X,\mu)$ will denote the collection of all measurable (classes of) functions f such that $\|f\|_p < +\infty$. The pair $(L^p(X,\mu), \|\cdot\|_p)$ becomes a separable Banach space for every $p \in [1,+\infty)$.

1.2.5 $L^\infty(X,\mu)$ space of functions

Notwithstanding, in order to study the case $p = +\infty$ we have to be more precise. If we try to reason as in the case of sequences of scalars, we see that the supremum over the whole space is not well-defined among classes of measurable functions. For instance, the functions $f(x) = 1$, and $g(x) = 1$ if $x \in [0,1] \setminus \{1/n\}$ $(n \in \mathbb{N})$ and $g(1/n) = n$ $(n \in \mathbb{N})$ represent the same class in $L^0([0,1], m)$ (where m denotes the

Lebesgue measure), but $\sup_{[0,1]} f = 1$ and $\sup_{[0,1]} g = +\infty$. So we need to consider a new concept, the essential supremum.

Let $f : X \to \mathbb{R}$ be a measurable function, and define the set S_f by

$$S_f := \big\{\alpha \in \mathbb{R} \,:\, \mu(f^{-1}((\alpha, +\infty])) = 0\big\}.$$

It turns out that S_f collects the bounds of f except for a set of measure zero, that is, $\alpha \in S_f$ if and only if $\mu\big(\{x \in X \,:\, f(x) > \gamma\}\big) = 0$ for all $\gamma > \alpha$, and, in particular, if f is bounded, $\sup f \in S_f$.

We define the *essential supremum* of the function f as

$$\operatorname{ess\,sup} f = \inf S_f.$$

If $S_f = \varnothing$, then $\operatorname{ess\,sup} f = +\infty$, and we always have $\operatorname{ess\,sup} f \leq \sup f$ (even if f is unbounded). Observe that, by the definition of $\beta := \operatorname{ess\,sup} f$, we have

$$\begin{aligned}
\mu\left(f^{-1}((\beta, +\infty])\right) &= \mu\left(\bigcup_{n=1}^{\infty} f^{-1}\left(\left(\beta + \frac{1}{n}, +\infty\right]\right)\right) \\
&\leq \sum_{n=1}^{\infty} \mu\left(f^{-1}\left(\left(\beta + \frac{1}{n}, +\infty\right]\right)\right) = 0.
\end{aligned}$$

Thus, $\mu\left(f^{-1}\left(\operatorname{ess\,sup} f, +\infty\right]\right) = 0$.

We have that $\operatorname{ess\,sup} f = \operatorname{ess\,sup} g$ for any $f, g \in \mathcal{L}^0(X, \mu)$ such that $f \sim g$. Indeed, if $f(x) = g(x)$ for any $x \in X \setminus A$, where $\mu(A) = 0$, then for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\mu\big(\{x \in X \,:\, |f(x)| > \alpha\}\big) &= \mu\big(\{x \in X \setminus A \,:\, |f(x)| > \alpha\}\big) + \mu\big(\{x \in A \,:\, |f(x)| > \alpha\}\big) \\
&= \mu\big(\{x \in X \setminus A \,:\, |g(x)| > \alpha\}\big) \\
&= \mu\big(\{x \in X \setminus A \,:\, |g(x)| > \alpha\}\big) + \mu\big(\{x \in A \,:\, |g(x)| > \alpha\}\big) \\
&= \mu\big(\{x \in X \,:\, |g(x)| > \alpha\}\big).
\end{aligned}$$

So, $S_f = S_g$ and $\operatorname{ess\,sup} f = \operatorname{ess\,sup} g$. In particular, the essential supremum is well-defined in $L^0(X, \mu)$.

We denote by $L^\infty(X,\mu)$ the space of all (classes of) measurable functions f on X such that $\|f\|_\infty := \operatorname{ess\,sup}|f| < +\infty$.

Observe that the pair $(L^\infty(X,\mu), \|\cdot\|_\infty)$ becomes a Banach space where its elements are usually called measurable functions essentially bounded on X. In fact, the functions of $L^\infty(X,\mu)$ are precisely those functions that are bounded except for a set of measure zero.

Proposition 1.3. *Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a continuous function. Then* $\operatorname{ess\,sup} f = \sup f$.

Proof. We know that $\operatorname{ess\,sup} f \leq \sup f$. Assume, by way of contradiction, that $\beta := \operatorname{ess\,sup} f < \sup f =: \gamma$. Then, there exists $\alpha \in (\beta, \gamma)$ such that $m\big(\{x \in X : f(x) > \alpha\}\big) = 0$ and $\alpha < \gamma$. Then, we also have that $m\big(\{x \in X : \gamma > f(x) > \alpha\}\big) = 0$. Observe that $\{x \in X : \gamma > f(x) > \alpha\} = f^{-1}((\alpha, \gamma))$, which is a non-empty open set in X. So, $m\big(f^{-1}((\alpha,\gamma))\big) \neq 0$, which is a contradiction. □

It is immediate that we also have:

Corollary 1.4. *Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a continuous function except for a finite number of points. Then* $\operatorname{ess\,sup} f = \sup f$.

1.3 Topology of sequence spaces

Let X be a topological vector space. We denote by $X^{\mathbb{N}}$ the set of all X-valued sequences, that is,

$$X^{\mathbb{N}} := \{(x_n)_n : x_n \in X \text{ for all } n \in \mathbb{N}\}.$$

Over this space we can define the product topology τ_Π as the strongest topology that makes every projection (that is $(x_n)_n \mapsto x_n$ for all $n \in \mathbb{N}$) continuous, and $V \subset X^{\mathbb{N}}$ is an open set for this topology if there are finitely many open sets $V_1, V_2, \dots, V_N$ of X such that

$$V_1 \times V_2 \times \cdots V_N \times X \times X \times \cdots \subset V.$$

If (X, d) is a complete metric space, then $X^{\mathbb{N}}$ (with the product topology) is completely metrizable. Indeed, if we define, for instance,

$$D((x_n)_n, (y_n)_n) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d(x_n, y_n)}{1 + d(x_n, y_n)},$$

for $(x_n)_n, (y_n)_n \in X^{\mathbb{N}}$, it is clear that $D((x_n)_n, (y_n)_n) = 0$ if and only if $d(x_n, y_n) = 0$ for any $n \in \mathbb{N}$; $D((x_n)_n, (y_n)_n) \geq 0$ and $D((x_n)_n, (y_n)_n) = D((y_n)_n, (x_n)_n)$. In order to see that D is a metric it only remains to show the triangular inequality. For this, let us consider the function $p : [0, +\infty) \to [0, +\infty)$ given by

$$p(x) = \frac{x}{1 + x}.$$

Clearly, $p(x)$ is differentiable on $[0, +\infty)$, and

$$p'(x) = \frac{1}{(1 + x)^2} > 0,$$

so $p(x)$ is increasing on $[0, +\infty)$. In particular, as $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$ for any $n \in \mathbb{N}$, we have

$$D((x_n)_n, (y_n)_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot p(d(x_n, y_n))$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot p(d(x_n, z_n) + d(z_n, y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d(x_n, z_n) + d(z_n, y_n)}{1 + d(x_n, z_n) + d(z_n, y_n)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d(x_n, z_n)}{1 + d(x_n, z_n) + d(z_n, y_n)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d(z_n, y_n)}{1 + d(x_n, z_n) + d(z_n, y_n)}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d(x_n, z_n)}{1 + d(x_n, z_n)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d(z_n, y_n)}{1 + d(z_n, y_n)}$$

$$= D((x_n), (z_n)_n) + D((z_n)_n, (y_n)_n).$$

Moreover, this metric is compatible for the product topology. Indeed, let V be a basic open set for $(X^{\mathbb{N}}, \tau_{\Pi})$, that is, $V = V_1 \times V_2 \times \cdots \times V_N \times X \times \cdots$, with $V_1, V_2, \ldots, V_n$ are open sets in (X, d). Pick $(x_n)_n \in V$, so, $x_n \in V_n$ for $1 \leq n \leq N$ and there is $\varepsilon_n > 0$ such that $B_d(x_n, \varepsilon_n) \subset V_n$. Let $\varepsilon := \min \left\{ \frac{1}{2^n} \cdot \varepsilon_n \,:\, 1 \leq n \leq N \right\}$. We claim that $B_D((x_n)_n, \varepsilon) \subset V$.

For this, let $(y_n)_n \in X^{\mathbb{N}}$ such that $D((x_n)_n, (y_n)_n) < \varepsilon$. For any $1 \leq n \leq N$,

$$\frac{1}{2^n} \cdot d(x_n, y_n) \leq D((x_n)_n, (y_n)_n) < \varepsilon \leq \frac{1}{2^n} \cdot \varepsilon_n,$$

thus, $y_n \in B_d(x_n, \varepsilon_n)$ $(1 \leq n \leq N)$ and $(y_n)_n \in V$.

Reciprocally, fix $(x_n)_n \in X^{\mathbb{N}}$ and $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\varepsilon}{2}$. Let $V = V_1 \times V_2 \times \cdots \times V_N \times X \times \cdots$, where $V_n := B_d\left(x_n, \frac{\varepsilon}{2N}\right)$ $(1 \leq n \leq N)$. We claim that $V \subset B_D((x_n)_n, \varepsilon)$.

For $(y_n)_n \in V$, we have that $d(x_n, y_n) < \frac{\varepsilon}{2N}$ $(1 \leq n \leq N)$, so

$$\sum_{n=1}^{N} \frac{1}{2^n} \cdot \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} \leq \sum_{n=1}^{N} d(x_n, y_n) < N \cdot \frac{\varepsilon}{2N} = \frac{\varepsilon}{2}. \tag{1.1}$$

On the other hand,

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} \cdot \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} < \frac{\varepsilon}{2}. \tag{1.2}$$

So, by (1.1) and (1.2) we have that $D((x_n)_n, (y_n)_n) < \varepsilon$ and we are done.

Finally, D is complete (see [53, Theorem 24.11]). Indeed, let $(x^k)_k \subset X^{\mathbb{N}}$ be a Cauchy sequence, where $x^k = (x_n^k)_n$ for every $k \in \mathbb{N}$. Since $d(x_n, y_n) \leq D((x_n)_n, (y_n)_n)$ for all $n \in \mathbb{N}$, we have that for any fixed $n \in \mathbb{N}$, $(x_n^k)_k$ is a Cauchy sequence in (X, d), so, because (X, d) is complete, there is $x_n \in X$ with $x_n^k \to x_n$ $(k \to \infty)$.

Let $x = (x_n)_n \in X^{\mathbb{N}}$ and let us see that $D(x^k, x) \to 0$ $(k \to \infty)$. For this, fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that

$$\frac{1}{2^N} < \frac{\varepsilon}{2}. \tag{1.3}$$

since $x_n^k \to x_n$ $(k \to \infty)$, for $1 \leq n \leq N$ there exists $K \in \mathbb{N}$ such that if $k \geq K$ we have that

$$d(x_n^k, x_n) < \frac{\varepsilon 2^n}{2N} \qquad (1 \leq n \leq N). \tag{1.4}$$

Now, for $k \geq K$,

$$\begin{aligned} D(x^k, x) &= \sum_{n=1}^{N} \frac{1}{2^n} \cdot \frac{d(x_n^k, x_n)}{1 + d(x_n^k, x_n)} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \cdot \frac{d(x_n^k, x_n)}{1 + d(x_n^k, x_n)} \\ &\leq \sum_{n=1}^{N} \frac{1}{2^n} \cdot d(x_n^k, x_n) + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &< \sum_{n=1}^{N} \frac{1}{2^n} \cdot \frac{\varepsilon 2^n}{2N} + \frac{1}{2^N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and we get that $x^k \to x$ $(k \to \infty)$.

Assume that (X, d) is a complete metric space. Following the cases of c_0, c_{00} and ℓ^∞, we denote by $c_0(X)$ the set of all sequences in $X^{\mathbb{N}}$ convergent to zero, that is,

$$c_0(X) := \big\{ (x_n)_n \in X^{\mathbb{N}} \,:\, d(x_n, 0) \to 0 \qquad (n \to \infty) \big\},$$

and by $c_{00}(X)$ the set of all sequences in $X^{\mathbb{N}}$ eventually vanishing, that is,

$$c_{00}(X) := \big\{ (x_n)_n \in X^{\mathbb{N}} \,:\, \text{there is } N \in \mathbb{N} \text{ such that } x_n = 0 \text{ for every } n > N \big\}.$$

We can see $c_0(X)$ as a subset of $(X^{\mathbb{N}}, \tau_\Pi)$, but if we consider in $c_0(X)$ the metric:

$$d_\infty((x_n)_n, (y_n)_n) := \sup_{n \in \mathbb{N}} d(x_n, y_n),$$

we have that $(c_0(X), d_\infty)$ is also a complete metric space.

In addition, the set $c_{00}(X)$ verifies the following assertions:

(1) $c_{00}(X)$ is dense in $(X^{\mathbb{N}}, \tau_{\Pi})$. Indeed, for any element $x = (x_n)_n \in X^{\mathbb{N}}$, the sequence $\omega^k = (\omega_n^k)_n = (x_1, x_2, \ldots, x_k, 0, 0, \ldots)$ $(k \in \mathbb{N})$ does the job.

(2) $c_{00}(X)$ is dense in $(c_0(X), d_\infty)$, since for any $(x_n)_n \in c_0(X)$, choosing the sequence ω^k as before, we have that $\omega^k \in c_{00}(X)$ for every $k \in \mathbb{N}$, and

$$d_\infty((\omega_n^k)_n, (x_n)_n) = \sup_{n \in \mathbb{N}} d(\omega_n^k, x_n) = \sup_{n \geq k+1} d(x_n, 0).$$

But $(x_n)_n \in c_0(X)$, so, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, 0) < \varepsilon$ for every $n \geq N$. Hence, $d_\infty((\omega_n^k)_n, (x_n)_n) < \varepsilon$ for any $k \geq N$, and we are done.

Chapter 2

Lineability

2.1 Introduction

Many mathematicians over History have been attracted and fascinated over the existence of different mathematical objects fulfilling properties that are, *a priori*, anti-intuitive. This thought was plausible with the idea that was held until the end of the 19th century, where the geometric behaviour of the graph of a given function was thought to characterize the analytic behaviour of the same function. It was then commonly believed that it was impossible to have a continuous function not being differentiable at some set of points "big enough". Nonetheless, this idea was disproved by the German mathematician K. Weierstrass [52], providing the existence of the so-called Weierstrass' Monsters.

A new trend that has been growing attention of mathematicians all around the globe for the past few decades focus on the study of large algebraic structures inside a (commonly) non-linear setting, where those "pathological" objects usually lie. In fact, in [30], Gurariy provided a first (and very famous) example of this new branch.

Theorem 2.1. *There exists an infinite dimensional linear space such that its non-null elements are continuous nowhere differentiable functions on* $[0, 1]$.

Observe that the above theorem states that what was initially supposed to be an isolated phenomenon, as Weierstrass pointed out with the existence of his Monsters, has actually a big linear structure supporting it.

This new approach receives the name of *Lineability*, and the main terminology was firstly introduced by Aron, Gurariy and Seoane-Sepúlveda in [7], [32], [51], although the essence of them can be found in [30] and [31]. In this new terminology Gurariy showed that the set of Monsters is lineable.

2.2 Main definitions

Vector spaces and linear algebras are an example of some mathematical structures which are, at first, unexpected to appear in spaces with a strong non-linear setting, that is, their existence within families with strange or pathological properties is rather unlikely. As commented before, Gurariy was the first one who provided such an example, with the existence of an infinite dimensional vector subspace in the family of continuous nowhere differentiable functions on $[0, 1]$. In this Section, the basic concepts and definitions about existence of such linear structures are presented. The concepts of lineability and spaceability are due to Aron, Gurariy and Seoane-Sepúlveda, although these notions were treated before, under another terminology (see [7], [32], [51]). The notion of maximal lineability (see Definition 2.2) was introduced by Bernal in [15]. Later, Gurariy partnered with Aron *et al.* coined the concepts of algebrability (see Definition 2.4), which has its roots in the papers [8], [9], [51]. Finally, this definitions were completed with the notion of strong algebrability, introduced by Bartoszewicz and Glab (see [10]).

Definition 2.2. *Let X be a vector space and let $A \subset X$ be a subset. Let κ be a (finite or infinite) cardinal number. We say that A is:*

(1) κ-lineable *if there is a vector space M such that $M \subset A \cup \{0\}$ and $dim(M) = \kappa$;*

(2) maximal lineable *if A is $dim(X)$-lineable.*

Although we speak about lineability of some subset $A \subset X$, this set does not need to have a linear structure itself. In fact, the neutral or 0 element of X, or even the sum of any of these objects would not have the "strange" behaviour.

The definitions of lineability can be completed if a topological structure is taken into account, as the next definition shows.

Definition 2.3. *Let X be a topological vector space and let $A \subset X$ be a subset. Let κ be a cardinal number. We say that A is:*

(1) spaceable *if there is a closed infinite dimensional vector space M such that $M \subset A \cup \{0\}$;*

(2) κ-dense-lineable *if there is a dense vector space M in X with $M \subset A \cup \{0\}$, and $dim(M) = \kappa$;*

(3) maximal dense-lineable *if A is $dim(X)$-dense-lineable.*

If we replace linear spaces by algebras we have the notion of algebrability.

Definition 2.4. *Let X be a vector space contained in some (linear) algebra. Let κ be a cardinal number. We say that $A \subset X$ is:*

(1) algebrable *if there is an algebra $\mathcal{B}$ so that $\mathcal{B} \subset A \cup \{0\}$ and $\mathcal{B}$ is infinitely generated, that is, the cardinality of any system of generators of $\mathcal{B}$ is infinite;*

(2) κ-algebrable *if there is a κ-generated algebra $\mathcal{B}$ with $\mathcal{B} \subset A \cup \{0\}$;*

(3) strongly κ-algebrable *if, in addition, the algebra $\mathcal{B}$ can be taken free.*

If the structure of the algebra is commutative, the notion of strong κ-algebrability is equivalent to the existence of a generating system $\mathcal{C}$ of the subalgebra $\mathcal{B} \subset A \cup \{0\}$ with $\mathrm{card}(\mathcal{C}) = \kappa$ and such that for any positive integer $N \in \mathbb{N}$, any non-constant polynomial P in N variables, and any distinct $f_1, f_2, \ldots, f_N \in \mathcal{C}$, we have

$$P(f_1, f_2, \ldots, f_N) \in \mathcal{B} \setminus \{0\}.$$

2.3 On the relation between lineability, spaceability and denseness

Given a subset $A \subset X$, whenever A is spaceable or algebrable, it is trivial that A is also lineable. For the other definitions of lineability, in 2009 Aron *et al.* [6] showed that there is actually no relation between the concepts of spaceability and dense-lineability. We include the proof as appears in [5, Theorem 7.2.1].

Theorem 2.5.

(a) *Let X be an infinite dimensional locally convex space. There exists a subset $A \subset X$ such that A is spaceable and dense in X, although it is not dense-lineable in X.*

(b) *Let X be an infinite dimensional F-space. There exists a subset $A \subset X$ which is lineable and dense in X, but which is not spaceable. If X is separable, then A can also be chosen to be dense-lineable in X.*

Proof. We will prove only the real case. For $\mathbb{K} = \mathbb{C}$, just change $\mathbb{Q}$ by $\mathbb{Q} + i\mathbb{Q}$ in the proofs.

(a) Take any vector $u \in X \setminus \{0\}$ and let $Z := \mathrm{span}(\{u\}) = \{cu : c \in \mathbb{R}\}$. Since $\dim(Z) = 1$ and X is infinite dimensional, there is an algebraic complement Y of Z

(that is, a linear subspace Y of X such that $Z \cap Y = \{0\}$ and $X = Z + Y$), satisfying that Y is a closed subspace. Take

$$A := Y + \{\lambda u \,:\, \lambda \in \mathbb{Q}\}.$$

We will now show that A is our candidate of dense and spaceable but not dense-lineable subset of X. Since $Y \subset A$ and Y is a closed infinite dimensional subspace of X (because X is infinite dimensional and $\dim(Z) = 1$), the set A is spaceable. The set A is also dense. Indeed, $\{\lambda u \,:\, \lambda \in \mathbb{Q}\}$ is dense in Z, so $A = Y + \{\lambda u \,:\, \lambda \in \mathbb{Q}\}$ is dense in $Y + Z = X$.

Finally, let us prove that A is not dense-lineable. By way of contradiction, suppose that M is a dense vector subspace of X contained in $A \cup \{0\} = A$. Take $w \in M$ and write

$$w = y + qu$$

with $y \in Y$ and $q \in \mathbb{Q}$. Suppose that $q \neq 0$. Then $\frac{\pi}{q} w \in M \subset A$; that is,

$$\frac{\pi}{q} w = y' + q'u$$

with $y' \in Y$ and $q' \in \mathbb{Q}$. Since $X = Y + Z$ and Z is the algebraic complement of Y, we infer that

$$y' = \frac{\pi}{q} y \qquad \text{and} \qquad q'u = \frac{\pi}{q} qu,$$

but this implies that $\pi = q' \in \mathbb{Q}$, which is clearly false. Thus, $q = 0$, and we get that $w = y + qu = y \in Y$, so, $M \subset Y$, which (again) is a contradiction because Y is closed and proper, and M is dense.

(b) Let us consider a Hamel basis $\mathcal{B}$ for X. Because $\dim(X) = \infty$, we always can take a countably infinite subset $\{b_n \,:\, n \in \mathbb{N}\}$ of $\mathcal{B}$. Let $Y = \operatorname{span}\{b_n \,:\, n \in \mathbb{N}\}$ and $Z = \operatorname{span}\big(\mathcal{B} \setminus \{b_n \,:\, n \in \mathbb{N}\}\big)$. It is clear that Z is the algebraic complement of Y and that $X = Y + Z$. Let

$$A := Y + \operatorname{span}_{\mathbb{Q}}\big(\mathcal{B} \setminus \{b_n \,:\, n \in \mathbb{N}\}\big),$$

where $\mathrm{span}_{\mathbb{Q}}(C)$ denotes the set of all finite linear combinations of elements of C with coefficients in $\mathbb{Q}$. We claim that A satisfies the required properties.

First, A is lineable since $Y \subset A$. Note that $\mathrm{span}_{\mathbb{Q}}\big(\mathcal{B} \setminus \{b_n : n \in \mathbb{N}\}\big)$ is dense in Z, so $A = Y + \mathrm{span}_{\mathbb{Q}}\big(\mathcal{B} \setminus \{b_n : n \in \mathbb{N}\}\big)$ is dense in $Y + Z = X$.

Finally, A is not spaceable. Indeed, suppose that M is an infinite dimensional closed vector subspace of X contained in $A \cup \{0\} = A$. Now, we proceed similarly as we did in part (a). Take $w \in M$ and write:

$$w = y + q_1 b_1' + \cdots + q_r b_r'$$

with $y \in Y$, $q_1, \ldots, q_r \in \mathbb{Q}$ and $b_1', \cdots, b_r' \in \mathcal{B} \setminus \{b_n : n \in \mathbb{N}\}$. Our aim is to show that $q_j = 0$ for every $j \in \{1, 2, \ldots, r\}$. Suppose not, and assume, without loss of generality, that $q_1 \neq 0$. Then $\frac{\pi}{q_1} w \in M \subset A$, that is,

$$\frac{\pi}{q_1} w = y' + q_1' b_1' + \cdots + q_s' b_s',$$

with $s \geq r$, $y' \in Y$, $q_1', \ldots, q_s' \in \mathbb{Q}$, and $b_1', \ldots, b_s' \in \mathcal{B} \setminus \{b_n : n \in \mathbb{N}\}$. Therefore,

$$y' = \frac{\pi}{q_1} y,$$

and

$$\pi b_1' + \frac{q_2 \pi}{q_1} b_2' + \cdots + \frac{q_r \pi}{q_1} b_r' = q_1' b_1' + \cdots + q_s' b_s',$$

obtaining again the contradiction that $\pi = q_1' \in \mathbb{Q}$. Thus, all q_j's are 0, and, consequently, $M \subset Y$. But this is impossible because $\dim(Y) = \aleph_0$ and the cardinality of any Hamel basis of M is uncountable (any infinite dimensional closed subspace of a complete metrizable space has the cardinality of continuum).

To finish the proof, notice that if X is separable, then we can choose Y to be dense in X, and therefore A is dense-lineable. $\square$

Nonetheless, if we ask for some conditions, it is possible to find a positive relation between lineability and dense-lineability. In order to do so, we introduce the following definition (see [6]).

Definition 2.6. *Let X be a vector space and let $A, B \subset X$ be two subsets. We say that A is* stronger *than B if $A + B \subset A$.*

Under this situation, if we have lineability of some subset $A \subset X$ and dense-lineability of a subset $B \subset X$, it is possible to transfer the density from B to A. This result was proved by Bernal [15]. Previously, Aron *et al.* [6] gave a version for separable Banach spaces and, in fact, is often presented in the current literature in many and vary forms (see [5], [20]). The proof included is an adaptation of the proof on [15].

Theorem 2.7. *Let X be a metrizable separable topological vector space and κ be an infinite cardinal. Let $A, B \subset X$ be two subsets such that A is κ-lineable and B is dense-lineable in X. If A is stronger than B and $A \cap B = \varnothing$, then A is κ-dense-lineable in X.*

Proof. Since X is separable, there exists a sequence $(x_n)_n \subset X$ such that the set $\{x_n : n \in \mathbb{N}\}$ is dense in X. Now, A is κ-lineable in X, so there exists a vector subspace A_1 of $A \cup \{0\}$ with $\dim(A_1) = \kappa$. Because $\dim(A_1) = \kappa$, there exists $\{v_i : i \in I\} \subset A_1 \setminus \{0\}$ linearly independent such that $A_1 = \operatorname{span}\{v_i : i \in I\}$ and $\operatorname{card}(I) = \kappa$. Moreover, κ is an infinite cardinal, which means that we can split I into infinitely many pairwise disjoint non-empty sets J_n ($n \in \mathbb{N}$), that is,

$$I = \bigcup_{n=1}^{\infty} J_n.$$

Fix $n \in \mathbb{N}$ and $i \in J_n$. Because multiplication by scalars is a continuous operation in a topological vector space, there exists $\varepsilon_i > 0$ such that $d(\varepsilon_i v_i, 0) < 1/n$, where d is the translation-invariant distance that generates the topology of X.

On the other hand, B is dense-lineable in X, so there exists a vector subspace B_1 such that $B_1 \subset B \cup \{0\}$ and B_1 is dense in X. For every $n \in \mathbb{N}$, the denseness of B_1 guarantees the existence of $y_n \in B_1$ such that $d(y_n, x_n) < 1/n$.

Now, we define the elements

$$x_{n,i} := y_n + \varepsilon_i v_i$$

for every $n \in \mathbb{N}$ and $i \in J_n$, and consider the vector space

$$M := \operatorname{span}\{x_{n,i} \,:\, n \in \mathbb{N}, i \in J_n\}.$$

We are going to see that M is dense in X, $M \subset A \cup \{0\}$ and $\dim(M) = \kappa$.

(1) Fix $n \in \mathbb{N}$ and take some $i_n \in J_n$. Let $u_n := x_{n,i_n}$. We have that

$$d(u_n, x_n) \leq d(u_n, y_n) + d(y_n, x_n) = d(y_n + \varepsilon_{i_n} v_{i_n}, y_n) + d(y_n, x_n)$$

$$= d(\varepsilon_{i_n} v_n, 0) + d(y_n, x_n) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \to 0 \qquad (n \to \infty).$$

But $(x_n)_n$ is dense in X, so $(u_n)_n$ is also dense, and the same is true for M.

(2) Fix $x \in M \setminus \{0\}$. There are scalars $c_1, c_2, \ldots, c_s$ with $c_s \neq 0$, and indices $j_r \in I$ $(r = 1, 2, \ldots, s)$ such that

$$x = c_1 x_{1,j_1} + c_2 x_{2,j_2} + \ldots + c_s x_{s,j_s}.$$

But by the definition of $(x_{n,i})_{n,i}$ we have that

$$x = c_1 y_1 + c_2 y_2 + \ldots + c_s y_s + c_1 \varepsilon_{j_1} v_{j_1} + c_2 \varepsilon_{j_2} v_{j_2} + \ldots + c_s \varepsilon_{j_s} v_{j_s} =: y_0 + z_0,$$

where $y_0 = c_1 y_1 + c_2 y_2 + \ldots + c_s y_s$ and $z_0 = c_1 \varepsilon_{j_1} v_{j_1} + c_2 \varepsilon_{j_2} v_{j_2} + \ldots + c_s \varepsilon_{j_s} v_{j_s}$. Recall that $y_1, y_2, \ldots, y_s \in B_1$, which is a vector space, so $y_0 \in B_1 \subset B \cup \{0\}$. Analogously, $v_{j_1}, v_{j_2}, \ldots, v_{j_s} \in A_1$, they are linearly independent, and $c_s \varepsilon_{j_s} \neq 0$, so $z_0 \in A_1 \setminus \{0\} \subset A$. Finally, $x = y_0 + z_0 \in (B \cup \{0\}) + A \subset A$ because A is stronger than B, and we have $M \subset A$.

(3) It only remains to prove that $\dim(M) = \kappa$. For this, it is clear that

$$\operatorname{card}\big(\{(n,i) \,:\, n \in \mathbb{N}, i \in J_n\}\big) = \operatorname{card}\left(\bigcup_{n=1}^{\infty} J_n\right) = \operatorname{card}(I) = \kappa.$$

So, if we prove that the vectors of $\{x_{n,i} : n \in \mathbb{N}, i \in J_n\}$ are linearly independent we are done. Indeed, assume by way of contradiction that $c_1 x_{1,j_1} + c_2 x_{2,j_2} + \ldots + c_s x_{s,j_s} = 0$ with $c_s \neq 0$. As done before (and following the same notation), we have that $y_0 + z_0 = 0$, where $y_0 \in B_1 \cup \{0\}$ and $z_0 \in A_1 \setminus \{0\}$. But then, $y_0 = -z_0 \in A_1 \setminus \{0\}$, since A_1 is a vector space. Hence, we have that

$$y_0 \in \big(A_1 \setminus \{0\}\big) \cap \big(B_1 \cup \{0\}\big) \subset A \cap B = \varnothing,$$

which is a contradiction. □

Recall that, given two vector spaces X and Y with $Y \subset X$, the codimension of Y, denoted by $\operatorname{codim}(Y)$, is defined as the dimension of the algebraic complement of Y in X, that is, the dimension of the vector space Z such that $Z + Y = X$ and $Z \cap Y = \{0\}$. Having this definition in mind, the next result by Bernal and Ordóñez-Cabrera (see [19]) assures that, in the separable case, there is nothing to add in order to obtain the dense-lineability from mere lineability. We include the proof as it appears in [5, Theorem 7.3.3].

Theorem 2.8. *Let X be a metrizable separable topological vector space and Y be a vector subspace of X. If $X \setminus Y$ is lineable, then $X \setminus Y$ is dense-lineable. Consequently, both properties of lineability and dense-lineability for $X \setminus Y$ are equivalent provided that X has infinite dimension.*

Proof. It is evident that $X \setminus Y$ is lineable if and only Y has infinite codimension. Since X is metrizable and separable, it has a countable open basis $\{G_n : n \in \mathbb{N}\}$. Assume that $X \setminus Y$ is lineable. In particular, Y is a proper vector space of X, so $\operatorname{int}(Y) = \varnothing$. Hence $X \setminus Y$ is dense and there is $x_1 \in G_1 \setminus Y$. Since $\operatorname{codim}(Y) = \infty$, we have $\operatorname{span}(Y \cup \{x_1\}) \subsetneq X$. Then, $\operatorname{int}(\operatorname{span}(Y \cup \{x_1\})) = \varnothing$, and it follows that there exists $x_2 \in G_2 \setminus \operatorname{span}(Y \cup \{x_1\})$. With this procedure, we get recursively a sequence of vectors $(x_n)_{n\in\mathbb{N}}$ satisfying

$$x_n \in G_n \setminus \operatorname{span}(Y \setminus \{x_1, \ldots, x_{n-1}\}) \qquad (n \in \mathbb{N}).$$

In particular, the set $\{x_n : n \in \mathbb{N}\}$ is dense. Now, if we define $M := \text{span}\{x_n : n \in \mathbb{N}\}$, then M is a dense vector space, and $M \subset (X \setminus Y) \cup \{0\}$. $\square$

However, even in the case of non-separability a similar result can be found in [19] (see also [5, Theorem 7.3.4]), for which the Continuum Hypothesis needs to be assumed.

Proposition 2.9. *Let X be a non-separable F-space and Y be a closed separable vector subspace of X. Then $X \setminus Y$ is maximal lineable.*

Proof. Indeed, let Z be a vector space that is an algebraic complement of Y, so that $Z \subset (X \setminus Y) \cup \{0\}$. Note that, since Y is separable and X is not,

$$\dim(Y) \leq \mathfrak{c} \leq \dim(X) = \dim(Y) + \dim(Z).$$

If $\dim(Z) \leq \aleph_0$, then Z, and so $X = Y + Z$, would be separable, which is a contradiction. Hence (assuming the Continuum Hypothesis), $\dim(Z) \geq \mathfrak{c}$, which implies $\dim(Z) = \dim(X)$ (see, for instance [5, Corollary II.3]), and the proof is finished. $\square$

On the previous results, different relations between lineability concepts have been studied. In order to get the lineability of a family with some “pathological” properties, there is not a sufficient criterion which gives us the solution. Usually, it is necessary to construct directly the basis of the vector space to preserve the pathological properties under sums and products by scalars, although this property will not be consistent under linear combinations. The case of spaceability is also studied in many cases by a constructive method. However, there exist some results that provide the spacebility directly. The main one was obtained by Kalton [37], and was later improved by Kitson and Timoney [39]. The proofs of these results are large and very technical, and we will not include them in this work. The interested reader can consult them in [5] and [37].

Theorem 2.10. *Let X be a Fréchet space and $Y \subset X$ be a closed linear space. Then, the complement $X \setminus Y$ is spaceable if and only if Y has infinite codimension.*

Theorem 2.11. *Let Z_n with $n \in \mathbb{N}$ be a collection of Banach spaces and X be a Fréchet space. Let $T_n : Z_n \to X$ be linear continuous mappings and Y be the linear subspace generated by $\bigcup_{n\in\mathbb{N}} T_n(Z_n)$. If Y is not closed in X, then the complement $X \setminus Y$ is spaceable.*

Chapter 3

Modes of convergence

3.1 Definitions and relations between convergences

When we study sequences of scalars $(x_n)_n \subset \mathbb{K}$, it is clear what does it mean for the sequence to be convergent to some $x \in \mathbb{K}$. Indeed, we just have that for every $\varepsilon > 0$ we can always find some natural number $N \in \mathbb{N}$ such that for every $n \geq N$ the elements x_n of the sequence are "near" the limit point x, that is $|x_n - x| < \varepsilon$. Even in the case of topological spaces, this concept of convergence remains the same, just by replacing $|x_n - x| < \varepsilon$ with the corresponding condition $x_n \in U$, where U is any open set of the topology such that $x \in U$.

However, if we consider sequences of functions $f_n : X \to \mathbb{K}$ $(n \in \mathbb{N})$, for some non-empty set X, the amount of ways to approach a certain limit function $f : X \to \mathbb{K}$ increases considerably. Every student of Mathematics has seen some time the concepts of pointwise or uniform convergence, but these are not the only ones. We will recall the different modes of convergence of sequences of functions that will be studied. Although there is a wide plethora of ways of convergence for a sequence of functions, we mainly focus our attention on the ones that are related to the topic of the following chapters, measure spaces. A natural question that arises is how can we study the different relations among them.

The most common known definitions of convergence are the pointwise and uniform convergence.

Definition 3.1. *Let $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be functions defined on a non-empty set X, and $A \subset X$. We say that:*

(1) $f_n \to f$ pointwise *on A if for every $x \in A$ we have that $f_n(x) \to f(x)$ $(n \to \infty)$ on A, that is, if for each $x \in A$, and for each $\varepsilon > 0$, there exists $N = N(x, \varepsilon) \in \mathbb{N}$ such that*

$$|f_n(x) - f(x)| < \varepsilon, \qquad (n \geq N).$$

(2) $f_n \to f$ uniformly *on A if for each $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, such that*

$$|f_n(x) - f(x)| < \varepsilon \qquad (n \geq N, x \in A).$$

Obviously, uniform convergence implies pointwise convergence, but the reciprocal is false, as we will see in Section 3.2. From the definition of uniform convergence the following equivalent characterization can be stated (see, for instance [3]).

Theorem 3.2 (Criterion of the Supremum). *Let $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be functions on X. Then, the sequence $f_n \to f$ uniformly on $A \subset X$ if and only if*

$$\sup_{x \in A} |f_n(x) - f(x)| \to 0 \qquad (n \to \infty).$$

These two modes of convergence only rely on the good structure of $\mathbb{K}$. But if we also care about some structure on X (as for example, a measure one), we get more modes of convergence, such as convergence in measure, almost uniform convergence, etc.

Definition 3.3. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Let* $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ *be measurable functions and* $A \in \mathcal{M}$*. We say that:*

(1) $f_n \to f$ pointwise almost everywhere (a.e.) *on* A*, if there exists a measurable subset* $E \subset A$ *with* $\mu(E) = 0$*, such that* $f_n \to f$ *pointwise on* $A \setminus E$*.*

(2) $f_n \to f$ uniformly almost everywhere (a.e.) *on* A*, if there exists a measurable subset* $E \subset A$ *with* $\mu(E) = 0$*, such that* $f_n \to f$ *uniformly on* $A \setminus E$*.*

(3) $f_n \to f$ almost uniformly *on* A*, if for every* $\varepsilon > 0$*, there exists a measurable set* $E \subset A$ *with* $\mu(E) < \varepsilon$ *such that* $f_n \to f$ *uniformly on* $A \setminus E$*.*

(4) $f_n \to f$ in measure *on* A*, if for every* $\varepsilon > 0$*, we have that*

$$\mu\big(\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}\big) \to 0 \qquad (n \to \infty).$$

In the same way that uniform convergence implies pointwise convergence, we have the same relation with uniform a.e. convergence and pointwise a.e. convergence. Furthermore, uniform convergence clearly implies uniform a.e. convergence, and pointwise convergence implies the pointwise a.e. convergence, just by choosing $E = \varnothing$, that is, we have the following (strict) implications:

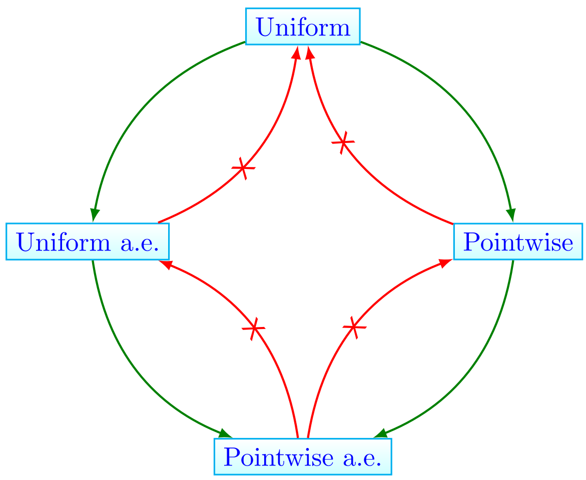


Figure 3.1: Relations between pointwise and uniform convergences

Yet, the reciprocal of this implications are not true (see Section 3.2 for counterexamples). In fact, when we speak about convergence a.e. we only have uniqueness of the limit if we understand it as for the exception of a measure zero subset.

Now, we would like to see if there is possible to find a relation among these new modes of convergence, as we have done with the uniformly and pointwise convergence. Recall that given a sequence $(E_n)_n \subset \mathcal{M}$, we define the limit superior and the limit inferior, respectively, as

$$\limsup_{n\to\infty} E_n := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n, \qquad \liminf_{n\to\infty} E_n := \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} E_n,$$

which are both measurable sets.

Theorem 3.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be measurable functions and $A \in \mathcal{M}$. We have the following implications:*

(1) *If $f_n \to f$ uniformly a.e. on A, then $f_n \to f$ almost uniformly on A.*

(2) *If $f_n \to f$ almost uniformly on A, then*

(2.1) *$f_n \to f$ pointwise a.e. on A;*

(2.2) *$f_n \to f$ in measure on A.*

Proof.

(1) Since $f_n \to f$ uniformly a.e. on A we have that there exists $E \in \mathcal{M}$ with $\mu(E) = 0$ such that $f_n \to f$ uniformly on $A \setminus E$. So, for any $\varepsilon > 0$ we have that $\mu(E) = 0 < \varepsilon$, and $f_n \to f$ uniformly on $A \setminus E$. Hence, $f_n \to f$ almost uniformly on A.

(2.1) Now, let's suppose that $f_n \to f$ almost uniformly on A. Then, for every $N \in \mathbb{N}$ there exists a measurable set $E_N \subset A$ with $\mu(E_N) < \frac{1}{2^N}$ such that $f_n \to f$ uniformly on $A \setminus E_N$.

Consider the set $F = \limsup_{N\to\infty} E_N$ and denote by $F_m := \bigcup_{N=m}^{\infty} E_N$, so $F = \bigcap_{m=1}^{\infty} F_m$.

Thanks to the countable subadditivity of the measure μ we obtain that

$$\mu(F_m) \leq \sum_{N=m}^{\infty} \mu(E_N) < \sum_{N=m}^{\infty} \frac{1}{2^N} = \frac{1}{2^{m-1}}.$$

Hence, $\mu(F) \leq \mu(F_m) < \frac{1}{2^{m-1}}$ for every $m \in \mathbb{N}$, which leads to $\mu(F) = 0$.

Furthermore, we also know that

$$A \setminus F = \liminf_{N\to\infty} (A \setminus E_N).$$

Since $f_n \to f$ uniformly on $A \setminus E_N$, and then, in particular, also converges pointwise to f on each $A \setminus E_N$, we conclude that $f_n \to f$ pointwise on $A \setminus F$.

(2.2) Now let's again assume that $f_n \to f$ almost uniformly on A. For the convergence in measure, given any $\varepsilon > 0$ we have to prove that

$$\mu\big(\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}\big) \to 0 \qquad (n \to \infty).$$

For this, fix $\varepsilon > 0$ and take $\delta > 0$. Since $f_n \to f$ almost uniformly on A, there exists some set $E \in \mathcal{M}$ with $\mu(E) < \delta$ such that $f_n \to f$ uniformly on $A \setminus E$.

Thus, for the prefixed $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for any $n \geq N$ and $x \in A \setminus E$. But this would imply, in particular, that:

$$\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\} \subset E \qquad \text{for every } n \geq N,$$

and hence

$$\mu\big(\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}\big) \leq \mu(E) < \delta.$$

Thus, we have shown that given any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have that

$$\mu\big(\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}\big) < \delta,$$

that is,

$$\mu\big(\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\}\big) \to 0 \qquad (n \to \infty),$$

and the convergence in measure is obtained. □

3.2 Counterexamples on the reciprocals

The reciprocals of the different implications among the modes of convergence that have been introduced are not true in general. This will be clarified during this Section with appropiated counterexamples on the modes of convergence. We start with the easiest one, namely that pointwise convergence does not imply uniform convergence.

Example 3.5 (Pointwise $\not\Rightarrow$ Uniform). Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ be the sequence of functions given by

$$f_n(x) := \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N}).$$

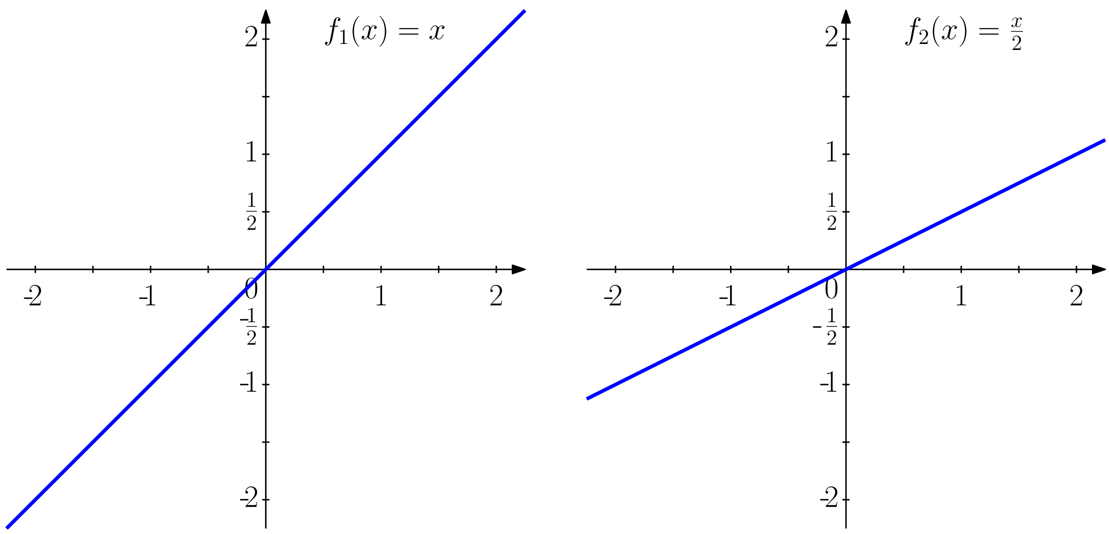


Figure 3.2: First terms of the sequence $f_n(x) = \dfrac{x}{n}$

Firstly, we have that $f_n(x) \to 0 =: f(x)$ $(n \to \infty)$ pointwise on $\mathbb{R}$. Yet, if we apply Theorem 3.2, we obtain that

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \frac{|x|}{n} \geq 1,$$

which means that we cannot have uniformly convergence to the null function on $\mathbb{R}$.

Although this is of the most typical examples of a sequence of functions converging pointwise (to zero) but not uniformly, we want to point out a different example, since it will be the germ of the proofs of some results in Chapter 4. Recall that if $(X, \mathcal{M}, \mu)$ is a measure space and $A \subset X$, the characteristic function of A is the function $\chi_A : X \to \mathbb{R}$ defined as:

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \neq A. \end{cases}$$

Clearly, χ_A is a measurable function if and only if $A \in \mathcal{M}$.

Example 3.6 (Pointwise $\not\Rightarrow$ Uniform). For fixed $n \in \mathbb{N}$, denote by $E_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$. Let $f_n : [0,1] \to \mathbb{R}$ $(n \in \mathbb{N})$ be the sequence of functions given by

$$f_n(x) := \chi_{E_n}(x) \qquad (x \in [0,1], n \in \mathbb{N}).$$

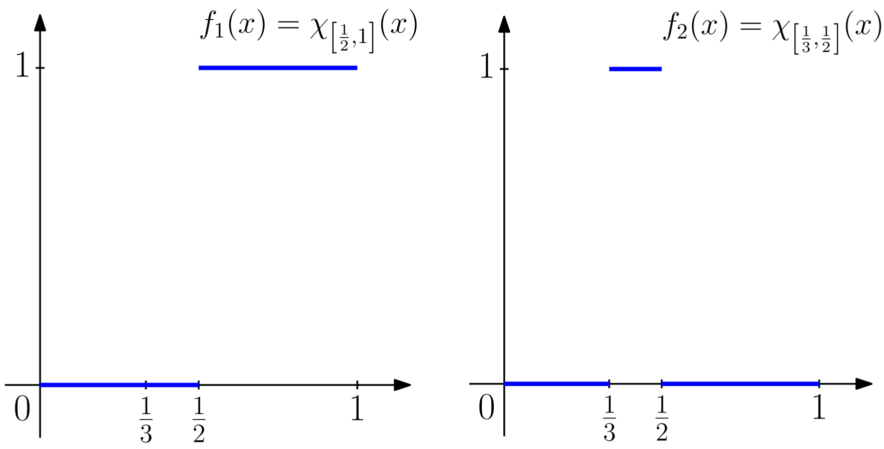


Figure 3.3: First terms of the sequence $f_n(x) = \chi_{\left[\frac{1}{n+1},\frac{1}{n}\right]}(x)$

Clearly, we have that $f_n(x) \to 0 =: f(x)$ $(n \to \infty)$ pointwise on $[0,1]$, because $f_n(0) = 0$ for all $n \in \mathbb{N}$, and for $x \in (0,1]$ there is $N_x \in \mathbb{N}$ such that $\frac{1}{N_x} < x$, thus $f_n(x) = 0$ for all $n \geq N_x$. However,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |\chi_{E_n}(x)| = 1,$$

and an application of Theorem 3.2 gives us the non-uniform convergence to the null function on $[0,1]$.

Now we will focus on the modes of convergence that arises from the framework of measures spaces. In order to present the different examples we will consider the measure space given by $(\mathbb{R}, \mathcal{L}, m)$, where $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets and m the Lebesgue measure on $\mathbb{R}$.

From Theorem 3.4 we obtained that almost uniform convergence is a mode of convergence between pointwise a.e. and uniformly a.e. convergence, but the reciprocal of none of these implications are true. This will be shown in the next two examples.

Example 3.7 (Pointwise a.e $\not\Rightarrow$ Almost uniform). Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ be the sequence of functions given by

$$f_n(x) := \frac{x}{n}\chi_{\mathbb{R}\setminus\mathbb{Q}}(x) + \chi_{\mathbb{Q}}(x), \qquad (x \in \mathbb{R}, n \in \mathbb{N}).$$

Clearly, we have that $f_n(x) \to 0$ pointwise on $\mathbb{R} \setminus \mathbb{Q}$, and since $m(\mathbb{Q}) = 0$, we can state the pointwise a.e. convergence of the sequence of functions $(f_n)_n$ to 0 on $\mathbb{R}$.

Now we will prove that the almost uniform convergence to the null function cannot be achieved. For this, let $E \in \mathcal{L}$ be any measurable set with $m(E) < +\infty$. Then, we have that $m(\mathbb{R}\setminus(E\cup\mathbb{Q})) = +\infty$. Hence, for any $n \in \mathbb{N}$, $m((\mathbb{R}\setminus(E\cup\mathbb{Q}\cup[-n,n])) = +\infty$, and in particular, there exists $x_n \in \mathbb{R} \setminus (E \cup \mathbb{Q})$ with $|x_n| > n$, so

$$\sup_{x\in\mathbb{R}\setminus E} |f_n(x)| \geq |f_n(x_n)| = \frac{|x_n|}{n} > 1.$$

Thus, $(f_n)_n$ cannot converge uniformly to 0 on $\mathbb{R}\setminus E$, which means that the almost uniformly convergence of $(f_n)_n$ to 0 on $\mathbb{R}$ cannot be obtained.

Example 3.8 (Almost uniformly $\not\Rightarrow$ Uniformly a.e). For fixed $n \in \mathbb{N}$, denote by $E_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$. Let $f_n : \mathbb{R} \to \mathbb{R}$ ($n \in \mathbb{N}$) be the sequences of functions given by

$$f_n(x) := \chi_{E_n}(x), \qquad (x \in \mathbb{R}, n \in \mathbb{N}).$$

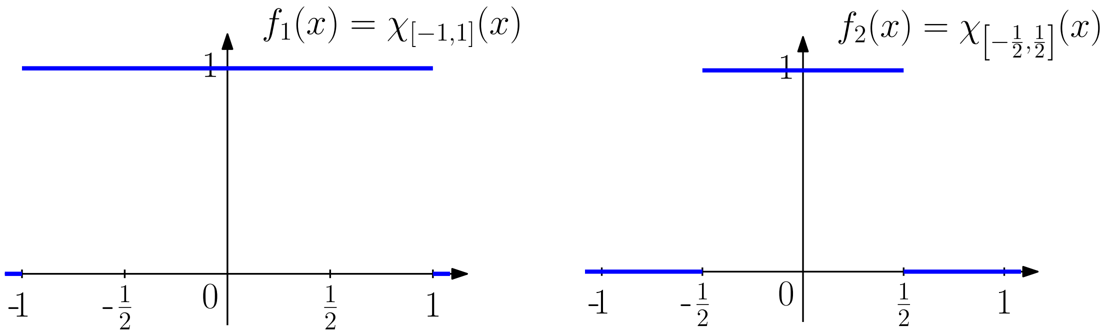


Figure 3.4: First terms of the sequence $f_n(x) = \chi_{[-\frac{1}{n},\frac{1}{n}]}(x)$

Trivially $f_n \to \chi_{\{0\}}$ pointwise on $\mathbb{R}$. Let's see that $f_n \to \chi_{\{0\}}$ almost uniformly but not uniformly a.e. on $\mathbb{R}$. Fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that $\dfrac{2}{N} \leq \varepsilon$, so that $m(E_N) = m\left(\left[-\frac{1}{N}, \frac{1}{N}\right]\right) = \frac{2}{N} < \varepsilon$.

Furthermore, for every $n \geq N$ it holds that $E_n \subset E_N$, and then $f_n(x) = 0$ for all $x \in \mathbb{R} \setminus E_N$. Thus, $f_n \to 0$ uniformly on $\mathbb{R} \setminus E_N$, and f_n converges almost uniformly to $\chi_{\{0\}}$ on $\mathbb{R}$ (observe that $\chi_{\{0\}} = 0$ on $\mathbb{R} \setminus E_N$).

On the other hand, if $f_n \to \chi_{\{0\}}$ uniformly a.e. on $\mathbb{R}$, there is $E \in \mathcal{L}$ with $m(E) = 0$ such that $f_n \to 0$ uniformly on $\mathbb{R} \setminus E$. In this case, for $\varepsilon \in (0, 1)$, there is $N \in \mathbb{N}$ such that for any $n \geq N$, $|f_n(x)| < \varepsilon < 1$ if $x \in \mathbb{R} \setminus E$. But $f_n(x) = \chi_{E_n}(x)$, so this implies that $|f_n(x)| = 0$, and, hence, $x \in \mathbb{R} \setminus E_n$. Thus

$$\mathbb{R} \setminus E \subset \bigcap_{n \geq N} (\mathbb{R} \setminus E_n) = \mathbb{R} \setminus E_N,$$

and $E_N \subset E$, which is a contradiction with $m(E) = 0$.

F. Riesz showed in 1909, and at the same time he introduced the convergence in measure, the existence of a relation between the convergence in measure and the pointwise a.e. convergence, as next the result states (see [48], [33] or [45, Theorem 21.9]).

Theorem 3.9. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be measurable functions. If $f_n \to f$ $(n \to \infty)$ in measure, then there exists a subsequence $(f_{n_k})_k \subset (f_n)_n$ such that $f_{n_k} \to f$ $(k \to \infty)$ pointwise a.e. on X.*

Since we can already obtain from the convergence in measure of a sequence $(f_n)_n$ to f the existence of some subsequence $(f_{n_k})_k$ such that it converges pointwise a.e. to the function f, our first logical question would be if this convergence could be extended to the whole sequence instead of just only for some subsequence. But the so-called "Typewriter sequence" already states that this cannot be done, that is, in general, the whole sequence $(T_n)_n \subset \mathcal{L}_0([0,1])$ cannot inherit the pointwise a.e. convergence to 0 of some subsequence $(T_{n_k})_k$.

Example 3.10 (**Measure $\not\Rightarrow$ Pointwise a.e. (Typewriter sequence)**)**.** Let divide the unit interval $[0,1]$ into infinitely many subintervals I_n of the form

$$I_n := \left[\frac{j_n}{2^{k_n}}, \frac{j_n + 1}{2^{k_n}}\right],$$

where for each $n \in \mathbb{N}$, the non-negative integers j_n and k_n are uniquely determined by $n = 2^{k_n} + j_n$ and $0 \leq j_n < 2^{k_n}$. We define the "Typewriter sequence" $T_n : [0,1] \to \mathbb{R}$ $(n \in \mathbb{N})$ by

$$T_n(x) := \chi_{I_n}(x) = \chi_{\left[\frac{j_n}{2^{k_n}}, \frac{j_n+1}{2^{k_n}}\right]}(x), \qquad (x \in [0,1], n \in \mathbb{N}). \tag{3.1}$$

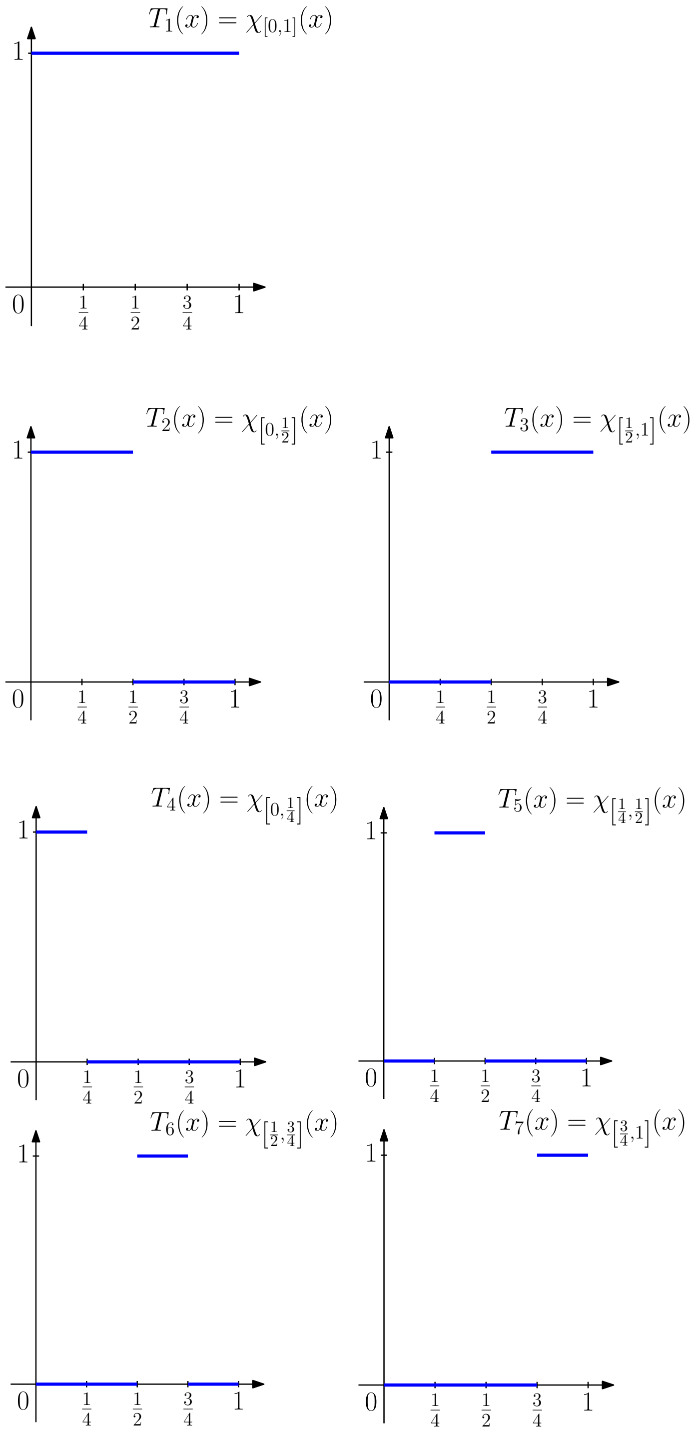


Figure 3.5: Firsts iterations of the "Typewriter sequence"

Consider $([0,1], \mathcal{L}, m)$. It is clear that each T_n is a measurable function for each $n \in \mathbb{N}$. Observe that

$$m\big(\{x \in [0,1] \,:\, T_n(x) \neq 0\}\big) = m(I_n) \to 0 \qquad (n \to \infty).$$

In particular, we obtain the convergence in measure of the "Typewriter sequence" to the null function.

Yet, for any fixed $x \in [0,1]$, the sequence $(T_n(x))_n$ takes infinitely many times the value 0 and infinitely many times the value 1. Hence, the sequence $(T_n(x))_n$ can not be convergent to zero, and thus, the "Typewriter sequence" cannot converge pointwise a.e. on $[0,1]$ to the null function. In fact, the sequence $(T_n)_n$ does not converge pointwise at any point on $[0,1]$.

Observe that as Riesz's Theorem assures, there is a subsequence of $(T_n)_n$ convergent pointwise a.e. to zero, just taking $(T_{n_k})_k$ for $n_k = 2^k$ $(k \in \mathbb{N})$. Indeed, $T_{n_k}(x) \to 0$ $(k \to \infty)$ for all $x \in (0,1]$ and $T_{n_k}(0) = 1$ for every $k \in \mathbb{N}$.

Example 3.11 (Pointwise a.e. $\not\Rightarrow$ Measure). Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ be the sequence of functions given by

$$f_n(x) := \chi_{[n,n+1]}(x) \qquad (x \in \mathbb{R}, n \in \mathbb{N}).$$

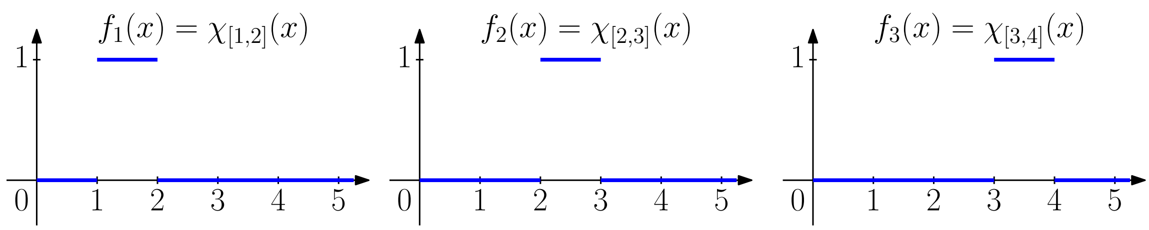


Figure 3.6: First terms of the sequence $f_n(x) = \chi_{[n,n+1]}(x)$

We clearly have that $f_n(x) \to 0$ $(n \to \infty)$ pointwise on $\mathbb{R}$, and hence pointwise a.e. on $\mathbb{R}$. On the other hand, for any $\varepsilon \in (0,1)$ and for any $n \in \mathbb{N}$,

$$m\big(\{x \in \mathbb{R} \,:\, |f_n(x)| \geq \varepsilon\}\big) = m\big([n, n+1]\big) = 1,$$

and $m\big(\{x \in \mathbb{R} : |f_n(x)| \geq \varepsilon\}\big) \not\to 0$ $(n \to \infty)$. Thus, $(f_n)_n$ does not converge to zero in measure.

In order to finish this Chapter, we will provide an useful diagram where all the different relations and implications among the studied concepts of convergence are established.

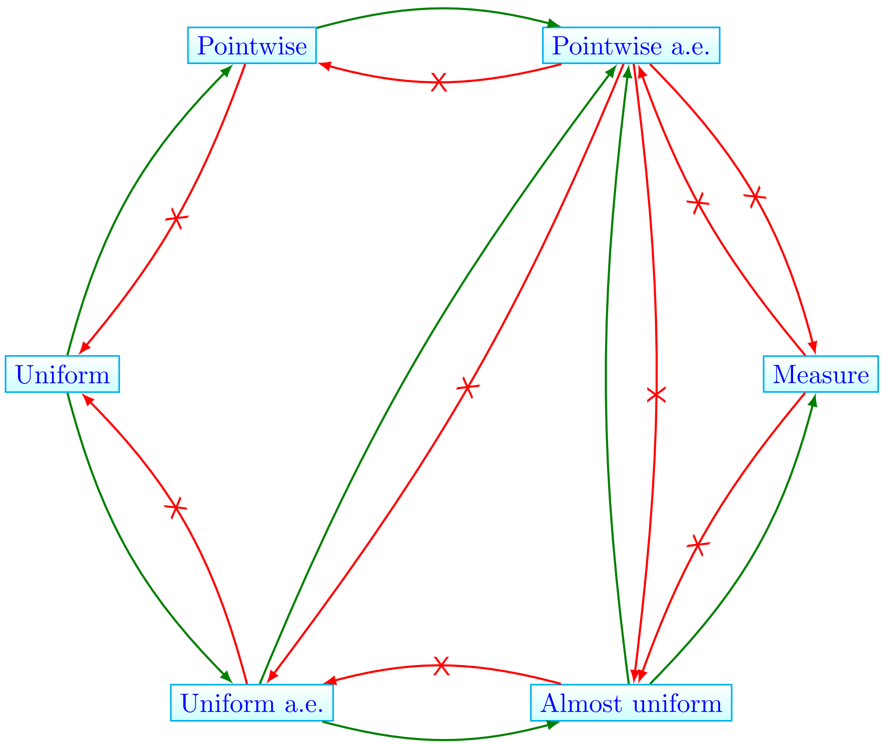


Figure 3.7: Relation among modes of convergences

Chapter 4

Lineability and modes of convergence

In this Chapter we continue with the study of the relations between the modes of convergence. Specifically, we focus our attention on the size of the sets of "counter-examples" of some of these relations. We divide the Chapter in two sections. The first one is devoted to continue the study of the family of sequence of functions that are convergent in measure to zero but not pointwise a.e. on $[0, 1]$, which was already carried out by Araújo *et al.* in [4]. They showed the existence of a dense vector space of dimension $\mathfrak{c}$, and we improve it to the existence of a closed infinite dimensional vector space and a $\mathfrak{c}$-dimensional free algebra. In the second Section, we will study precisely the family of sequences of continuous functions on the unit interval that are pointwise convergent to zero but not uniformly.

4.1 Measure vs. Pointwise a.e. Convergence

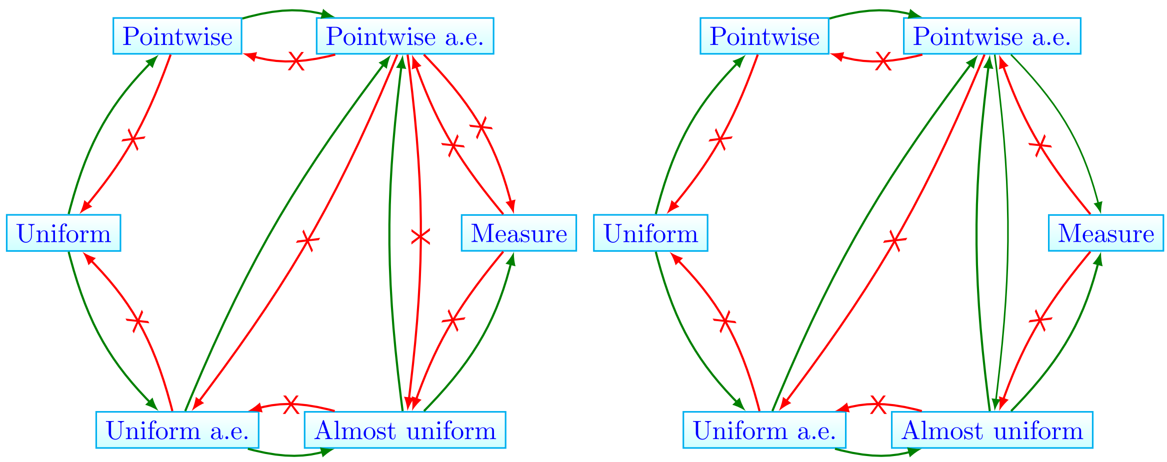
As we have seen in Chapter 3 (see Examples 3.10 and 3.11), there is no relation between convergence in measure and pointwise a.e. convergence. Nonetheless, we have already seen, thanks to Riesz's Theorem (see Theorem 3.9), that the gap between these modes of convergence is not as big as it appears to be.

In the case of a finite dimensional space X, this relation can be improved, as we can see from the next result (see [46, Theorem 8.3]) due to Egorov.

Theorem 4.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be measurable functions. Then $f_n \to f$ $(n \to \infty)$ pointwise a.e. on X if and only if $f_n \to f$ $(n \to \infty)$ almost uniformly on X.*

From Theorem 3.4 we have that almost uniform convergence implies both pointwise a.e. convergence and convergence in measure, where the reciprocal are not true in general (see Example 3.7). However, in the finite dimensional case, Egorov's Theorem states one of the above reciprocal, namely that pointwise a.e. convergence implies almost uniform convergence. In that case, from the relations among these modes of convergence, we also infer that pointwise a.e. convergence implies convergence in measure.

In this Section we focus our attention on the unit interval, $\mathcal{M} = \mathcal{L}$ and $\mu = m$, that is, $X = [0, 1]$ will be a space of finite measure, and, thanks to Egorov's Theorem, pointwise a.e. convergence is stronger than convergence in measure. Moreover, in the setting of the space $L^0([0, 1])$ of all (classes of) functions $f : [0, 1] \to \mathbb{K}$ that are Lebesgue measurable, the Typewritter sequence $(T_n)_n \subset L^0([0, 1])$ (see Example 3.10), gives us an example of a sequence of functions converging in measure to 0 but not pointwise a.e. on $[0, 1]$, so it is natural to ask if this is just an isolated phenomenon, or if it is possible to construct an algebraic structure for these sequences of functions.



(a) $\mu(X) = \infty$ **(b)** $\mu(X) < \infty$

Figure 4.1: Relations among convergences in finite and infinite dimensional spaces

We will start by defining a proper metric in $L^0([0,1])$ that will be useful later (see [4], [45]). Consider the application $\rho : L^0([0,1]) \times L^0([0,1]) \to [0,+\infty)$ given by

$$\rho(f,g) = \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\, dx.$$

We have that $\rho(f,g) \geq 0$ for every $f,g \in L^0([0,1])$, and the equality holds if and only if

$$\frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} = 0 \qquad \text{a.e. on } [0,1],$$

or, equivalently, $f = g$ a.e. on $[0,1]$, and in the setting of classes of functions, $f = g$.

It is evident that $\rho(f,g) = \rho(g,f)$ for every $f,g \in L^0([0,1])$, so it only remains to prove the triangle inequality. For this, observe that as done in Section 1.3, the function $p(x) = \dfrac{x}{1+x}$ is strictly increasing on $[0,+\infty)$. So, since $|f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|$, we obtain that $p\big(|f(x)-g(x)|\big) \leq p\big(|f(x)-h(x)|+|h(x)-g(x)|\big)$, which leads to

$$\rho(f,g) = \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx \leq \int_0^1 \frac{|f(x)-h(x)|+|h(x)-g(x)|}{1+|f(x)-h(x)|+|h(x)-g(x)|}\,dx$$

$$\leq \int_0^1 \frac{|f(x)-h(x)|}{1+|f(x)-h(x)|}\,dx + \int_0^1 \frac{|h(x)-g(x)|}{1+|h(x)-g(x)|}\,dx$$

$$= \rho(f,h) + \rho(h,g).$$

Thus, ρ is a metric over $L^0([0,1])$. Furthermore, the convergence in this metric ρ is precisely the natural convergence for the measurable functions, that is, the convergence in measure.

Lemma 4.2. *Let $\rho : L^0([0,1]) \times L^0([0,1]) \to [0,+\infty)$ be the metric defined as*

$$\rho(f,g) = \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx.$$

Then, $\rho(f_n,f) \to 0$ $(n\to\infty)$ if and only if $f_n \to f$ $(n\to\infty)$ in measure.

Proof. For every $n \in \mathbb{N}$ define the integral I_n by

$$I_n := \int_0^1 \frac{|f_n(x)|}{1+|f_n(x)|}\,dx.$$

Without loss of generality, we can assume that $f=0$. Hence, it suffices to prove that $I_n \to 0$ $(n\to\infty)$ if and only if, for every $\varepsilon>0$,

$$m\big(\{x\in[0,1] \,:\, |f_n(x)| \geq \varepsilon\}\big) \to 0 \qquad (n\to\infty).$$

Firstly, if $f_n \to 0$ $(n\to\infty)$ in measure, by definition we have that, for fixed $\varepsilon>0$,

$$m\big(\{x\in[0,1] \,:\, |f_n(x)| \geq \varepsilon\}\big) \to 0 \qquad (n\to\infty).$$

Hence,

$$0 \le I_n = \int_{[0,1]} \frac{|f_n(x)|}{1+|f_n(x)|}\,dx = \int_{|f_n|\ge\varepsilon} \frac{|f_n(x)|}{1+|f_n(x)|}\,dx + \int_{|f_n|<\varepsilon} \frac{|f_n(x)|}{1+|f_n(x)|}\,dx$$

$$\le \int_{|f_n|\ge\varepsilon} 1\,dx + \int_{|f_n|<\varepsilon} \frac{|f_n(x)|}{1+|f_n(x)|}\,dx$$

$$= m\big(\{x\in[0,1] : |f_n(x)|\ge\varepsilon\}\big) + \int_{|f_n|<\varepsilon} \frac{|f_n(x)|}{1+|f_n(x)|}\,dx$$

But the function $p(x) = \dfrac{x}{1+x}$ is increasing on $[0,+\infty)$, so

$$p(|f_n(x)|) = \frac{|f_n(x)|}{1+|f_n(x)|} < \frac{\varepsilon}{1+\varepsilon} = p(\varepsilon)$$

for every $x\in[0,1]$ such that $|f_n(x)|<\varepsilon$, and we have

$$0 \le I_n \le m\big(\{x\in[0,1] : |f_n(x)|\ge\varepsilon\}\big) + \frac{\varepsilon}{1+\varepsilon}\cdot m\big(\{x\in[0,1] : |f_n(x)|<\varepsilon\}\big)$$

$$\le m\big(\{x\in[0,1] : |f_n(x)|\ge\varepsilon\}\big) + \frac{\varepsilon}{1+\varepsilon}.$$

Now, by taking limit as $n\to\infty$, we obtain

$$0 \le \liminf_{n\to\infty} I_n \le \limsup_{n\to\infty} I_n \le \frac{\varepsilon}{1+\varepsilon}$$

for each $\varepsilon>0$, thus $I_n\to 0$ $(n\to\infty)$.

Reciprocally, if $I_n\to 0$ $(n\to\infty)$, and again by the increasing condition of the function $p(x)$, for each $\varepsilon>0$, we have

$$I_n = \int_{[0,1]} \frac{|f_n(x)|}{1+|f_n(x)|}\,dx \ge \int_{|f_n|\ge\varepsilon} \frac{|f_n(x)|}{1+|f_n(x)|}\,dx \ge \int_{|f_n|\ge\varepsilon} \frac{\varepsilon}{1+\varepsilon}\,dx$$

$$= \frac{\varepsilon}{1+\varepsilon}\cdot m\big(\{x\in[0,1] : |f_n(x)|\ge\varepsilon\}\big).$$

So, by taking limit when $n\to\infty$, we have that

$$m\big(\{x\in[0,1] : |f_n(x)|\ge\varepsilon\}\big)\to 0 \qquad (n\to\infty),$$

and we obtained the convergence in measure as desired. □

From now on, in this Section we will work with sequences of (classes of) functions on the unit interval, that is, our framework will be the space $L^0([0,1])^{\mathbb{N}}$. Recall from Section 1.3 that the metric D defined on $L^0([0,1])^{\mathbb{N}}$ by

$$D((f_n)_n, (g_n)_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho(f_n, g_n)}{1 + \rho(f_n, g_n)}$$

is compatible with the product topology in $L^0([0,1])^{\mathbb{N}}$. Recall that

$$\begin{aligned} c_0(L^0([0,1])) &= \{(f_n)_n \in L^0([0,1])^{\mathbb{N}} \,:\, \rho(f_n, 0) \to 0 \qquad (n \to \infty)\} \\ &= \{(f_n)_n \in L^0([0,1])^{\mathbb{N}} \,:\, f_n \to 0 \text{ in measure}\} \end{aligned}$$

Define the family $n\mathcal{P}_{ae}\mathcal{M}([0,1])$ of all sequences of functions in $L^0([0,1])^{\mathbb{N}}$ converging to zero in measure but not pointwise a.e. on $[0,1]$, that is,

$$n\mathcal{P}_{ae}\mathcal{M}([0,1]) := \left\{(f_n)_n \in c_0(L^0([0,1])) \,:\, f_n \not\to 0 \text{ pointwise a.e. on } [0,1]\right\}.$$

In 2017, Araújo *et al.* [4] provided a first result about the topological and linear size of $n\mathcal{P}_{ae}\mathcal{M}([0,1])$. We include here the original proof for the sake of completeness.

Theorem 4.3. *The family $n\mathcal{P}_{ae}\mathcal{M}([0,1])$ is maximal dense-lineable in $L^0([0,1])^{\mathbb{N}}$ with the product topology.*

Proof. Let $(T_n)_n \in L^0([0,1])^{\mathbb{N}}$ be the "Typewriter sequence" defined in Example 3.10. Recall that we already know that $(T_n)_n \in n\mathcal{P}_{ae}\mathcal{M}([0,1])$. In order to find our desired vector space we will start by extending each T_n from the unit interval $[0,1]$ into the real line $\mathbb{R}$, just by defining $T_n(x) = 0$ for every $x \notin [0,1]$. Now, for every $n \in \mathbb{N}$, consider the translated-dilated sequence $(T_{n,t})_n$ given by

$$T_{n,t}(x) := T_n(2(x-t)), \qquad \left(0, \tfrac{1}{2}\right), x \in \mathbb{R}).$$

It is clear that $(T_{n,t})_n \in L^0([0,1])^{\mathbb{N}}$, and that also converges to zero in measure for every $t \in (0, 1/2)$. Let us consider the vector space M given by

$$M := \operatorname{span}\left\{(T_{n,t})_n \,:\, t \in \left(0, \tfrac{1}{2}\right)\right\}.$$

In a first step we are going to show that $M \subset n\mathcal{P}_{ae}\mathcal{M}([0,1]) \cup \{0\}$, and $\dim(M) = \dim(L^0([0,1])^{\mathbb{N}}) = \mathfrak{c}$ (and the last equality holds because $L^0([0,1])^{\mathbb{N}}$ is a complete metrizable topological vector space, hence a Baire space). In order to see that the elements of M are linearly independent, assume that there exist $0 < t_1 < t_2 < \cdots < t_s < \frac{1}{2}$ and real numbers $c_1, c_2, \ldots, c_s$ not simultaneously zero (take $c_s \neq 0$ for example), such that

$$c_1 T_{n,t_1} + c_2 T_{n,t_2} + \cdots + c_s T_{n,t_s} = 0, \qquad (n \in \mathbb{N}).$$

In particular, by choosing $n = 1$ we have that

$$c_1 T_{1,t_1}(x) + c_2 T_{1,t_2}(x) + \cdots + c_s T_{1,t_s}(x) = 0,$$

for almost every $x \in [0,1]$. Since

$$T_{1,t}(x) = T_1(2(x-t)) = \chi_{[0,1]}(2(x-t)) = \chi_{[t,t+1/2]}(x),$$

we have that the last equation can be written as

$$c_1 \chi_{\left[t_1, t_1 + \frac{1}{2}\right]} + c_2 \chi_{\left[t_2, t_2 + \frac{1}{2}\right]} + \cdots + c_s \chi_{\left[t_s, t_s + \frac{1}{2}\right]} = 0, \qquad \text{for almost every } x \in [0,1].$$

But, for (almost) every $x \in \left(t_{s-1} + \frac{1}{2}, t_s + \frac{1}{2}\right]$, we would obtain that

$$c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_{s-1} \cdot 0 + c_s \cdot 1 = c_s = 0,$$

which would be a contradiction with $c_s \neq 0$. This shows the required linear independence. As $\dim\left(0, \frac{1}{2}\right) = \mathfrak{c}$, we obtain that $\dim(M) = \mathfrak{c}$. Moreover, since $L^0([0,1])^{\mathbb{N}}$ is a topological vector space endowed with the topology of convergence in measure, and $(T_{n,t})_n$ converges to zero in measure for each $t \in \left(0, \frac{1}{2}\right)$, we get that every element

$$(F_n)_n := (c_1 T_{n,t_1} + c_2 T_{n,t_2} + \cdots + c_s T_{n,t_s})_n \in M,$$

is a sequence converging to zero in measure. Next, fix any sequence $(F_n)_n \in M$ as above, with $0 < t_1 < t_2 < \cdots < t_s < \frac{1}{2}$ and $c_s \neq 0$. For all $x \in \left(t_{s-1} + \frac{1}{2}, t_s + \frac{1}{2}\right]$, we have that

$$F_n(x) = \sum_{j=1}^{s} c_j T_{n,t_j}(x) = \sum_{j=1}^{s} c_j T_n(2(x-t_j)) = c_s T_n(2(x-t_s)).$$

For each $x \in (t_{s-1}+1/2, t_s+1/2]$, we have $w := 2(x-t_s) \in (2(t_{s-1}-t_s)+1, 1] \subset [0,1]$. Hence, since $c_s \neq 0$ and the sequence $(T_n(w))_n$ does not converge (as a sequence of scalars) for any $w \in [0,1]$, we derive that, for each $x \in \left(t_{s-1}+\frac{1}{2}, t_s+\frac{1}{2}\right]$, the sequence $(F_n)_n$ does not converge pointwise a.e. to zero. This shows that $M \subset n\mathcal{P}_{ae}\mathcal{M}([0,1]) \cup \{0\}$. Thus $n\mathcal{P}_{ae}\mathcal{M}([0,1])$ is maximal lineable.

Moreover, since the set $c_{00}(L^0([0,1]))$ is dense in $(L^0([0,1])^{\mathbb{N}}, D)$ (see Section 1.3), $n\mathcal{P}_{ae}\mathcal{M}([0,1]) \cap c_{00}(L^0([0,1])) = \varnothing$ (because the elements of $c_{00}(L^0([0,1]))$ are pointwise convergent to 0 on $[0,1]$), and $n\mathcal{P}_{ae}\mathcal{M}([0,1])$ is stronger than $c_{00}(L^0([0,1]))$, an application of Theorem 2.7 with $A = n\mathcal{P}_{ae}\mathcal{M}([0,1])$, $B = c_{00}(L^0([0,1]))$ and $\kappa = \mathfrak{c}$ puts an end on the proof. □

Recall that $c_{00}(L^0([0,1])$ is also dense in $c_0(L^0([0,1])$ when endowed with the supremum metric d_∞. In particular, analogously we can state:

Theorem 4.4. *The family $n\mathcal{P}_{ae}\mathcal{M}([0,1])$ is maximal dense-lineable in $(c_0(L^0([0,1])), d_\infty)$.*

4.1.1 Algebrability and spaceability

With the maximal dense-lineability of this family we already have that these sequences of functions maintain a good structure under linear combinations. In the next result we find that it also has a nice algebraic structure. But firstly, we will introduce some useful notation and two technical lemmas.

Let us represent each N-tuple $(r_1, r_2, \cdots, r_N) \in \mathbb{R}^N$ by $\mathbf{r}$, and set $|\mathbf{r}| := r_1 + r_2 + \cdots + r_N$ and $\mathbf{r} \cdot \mathbf{s} := r_1 s_1 + r_2 s_2 + \cdots + r_N s_N$.

Lemma 4.5. *Let $I \subset \mathbb{R}$ be an interval, and let $k_1, k_2, \ldots, k_N$ be mutually different real numbers ($N \in \mathbb{N}$). Then, the set of exponential functions*

$$\{e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_N x}\} \qquad (x \in I)$$

is a linearly independent set of functions.

Proof. Observe that we can assume without loss of generality that $0 \in I$ (if not, just use a translation). Assume that a linear combination of these functions is zero, that is, there exist $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}$ such that

$$\lambda_1 e^{k_1 x} + \lambda_2 e^{k_2 x} + \cdots + \lambda_N e^{k_N x} = 0.$$

If we derive this equation $N-1$ times, we get

$$\lambda_1 k_1^p e^{k_1 x} + \lambda_2 k_2^p e^{k_2 x} + \cdots + \lambda_N k_N^p e^{k_N x} = 0, \qquad (0 \leq p \leq N-1).$$

Now, by taking $x = 0$, we get the following Vandermonde system

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_N & = & 0 \\ \lambda_1 k_1 + \lambda_2 k_2 + \cdots + \lambda_N k_N & = & 0 \\ \vdots & & \vdots \\ \lambda_1 k_1^{N-1} + \lambda_2 k_2^{N-1} + \cdots + \lambda_N k_N^{N-1} & = & 0 \end{cases}$$

Thus, since $\{k_1, k_2, \ldots, k_N\}$ are mutually distinct, the only solution of the above system is the trivial one and we are done. □

Lemma 4.6. *Let $I \subset \mathbb{R}$ be an interval. Let $H \subset (0, +\infty)$ be a $\mathbb{Q}$-linearly independent set. Then, the algebra $\mathcal{A}$ generated by the set of exponential functions*

$$\{e^{-cx} \, : \, c \in H\} \qquad (x \in I)$$

is free.

Proof. Let F be an element of $\mathcal{A} \setminus \{0\}$. Then, there exists a non-zero polynomial P in N variables without constant term and mutually different $c_1, c_2, \ldots, c_N \in H$ such that

$$F(x) = P(e^{-c_1 x}, e^{-c_2 x}, \ldots, e^{-c_N x}).$$

Specifically, there exist a natural number $N \in \mathbb{N}$, a non-empty finite set $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\ldots}, 0)\}$, scalars $\alpha_{\mathbf{j}} \in \mathbb{R} \setminus \{0\}$, for $\mathbf{j} = (j_1, j_2, \ldots, j_N) \in J$, and $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in H^N$ such that

$$F(x) = \sum_{\mathbf{j} \in J} \alpha_{\mathbf{j}} (e^{-c_1 x})^{j_1} \cdot (e^{-c_2 x})^{j_2} \cdots (e^{-c_N x})^{j_N} = \sum_{\mathbf{j} \in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c} \cdot \mathbf{j}) x}$$

Since H is $\mathbb{Q}$-linearly independent and $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\ldots}, 0)\}$, the numbers $\{\mathbf{c} \cdot \mathbf{j} : \mathbf{j} \in J, \mathbf{c} \in H^N\}$ are non-null and mutually different. But Lemma 4.5 assures that the family $\{e^{-(\mathbf{c}\cdot\mathbf{j})x} : \mathbf{j} \in J, \mathbf{c} \in H^N\}$ is linearly independent, so $F \not\equiv 0$ (because $\alpha_{\mathbf{j}} \neq 0$ for all $\mathbf{j} \in J$). □

It is well-known that the field $\mathbb{R}$, as seen as a vector space over $\mathbb{Q}$, has dimension $\mathfrak{c}$. Hence, we always can find a $\mathbb{Q}$-linearly independent set in $\mathbb{R}$ (or even inside $(0, +\infty)$) with cardinal $\mathfrak{c}$.

Theorem 4.7. *The family $n\mathcal{P}_{ae}\mathcal{M}([0,1])$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $H \subset (0, +\infty)$ be a linearly $\mathbb{Q}$-independent set with $\mathrm{card}(H) = \mathfrak{c}$. For each $c \in H$, we define the sequence $f_c = (f_{c,n})_n$ as

$$f_{c,n}(x) := e^{-cx} \cdot T_n(x), \qquad (x \in [0,1], n \in \mathbb{N}),$$

where $(T_n)_n \in L^0([0,1])^{\mathbb{N}}$ is the "Typewriter sequence" defined in Example 3.10. Recall that

$$m\big(\{x \in [0,1] : T_n(x) \neq 0\}\big) \to 0 \qquad (n \to \infty), \tag{4.1}$$

and that $(T_n)_n$ does not converge to 0 pointwise a.e. on $[0,1]$.

Let $\mathcal{B}$ be the algebra generated by the family of sequences $\{f_c : c \in H\}$, that is, $\mathcal{B}$ is the family of all sequences $(F_n)_n$ for which there exists $N \in \mathbb{N}$, mutually different $c_1, c_2, \ldots, c_N \in H$ and a non-zero polynomial P in N real variables without constant term such that

$$F_n = P(f_{c_1,n}, f_{c_2,n}, \ldots, f_{c_N,n}), \qquad (n \in \mathbb{N}).$$

Specifically, there exist a natural number $N \in \mathbb{N}$, a non-empty finite set $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\ldots}, 0)\}$, scalars $\alpha_{\mathbf{j}} \in \mathbb{R}\setminus\{0\}$, for $\mathbf{j} = (j_1, j_2, \ldots, j_N) \in J$, and $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in H^N$

such that

$$\begin{aligned} F_n(x) &= \sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} f_{c_1,n}(x)^{j_1} \cdots f_{c_N,n}(x)^{j_N} \\ &= \sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c}\cdot\mathbf{j})x} T_n(x)^{|\mathbf{j}|} = \left(\sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c}\cdot\mathbf{j})x} \right) T_n(x), \end{aligned} \tag{4.2}$$

where in the last equality is crucial the fact that $T_n(x)$ is a characteristic function, so $T_n(x)^\beta = T_n(x)$ for any $\beta > 0$.

For each subset $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\ldots}, 0)\}$ and each $\mathbf{c} \in H^N$, consider the function $\varphi_{\mathbf{c},J} : [0,1] \to \mathbb{R}$ given by

$$\varphi_{\mathbf{c},J}(x) := \sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c}\cdot\mathbf{j})x}.$$

Since H is a $\mathbb{Q}$-linearly independent set and $\alpha_{\mathbf{j}} \neq 0$ ($\mathbf{j} \in J$), by Lemma 4.6, $\varphi_{\mathbf{c},J}(x)$ is a non-null function. So, by (4.2), F_n is non-null for each $n \in \mathbb{N}$, and the algebra $\mathcal{B}$ is free.

It only rests to prove that every non-zero element of our algebra $\mathcal{B}$ converges in measure to zero but not pointwise a.e. on $[0,1]$. Observe that the sequence $(F_n(x))_n = (\varphi_{\mathbf{c},J}(x)T_n(x))_n$ converges to zero in measure because, for any $\varepsilon > 0$, we have the following inclusion

$$\{x \in [0,1] \,:\, |\varphi_{\mathbf{c},J}(x)T_n(x)| > \varepsilon\} \subset \{x \in [0,1] \,:\, T_n(x) \neq 0\}.$$

By (4.1), the measures of the above sets go to zero when $n \to \infty$, and then $F_n \to 0$ in measure. Finally, as $\varphi_{\mathbf{c},J}(x)$ is a finite linear combination of exponentials, we have that $\varphi_{\mathbf{c},J}(x) = 0$ for finitely many points (specifically, at most $\mathrm{card}(J) - 1$ points, see for instance, [47, p. 46]). Hence, (4.2) and the non-pointwise a.e. convergence of $(T_n)_n$ to zero give us that $(F_n)_n$ does not converge to zero pointwise a.e. on $[0,1]$. □

Observe that while searching for a free algebra inside our family of sequences of functions, we did not need to pay much attention to the topological structure of

$L^0([0,1])^{\mathbb{N}}$. This is not the case anymore if we speak about dense-lineability, as Araújo *et al.* did previously in [4]. Now, by taking into account the topological structure of $L^0([0,1])^{\mathbb{N}}$, we are able to get spaceability of our family of sequences of functions.

Theorem 4.8. *The family $n\mathcal{P}_{ae}\mathcal{M}([0,1])$ is spaceable in $(c_0(L^0([0,1])), d_\infty)$.*

Proof. The proof presented here is a constructive proof, which also heavily relies on the "Typewriter sequence" defined previously on Example 3.10.

First, we start by dividing the set $\mathbb{N}$ into infinitely many strictly increasing and disjoints subsequences $(i(k,n))_{n\in\mathbb{N}}$ $(k \in \mathbb{N})$ such that

$$\mathbb{N} = \bigcup_{k,n\in\mathbb{N}} \{i(k,n)\}.$$

We can take for instance the subsequences $(i(k,n))_n$ given by

$$i(k,n) := \frac{k(k+1)}{2} + (n-1)k, \qquad (k,n \in \mathbb{N}).$$

For each $k \in \mathbb{N}$, define the sequence $T(k) = (T(k,n))_n$ as follows

$$T(k,n)(x) := \begin{cases} \chi_{\left[\frac{j}{2^{i(k,m)}}, \frac{j+1}{2^{i(k,m)}}\right]}(x) & \text{if } n = j + 2^{i(k,m)}, 0 \le j < 2^{i(k,m)}, \\ 0 & \text{elsewhere.} \end{cases}$$

Roughly speaking, for fixed $k \in \mathbb{N}$, we preserve every term of the "Typewriter sequence" where the support has length $\frac{1}{2^{i(k,m)}}$ for $m \in \mathbb{N}$, and change the rest to be 0. Similarly as the "Typewriter sequence", it is straightforward that every sequence $T(k)$ is convergent to zero in measure, since for every $\varepsilon > 0$ it holds

$$\{x \in [0,1] \, : \, |T(k,n)(x)| > \varepsilon\} \subset \{x \in [0,1] \, : \, T(k,n)(x) \neq 0\},$$

and the measure of the lasts sets goes to zero when $n \to \infty$. Moreover, by construction of the sequences $T(k)$, given any $x \in [0,1]$, there are infinitely many terms of $T(k)(x)$ where the sequence takes the value 0, and infinitely many terms where it takes the value 1, thus making it impossible for the sequence $(T(k,n)(x))_n$ to be convergent to zero.

By construction, if $k \neq k'$, then the sequences $T(k)$ and $T(k')$ cannot be both simultaneously non-zero. This implies that, if we choose scalars $\lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{R}$, and consider the linear combination

$$\lambda_1 T(k_1) + \lambda_2 T(k_2) + \cdots + \lambda_N T(k_N) = 0,$$

then for any $1 \leq j_0 \leq N$ and $x \in [0,1]$ we can always find some $n_{j_0} \in \mathbb{N}$ such that $T(k_{j_0}, n_{j_0})(x)$ is non-zero but $T(k_j, n_{j_0}) = 0$ for $j \neq j_0$. So, writing the linear combination at the n_{j_0}-coordinate, we get that $\lambda_{j_0} T(k_{j_0}, n_{j_0})(x) = 0$, whence $\lambda_{j_0} = 0$, and we get that the set $\{T(k) : k \in \mathbb{N}\}$ is linearly independent.

Now, in order to prove the spaceability of our family of sequences of functions we need to find some closed infinite dimensional vector space inside our family. For this, let us define

$$M := \overline{\operatorname{span}}\{T(k) : k \in \mathbb{N}\}.$$

It is clear that M is a closed infinite dimensional subspace of $(c_0(L^0([0,1])), d_\infty)$.

It only rests to prove the non-pointwise a.e. convergence to zero of the members of M. For that purpose, observe that every non-zero member of M is a finite or infinite linear combination of sequences $T(k)$. More precisely, if $F \in M \setminus \{0\}$, there exists a subset $J \subset \mathbb{N}$ and scalars $\lambda_j \in \mathbb{R} \setminus \{0\}$ for every $j \in J$, such that we can write

$$F = \sum_{j \in J} \lambda_j T(k_j).$$

Fix $j_0 \in J$ and let

$$J_0 := \left\{2^{i(k_{j_0}, m)} + j \, : \, m \in \mathbb{N}, 0 \leq j < 2^{i(k_{j_0}, m)}\right\}.$$

By construction, $F_n = \lambda_{j_0} T(k_{j_0}, n)$ for every $n \in J_0$, and so, for fixed $x \in [0,1]$, there are infinitely many natural numbers (at least every number in J_0) such that

$F_n(x) = \lambda_{j_0} \neq 0$, and infinitely many natural numbers such that $F_n(x) = 0$. Hence $(F_n)_n$ is not pointwise a.e. convergent to zero. $\square$

4.2 Pointwise vs. Uniform Convergence

4.2.1 Definitions and first examples

In this Section we focus on two well-known different types of convergence, the pointwise and the uniform convergence on $[0,1]$.

The fact that these two types of convergence are the most commonly studied when working with sequences of functions allows us to provide easily many examples of sequences of functions converging pointwise but not uniformly on $[0,1]$. For instance, we can just recall the sequence $(f_n)_n \subset L^0([0,1])$ defined on Example 3.6 given by

$$f_n(x) = \chi_{E_n}(x), \qquad \text{for every } n \in \mathbb{N},$$

where $E_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$. Observe that, in particular, $\limsup\limits_{n\to\infty} E_n = \varnothing$.

This is not only a coincidence for the sequence of functions $(f_n)_n$, but rather a shared property of many sequences of scalar multiples of characteristics functions that are pointwise convergent to zero but not uniformly convergent on $[0,1]$.

Proposition 4.9. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $(\alpha_n)_n$ be a sequence of non-zero real numbers such that either $(\alpha_n)_n \in c_0$ or there exists $M > 0$ such that $|\alpha_n| > M$ for n large enough. Let $E_n \in \mathcal{M} \setminus \{\varnothing\}$ and $f_n = \alpha_n \chi_{E_n}$ $(n \in \mathbb{N})$. Then:*

(1) *$f_n \to 0$ pointwise on X if and only if $(\alpha_n)_n \in c_0$ or $\limsup\limits_{n\to\infty} E_n = \varnothing$.*

(2) *$f_n \to 0$ pointwise a.e. on X if and only if $(\alpha_n)_n \in c_0$ or $\mu\left(\limsup\limits_{n\to\infty} E_n\right) = 0$.*

(3) *$f_n \to 0$ uniformly on X if and only if $(\alpha_n)_n \in c_0$.*

Proof.

(1) Suppose that $f_n \to 0$ pointwise on X and $\limsup_{n\to\infty} E_n \neq \varnothing$. Let $x_0 \in \limsup_{n\to\infty} E_n$. Then, for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ with $m_n \geq n$ and $x_0 \in E_{m_n}$. So,

$$|\alpha_{m_n}| = |f_{m_n}(x_0)| \to 0 \qquad (n \to \infty). \tag{4.3}$$

Hence $(\alpha_n)_n$ cannot be far from zero, and by hypothesis, must converge to 0.

Conversely, for any $x \in X$, $|f_n(x)| = |\alpha_n|\chi_{E_n}(x) \leq |\alpha_n|$. So, if $(\alpha_n)_n \in c_0$, then $f_n(x) \to 0$. On the other hand, if $\limsup_{n\to\infty} E_n = \varnothing$, then there is $n_0 \in \mathbb{N}$ such that $x \notin E_n$ for all $n \geq n_0$; hence, $f_n(x) = 0$ for all $n \geq n_0$ and we are done.

(2) Assume that $f_n \to 0$ pointwise a.e. on X and $\mu\left(\limsup_{n\to\infty} E_n\right) > 0$. Then, we always can find $x_0 \in \limsup_{n\to\infty} E_n$ and $f_n(x_0)$ converges to zero and we can finish as in (1).

In fact, in (1) we have shown that if $x \notin \limsup_{n\to\infty} E_n$, then $f_n(x) \to 0$ $(n \to \infty)$, and if $x \in \limsup_{n\to\infty} E_n$, then $f_n(x) \to 0$ $(n \to \infty)$ if and only if $(\alpha_n)_n \in c_0$. So, it is clear that if $(\alpha_n)_n \in c_0$ or $\mu\left(\limsup_{n\to\infty} E_n\right) = 0$, then $f_n \to 0$ pointwise a.e. on X.

(3) Suppose that $f_n \to 0$ uniformly on X. Assume, by way of contradiction, that $\alpha_n \notin c_0$. Then, by hypothesis, there exist $M > 0$ and $n_0 \in \mathbb{N}$ such that $|\alpha_n| > M > 0$ for all $n \geq n_0$. But the uniform convergence of f_n allows us to get $m \geq n_0$ such that $|f_n(x)| < \frac{M}{2}$ for all $x \in X$ and $n \geq m$. Therefore, $f_n = 0$, and so $E_n = \varnothing$ for all $n \geq m$, which is impossible by hypothesis.

The reciprocal is immediate because of the fact that $|f_n(x)| \leq |\alpha_n|$ for all $n \in \mathbb{N}$ and every $x \in X$. □

Remark 4.10. Observe that, similarly as in part (1) of the above Proposition, if we have any sequence $(\varphi_n)_n$ of measurable functions, it can be proved that $(\varphi_n \cdot \chi_{E_n})_n$ converges pointwise to zero on X if $\limsup_{n\to\infty} E_n = \varnothing$. Indeed, it would exists $n_0 \in \mathbb{N}$ such that $x \notin E_n$ for every $n \geq n_0$. Hence, $\varphi_n(x) \cdot \chi_{E_n}(x) = 0$ for all $n \geq n_0$.

Thanks to Proposition 4.9 we can construct a huge number of sequences of measurable functions being pointwise convergent but not uniformly convergent on X. In particular, we will focus our attention again on the space $L^0([0,1])^{\mathbb{N}}$. Furthermore, not only are we interested in the existence of such sequences of functions, but we also want to precise the amount of these sequences of functions in $L^0([0,1])^{\mathbb{N}}$, in both a linear and algebraic sense. In fact, although in Proposition 4.9 we were initially interested in the uniform convergence versus the pointwise convergence, we will ask for stronger modes of convergence than this last one.

Recall (see Definition 3.3) that given $(f_n)_n \in \mathcal{L}^0([0,1])^{\mathbb{N}}$ and $f \in \mathcal{L}^0([0,1])$, we say that $f_n \to f$ almost uniformly on $[0,1]$ if for every $\varepsilon > 0$ there exists a set $E \in [0,1]$ with $m(E) < \varepsilon$ such that $f_n \to f$ uniformly on $[0,1] \setminus E$; and $f_n \to f$ uniformly a.e. on $[0,1]$ if there is $E \subset [0,1]$ with $m(E) = 0$ such that $f_n \to f$ uniformly on $[0,1] \setminus E$. Uniform a.e. convergence can be trivially adapted to $L^0([0,1])^{\mathbb{N}}$, but almost uniform convergence should be slightly adapted to classes of functions.

Since the sets of null measure are not important for the classes of functions, and the concept of essential supremum (see Subsection 1.2.5) gives us the supremum except for sets of null measure, in $L^0([0,1])$ we can rewrite the almost uniform convergence in terms of the essential supremum (see condition (b) in the next definition).

We define the family of sequences of functions which we are interested in.

Definition 4.11. *A sequence of measurable functions $f_n : [0,1] \to \mathbb{R}$ $(n \in \mathbb{N})$ is said to belong to the family $n\mathcal{UP}_{ae}([0,1])$, whenever it enjoys the next properties:*

(a) *$f_n \to 0$ pointwise a.e. on $[0,1]$,*

(b) *for any $\varepsilon > 0$ there is a measurable set $E \subset [0,1]$ such that $m(E) < \varepsilon$ and $\operatorname*{ess\,sup}_{[0,1]\setminus E} |f_n| \to 0$ $(n \to \infty)$,*

(c) *$(f_n)_n$ does not converge (to zero) uniformly a.e. on $[0,1]$.*

Observe that the sequence defined in Example 3.6, that is,

$$f_n(x) = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}(x), \qquad (x \in [0,1], n \in \mathbb{N}),$$

is an example of a sequence of functions in $n\mathcal{UP}_{ae}([0,1])$. We know that it is pointwise but not uniformly convergent to zero on $[0,1]$, but it also satisfies condition (b) in Definition 4.11, because for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$. So, for any $n \geq n_0$ and $x > \frac{1}{n}$, $f_n(x) = 0$, and $\operatorname{ess\,sup}_{x\in\left(\frac{\varepsilon}{2},1\right]} f_n(x) = 0$ (and $m\left(\left[0, \frac{\varepsilon}{2}\right]\right) = \frac{\varepsilon}{2} < \varepsilon$).

4.2.2 Lineability results

We already know that $n\mathcal{UP}_{ae}([0,1]) \neq \varnothing$, so it is natural to ask for its size in terms of Lineability. For this, the next result states that not only are there many elements in the family $n\mathcal{UP}_{ae}([0,1])$, but they also behave nicely under algebraic combinations.

Theorem 4.12. *The family $n\mathcal{UP}_{ae}([0,1])$ is strongly $\mathfrak{c}$-algebrable.*

Proof. As in the proof of Theorem 4.7, let $H \subset (0, +\infty)$ a $\mathbb{Q}$-linearly independent set with $\mathrm{card}(H) = \mathfrak{c}$. For each $c \in H$ we define the sequence $F(c) = (F(c,n))_n$ by

$$F(c,n)(x) := e^{-cn[(n+1)x-1]} \cdot \chi_{E_n}(x),$$

where $E_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$ as in Example 3.6.

Let $\mathcal{B}$ be the algebra generated by the family of sequences $\{F(c) : c \in H\}$. Now, because each χ_{E_n} is a characteristic function, any non-zero member $(F_n)_n$ of $\mathcal{B}$ is of the form

$$F_n(x) = \left(\sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c}\cdot\mathbf{j})n[(n+1)x-1]} \right) \chi_{E_n}(x), \tag{4.4}$$

where, for some $N \in \mathbb{N}$, $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\ldots}, 0)\}$ is a non-empty finite set, $\alpha_{\mathbf{j}} \in \mathbb{R} \setminus \{0\}$ for $\mathbf{j} \in J$ and $\mathbf{c} \in H^N$.

It is trivial that finite linear combinations of exponentials are measurable, so from (4.4) and Remark 4.10, it follows that $F_n(x)$ is pointwise convergent to zero. Moreover, condition (b) of Definition 4.11 also holds. For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$ for any $n \geq n_0$. Hence, for any $x \in \left(\frac{\varepsilon}{2}, 1\right)$, $F_n(x) = 0$. So, $\operatorname*{ess\,sup}_{(\frac{\varepsilon}{2},1]} |f_n| = 0$ for any $n \geq n_0$ and we get (b).

Finally, observe that for fixed $n \in \mathbb{N}$, we have that $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$ if and only if $w := n\big((n+1)x - 1\big) \in [0,1]$. Recall that the essential supremum of a continuous functions, except finitely many points, coincides with its supremum (see Corollary 1.4). Then:

$$\operatorname*{ess\,sup}_{0\leq x\leq 1} |F_n(x)| = \sup_{0\leq x\leq 1} \left| \left(\sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c}\cdot\mathbf{j})n[(n+1)x-1]} \right) \chi_{E_n}(x) \right|$$

$$= \sup_{\frac{1}{n+1}\leq x\leq \frac{1}{n}} \left| \sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c}\cdot\mathbf{j})n[(n+1)x-1]} \right| = \sup_{0\leq w\leq 1} \left| \sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}} e^{-(\mathbf{c}\cdot\mathbf{j})w} \right|.$$

But this last amount does not depend on n and, in addition, it is positive, because $\alpha_{\mathbf{j}} \neq 0$ ($\mathbf{j} \in J$) and, by Lemma 4.6, the $\mathbb{Q}$-linearly independence of H implies the linear independence of the set $\{e^{-(\mathbf{c}\cdot\mathbf{j})x} : c \in H, \mathbf{j} \in J\}$. Thus, $(F_n)_n$ does not converge (to zero) uniformly a.e. on $[0,1]$ and the proof is finished. □

Theorem 4.13. *The family* $n\mathcal{UP}_{ae}([0,1])$ *is spaceable in* $L^0([0,1])^{\mathbb{N}}$.

Proof. Consider the sets $E_n := \left[\frac{1}{n+1}, \frac{1}{n}\right]$. Let us divide $\mathbb{N}$ into infinitely many pairwise disjoint subsequences $\{(i(k,n))_n : k \in \mathbb{N}\}$, such that $i(k,n) < i(k',n)$ for $k < k'$ (again, as in the proof of Theorem 4.8, $i(k,n) := \frac{k(k+1)}{2} + (n-1)k$ does the job). For each $k \in \mathbb{N}$, define the sequence $S(k) = (S(k,n))_n := \left(\chi_{E_{i(k,n)}}\right)_n$.

First of all, we are going to prove that $\{S(k) : k \in \mathbb{N}\}$ is a linearly independent set. Indeed, let $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ and pairwise different $k_1, k_2, \ldots, k_N \in \mathbb{N}$ such that

$\sum_{j=1}^{N} \lambda_j S(k_j)$ is the null sequence. Then, for every $n \in \mathbb{N}$ and every $x \in [0,1]$, we have

$$\lambda_1 \chi_{E_{i(k_1,n)}}(x) + \cdots + \lambda_N \chi_{E_{i(k_N,n)}}(x) = 0. \tag{4.5}$$

But, by construction, if $(k,n) \neq (k',n')$ then $i(k,n) \neq i(k',n')$, so $E_{i(k,n)} \cap E_{i(k',n')}$ is either empty or a singleton. Then, for $1 \leq j \leq N$ we always can find

$$x_j \in E_{i(k_j,n)} \setminus \bigcap_{\substack{1 \leq \nu \leq N \\ \nu \neq j}} E_{i(k_\nu,n)}$$

and applying (4.5) at $x = x_j$, we get that $\lambda_j = 0$ for $1 \leq j \leq N$.

Let $M := \overline{\operatorname{span}}\{S(k) : \ k \in \mathbb{N}\}$. It is clear that M is an infinite dimensional closed subspace of $L^0([0,1])^{\mathbb{N}}$. We claim that every non-zero member of M enjoys properties (a), (b) and (c) of Definition 4.11.

Given $F = (F_n)_n \in M \setminus \{0\}$, there exist a strictly increasing sequence $(k_j)_j \subset \mathbb{N}$ and a sequence $(\alpha_j)_j \subset \mathbb{R}$ (not identically zero), such that $F = \sum_{j=1}^{\infty} \alpha_j S(k_j)$ (because F is not really a series, since all $S(k_j)$ have pairwise disjoint support) and, without loss of generality, we may assume that $\alpha_1 \neq 0$.

For every $j, n \in \mathbb{N}$, it is clear that

$$0 < \frac{1}{i(k_j,n)} < \frac{1}{i(k_1,n)} \to 0 \qquad (n \to \infty). \tag{4.6}$$

So, for any $x \in [0,1]$, there is a number $n_0 \in \mathbb{N}$ such that $F_n(x) = 0$ for all $n \geq n_0$. Hence, F_n is convergent to 0 in $[0,1]$ and we have (a). Moreover, given $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ such that $E_{i(k_1,n_1)} \subset [0, \varepsilon/2]$. Since $i(k_1,n) < i(k_j,n)$ for every $k_1 < k_j$ and $n \in \mathbb{N}$, and from (4.6), we obtain that $E_{i(k_j,n)} \subset [0,\varepsilon/2]$ for every $n \geq n_1$ and every $j \in \mathbb{N}$. Hence, $F_n(x) = 0$ for every $n \geq n_1$ and every $x \in (\varepsilon/2, 1]$, and we get (b).

Finally, (c) holds because, for every $n \in \mathbb{N}$, we have that

$$\operatorname*{ess\,sup}_{0 \leq x \leq 1} |F_n(x)| \geq \operatorname*{ess\,sup}_{x \in E_{i(k_1,n)}} |F_n(x)| = \operatorname*{ess\,sup}_{x \in E_{i(k_1,n)}} |\alpha_1 S(k_1,n)| = |\alpha_1| \neq 0$$

□

It is a direct consequence of both Theorems 4.12 and 4.13 that the family $n\mathcal{UP}_{ae}([0,1])$ is $\mathfrak{c}$-lineable. But again bringing up the topological structure of $L^0([0,1])^{\mathbb{N}}$, and taking into account that any complete separable metric topological vector space has dimension at most $\mathfrak{c}$, with an application of Theorem 2.7 we can prove also the maximal dense-lineability of this family.

Theorem 4.14. *The family $n\mathcal{UP}_{ae}([0,1])$ is maximal dense-lineable in $L^0([0,1])^{\mathbb{N}}$.*

Proof. Following the notation of Theorem 2.7, let $A = n\mathcal{UP}_{ae}([0,1])$, $B = c_{00}(L^0([0,1]))$, and $\kappa = \mathfrak{c}$. We already have that A is maximal lineable, and B is dense-lineable in $L^0([0,1])^{\mathbb{N}}$ (see Section 1.3). Moreover, $A \cap B = \varnothing$, because the elements of B are uniformly convergent to zero on $[0,1]$, and A is stronger than B, because adding elements of B does not altere the sequence for n large enough. Now, an application of Theorem 2.7 finishes the proof. □

Chapter 5

The space $L^1([0, +\infty))$

In this Chapter we will focus our attention on the space $L^1([0, +\infty))$ of all (classes of) Lebesgue measurable functions that are integrable over $[0, +\infty)$. Recall that it becomes a Banach space with the L^1-norm (see Subsection 1.2.3). In Section 5.1 we are interested in the behaviour of the L^1-norm of a sequence of functions "in contrast" to other modes of convergence defined in the last chapters. In Section 5.2 we provide linear structures of integrable functions which are continuous but not bounded. In Section 5.3 we translate the study of lineability to sequences of the previous functions with an adequate and natural convergence. The last Section focus on some final remarks about the best possible convergence of the sequences, its growth and smoothness.

5.1 Convergence in $L^1(X,\mu)$

If we consider a general measure space, we define the L^1-convergence as follows.

Definition 5.1. *Let $(X,\mathcal{M},\mu)$ be a measure space. Let $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be measurable functions. We say that $f_n \to f$ $(n \to \infty)$ in L^1-norm on X, if*

$$\|f_n - f\|_1 = \int_X |f_n - f|\, d\mu \to 0 \qquad (n \to \infty).$$

In this section we will study the relationship among the convergence in L^1-norm and other modes of convergence.

5.1.1 Measure vs. L^1 convergence

As next result shows, the convergence in L^1-norm is stronger than the convergence in measure.

Theorem 5.2. *Let $(X,\mathcal{M},\mu)$ be a measure space. Let $f_n, f : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be measurable functions. If $f_n \to f$ $(n \to \infty)$ in L^1-norm, then $f_n \to f$ $(n \to \infty)$ in measure on X.*

Proof. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Then

$$0 \le \mu\big(\{x \in X \,:\, |f_n(x) - f(x)| \ge \varepsilon\}\big) = \int_{\{|f_n - f| \ge \varepsilon\}} 1\, d\mu$$

$$\le \int_{\{|f_n - f| \ge \varepsilon\}} \frac{1}{\varepsilon} \cdot |f_n - f|\, d\mu \le \frac{1}{\varepsilon}\int_X |f_n - f|\, d\mu$$

$$= \frac{1}{\varepsilon}\|f_n - f\|_{L^1}.$$

Thus, if $f_n \to f$ $(n \to \infty)$ in L^1-norm we have that

$$\mu\big(\{x \in X \,:\, |f_n(x) - f(x)| \ge \varepsilon\}\big) \to 0 \qquad (n \to \infty),$$

that is, $f_n \to f$ $(n \to \infty)$ in measure on X. □

The reciprocal is not true, as the next example shows.

Example 5.3 (**Measure $\not\Rightarrow$ L^1-norm**)**.** Let $f_n : [0,+\infty) \to \mathbb{R}$ $(n \in \mathbb{N})$ be the sequence of functions given by

$$f_n(x) := n\chi_{\left[\frac{1}{n},\frac{2}{n}\right]}(x), \qquad (x \in [0,+\infty), n \in \mathbb{N}).$$

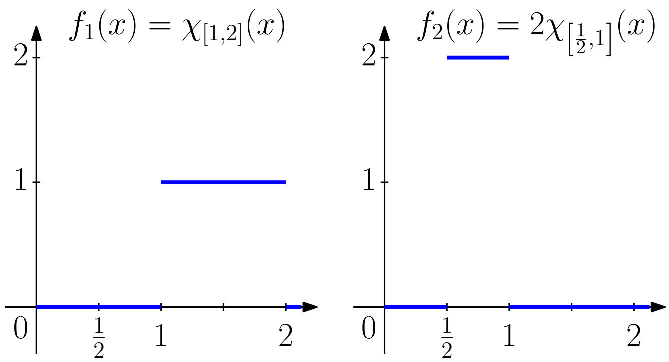


Figure 5.1: First terms of the sequence $f_n(x) = n\chi_{\left[\frac{1}{n},\frac{2}{n}\right]}(x)$

Clearly, $f_n \to 0 =: f$ pointwise on $[0,+\infty)$. Now, given any $\varepsilon \in (0,1)$, we have that

$$m\big(\{x \in [0,+\infty) : |f_n(x) - f(x)| \geq \varepsilon\}\big) \leq m\left(\left[\tfrac{1}{n},\tfrac{2}{n}\right]\right) = \frac{1}{n}.$$

So, we also have that $f_n \to 0$ in measure. Nonetheless, let us compute its L^1-norm:

$$\|f_n - f\|_1 = \int_0^{+\infty} |f_n(x)|\,dx = \int_{\frac{1}{n}}^{\frac{2}{n}} n\,dx = n \cdot \frac{1}{n} = 1.$$

Thus, we do not have convergence of the sequence $(f_n)_n$ to 0 in the L^1-norm.

5.1.2 Uniform vs L^1 convergence

Observe that there is no relation between uniform convergence and convergence in L^1-norm, as the next two examples show.

Example 5.4 (L^1**-norm** $\not\Rightarrow$ **Uniform**)**.** Let $f_n : [0,1] \to \mathbb{R}$ ($n \in \mathbb{N}$) be the sequence of functions of Example 3.6 given by

$$f_n(x) := \chi_{E_n}(x) \qquad (n \in \mathbb{N}, x \in [0,1]),$$

where $E_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$ for all $n \in \mathbb{N}$. We already know that this sequence converges pointwise but not uniformly to $f := 0$ on $[0,1]$. But,

$$\|f_n - f\|_1 = \int_{\frac{1}{n+1}}^{\frac{1}{n}} 1\,dx = m\left(\left[\tfrac{1}{n+1}, \tfrac{1}{n}\right]\right) = \frac{1}{n(n+1)} \to 0, \qquad (n \to \infty).$$

So, we obtain that $f_n \to 0$ in L^1-norm.

For the other relation, observe that this cannot happen in a finite measure setting. Indeed, if we have that $f_n \to f$ uniformly on X with $\mu(X) < +\infty$, then we also have the convergence of the integrals, hence the L^1-norm convergence. So, we will work with measurable functions defined on $[0,+\infty)$.

Example 5.5 (**Uniform** $\not\Rightarrow$ L^1**-norm on infinite measure spaces**)**.** Consider the sequence of functions $f_n : [0,+\infty) \to \mathbb{R}$ ($n \in \mathbb{N}$) given by

$$f_n(x) := \frac{1}{n}\chi_{[0,n]}(x), \qquad (x \geq 0, n \in \mathbb{N}).$$

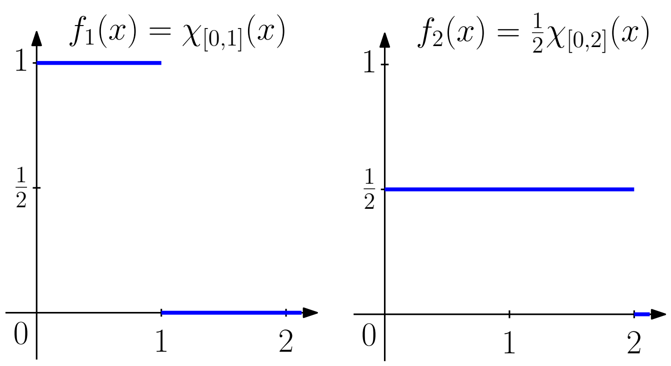


Figure 5.2: First terms of the sequence $f_n(x) = \frac{1}{n}\chi_{[0,n]}(x)$

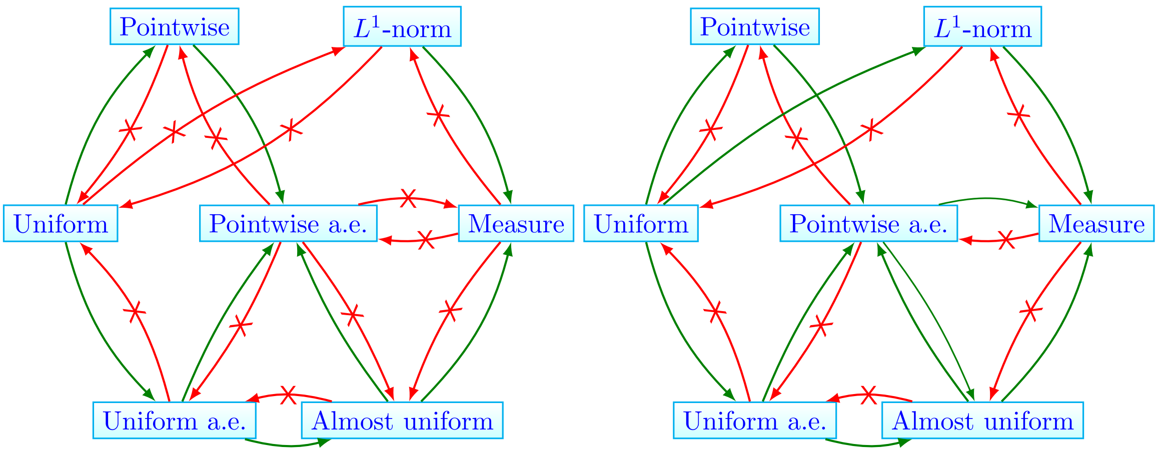
Each f_n is a measurable function, and $f_n \to 0$ uniformly on $[0,+\infty)$ (just take into account that $|f_n(x)| \leq 1/n$ for all $x \geq 0$). Furthermore, if we compute the L^1-norm of each f_n, we obtain,

$$\|f_n\|_{L^1} = \int_0^{+\infty} |f_n(x)|\,dx = \int_0^n \frac{1}{n}\,dx = 1,$$

and f_n cannot converge to zero in L^1-norm.

The last sequence is a classical example of a sequence of functions in $L^0([0,+\infty))^{\mathbb{N}}$ converging uniformly to zero but not in L^1-norm, and it will be the germ of the proofs of all results in this Subsection.

But before, let us collect all the relations between the different convergences.



(a) $\mu(X) = \infty$ **(b)** $\mu(X) < \infty$

Define the family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ of all sequences of (classes of) Lebesgue measurable functions that are uniformly convergent to zero on $[0,+\infty)$ but not in L^1-norm, that is,

$$n\mathcal{L}^1\mathcal{U}([0,+\infty)) := \big\{(f_n)_n \in L^0([0,+\infty))^N \ : \ f_n \to 0 \text{ uniformly on } [0,+\infty) \text{ and } \|f_n\|_1 \not\to 0 \ (n\to\infty)\big\}.$$

From the previous Example 5.5, we already have that this family is non-empty. In the next results we will state which is its algebraic size in terms of Lineability. But before we need an auxiliary result.

Lemma 5.6. *For any non-void set $H \subset (0,+\infty)$ the family of sequences*

$$\left\{ \left(\frac{1}{n^c} \right)_n : c \in H \right\}$$

is linearly independent.

Proof. Take a finite linear combination of elements of the family, that is, fix $N \in \mathbb{N}$, $c_1, c_2, \ldots, c_N \in H$ ($c_1 < c_2 < \cdots < c_N$) and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}$, and assume that, for all $n \in \mathbb{N}$,

$$\lambda_1 \frac{1}{n^{c_1}} + \lambda_2 \frac{1}{n^{c_2}} + \cdots + \lambda_N \frac{1}{n^{c_N}} = 0. \tag{5.1}$$

But it is a well-known result of Pólya (see for instance [34, Corollary 3.2] or [47, pp. 46–47]) that the "generalized polynomial"

$$p(x) := \lambda_1 x^{c_1} + \lambda_2 x^{c_2} + \cdots + \lambda_N x^{c_N}$$

has a finite number of zeros (at most $N-1$). So, the only possibility for (5.1) to be true for all $n \in \mathbb{N}$ is that $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$. □

Theorem 5.7. *The family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $H \subset (0,+\infty)$ be a $\mathbb{Q}$-linearly independent set with $\operatorname{card}(H) = \mathfrak{c}$. For every $c \in H$, consider the sequence of functions $f_c := (f_{c,n})_n \in L^0([0,+\infty))^{\mathbb{N}}$ given by

$$f_{c,n}(x) := \frac{1}{n^c} \chi_{[0,e^n]}(x) \qquad (x \geq 0, n \in \mathbb{N}). \tag{5.2}$$

For any $c \in H$,

$$\sup_{x \in [0,+\infty)} |f_{c,n}(x)| = \sup_{x \in [0,+\infty)} \left| \frac{1}{n^c} \chi_{[0,e^n]}(x) \right| = \frac{1}{n^c} \to 0 \qquad (n \to \infty),$$

and applying Theorem 3.2, the sequence f_c is uniformly convergent to zero. Furthermore, it does not converge to zero in L^1-norm, since for every $c \in H$, we have

$$\|f_{c,n}\|_{L^1} = \int_0^{+\infty} \left| \frac{1}{n^c} \chi_{[0,e^n]}(x) \right| dx = \frac{e^n}{n^c} \to +\infty \qquad (n \to \infty).$$

Let $\mathcal{B}$ be the algebra generated by the family $\{f_c : c \in H\}$, which is a linearly independent family by Lemma 5.6. We claim that $\mathcal{B}$ is a free algebra such that any non-zero member is a sequence uniformly convergent to zero in $[0,+\infty)$ but not in L^1-norm.

Let $F = (F_n)_n \in \mathcal{B} \setminus \{0\}$. Similarly to Theorems 4.7 and 4.12, there exist a natural number $N \in \mathbb{N}$, a non-empty finite set $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\ldots}, 0)\}$ and scalars $\alpha_{\mathbf{j}} \in \mathbb{R} \setminus \{0\}$, for $\mathbf{j} = (j_1, j_2, \ldots, j_N) \in J$ such that, for every $n \in \mathbb{N}$ and $x \geq 0$,

$$F_n(x) = \sum_{\mathbf{j} \in J} \alpha_{\mathbf{j}} f_{c_1,n}(x)^{j_1} \cdots f_{c_N,n}(x)^{j_N} = \left(\sum_{\mathbf{j} \in J} \alpha_{\mathbf{j}} \frac{1}{n^{\mathbf{c}\cdot\mathbf{j}}} \right) \chi_{[0,e^n]}(x), \tag{5.3}$$

where the fact of $\chi_{[0,e^n]}$ being a characteristic function is crucial again in the last equality.

By the $\mathbb{Q}$-linearly independence of H, the numbers $\mathbf{c} \cdot \mathbf{j}$ are mutually distinct and non-null. Moreover, $\alpha_{\mathbf{j}} \neq 0$ ($\mathbf{j} \in J$), hence by Lemma 5.6 the numbers

$$\sum_{\mathbf{j} \in J} \alpha_{\mathbf{j}} \frac{1}{n^{\mathbf{c}\cdot\mathbf{j}}}$$

can be zero at most for a finite number of n. In particular, $(F_n)_n$ is a non-null sequence and the algebra is free. Furthermore,

$$\sup_{x \geq 0} |F_n(x)| \leq \sum_{\mathbf{j} \in J} |\alpha_{\mathbf{j}}| \frac{1}{n^{\mathbf{c}\cdot\mathbf{j}}} \to 0 \qquad (n \to \infty).$$

So, $F = (F_n)_n$ is uniformly convergent to zero on $[0,+\infty)$. Finally,

$$\|F_n\|_1 = \int_0^{+\infty} \left| \left(\sum_{\mathbf{j} \in J} \alpha_{\mathbf{j}} \frac{1}{n^{\mathbf{c}\cdot\mathbf{j}}} \right) \chi_{[0,e^n]}(x) \right| dx = \left| \sum_{\mathbf{j} \in J} \alpha_{\mathbf{j}} \frac{1}{n^{\mathbf{c}\cdot\mathbf{j}}} \right| e^n \to +\infty \qquad (n \to \infty),$$

which concludes the proof. □

Since $L^0([0,+\infty))^{\mathbb{N}}$ with the product topology is also a complete metrizable (hence Baire) space and the family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ is $\mathfrak{c}$-algebrable, we obtain the next corollary about its linear size.

Corollary 5.8. *The family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ is maximal lineable.*

Observe that, although every non-zero sequence of the algebra $\mathcal{B}$ constructed in the proof of the previous theorem does not converge to zero in L^1-norm, all the functions are, in fact, integrable on $[0,+\infty)$. Hence, $\mathcal{B}\setminus\{0\}\subset L^1([0,+\infty))^{\mathbb{N}}$. Observe also that for any $(F_n)_n\in\mathcal{B}\setminus\{0\}$ we have $\sup_{n\in\mathbb{N}}\|F_n\|_1=+\infty$.

In 2014, Bernal and Ordóñez [19] considered the space CBL_s of all sequences of continuous, bounded and integrable functions $f_n:\mathbb{R}\to\mathbb{R}$ $(n\in\mathbb{N})$ such that $\|f_n\|_\infty\to 0$ $(n\to\infty)$ and $\sup_{n\in\mathbb{N}}\|f_n\|_1<+\infty$, that is, their supremum norm converges to zero and the sequence of L^1-norms is uniformly bounded. Moreover, they proved that

$$\|(f_n)_n\|:=\sup_{n\in\mathbb{N}}\|f_n\|_\infty+\sup_{n\in\mathbb{N}}\|f_n\|_1$$

defines a norm on CBL_s, under which it becomes a Banach space. Now, consider the family $\mathcal{F}$ of all sequences of functions $(f_n)_n$ of CBL_s such that $\|f_n\|_1\not\to 0$ $(n\to\infty)$. In [19, Theorem 4.16] Bernal and Ordóñez establish the spaceability of the family $\mathcal{F}$ in CBL_s

As a consequence, this family is also maximal lineable. However, the point with the family $\mathcal{F}$ that Bernal and Ordóñez considered is that this family turns out to be smaller than the family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$, that is, $\mathcal{F}\subset n\mathcal{L}^1\mathcal{U}([0,+\infty))$, while the topology of CBL_s is finer than the product topology of $L^1([0,+\infty))^{\mathbb{N}}$. As a consequence, the spaceability of the family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ cannot be directly derived from [19, Theorem 4.16], because not every open set in $L^1([0,+\infty))^{\mathbb{N}}$ turns out to be also open in CBL_s.

Again in [19], Bernal and Ordóñez, just by considering the map $T : \ell^\infty \to CBL_s$ given by

$$T(a_n) = \begin{cases} \frac{2a_n}{n}(x-k+1) & \text{if } k-1 \leq x < k+\frac{1}{2} \qquad (1 \leq k \leq n), \\ \frac{2a_n}{n}(x-k) & \text{if } k+\frac{1}{2} \leq xk \qquad (1 \leq k \leq n), \\ 0 & \text{otherwise,} \end{cases}$$

for any sequence $(a_n)_n \in \ell^\infty$, constructed an isomorphism T between the non-separable space ℓ^∞ and its image. Thus, $T(\ell^\infty)$, and so CBL_s, are not separable, which means that the maximal dense-lineability of the family $\mathcal{F}$ cannot be obtained as an application of Theorem 2.7. But, in this setting, taking into account that the product topology of $L^1([0,+\infty))^{\mathbb{N}}$ actually makes $L^1([0,+\infty))^{\mathbb{N}}$ separable, the next result can be proven.

Theorem 5.9. *The family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ is maximal dense-lineable in $L^1([0,+\infty))^{\mathbb{N}}$ with the product topology.*

Proof. Both from Bernal and Ordóñez result or directly from Theorem 5.7 (at this moment we do not care about the topology), we know that the family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ of the hypothesis is $\mathfrak{c}$-lineable. The space $L^1([0,+\infty))^{\mathbb{N}}$, endowed with the product topology, is a separable complete metrizable topological vector space, so its dimension is $\mathfrak{c}$.

Moreover, as seen in Section 1.3, the vector space $c_{00}(L^1([0,+\infty)))$ of vanishing sequences of functions of $L^1([0,+\infty))$ is dense in $L^1([0,+\infty))^{\mathbb{N}}$ with the product topology. In addition, $n\mathcal{L}^1\mathcal{U}([0,+\infty)) \cap c_{00}(L^1([0,+\infty))) = \varnothing$ because every sequence in $c_{00}(L^1([0,+\infty)))$ converges to zero in L^1-norm, and $n\mathcal{L}^1\mathcal{U}([0,+\infty)) + c_{00}(L^1([0,+\infty))) \subset n\mathcal{L}^1\mathcal{U}([0,+\infty))$.

Now, applying Theorem 2.7 with $A = n\mathcal{L}^1\mathcal{U}([0,+\infty))$, $B = c_{00}(L^1([0,+\infty)))$, and $\kappa = \mathfrak{c}$, we get the maximal dense-lineability. □

In order to state the spaceability, it turns out that we must take into account not only the product topology but also the property of the sequences of being uniformly convergent to zero on $[0,+\infty)$. For this purpose, let

$$Z_1 := \left\{ (f_n)_n \in L^1([0,+\infty))^{\mathbb{N}} \,:\, f_n \to 0 \text{ uniformly on } [0,+\infty) \right\}$$

endowed with the next F-norm

$$\|(f_n)_n\|_{Z_1} := \sup_{n\in\mathbb{N}} \|f_n\|_\infty + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f_n\|_1}{1+\|f_n\|_1}.$$

It is straightforward that $(Z_1, \|\cdot\|_{Z_1})$ becomes a complete separable metrizable locally convex topological vector space, hence a Fréchet space, whose topology recover both the product topology inherited by $\|\cdot\|_1$ and the uniform convergence to zero. This is the right framework to state the spaceability.

Theorem 5.10. *The family $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ is spaceable in $(Z_1, \|\cdot\|_{Z_1})$.*

Proof. Let us divide the interval $[0,+\infty)$ into infinitely many sequences of pairwise disjoint intervals (except, possibly, for the extremes). For every $N \in \mathbb{N}$ and every $M = 1, \ldots, N$, let

$$I_{N,M} := \left[\sum_{j=1}^{N-1} j(N-j) + \frac{M(M-1)}{2}, \sum_{j=1}^{N-1} j(N-j) + \frac{M(M+1)}{2} \right].$$

Observe that for each $M \in \mathbb{N}$, the interval $I_{N,M}$ has always length M.

For every $k \in \mathbb{N}$, define the sequence $G(k) = (G(k,n))_n := \left(\frac{1}{n}\chi_{I_{k+n-1,n}}\right)_n$. It is straightforward that every sequence $G(k)$ converges to zero uniformly in $[0,+\infty)$ but not in L^1-norm (observe that $\|G(k,n)\|_1 = 1$ for all $n \in \mathbb{N}$). Moreover, the family $\{G(k) : k \in \mathbb{N}\}$ is linearly independent because of the disjointness of the (interiors of the) supports of all functions included in it.

Now, an application of Kalton criterion of spacebility (see Theorem 2.10) with $X = Z_1$ and $Y = \{(f_n)_n \in X \,:\, \|f_n\|_1 \to 0 \ (n \to \infty)\}$ (which is closed in X by a standard topological argument) finishes the proof. □

Remark 5.11. Sequences of $n\mathcal{L}^1\mathcal{U}([0,+\infty))$ are, by definition, not convergent to zero in L^1-norm. But, in fact, by the construction of the algebra $\mathcal{B}$ of the proof of Theorem 5.7, the sequences of L^1-norms of every non-zero member of $\mathcal{B}$ diverges to infinity. However, every sequence of the closed vector space given by the spaceability of CBL_s is uniformly bounded in L^1-norm. Thus, it is natural to ask about the algebraic genericity of the family of sequences $(f_n)_n \in L^1([0,+\infty))^{\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_1 < +\infty$, $f_n \to 0$ uniformly on $[0,+\infty)$ and $\|f_n\|_1 \not\to 0$ $(n \to \infty)$.

5.2 Unbounded functions in $CL^1([0,+\infty))$

In this Section we will focus our attention on the space $CL^1([0,+\infty))$ of all continuous and integrable (classes of) functions on $[0,+\infty)$, that is,

$$CL^1([0,+\infty)) = L^1([0,+\infty)) \cap C([0,+\infty)).$$

This space becomes a topological metrizable vector space when endowed with the natural translation-invariant metric given by

$$d_{CL^1}(f,g) := \|f-g\|_1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{\infty,n}}{1+\|f-g\|_{\infty,n}}, \tag{5.4}$$

Note that d_{CL^1}-convergence means L^1-convergence plus uniform convergence on compacta.

The necessary condition of convergence of series of scalars states that if a series is convergent, then its general term must tend to zero. Now, turning into the setting of integrable (and even continuous) functions, one may think that something similar should happen, namely, that if f is integrable on an unbounded interval (say $[0,+\infty)$), then it must tend to zero somehow. However, this is far from being true. Our intuition would be actually correct if the function f also verifies that the limit of $f(x)$ $(x \to +\infty)$ exists, or even if the function is decreasing over the interval. Nonetheless, this affirmation is false in general.

In the next example we show a classical undergraduate construction of an unbounded, continuous and integrable function on $[0,+\infty)$ that will play a fundamental role during this section.

Example 5.12 (Triangular Function). For each $n \in \mathbb{N}$, consider the triangular functions $T_n : [0,+\infty) \to \mathbb{R}$ given by

$$T_n(x) := \begin{cases} n(2^{n+1}x + (1 - n2^{n+1})) & \text{if } x \in \left[n - \frac{1}{2^{n+1}}, n\right), \\ n(-2^{n+1}x + (1 + n2^{n+1})) & \text{if } x \in \left[n, n + \frac{1}{2^{n+1}}\right], \\ 0 & \text{otherwise,} \end{cases} \tag{5.5}$$

Observe that, for each $n \in \mathbb{N}$, the function $T_n(x)$ "draws" the isoceles triangle of height n and basis $\frac{1}{2^n}$ centred at the point $x = n$.

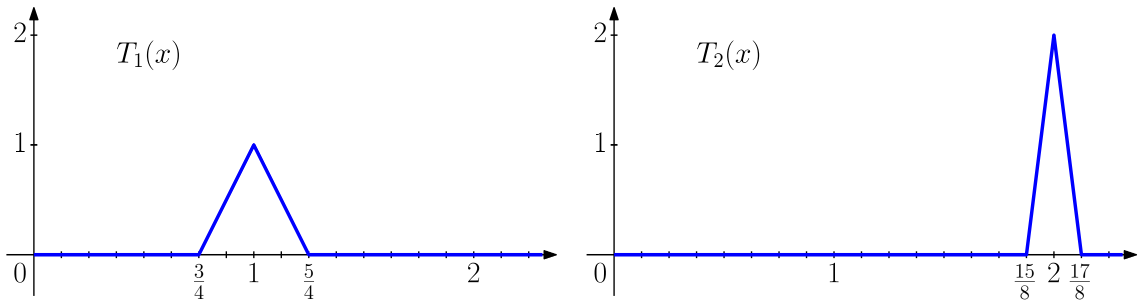


Figure 5.4: First terms of the "triangles" $T_n(x)$

Now, we will define the function $f : [0,+\infty) \to \mathbb{R}$ by joining all the previous "triangles" of the functions T_n, that is,

$$f(x) := \sum_{n=1}^{\infty} T_n(x). \tag{5.6}$$

This is a formal infinite series, but due to the disjointness of the supports of the triangular functions for fixed $x \geq 0$, there is at most one non-zero summand. Observe that now, the functions $f(x)$ "draws" all the triangles given by the functions T_n, for $n \in \mathbb{N}$, together.

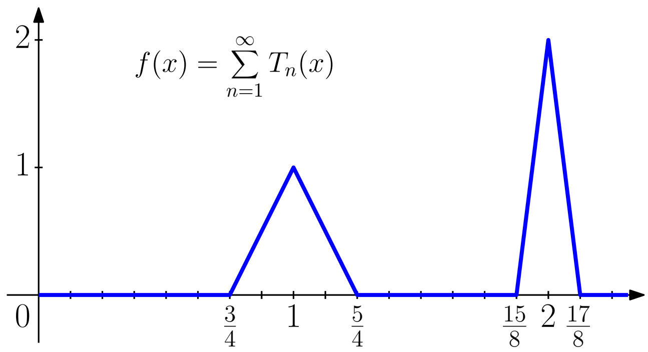


Figure 5.5: Triangular function $f(x)$

Clearly, the continuity of each T_n ($n \in \mathbb{N}$) and the construction of f allows us to state the continuity of the function f. Furthermore, f is integrable on $[0, +\infty)$, since its L^1-norm can be computed as follows:

$$\|f\|_{L^1} = \int_0^{+\infty} \left| \sum_{n=1}^{\infty} T_n(x) \right| dx = \sum_{n=1}^{\infty} \int_0^{+\infty} T_n(x)\, dx = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1 < +\infty.$$

Hence, we have that $f \in CL^1([0, +\infty))$. Finally, if we consider the sequence $(x_n := n)_n \subset \mathbb{R}$ of all positive integers, we have that

$$f(x_n) = f(n) = n \to +\infty \qquad (n \to \infty),$$

so,

$$\limsup_{x \to +\infty} |f(x)| = +\infty.$$

Thus, $f(x)$ is an example of a continuous, unbounded and integrable function on $[0, +\infty)$.

Remark 5.13. Observe that, with a similar construction of the triangles in (5.5) we could prefix the value of $\|f\|_1$ in $(0, +\infty)$, just adjusting the size of the triangles. Indeed, given $\alpha > 0$, we can consider the functions $T_{n,\alpha}(x) := \alpha \cdot T_n(x)$ for every $x \geq 0$ and $n \in \mathbb{N}$.

Since $\|T_{n,\alpha}\|_1 = \alpha \cdot \|T_n\|_1$, if we define the function F_α as in (5.6), we obtain that $\|F_\alpha\|_1 = \alpha$.

In view of Example 5.12, we introduce the family $n\mathcal{BCL}^1([0,+\infty))$ of all continuous, unbounded and integrable functions on $[0,+\infty)$, that is,

$$n\mathcal{BCL}^1([0,+\infty)) := \left\{ f \in CL^1([0,+\infty)) \,:\, \limsup_{x\to+\infty} |f(x)| = +\infty \right\}.$$

Observe that from Example 5.12 we already have that $n\mathcal{BCL}^1([0,+\infty)) \neq \varnothing$. Furthermore, given any $\alpha > 0$, Remark 5.13 assures the existence of a function $F_\alpha \in n\mathcal{BCL}^1([0,+\infty))$, which provides us with $\text{card}(n\mathcal{BCL}^1([0,+\infty))) = \mathfrak{c}$ (observe that, if $\alpha \neq \beta$, then $F_\alpha \neq F_\beta$). So, in terms of cardinality we know that there are many functions in the family $n\mathcal{BCL}^1([0,+\infty))$ but, although $n\mathcal{BCL}^1([0,+\infty))$ has as many elements as $CL^1([0,\infty))$, it is not a vector space itself. For this, let us just take $f(x)$ and $e^{-x} - f(x)$, where $f \in n\mathcal{BCL}^1([0,+\infty))$. This linear combination would result in

$$f(x) + (e^{-x} - f(x)) = e^{-x} \notin n\mathcal{BCL}^1([0,+\infty)).$$

Thus, the natural question that now arises is what the meaning of "big" would be when looking for lineability within $n\mathcal{BCL}^1([0,+\infty))$.

The following result gives us a positive answer. Remember that, given a vector subspace $M \subset n\mathcal{BCL}^1([0,+\infty)) \cup \{0\}$, its maximal dimension is $\mathfrak{c}$.

Theorem 5.14. *The family $n\mathcal{BCL}^1([0,+\infty))$ is maximal lineable.*

Proof. We are looking for a vector subspace M of dimension $\mathfrak{c}$ such that $M \subset n\mathcal{BCL}^1([0,+\infty)) \cup \{0\}$.

Consider the "triangles" $(T_n)_n$ given in Example 5.12. Observe that the minimum distance between (the supports of) any two triangles T_n and T_{n+1} is $\frac{5}{8}$, and the maximum length for the support of T_n is $\frac{1}{2}$.

So, for any $s, t \in \left[0, \frac{1}{8}\right)$, we have that the supports of the corresponding triangles $T_n(x-t)$ and $T_m(x-s)$ are disjoint for any pair $n, m \in \mathbb{N}$, $n \neq m$.

On the other hand, for any $s,t \in \left[0,\frac{1}{8}\right)$ with $s<t$, there exists $N \in \mathbb{N}$ such that $\frac{1}{2^n} < t-s$ for any $n \geq N$. In particular,

$$n+\frac{1}{2^{n+1}}+s<n-\frac{1}{2^{n+1}}+t$$

and the supports of the triangles $T_n(x-t)$ and $T_n(x-s)$ are disjoint for any $n \geq N$.

For each $t \in \left[0,\frac{1}{8}\right)$ we define the function $f_t : [0,+\infty) \to \mathbb{R}$ by

$$f_t(x) := \sum_{n=1}^{\infty} T_n(x-t),$$

where we understand $T_n(x-t)=0$ for $x-t<0$.

As in Example 5.12, we have that $f_t \in n\mathcal{BCL}^1([0,+\infty))$ for any $t \in \left[0,\frac{1}{8}\right)$ and, so,

$$M := \operatorname{span}\left\{f_t \,:\, t \in \left[0,\tfrac{1}{8}\right)\right\}$$

is a vector subspace in $CL^1([0,+\infty))$. It only remains to prove that $\dim(M)=\mathfrak{c}$ and the unboundedness of each $F \in M \setminus \{0\}$. Let $F \in M$, that is,

$$F(x)=c_1 f_{t_1}(x)+c_2 f_{t_2}(x)+\cdots+c_s f_{t_s}(x), \qquad (x \geq 0),$$

where $0 \leq t_1 < t_2 < \cdots < t_s < 1/8$ and $c_1, c_2, \ldots, c_s \in \mathbb{R}$. Assume that some c_j is non-null (without loss of generality we can assume $c_s \neq 0$).

Let $N \in \mathbb{N}$ be large enough to get the disjointness of $T_n(x-t_s)$ with any other $T_n(x-t_i)$, $i=1,2,\ldots,s-1$, $n \geq N$. Then, for any $x_n := n+t_s$ $(n \geq N)$,

$$F(x_n)=c_1 f_{t_1}(x_n)+c_2 f_{t_2}(x_n)+\cdots+c_s f_{t_s}(x_n)=c_s \cdot n.$$

Hence, $F \not\equiv 0$, the set $\left\{f_t \,:\, t \in \left[0,\frac{1}{8}\right)\right\}$ is linearly independent, and $\dim(M)=\mathfrak{c}$.

But also,

$$|F_n(x_n)|=|c_s|n \to +\infty \qquad (n \to \infty).$$

Thus, F is unbounded and the proof is finished. □

In order to determine the dense-lineability of the family $n\mathcal{BCL}^1([0,+\infty))$, the topological structure of $CL^1([0,+\infty))$ needs to be taken into account. Recall that a polygonal in $[0,+\infty)$ is a continuous function consisting of finitely many affine linear mappings on compact subintervals of $[0,+\infty)$. We state the next auxiliary lemma.

Lemma 5.15. *The family B of all functions of the form*

$$b_{p,n,\gamma}(x) = \begin{cases} p(x) & \text{if } 0 \le x \le n, \\ \frac{p(n)}{\gamma}(n+\gamma-x) & \text{if } n < x \le n+\gamma, \\ 0 & \text{if } x > n+\gamma, \end{cases}$$

where $p(x)$ is a polygonal in $[0,+\infty)$, $n \in \mathbb{N}$ and $\gamma > 0$, is dense in $(CL^1([0,+\infty)), d_{CL^1})$.

Proof. It is obvious that $B \subset CL^1([0,\infty))$. We are going to see that B is dense in $CL^1([0,+\infty))$. Let $f \in CL^1([0,+\infty))$ and $\varepsilon > 0$. Since f is integrable, there is $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{6} \qquad \text{and} \qquad \int_N^{\infty} |f(x)|\,dx < \frac{\varepsilon}{6}. \tag{5.7}$$

By using uniform continuity, it is easy to see that the set of all polygonals is dense in $(\mathcal{C}([0,N]), \|\cdot\|_\infty)$, even with the property that the approximating polygonal matches with the approximation function at the extremes of the interval. Consequently, we can take a polygonal $p(x)$ in $[0,N]$ such that

$$p(N) = f(N) \qquad \text{and} \qquad \|f-p\|_{\infty,N} < \frac{\varepsilon}{6N}. \tag{5.8}$$

Now define $\gamma := \dfrac{\varepsilon}{6(1+|f(N)|)}$. Then,

$$\begin{aligned} d_{CL^1}(f, b_{p,N,\gamma}) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f-b_{p,N,\gamma}\|_{\infty,n}}{1+\|f-b_{p,N,\gamma}\|_{\infty,n}} + \|f-b_{p,N,\gamma}\|_1 \\ &\le \|f-p\|_{\infty,N} \sum_{k=1}^{N} \frac{1}{2^k} + \sum_{k=N+1}^{\infty} \frac{1}{2^k} + \int_0^N |f(x)-p(x)|\,dx \\ &\quad + \int_N^{N+\gamma} \left| f(x) - \tfrac{f(N)}{\gamma}(N-x+\gamma) \right| dx + \int_{N+\gamma}^{\infty} |f(x)|\,dx \end{aligned} \tag{5.9}$$

But, by (5.8),

$$\begin{aligned}\int_N^{N+\gamma}\left|f(x)-\frac{p(N)}{\gamma}(N-x+\gamma)\right|dx &= \int_N^{N+\gamma}\left|f(x)-\frac{f(N)}{\gamma}(N-x+\gamma)\right|dx \\ &\leq \int_N^{N+\gamma}|f(x)|\,dx+\int_N^{N+\gamma}\frac{|f(N)|}{\gamma}|N-x+\gamma|\,dx \\ &\leq \int_N^{+\infty}|f(x)|\,dx+\frac{|f(N)|}{\gamma}\cdot\gamma\cdot\gamma \qquad (5.10)\end{aligned}$$

Now, by (5.7), (5.8), (5.9) and (5.10),

$$d_{CL^1}(f,b_{p,N,\gamma})\leq\frac{\varepsilon}{6N}\cdot\sum_{k=1}^{N}\frac{1}{2^k}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6N}\cdot N+\frac{\varepsilon}{6}+|f(N)|\cdot\gamma+\frac{\varepsilon}{6}<6\cdot\frac{\varepsilon}{6}=\varepsilon,$$

because $\sum_{k=1}^{N}\frac{1}{2^k}\leq N$, and by the definition of γ, we have $|f(N)|\cdot\gamma\leq\frac{\varepsilon}{6}$. Hence, the set B is dense in $CL^1([0,+\infty))$. $\square$

Now, with the application again of Theorem 2.7, we can obtain the maximal dense-lineability of our family $n\mathcal{BCL}^1([0,+\infty))$.

Theorem 5.16. *The family* $n\mathcal{BCL}^1([0,+\infty))$ *is maximal dense-lineable in* $CL^1([0,+\infty))$.

Proof. The family B defined in Lemma 5.15 is a vector space (indeed, any linear combination of polygonals is again a polygonal, even if the vertex do not match) and is dense in $CL^1([0,+\infty))$, so, in particular, B is dense-lineable in $CL^1([0,+\infty))$. By Theorem 5.14 we already have that the family $n\mathcal{BCL}^1([0,+\infty))$ is maximal lineable. By the construction of the family B, any function of B is bounded, so $n\mathcal{BCL}^1([0,+\infty))\cap B=\varnothing$. Moreover, $n\mathcal{BCL}^1([0,+\infty))+B\subset n\mathcal{BCL}^1([0,+\infty))$, because any function in B has compact support. Finally, an application of Theorem 2.7 with $A=n\mathcal{BCL}^1([0,+\infty))$, B and $\kappa=\mathfrak{c}$ gives us the maximal dense-lineability of $n\mathcal{BCL}^1([0,+\infty))$. $\square$

We have already seen that there exists a dense $\mathfrak{c}$-dimensional vector space $M\subset n\mathcal{BCL}^1([0,+\infty))\cup\{0\}$, which means, that our family $n\mathcal{BCL}^1([0,+\infty))$ behaves “well” under linear combinations. Let us see that this can also be extended to algebraic combinations, which would lead into the algebrability of the family $n\mathcal{BCL}^1([0,+\infty))$. But, before we start with this, we need some notation, since the argumentation here differs a little bit from the one carried out in the proofs of Theorems 4.7 and 4.12.

Definition 5.17. *Let $N \in \mathbb{N}$. Given a (monic) monomial in N variables*

$$m(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} x_i^{\alpha_i},$$

where $\alpha_i \in \mathbb{N} \cup \{0\}$ for all $i = 1, 2, \ldots, N$, and an increasing sequence of prime numbers $\mathbb{P} = (p_i)_i$, we define the $\mathbb{P}$-index of m by

$$ind_{\mathbb{P}}(m) := m(p_1, p_2, \ldots, p_N) = \prod_{i=1}^{N} p_i^{\alpha_i}.$$

Remark 5.18. Observe that we can always state a bijection between any increasing sequence of prime numbers and the whole sequence of primes, so, by the uniqueness of the factorization theorem, given a natural number $n \in \mathbb{N}$, there is only one way of expressing it as product of powers of prime numbers, that is, there is an unique factorization

$$n = \prod_{i=1}^{N} p_i^{\alpha_i},$$

and this number is the $\mathbb{P}$-index of the monomial $m(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} x_i^{\alpha_i}$. This fact allows us to state a bijection between $\mathbb{N}$ and the set of all monic monomials. So, given a monomial m, it is uniquely described by any of its $\mathbb{P}$-index.

Thanks to this, we will be able to find an algebra of dimension $\aleph_0$ inside the family $n\mathcal{BCL}^1([0,+\infty))$.

Theorem 5.19. *The family $n\mathcal{BCL}^1([0,+\infty))$ is strongly $\aleph_0$-algebrable.*

Proof. For any $n \in \mathbb{N}$ and $p \geq 1$, we consider the "triangles" on $[0,+\infty)$ given by

$$T_{n,p}(x) = \begin{cases} n^p(2^{n+1}x + (1 - n2^{n+1}))^p & \text{if } x \in \left[n - \frac{1}{2^{n+1}}, n\right), \\ n^p(-2^{n+1}x + (1 + n2^{n+1}))^p & \text{if } x \in \left[n, n + \frac{1}{2^{n+1}}\right], \\ 0 & \text{otherwise,} \end{cases}$$

and we define the functions $g_p : [0,+\infty) \to \mathbb{R}$ as:

$$g_p(x) = \sum_{n=1}^{\infty} T_{n,p}(x).$$

In fact, each $T_{n,p}$ draws a "curved triangle" whose basis lies in the x-axis, has height n^p, and the other sides are convex $(p \geq 1)$ functions.

Because of the disjointness of the supports of the triangles $T_{n,p}(x)$ $(n \in \mathbb{N})$, the functions g_p are well defined and continuous on $[0,+\infty)$. Furthermore, we can easily bound their L^1-norm. Observe that, for each $p \geq 1$ and $n \in \mathbb{N}$, the "triangle" $T_{n,p}$ is continuous on $\left[n - \frac{1}{2^{n+1}}, n + \frac{1}{2^{n+1}}\right]$ and infinite differentiable on $\left(n - \frac{1}{2^{n+1}}, n + \frac{1}{2^{n+1}}\right) \setminus \{n\}$. In fact, if $x \in \left(n - \frac{1}{2^{n+1}}, n\right)$,

$$\begin{aligned} T'_{n,p}(x) &= pn^p 2^{n+1}(2^{n+1}x + (1 - n2^{n+1}))^{p-1} > 0, \\ T''_{n,p}(x) &= p(p-1)n^p 2^{2n+2}(2^{n+1}x + (1 - n2^{n+1}))^{p-2} > 0, \end{aligned}$$

and if $x \in \left(n, n + \frac{1}{2^{n+1}}\right)$,

$$\begin{aligned} T'_{n,p}(x) &= -pn^p 2^{n+1}(-2^{n+1}x + (1 + n2^{n+1}))^{p-1} < 0, \\ T''_{n,p}(x) &= p(p-1)n^p 2^{2n+2}(-2^{n+1}x + (1 + n2^{n+1}))^{p-2} > 0, \end{aligned}$$

so the triangles $T_{n,p}(x)$ are always "inside" the real triangles of basis $\left[n - \frac{1}{2^{n+1}}, n + \frac{1}{2^{n+1}}\right]$ and height $T_{n,p}(n) = n^p$. Hence

$$\|g_p\|_{L^1} = \sum_{n=1}^{\infty} \int_0^{\infty} T_{n,p}(x)dx \leq \sum_{n=1}^{\infty} \frac{n^p}{2^{n+1}} < +\infty.$$

So, $g_p \in CL^1([0,\infty))$. Now, by evaluating g_p on the sequence of positive integers $x_n = n$ for every $n \in \mathbb{N}$, we get that

$$g_p(x_n) = g_p(n) = n^p \to +\infty \qquad (n \to \infty)$$

which yields $g_p \in n\mathcal{BCL}^1([0,+\infty))$ for all $p \geq 1$.

Let $\mathbb{P} = (p_j)_j$ be the increasing sequence of all prime numbers which are greater than 3, and let us define for each $j \in \mathbb{N}$ the function F_j as

$$F_j(x) := \sum_{n=1}^{\infty} T_{n,\log p_j}(x) = g_{\log p_j}(x).$$

Note that $(F_j)_j \subset n\mathcal{BCL}^1([0,+\infty))$. Let $\mathcal{B}$ be the algebra generated by $(F_j)_j$, that is,

$$\mathcal{B} = \big\{ P(F_1, \ldots, F_N) : \ P \text{ is a polynomial in } N \text{ variables without constant term, } N \in \mathbb{N} \big\}.$$

We are going to prove that $\mathcal{B}$ is the desired infinitely generated algebra. Let $m(x_1, \ldots, x_N) = \prod_{i=1}^{N} x_i^{\alpha_i}$ $(\alpha_i \in \mathbb{N} \cup \{0\})$ be a non-constant monomial. Hence

$$m(F_1, \ldots, F_N)(x) = \prod_{i=1}^{N} F_i(x)^{\alpha_i} = \prod_{i=1}^{N} \left(\sum_{n=1}^{\infty} T_{n,\log(p_i)}(x) \right)^{\alpha_i}.$$

For each $n \in \mathbb{N}$ and each $x \in \left[n - \frac{1}{2^{n+1}}, n\right)$ we have

$$\begin{aligned} m(F_1, \ldots, F_N)(x) &= \prod_{i=1}^{N} T_{n,\log(p_i)}(x)^{\alpha_i} \\ &= \prod_{i=1}^{N} \left(\left[n \left(2^{n+1}x + (1 - n2^{n+1}) \right) \right]^{\log(p_i)} \right)^{\alpha_i} \\ &= \left[n \left(2^{n+1}x + (1 - n2^{n+1}) \right) \right]^{\sum_{i=1}^{N} \log(p_i^{\alpha_i})} \\ &= \left[n \left(2^{n+1}x + (1 - n2^{n+1}) \right) \right]^{\log(\mathrm{ind}_{\mathbb{P}}(m))} \end{aligned}$$

Analogously, for each $x \in \left[n, n+\frac{1}{2^{n+1}}\right]$, we get

$$m(F_1,\ldots,F_N)(x) = \left[n\left(-2^{n+1}x + (1+n2^{n+1})\right)\right]^{\log(\mathrm{ind}_{\mathbb{P}}(m))}.$$

So

$$m(F_1,\ldots,F_N) = g_{\log(\mathrm{ind}_{\mathbb{P}}(m))}. \tag{5.11}$$

In particular, $m(F_1,\ldots,F_N) \in n\mathcal{BCL}^1([0,+\infty))$, and trivially $\mathcal{B} \subset CL^1([0,+\infty))$ (because continuity and integrability are stable under finite linear combinations). It only remains to prove that $(F_j)_j$ forms an algebraic independent set and that any element of $\mathcal{B}$ is not bounded. In order to prove these properties, let us take a non-trivial algebraic combination

$$F(x) := P(F_1,\ldots,F_N)(x) = \sum_{i=1}^{l} \lambda_i m_i(F_1,\ldots,F_N)(x).$$

For each $n \in \mathbb{N}$, we have by (5.11) that

$$F(n) = P(F_1,\ldots,F_N)(n) = \sum_{i=1}^{l} \lambda_i n^{\log(\mathrm{ind}_p(m_i))}. \tag{5.12}$$

By Remark 5.18, all exponents of the right hand part of (5.12) are positive and pairwise different, so $|F(n)| \to +\infty$ $(n \to \infty)$. In particular, $F \equiv 0$ if and only if $\lambda_i = 0$ for every $i = 1,\ldots,l$, and the algebra $\mathcal{B}$ is free. Hence, $F \in n\mathcal{BCL}^1([0,+\infty))$, and $\mathcal{B} \subset n\mathcal{BCL}^1([0,+\infty)) \cup \{0\}$. Thus, $n\mathcal{BCL}^1([0,+\infty))$ is strongly $\aleph_0$-algebrable. □

We finish this section stating that the family $n\mathcal{BCL}^1([0,+\infty))$ is not only algebraically large, but also large in a pure topological sense.

Theorem 5.20. *The family $n\mathcal{BCL}^1([0,+\infty))$ is residual in $CL^1([0,+\infty))$.*

Proof. Note that $CL^1([0,\infty))$ is Baire because, as it is easy to see, the distance d_{CL^1} given in (5.4) is complete. Indeed,

$$CL^1([0,\infty)) \setminus n\mathcal{BCL}^1([0,+\infty)) = \bigcup_{n=1}^{\infty} \mathcal{A}_n,$$

where for each $n \in \mathbb{N}$,

$$\mathcal{A}_n := \left\{ f \in CL^1([0,\infty)) : |f(x)| \leq n \text{ for all } x \geq 0 \right\}.$$

Since uniform convergence on compacta implies pointwise convergence, it is clear that each $\mathcal{A}_n$ is closed in $CL^1([0,\infty))$. Moreover, given $f \in \mathcal{A}_n$ and $\varepsilon > 0$, it is also easy but cumbersome to construct a function $g \in CL^1([0,+\infty))$ such that $d_{CL^1}(f,g) < \varepsilon$ but $|g(x_0)| > n$ for some $x_0 > 0$ (just "cut" a small enough part of g around x_0 and substitute it by a linear affine function joining continuously g to the point $(x_0, n+1)$). This implies that every $\mathcal{A}_n$ has empty interior, so that $CL^1([0,+\infty)) \setminus n\mathcal{BCL}^1([0,+\infty))$ is of first category, which proves that $n\mathcal{BCL}^1([0,+\infty))$ is residual. □

5.3 Sequences of continuous, unbounded and integrable functions

Until this point we have focused our attention on the algebraic structure of the family $n\mathcal{BCL}^1([0,+\infty))$ of continuous, unbounded, and integrable functions. In this Section we will step up and consider sequences of functions of the family $n\mathcal{BCL}^1([0,+\infty))$ with additional properties on its convergence. For this, let us introduce some useful notation. The family $n\mathcal{BCL}^1_0([0,+\infty))$ will denote all the sequences of continuous, unbounded and integrable functions on $[0,+\infty)$ converging to zero in the metric d_{CL^1}, that is:

$$\begin{aligned} n\mathcal{BCL}^1_0([0,+\infty)) := \Big\{ (f_n)_n \in CL^1([0,+\infty))^{\mathbb{N}} \,:\, & f_n \in n\mathcal{BCL}^1([0,+\infty)) \text{ for every } n \in \mathbb{N} \\ & \text{and } d_{CL^1}(f_n, 0) \to 0 \ (n \to \infty) \Big\} \\ = \Big\{ (f_n)_n \in c_0(CL^1([0,+\infty))) \,:\, & \lim_{x \to +\infty} |f_n(x)| = +\infty \text{ for all } n \in \mathbb{N} \Big\}. \end{aligned}$$

Recall that convergence to zero in the metric d_{CL^1} means that

$$\|f_n\|_1 + \sum_{k=1}^{+\infty} \frac{1}{2^k} \cdot \frac{\|f_n\|_{\infty,k}}{1 + \|f_n\|_{\infty,k}} \to 0 \qquad (n \to \infty)$$

that is, we have convergence to zero in the L^1-norm and uniformly convergence on compacta.

The next example shows that the family $n\mathcal{BCL}_0^1([0,+\infty))$ is not empty.

Example 5.21. Let T_n $(n \in \mathbb{N})$ be the triangular function already defined in Example 5.12. In order to construct our sequence of functions $(f_n)_n \in n\mathcal{BCL}_0^1([0,+\infty))$ we will truncate the series given by the triangles, that is, for every $n \in \mathbb{N}$ let $f_n : [0,+\infty) \to \mathbb{R}$ be defined as:

$$f_n(x) := \sum_{k=n}^{\infty} T_k(x) \qquad (x \geq 0).$$

Clearly, $(f_n)_n \subset n\mathcal{BCL}^1([0,+\infty))$. On the other hand, since each $f_n(x)$ is the tail of a convergent series in $CL^1([0,+\infty))$ (see Example 5.12), we have that $(f_n)_n$ converges to zero in the metric d_{CL^1} as $n \to \infty$. Thus, $(f_n)_n \in n\mathcal{BCL}_0^1([0,+\infty))$.

As in the case of single functions, we can prove the existence of a free-generated infinite dimensional algebra inside our family $n\mathcal{BCL}_0^1([0,+\infty))$.

Theorem 5.22. *The family of sequences $n\mathcal{BCL}_0^1([0,+\infty))$ is strongly $\aleph_0$-algebrable.*

Proof. Using the notation of the proof of Theorem 5.19, for each $j,n \in \mathbb{N}$, let

$$F_{j,n}(x) := \sum_{k=n}^{\infty} T_{k,\log(p_j)}(x).$$

Let $\mathcal{B}_0$ be the algebra generated by the sequences $\{(F_{j,n})_n : j \in \mathbb{N}\}$. Following the same argument as in Theorem 5.19 and taking the monomial

$$m(x_1,\ldots,x_N) = \prod_{i=1}^{N} x_i^{\alpha_i},$$

we have for each $n \in \mathbb{N}$ and $x \geq 0$, that

$$m(F_{1,n},\ldots,F_{N,n})(x) = \sum_{k=n}^{\infty} T_{k,\log(\mathrm{ind}_{\mathbb{P}}(m))}(x). \tag{5.13}$$

Therefore, the sequence $m((F_{1,n})_n, \ldots, (F_{N,n})_n)$ is the sequence of tails of a convergent series (in the topology generated by d_{CL^1}) and we have that

$$d_{CL^1}(m(F_{1,n}, \ldots, F_{N,n}), 0) \to 0 \qquad (n \to \infty).$$

In particular, $m((F_{1,n})_n, \ldots, (F_{N,n})_n) \in n\mathcal{BCL}_0^1([0,+\infty))$.

Finally, let $(F_n)_n$ be a non-trivial algebraic combination of the sequences $(F_{j,n})_n$ ($j \in \mathbb{N}$), that is, for each non-zero polynomial P in N variables without constant term and each $n \in \mathbb{N}$, we consider

$$F_n(x) := P(F_{1,n}, \ldots, F_{N,n})(x) = \sum_{i=1}^{l} \lambda_i m_i(F_{1,n}, \ldots, F_{N,n})(x),$$

where the λ_i's are not simultaneously zero.

For each $n \in \mathbb{N}$ and each $k \geq n$, by taking (5.13) into account, we obtain

$$F_n(k) = P(F_{1,n}, \ldots, F_{N,n})(k) = \sum_{i=1}^{l} \lambda_i k^{\log(\mathrm{ind}_{\mathbb{P}}(m_i))}.$$

Now, continuing in the same way as on the proof of Theorem 5.19, we get that $|F_n(k)| \to +\infty$ $(k \to \infty)$, hence $\mathcal{B}_0 \subset n\mathcal{BCL}_0^1([0,+\infty)) \cup \{0\}$ and $\mathcal{B}_0$ is a free algebra. Thus $n\mathcal{BCL}_0^1([0,+\infty))$ is strongly $\aleph_0$-algebrable. $\square$

An immediate consequence from the algebrability of $n\mathcal{BCL}_0^1([0,+\infty))$ is that mere lineability can be easily obtained.

Corollary 5.23. *The family of sequences* $n\mathcal{BCL}_0^1([0,+\infty))$ *is* $\aleph_0$*-lineable.*

In this case we could only infer the $\aleph_0$-lineability, since we obtained $\aleph_0$-algebrability. Nonetheless, by a direct approach, it is easy to attain the maximal dimension of the space $CL^1([0,+\infty))^{\mathbb{N}}$.

Theorem 5.24. *The family of sequences $n\mathcal{BCL}_0^1([0,+\infty))$ is maximal lineable.*

Proof. As in the proof of Theorem 5.19, for each $t \in \left[0, \frac{1}{8}\right)$ and each $n \in \mathbb{N}$ we define the functions

$$f_{n,t}(x) := \sum_{k=n}^{\infty} T_k(x-t) \qquad (x \geq 0),$$

and consider the set M_0 given by

$$M_0 := \text{span}\left\{(f_{n,t})_n \ : t \in \left[0, \tfrac{1}{8}\right)\right\}.$$

Following the same argument as in Theorem 5.14, we have that each $f_{n,t}$ is a continuous, unbounded and integrable function on $[0,+\infty)$, that is, $f_{n,t} \in n\mathcal{BCL}^1([0,+\infty))$ for all $t \in \left[0, \frac{1}{8}\right)$ and $n \in \mathbb{N}$, and that the sequences $\left\{(f_{n,t})_n \, : \, t \in \left[0, \frac{1}{8}\right)\right\}$ are linearly independent.

In addition, it is clear that the whole series $f_t(x) := \sum_{k=1}^{\infty} T_k(x-t)$ is convergent, both in L_1-norm and uniformly on compact sets of $[0,+\infty)$, so in d_{CL^1}. Hence $d_{CL^1}(f_{n,t}, 0) \to 0$ $(n \to \infty)$ for all $t \in \left[0, \frac{1}{8}\right)$, $M_0 \subset n\mathcal{BCL}_0^1([0,+\infty)) \cup \{0\}$ and $n\mathcal{BCL}_0^1([0,+\infty))$ is maximal lineable. □

Yet, in order to obtain the maximal dense-lineability, we need to establish a proper framework and the topology to be considered there. We consider the sequence space

$$c_0(CL^1([0,+\infty))) := \{(f_n)_n \subset CL^1([0,+\infty)) \, : \, d_{CL^1}(f_n, 0) \to 0 \ (n \to \infty)\},$$

endowed with the distance d_∞ given by

$$d_\infty((f_n)_n, (g_n)_n) := \sup_{n\in\mathbb{N}} d_{CL^1}(f_n, g_n), \tag{5.14}$$

for all $(f_n)_n, (g_n)_n \in c_0(CL^1([0,+\infty)))$. We know that $(c_0(CL^1([0,+\infty))), d_\infty)$ is a complete metric topological vector space, and thus, using again Baire's Theorem, we have that its dimension is $\mathfrak{c}$.

We define the set B_{00} as

$$B_{00} := \big\{(b_n)_n : \text{exists } n_0 \in \mathbb{N} \text{ such that } b_n \in B \text{ for all } n \leq n_0 \text{ and } b_n = 0 \text{ for all } n > n_0\big\},$$

where B is the dense subset of $CL^1([0,+\infty))$ defined in Lemma 5.15.

Lemma 5.25. *The space B_{00} is dense in $(c_0(CL^1([0,+\infty))), d_\infty)$.*

Proof. First, it is clear that $B_{00} \subset c_0(CL^1([0,+\infty)))$, since $B \subset CL^1([0,+\infty))$. Now, let $(f_n)_n \in c_0(CL^1([0,+\infty)))$. Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ with $d_{CL^1}(f_n, 0) < \varepsilon$ for all $n \geq n_0$. Furthermore, the denseness of B in $CL^1([0,+\infty))$ guarantees the existence of $b_1, \ldots, b_{n_0-1} \in B$ such that $d_{CL^1}(f_n, b_n) < \varepsilon$ $(1 \leq n \leq n_0 - 1)$. Finally, define $b_n := 0$ for $n \geq n_0$. It is clear that $(b_n)_n \in B_{00}$ and, by construction

$$d_\infty((f_n)_n, (b_n)_n) = \sup_{n\in\mathbb{N}} d_{CL^1}(f_n, b_n) = \max\left\{\max_{1\leq n\leq n_0-1} d_{CL^1}(f_n, b_n), \sup_{n\geq n_0} d_{CL^1}(f_n, 0)\right\} < \varepsilon,$$

and we are done. □

Theorem 5.26. *The family of sequences $n\mathcal{BCL}_0^1([0,+\infty))$ is maximal dense-lineable in $c_0(CL^1([0,+\infty)))$.*

Proof. We already have the maximal lineability of the family $n\mathcal{BCL}_0^1([0,+\infty))$ from Theorem 5.24 and, by Lemma 5.25, B_{00} is a dense vector subspace of $c_0(CL^1([0,+\infty)))$ (hence, dense-lineable). Moreover, each function b_n of a sequence $(b_n)_n \in B_{00}$ is bounded and $b_n = 0$ for n big enough, so $B_{00} + n\mathcal{BCL}_0^1([0,+\infty)) \subset n\mathcal{BCL}_0^1([0,+\infty))$, and $B_{00} \cap n\mathcal{BCL}_0^1([0,+\infty)) = \varnothing$. Finally, Theorem 2.7 tells us that $n\mathcal{BCL}_0^1([0,+\infty))$ is maximal dense-lineable in $c_0(CL^1([0,+\infty)))$. □

5.4 Final remarks

5.4.1 Maximal possible convergence

Observe that, in the above proofs, when we deal with the family $n\mathcal{BCL}_0^1([0,+\infty))$, and we consider the sequences $(f_{n,t})_n$ given by

$$f_{n,t}(x) = f_n(x-t) = \sum_{m=n}^{\infty} T_m(x-t),$$

for $t \in \left[0, \frac{1}{8}\right)$, $x \in [0,+\infty)$ and $n \in \mathbb{N}$, we only checked them to be in our space of functions $c_0(CL_0^1([0,+\infty)))$, that is, they had to converge to zero in L^1-norm plus uniformly on compact sets of $[0,+\infty)$. We want to comment that the convergence to zero can be strengthen.

Let $([0,+\infty), \mathcal{L}, m)$ be the Lebesgue measure space. Recall (see Definition 3.3) that a sequence of functions $(f_n)_n$ converges almost uniformly (on $[0,+\infty)$) to a function f if, for every $\varepsilon > 0$, there exists a set $E \subset [0,+\infty)$ with $m(E) < \varepsilon$ such that $f_n \to f$ uniformly on $[0,+\infty) \setminus E$.

In this line, all the sequences $(f_{n,t})_n$ constructed in the proof of Theorem 5.19 and Theorem 5.22 not only converge to zero uniformly on compacta on $[0,+\infty)$, but also converge almost uniformly to zero on $[0,+\infty)$. Indeed, following the notation used in this proof, and denoting by E_n the support of each function $f_{n,t}$, we obtain, because of the pairwise disjointness of the supports of the T_k's, that,

$$E_n := \operatorname{supp}(f_{n,t}) = \operatorname{supp}\left(\sum_{k=n}^{\infty} T_k(x-t)\right) = \bigcup_{k=n}^{\infty} \operatorname{supp}(T_k(x-t))$$

$$= \bigcup_{k=n}^{\infty} \left(k+t-\frac{1}{2^{k+1}}, k+t+\frac{1}{2^{k+1}}\right)$$

Clearly, for each natural number $n \in \mathbb{N}$ we have that $E_{n+1} \subset E_n$. Furthermore,

$$0 \le m(E_n) \le \sum_{k=n}^{\infty} \frac{1}{2^k} \to 0 \qquad (n \to \infty).$$

So, given any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $m(E_N) < \varepsilon$ and $f_{n,t}(x) = 0$ for all $n \geq N$ and all $x \in [0,+\infty) \setminus E_N$, that is, the sequence $(f_{n,t})_n$ converges almost uniformly to zero on $[0,+\infty)$.

On the other hand, it is clear that we cannot ask for the uniform convergence to zero, because each $(f_{n,t})_n$ is unbounded. The next proposition will tell us that almost uniform convergence is the "highest" level of convergence we can get. Remember that, in order to have the uniform a.e. convergence of $(f_n)_n$, we need to find a measurable set $E \subset [0,+\infty)$ with $m(E) = 0$ such that $(f_n)_n$ converges uniformly on $[0,+\infty) \setminus E$.

Proposition 5.27. *Let $(f_n)_n \subset C([0,+\infty))$ such that $f_n \to 0$ $(n \to \infty)$ pointwise on $[0,+\infty)$ and each f_n is unbounded. Then $f_n \not\to 0$ $(n \to \infty)$ uniformly a.e. on $[0,+\infty)$.*

Proof. By way of contradiction, assume that there exists a set $E \subset [0,+\infty)$ with $m(E) = 0$ such that $f_n \to 0$ $(n \to \infty)$ uniformly in $[0,+\infty) \setminus E$. Then, for $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $x \in [0,+\infty) \setminus E$ it holds that $|f_n(x)| \leq 1$.

Since f_n is unbounded for each $n \in \mathbb{N}$, there exists $x_n \in [0,+\infty)$ such that $|f_n(x_n)| > 2$, so $x_n \in E$, for $n \geq n_0$. Now, the continuity of each f_n and the fact that $m(E) = 0$ guarantee the existence of points $w_n \notin E$ but near enough to x_n (note that $m(E) = 0$ implies the denseness of $[0,+\infty) \setminus E$) such that $|f_n(w_n) - f_n(x_n)| < \frac{1}{2}$. But then, for $n \geq n_0$ we get

$$1 \geq |f_n(w_n)| \geq |f_n(x_n)| - |f_n(x_n) - f_n(w_n)| > 2 - \frac{1}{2} = \frac{3}{2},$$

which is a contradiction. □

As a consequence, we cannot obtain uniform a.e. convergence to zero of any sequence of $n\mathcal{BCL}_0^1([0,+\infty))$. Observe that Theorems 5.22 and 5.16 could be rewritten as follows.

Theorem 5.28. *The family of sequences* $(f_n)_n \subset CL^1([0,+\infty))$ *such that*

(a) f_n *is unbounded on* $[0,+\infty)$ *for every* $n \in \mathbb{N}$*, that is,*

$$\limsup_{x\to\infty} |f_n(x)| = +\infty, \qquad (n \in \mathbb{N}),$$

(b) $f_n \to 0$ $(n \to \infty)$ *almost uniformly on* $[0,+\infty)$*,*

(c) $f_n \not\to 0$ $(n \to \infty)$ *in* L^1*-norm* $[0,+\infty)$*,*

is strongly $\aleph_0$*-algebrable and maximal dense-lineable in* $c_0(CL^1([0,+\infty)))$*.*

We have that condition (a) implies in particular that the sequence $(f_n)_n$ does not converge uniformly to zero on $[0,+\infty)$, and by Proposition 5.27 the uniformly a.e. convergence is also not possible, that is, if we have convergence in d_{CL^1} to zero of a sequence of unbounded functions $(f_n)_n \subset CL^1([0,+\infty))$, the maximum degree of convergence can be the almost uniform convergence to zero on $[0,+\infty)$. So, in this sense, this result is sharp.

5.4.2 On the growth

When we deal with the families $n\mathcal{BCL}^1([0,+\infty))$ and $n\mathcal{BCL}_0^1([0,+\infty))$ of functions and sequences of functions, respectively, in the proofs of the results we look for suitable triangles: for their height and for their basis. Roughly speaking, the height let us control the unbounded behaviour, and the basis maintain the integrability. Now, the question is if we could ask for a stronger unboundedness. Recall that, given a continuous and non-decreasing function $\alpha : [0,+\infty) \to [1,+\infty)$, we say that a function $f \in C([0,+\infty))$ has growth α if

$$\limsup_{x\to+\infty} \frac{|f(x)|}{\alpha(x)} = +\infty.$$

We define the family $n\mathcal{BCL}^{1,\alpha}([0,+\infty))$ as the collection of all unbounded functions $f \in CL^1([0,\infty))$ that have growth α, that is,

$$n\mathcal{BCL}^{1,\alpha}([0,+\infty)) := \{f \in n\mathcal{BCL}^1([0,+\infty)) \,:\, f \text{ has growth } \alpha\}.$$

We can easily construct an example of this family, just by modifiying the triangular function of Example 5.12.

Example 5.29. Let $\alpha : [0,+\infty) \to [1,+\infty)$ be a continuous and non-decreasing function. For each $n \in \mathbb{N}$, we consider the triangular functions $T_{n,\alpha} : [0,+\infty) \to \mathbb{R}$ given by

$$T_{n,\alpha}(x) := \begin{cases} n\alpha(n)(2^{n+1}\alpha(n)x + (1 - n\alpha(n)2^{n+1})) & \text{if } x \in \left[n - \dfrac{1}{\alpha(n)2^{n+1}}, n\right), \\ n\alpha(n)(-2^{n+1}\alpha(n)x + (1 + n\alpha(n)2^{n+1})) & \text{if } x \in \left[n, n + \dfrac{1}{\alpha(n)2^{n+1}}\right], \\ 0 & \text{otherwise,} \end{cases}$$

Define the function $f_\alpha : [0,+\infty) \to \mathbb{R}$ by joining all the previous "triangles" of the functions $T_{n,\alpha}$, that is

$$f_\alpha(x) := \sum_{n=1}^{\infty} T_{n,\alpha}(x).$$

Similarly to Example 5.12, f_α is continuous and integrable. In fact,

$$\|f_\alpha\|_{L^1} := \sum_{n=1}^{\infty} \int_0^{+\infty} T_{n,\alpha}(x)\,dx = \sum_{n=1}^{\infty} \frac{n\alpha(n)}{\alpha(n)2^{n+1}} = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} < +\infty.$$

Moreover, if we consider the sequence $(x_n)_n \subset \mathbb{R}$ given by $x_n = n$ for every $n \in \mathbb{N}$, we have that

$$\frac{f_\alpha(x_n)}{\alpha(x_n)} = \frac{n\alpha(n)}{\alpha(n)} = n \to +\infty \qquad (n \to \infty),$$

which means that

$$\limsup_{x\to+\infty} \frac{|f_\alpha(x)|}{\alpha(x)} = +\infty.$$

Thus, the above constructed function f_α is an example of a continuous, unbounded and integrable function on $[0,+\infty)$ with growth α.

Adapting the proofs of Theorems 5.14 and 5.16 with the function f_α of the above example, we get the next result about the linear structure of $n\mathcal{BCL}^{1,\alpha}([0,+\infty))$.

Theorem 5.30. *Let $\alpha : [0,+\infty) \to [1,+\infty)$ be a continuous and non-decreasing function. The family $n\mathcal{BCL}^{1,\alpha}([0,+\infty))$ of continuous and integrable functions on $[0,+\infty)$ with growth α is maximal dense-lineable in $CL^1([0,+\infty))$.*

Proof. Let $\alpha : [0,+\infty) \to [1,+\infty)$ be a continuous and non-decreasing function. We look for a vector subspace M of dimension $\mathfrak{c}$ such that $M \subset n\mathcal{BCL}^{1,\alpha}([0,+\infty)) \cup \{0\}$. Following the steps of the proof in Theorem 5.16, consider the triangular function f_α given in the Example 5.29 and the vector subspace

$$M := \text{span}\left\{f_{\alpha,t} \ : \ t \in \left[0, \tfrac{1}{8}\right)\right\},$$

where $f_{\alpha,t}$ are the modified functions. The linearly independence and the integrability of each member of M follows the same argument as the previous proofs. It only remains to show that every finite linear combination has still growth α, and the unboundedness of each mentioned linear combination $c_1 f_{\alpha,t_1} + \cdots + c_s f_{\alpha,t_s}$. Note that we can assume $c_s \neq 0$. Since $t_s > t_i$ for all $i = 1, 2, \cdots, s-1$ and $\alpha(n) \geq 1$, there is $N \in \mathbb{N}$ such that the support of the triangles $T_{n,\alpha}(x - t_s)$ is disjoint with the support of the rest of triangles $T_{n,\alpha}(x - t_i)$ for any $n \geq N$, $i = 1, \ldots, s-1$.

Thus, by taking for any $n > N$ the point $x_n := n + t_s$, we have

$$|c_1 f_{\alpha,t_1}(x_n) + c_2 f_{\alpha,t_2}(x_n) + \cdots + c_s f_{\alpha,t_s}(x_n)| = |c_s| n\alpha(n) \to +\infty \qquad (n \to \infty).$$

Thus, we obtain that

$$\frac{|c_1 f_{\alpha,t_1}(x_n) + c_2 f_{\alpha,t_2}(x_n) + \cdots + c_s f_{\alpha,t_s}(x_n)|}{\alpha(x_n)} = \frac{|c_s| n\alpha(n)}{\alpha(n)} = |c_s| n \to +\infty \qquad (n \to \infty).$$

Hence, the family $n\mathcal{BCL}^{1,\alpha}([0,+\infty))$ is maximal-lineable.

In order to obtain the maximal dense-lineability we just have to consider $A = n\mathcal{BCL}^{1,\alpha}([0,+\infty))$, B as the dense subset given in the proof of Theorem 5.16 and $\kappa = \mathfrak{c}$, and apply Theorem 2.7. □

Theorem 5.31. *Let $\alpha : [0,+\infty) \to [1,+\infty)$ be a continuous and non-decreasing function. The family $n\mathcal{BCL}^{1,\alpha}([0,+\infty))$ is strongly $\aleph_0$-algebrable.*

Proof. Let $\alpha : [0,+\infty) \to [1,+\infty)$ be a continuous and non-decreasing function. For any $n \in \mathbb{N}$, $p \geq 1$, we consider the "triangles" on $[0,+\infty)$ given by

$$T_{\alpha,n,p}(x) = \begin{cases} (n\alpha(n))^p(\alpha(n)2^{n+1}x + (1 - n\alpha(n)2^{n+1}))^p & \text{if } x \in \left[n - \frac{1}{\alpha(n)2^{n+1}}, n\right), \\ (n\alpha(n))^p(-\alpha(n)2^{n+1}x + (1 + n\alpha(n)2^{n+1}))^p & \text{if } x \in \left[n, n + \frac{1}{\alpha(n)2^{n+1}}\right], \\ 0 & \text{otherwise,} \end{cases}$$

and we define the functions $g_{\alpha,p} : [0,+\infty) \to \mathbb{R}$ as:

$$g_{\alpha,p}(x) = \sum_{n=1}^{\infty} T_{\alpha,n,p}(x).$$

Let $\mathbb{P} = (p_j)_j$ be the increasing sequence of prime numbers greater than 3, and let us define for each $j \in \mathbb{N}$ the function $F_{\alpha,j}$ by

$$F_{\alpha,j}(x) := \sum_{n=1}^{\infty} T_{\alpha,n,\log p_j}(x) = g_{\alpha,\log p_j}(x).$$

Note that $(F_{\alpha,j})_j \subset n\mathcal{BCL}^{1,\alpha}([0,+\infty))$. Let $\mathcal{B}_\alpha$ be the algebra generated by $(F_{\alpha,j})_j$, that is,

$$\mathcal{B}_\alpha = \big\{P(F_{\alpha,1},\ldots,F_{\alpha,N}) : \ P \text{ is a polynomial in } N \text{ variables without constant term, } N \in \mathbb{N}\big\}.$$

Following the same steps as in Theorem 5.19 we obtain that $\mathcal{B}_\alpha$ is the desired infinitely generated algebra in $n\mathcal{BCL}^{1,\alpha}([0,+\infty))$. □

Observe that these two last theorems are both a generalization of Theorem 5.16 and Theorem 5.19, just by taking the growth function as $\alpha(x) \equiv 1$. Furthermore, we can now select a wide plethora of families of continuous and integrable functions with a nice algebraic structure that grow exponentially or even faster, for which we just have to choose $\alpha(x) = e^x$ or $\alpha(x) = e^{e^x}$ or even "bigger".

This generalization is not only focused on the family $n\mathcal{BCL}^1([0,+\infty))$ of functions of Section 5.2, but also for the family of sequences of functions $n\mathcal{BCL}_0^1([0,+\infty))$ converging to zero that have growth α.

Furthermore, from Proposition 5.27, we know that the optimal mode of convergence that we can achieve for these sequences is the almost uniform convergence (to zero) on $[0,+\infty)$, and we can state the next result.

Theorem 5.32. *Let $\alpha : [0,+\infty) \to [1,+\infty)$ be a continuous and non-decreasing function. The family of sequences $(f_n)_n$ of continuous and integrable functions on $[0,+\infty)$ with growth α such that $f_n \to 0$ $(n \to \infty)$ in L^1-norm and almost uniformly on $[0,+\infty)$ is maximal dense-lineable in $c_0(CL^1([0,+\infty)))$ and strongly $\aleph_0$-algebrable.*

5.4.3 Smoothness

In the previous sections of this Chapter we have been choosing triangles at some specific points, and defining them as zero elsewhere in order to obtain continuity of our functions and sequences of functions. For this purpose we can use instead of triangular functions the next ones:

$$\Phi_n(x) := \begin{cases} n \cdot e^{1-\frac{1}{1-(2^{n+1}(x-n))^2}} & \text{if } x \in \left(n - \frac{1}{2^{n+1}}, n + \frac{1}{2^{n+1}}\right) \\ 0 & \text{otherwise,} \end{cases}$$

For every $n \in \mathbb{N}$, each Φ_n is integrable, $|\Phi_n(x)| \leq n$ in $\left(n - \frac{1}{2^{n+1}}, n + \frac{1}{2^{n+1}}\right)$, $\Phi_n(n) = n$, and of class $\mathcal{C}^\infty$ on $[0,+\infty)$. So, we can then define the family

$$n\mathcal{BC}^\infty\mathcal{L}^1([0,+\infty)) := n\mathcal{BCL}^1([0,+\infty)) \cap C^\infty([0,+\infty)).$$

Again, slightly modifications on the triangular functions allow us to get "smooth" functions and not only continuous (observe that every function in Section 5.2 is not differentiable, for example, at any vertex of the triangles).

Theorem 5.33. *The family $n\mathcal{BC}^\infty\mathcal{L}^1([0,+\infty))$ of unbounded and integrable functions on $[0,+\infty)$ of class $\mathcal{C}^\infty$ is maximal lineable.*

Proof. Following the steps of the proof of Theorem 5.14, consider the vector space M given by

$$M := \operatorname{span}\left\{f_t \,:\, t \in \left[0, \tfrac{1}{8}\right)\right\},$$

where f_t is the infinite sum of the Φ_n's traslated to the right by the factor $t \in \left[0\tfrac{1}{8}\right)$. It is obvious that every finite linear combination of elements of M is of class $\mathcal{C}^\infty$ and integrable, hence it only remains to prove the unboundedness of each member of M. Let us assume that there are $0 \le t_1 < t_2 < \cdots < t_s < \frac{1}{8}$ and scalars $c_1, c_2, \ldots, c_s \in \mathbb{R}$ such that $c_s \neq 0$, and there exists $N \in \mathbb{N}$ with $\min\{t_s - t_i \,:\, i = 1, 2, \ldots, s-1\} > \frac{1}{2^N}$. Hence, by taking for any $n > N$ the point $x_n = n + t_s$, we have

$$|c_1 f_{t_1}(x_n) + c_2 f_{t_2}(x_n) + \cdots + c_s f_{t_s}(x_n)| = |c_s| n e^{1-\frac{1}{1-(2^{n+1}t_s)^2}} \to +\infty \ (n \to \infty).$$

Thus, the family $n\mathcal{BC}^\infty\mathcal{L}^1([0,+\infty))$ is maximal lineable. □

This result can be also extended to sequences of functions, using a similar construction as in the one provided in Theorem 5.24.

Theorem 5.34. *The family $n\mathcal{BC}^\infty\mathcal{L}^1_0([0,+\infty))$ of sequences $(f_n)_n$ of unbounded and integrable functions on $[0,+\infty)$ of class $\mathcal{C}^\infty$ such that $f_n \to 0$ $(n \to \infty)$ in L^1-norm and almost uniformly on $[0,+\infty)$ is maximal lineable.*

Proof. Following the same steps as in Theorem 5.24, and using the exponential-like functions defined at the beginning, we define the sequence $f_{n,t}$ for every $n \in \mathbb{N}$ and $t \in \left[0, \frac{1}{8}\right)$ as

$$f_{n,t}(x) = \sum_{k=n}^{\infty} \Phi_k(x - t).$$

So, if we consider the vector subspace

$$M_0 := \operatorname{span}\left\{(f_{n,t})_n \,:\, t \in \left[0, \tfrac{1}{8}\right)\right\},$$

we have that each $f_{n,t}$ is of class $\mathcal{C}^\infty$, unbounded and integrable on $[0,+\infty)$. In addition, it is clear that the whole series $\sum_{k=1}^{\infty} \Phi_k(x)$ converges to zero both in L^1-norm and almost uniformly on $[0,+\infty)$. Thus, $n\mathcal{BC}^\infty\mathcal{L}_0^1([0,+\infty))$ is maximal lineable. $\square$

Chapter 6

Anti M-Weierstrass sequences of functions

6.1 Concepts and examples

In the previous chapters we have considered sequences of functions with different properties, and we studied the linear and algebraic size of these families. Now, we will turn our attention into series of functions.

Let $X \neq \varnothing$ be a non-empty set, and $f_n : X \to \mathbb{K}$ $(n \in \mathbb{N})$ be a sequence of functions. Usually, when the uniform convergence of the series of functions $\sum_{n=1}^{\infty} f_n(x)$ has to be studied, the first tool one thinks about is the well-known Weierstrass M-test (see for instance [3, §9.6]).

Theorem 6.1 (Weierstrass M-test). *In the above conditions, if there exists a sequence* $(M_n)_n \subset (0, +\infty)$ *such that* $\sum_{n=1}^{\infty} M_n < +\infty$ *and* $|f_n(x)| \leq M_n$ *for every* $x \in X$ *and every* $n \in \mathbb{N}$*, the series* $\sum_{n=1}^{\infty} f_n(x)$ *is absolutely and uniformly convergent on* X*.*

However, the reciprocal is false in general, that is, it is possible to find series of functions that are uniformly convergent but do not satisfy all the hypothesis of the Weierstrass M-test. More precisely, the condition about the majorant sequence can be dropped without losing the uniform convergence of the series of functions. The upcoming example (see [23, Chapter 1, Example 10]) will show this situation.

Example 6.2. Let $f_n : [0,1] \to \mathbb{R}$ ($n \in \mathbb{N}$) be the sequence of (continuous) functions given by

$$f_n(x) := \begin{cases} \dfrac{1}{n}\sin^2(2^{n+1}\pi x) & \text{if } x \in \left(\dfrac{1}{2^{n+1}}, \dfrac{1}{2^n}\right), \\ 0 & \text{if } x \in \left[0, \dfrac{1}{2^{n+1}}\right] \cup \left[\dfrac{1}{2^n}, 1\right]. \end{cases}$$

Since $f_n(x) \geq 0$ for all $x \in [0,1]$, the normal convergence of the series and the absolute convergence are equal for this series.

If $x \in \left(0, \frac{1}{2}\right) \setminus \{2^{-n} : n \in \mathbb{N}\}$, there is a unique $n_0 \in \mathbb{N}$ such that $x \in \left(\frac{1}{2^{n_0+1}}, \frac{1}{2^{n_0}}\right)$. So,

$$\sum_{n=1}^{\infty} f_n(x) = f_{n_0}(x) = \frac{1}{n_0}\sin^2(2^{n_0+1}\pi x).$$

As $f_n(x) = 0$ elsewhere, we get the absolute convergence of our series for any $x \in [0,1]$.

In order to obtain the uniform convergence we will apply the Cauchy's Criterion for series (see for instance [3, §8.8]). Fix $n, p \in \mathbb{N}$ and $x \in [0,1]$. As for the whole series, there is at most one $n_x \in \mathbb{N}$ with $n+1 \leq n_x \leq n+p$ such that $f_{n_x}(x) \neq 0$, and, in this case,

$$\left|\sum_{k=n+1}^{n+p} f_k(x)\right| < |f_{n_x}(x)| \leq \frac{1}{n_x} \leq \frac{1}{n+1} < \frac{1}{n}.$$

for every $n, p \in \mathbb{N}$ and every $x \in [0,1]$.

Now, given any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\dfrac{1}{n} < \varepsilon$ for every $n \geq N$. So, for every $n, p \in \mathbb{N}$ with $n \geq N$,

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| = \frac{1}{n} < \varepsilon,$$

and we have the uniform convergence of the series on $[0, 1]$.

Finally, for any $n \in \mathbb{N}$,

$$f_n \left(\frac{3}{2^{n+2}} \right) = \frac{1}{n} \sin^2 \left(\frac{3}{2} \pi \right) = \frac{1}{n},$$

so $\|f_n\|_\infty = \frac{1}{n}$, and $\sum_{n=1}^{\infty} \|f_n\|_\infty = +\infty$.

A sequence of functions as given in Example 6.2 will be called an *Anti M-Weierstrass* sequence. Concretely:

Definition 6.3. *Let $f_n : [a, b] \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence of continuous functions on $[a, b]$. We will denote by $\mathcal{AMW}([a, b])$ (or just $\mathcal{AMW}$, if there is no possibility of confusion on the interval) the family of Anti M-Weierstrass sequences on $[a, b]$, that is, the family of sequences $(f_n)_n$ such that $\sum_{n=1}^{\infty} f_n(x)$ is absolutely and uniformly convergent on $[a, b]$, but $\sum_{n=1}^{\infty} \|f_n\|_\infty$ diverges.*

Thanks to Example 6.2 we already know that $\mathcal{AMW}([0, 1]) \neq \varnothing$. In order to look for more general examples we introduce the following helpful family.

Definition 6.4. *Let $\mathcal{F}$ be the family of all sequences of continuous functions $u_n : [a, b] \to \mathbb{R}$ such that:*

(a) *The supports are pairwise disjoint, that is,*

$$supp(u_n) \cap supp(u_m) = \varnothing, \qquad n \neq m,$$

(b) *The sequence* $(||u_n||_\infty)_n$ *is bounded and far from zero, that is,*

$$0 < \inf_{n\in\mathbb{N}} ||u_n||_\infty \leq \sup_{n\in\mathbb{N}} ||u_n||_\infty < +\infty.$$

Observe that any element of this family $\mathcal{F}$ allows us construct a series of functions with a very concrete convergence.

Lemma 6.5. *Let* $(u_n)_n \in \mathcal{F}$ *and* $(a_n)_n \subset \mathbb{R}$*. Then:*

(a) *The series* $\sum_{n=1}^{\infty} a_n u_n(x)$ *converges absolutely on* $[a,b]$*.*

(b) *The series* $\sum_{n=1}^{\infty} a_n u_n(x)$ *converges uniformly on* $[a,b]$ *if and only if* $(a_n)_n \in c_0$*.*

(c) *The series* $\sum_{n=1}^{\infty} \|a_n u_n\|_\infty < +\infty$ *if and only if* $(a_n)_n \in \ell_1$*.*

Proof.

(a) The absolute convergence of the series is immediate, since the disjointness of the supports of the u_n's implies that, for a fixed $x_0 \in [a,b]$, either $u_n(x_0) = 0$ for all $n \in \mathbb{N}$, or there exists only one $n_0 \in \mathbb{N}$ such that $x_0 \in \operatorname{supp}(u_{n_0})$, and

$$\sum_{n=1}^{\infty} |a_n u_n(x_0)| = |a_{n_0} u_{n_0}(x_0)| \qquad (< +\infty).$$

(b) Firstly, consider the case when $(a_n)_n \in c_0$. Because $(u_n)_n \in \mathcal{F}$, then $M := \sup_{n\in\mathbb{N}} ||u_n||_\infty \in (0, +\infty)$. Given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n| < \frac{\varepsilon}{M}$ for any $n \geq N$.

Thus, as for each $x \in [a,b]$ there is at most one $n_0 \geq N$ such that $x \in \operatorname{supp}(u_{n_0})$,

$$\left|\sum_{n=N}^{\infty} a_n u_n(x)\right| = |a_{n_0} u_{n_0}(x)| \leq \frac{\varepsilon}{M} \cdot M = \varepsilon.$$

Hence, the uniform convergence of the series on $[a,b]$ is obtained.

Reciprocally, as the series is uniformly convergent, we have $a_n u_n(x) \to 0$ $(n \to \infty)$ uniformly on $[a,b]$. Because $(u_n)_n \in \mathcal{F}$, then $L := \inf_{n\in\mathbb{N}} ||u_n||_\infty > 0$. Given any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n u_n(x)| < \varepsilon \cdot L$ for every $x \in [a,b]$ and every $n \geq n_0$. But for any $n \geq n_0$, there exists $x_n \in [a,b]$ such that $u_n(x_n) \geq L$, hence

$$|a_n| = \frac{|a_n u_n(x_n)|}{|u_n(x_n)|} \leq \frac{\varepsilon \cdot L}{L} = \varepsilon,$$

and $a_n \to 0$ $(n \to \infty)$.

(c) As $(u_n)_n \in \mathcal{F}$, we have

$$0 < L := \inf_{n\in\mathbb{N}} ||u_n||_\infty \leq \sup_{n\in\mathbb{N}} ||u_n||_\infty =: M < +\infty,$$

and then,

$$0 \leq L \cdot |a_n| \leq \|a_n u_n\|_\infty \leq M \cdot |a_n| < +\infty, \qquad (n \in \mathbb{N}).$$

Thus, by comparison test, $(a_n)_n \in \ell_1$ if and only if $(||a_n u_n||_\infty)_n \in \ell_1$. □

As a consequence of this lemma, whenever we have a sequence of functions $(u_n)_n \in \mathcal{F}$ and a sequence of coefficients $(a_n)_n \in c_0 \setminus \ell_1$, the sequence of functions $(a_n u_n)_n$ belongs to the family $\mathcal{AMW}([a,b])$. In particular, it allows us to provide a wide plethora of sequences of functions with this behaviour.

Examples 6.6.

1. The first example of an Anti M-Weierstrass sequence given in Example 6.2 can be rewritten in terms of Lemma 6.5. For this, for every $n \in \mathbb{N}$ and $x \in [0,1]$ we just have to consider

$$a_n = \frac{1}{n} \qquad \text{and} \qquad u_n(x) = \sin^2(2^{n+1}\pi x)\chi_{\left(\frac{1}{2^{n+1}},\frac{1}{2^n}\right)}(x).$$

2. For any interval $[a,b]$ we can also adapt this example. For this, consider the sequence $u_n : [a,b] \to \mathbb{R}$ $(n \in \mathbb{N})$ of continuous functions given by

$$u_n(x) = \begin{cases} \sin\left(2^n\pi\left(\dfrac{x-a}{b-a}\right) - \pi\right) & \text{if } x \in I_n, \\ \\ 0 & \text{elsewhere,} \end{cases}$$

where $I_n = \left(\frac{(2^n-1)a+b}{2^n}, \frac{(2^{n-1}-1)a+b}{2^{n-1}}\right)$ for every $n \in \mathbb{N}$.

The I_n's are pairwise disjoint, and so the u_n's have disjoint support. Furthermore, since $\|u_n\|_\infty = 1$ for all $n \in \mathbb{N}$, we immediately obtain that $(u_n)_n \in \mathcal{F}$.

Finally, thanks to Lemma 6.5, whenever we choose a sequence $a = (a_n)_n \in c_0 \setminus \ell_1$ we will have absolute and uniform convergence on $[a, b]$ of the series $\sum_{n=1}^{\infty} a_n u_n(x)$, and there will not be a mayorant sequence, since for

$$x_n = \frac{(2^{n-2} - 1)a + b}{2^{n-1}} \in I_n,$$

we have that

$$a_n u_n(x_n) = a_n \sin\left(\frac{\pi}{2}\right) = a_n,$$

and the series $\sum_{n=1}^{\infty} a_n$ diverges. Thus, the sequence $(a_n u_n)_n \in \mathcal{AMW}([a, b])$.

In the above examples, the element of $\mathcal{F}$ comes from the sinus function, but it is possible to consider any other continuous functions.

Example 6.7. Let $f \in C([a, b]) \setminus \{0\}$. Let $\Lambda := (\alpha_n)_n$ be any sequence of scalars such that

$$a = \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < \cdots \to b \qquad (n \to \infty).$$

For each $n \in \mathbb{N}$ we define the function $u_n^{\Lambda,f} : [a, b] \to \mathbb{R}$ by

$$u_n^{\Lambda,f}(x) := \begin{cases} f(a) \cdot \dfrac{x - \alpha_{3n-2}}{\alpha_{3n-1} - \alpha_{3n-2}} & \text{if } \ x \in [\alpha_{3n-2}, \alpha_{3n-1}] \\[2ex] f\left(a + (b-a) \cdot \dfrac{x - \alpha_{3n-1}}{\alpha_{3n} - \alpha_{3n-1}}\right) & \text{if } \ x \in [\alpha_{3n-1}, \alpha_{3n}] \\[2ex] f(b) \cdot \dfrac{\alpha_{3n+1} - x}{\alpha_{3n+1} - \alpha_{3n}} & \text{if } \ x \in [\alpha_{3n}, \alpha_{3n+1}] \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Observe that with this construction we are adjusting the graph of f into the interval $[\alpha_{3n-1}, \alpha_{3n}]$, and extending it continuously and affine-linearly to zero in the left and right intervals. It is clear that $u_n^{\Lambda,f} \in C([a,b])$; $\mathrm{supp}(u_n^{\Lambda,f}) \subset (\alpha_{3n-2}, \alpha_{3n+1})$ for each $n \in \mathbb{N}$, so the supports of the $u_n^{\Lambda,f}$'s are pairwise disjoint; and $||u_n^{\Lambda,f}||_\infty = ||f||_\infty \in (0, +\infty)$ for any $n \in \mathbb{N}$. So, trivially, $(u_n^{\Lambda,f})_n \in \mathcal{F}$. Thus, whenever we choose a sequence of coefficients $(a_n)_n \in c_0 \setminus \ell_1$, the sequence of functions $(a_n u_n^{\Lambda,f})_n \in \mathcal{AMW}([a,b])$ for every continuous function f. In particular, $\mathrm{card}(\mathcal{AMW}([a,b])) = \mathfrak{c}$.

Observe that if $f \equiv 0$, with the same definition, $u_n^{\Lambda,f} \equiv 0$ for any $n \in \mathbb{N}$. But in this case, $(u_n^{\Lambda,f})_n \notin \mathcal{F}$. The above example let us define the following application.

Proposition 6.8. *Let $\Lambda := (\alpha_n)_n$ be any sequence of scalars such that*

$$a = \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < \cdots \to b \qquad (n \to \infty).$$

Then, the application

$$\begin{array}{rccc} \mathcal{J}_\Lambda : & C([a,b]) & \to & \mathcal{F} \cup \{0\} \\ & f & \mapsto & \mathcal{J}_\Lambda(f) := (u_n^{\Lambda,f})_n, \end{array}$$

is well-defined, linear, injective, and satisfies the following properties:

(1) $u_n^{\Lambda,f}(\alpha_{3n-1}) = f(a)$ *for any* $n \in \mathbb{N}$ *and any* $f \in C([a,b])$*;*

(2) *supp*$(u_n^{\Lambda,f}) \subset (\alpha_{3n-2}, \alpha_{3n+1})$ *for any* $f \in C([a,b])$ *and any* $n \in \mathbb{N}$.

(3) *For any $n \in \mathbb{N}$ there exists a linear affine transformation τ_n such that $\tau_n([\alpha_{3n-1}, \alpha_{3n}]) = [a,b]$ and $u_n^{\Lambda,f} = f \circ \tau_n$ for each $x \in [\alpha_{3n-1}, \alpha_{3n}]$ and each $f \in C([a,b])$.*

(4) $||u_n^{\Lambda,f}||_\infty = ||f||_\infty$ *for any* $n \in \mathbb{N}$ *and any* $f \in C([a,b])$.

Proof. By Example 6.7 we have that $\mathcal{J}_\Lambda$ is well-defined (even for $f \equiv 0$), and we have the properties (1) to (4). The injectivity follows from the definition of $u_n^{\Lambda,f}$ in $[\alpha_{3n-1}, \alpha_{3n}]$. Finally, the linearity is clear, because of the linearity of the piecewise linear affine transformation, concretely, for any $f, g \in C([a,b])$, $\lambda, \mu \in \mathbb{R}$, $c \in [a,b]$ and $\tau(x) = sx + t$, we have

$$(\lambda f + \mu g)(c) \cdot (sx + t) = \lambda(f(c)) \cdot (sx + t)) + \mu(g(c)) \cdot (sx + t).$$

□

From Proposition 6.8, if $\mathcal{C}$ is a linear vector subspace in $C([a,b])$ with dimension κ, then $\mathcal{J}_\Lambda(\mathcal{C})$ is a linear vector subspace in $\mathcal{F} \cup \{0\}$ with dimension κ.

6.2 Lineability within $\mathcal{AMW}$

Observe that in Example 6.6, we were able to find many Anti M-Weiertrass sequences just by changing the sequence of coefficients $(a_n)_n \in c_0 \setminus \ell_1$ or the sequence of functions $(u_n)_n \in \mathcal{F}$. In particular, the possibility to focus on either the coefficients or the functions will allows us to find concrete linear structures in both cases.

Theorem 6.9. *Let M be a linear vector subspace such that $M \subset (c_0 \setminus \ell_1) \cup \{0\}$ and dim$(M) = \kappa$. Let $\{(a_n^i)_n\}_{i \in I}$ be a generator system of M with* card$(I) = \kappa$. *Then for any prefixed sequence of functions $(u_n)_n \in \mathcal{F}$, the subspace*

$$\mathcal{M} := \big\{(a_n u_n)_n : \ (a_n)_n \in M\big\}$$

is a linear vector subspace of dimension κ, generated by $\{(a_n^i u_n)_n\}_{i \in I}$, such that $\mathcal{M} \subset \mathcal{AMW} \cup \{0\}$.

Proof. Observe that for any sequence $(a_n u_n)_n \in \mathcal{M}$, there exists $\{\lambda_i\}_{i \in I} \subset \mathbb{R}$ such that, for any $n \in \mathbb{N}$,

$$a_n = \sum_{i \in I} \lambda_i a_n^i.$$

So,

$$a_n u_n(x) = \left(\sum_{i \in I} \lambda_i a_n^i\right) u_n(x) = \sum_{i \in I} \lambda_i a_n^i u_n(x) \qquad (x \in [a,b]). \tag{6.1}$$

But $(u_n)_n \in \mathcal{F}$, so $||u_n||_\infty > 0$ for all $n \in \mathbb{N}$. In particular, $(a_n u_n)_n \equiv 0$ if and only if $\lambda_i = 0$ for any $i \in I$. Hence (6.1) and Lemma 6.5 give us trivially the statement. □

Theorem 6.10. *Let $(a_n)_n \in c_0 \setminus \ell_1$ be a prefixed sequence of scalars. Let U be a linear vector subspace of dimension κ, generated by $\{(u_n^i)_n\}_{i\in I}$ with $\mathrm{card}(I) = \kappa$, such that $U \subset \mathcal{F} \cup \{0\}$. Then*

$$\mathcal{U} := \{(a_n u_n)_n : \ (u_n)_n \in U\}$$

is a linear vector subspace of dimension κ, generated by $\{(a_n u_n^i)_n\}_{i\in I}$, such that $\mathcal{U} \subset \mathcal{AMW} \cup \{0\}$.

Proof. We have that any sequence $(a_n u_n)_n \in \mathcal{U}$ can be written as

$$a_n u_n(x) = a_n \cdot \sum_{i\in I} \lambda_i u_n^i(x) = \sum_{i\in I} \lambda_i a_n u_n^i(x) \qquad (x \in [a,b],\ n \in \mathbb{N}),$$

where $\{\lambda_i\}_{i\in I} \subset \mathbb{R}$. Observe that because $(a_n)_n \notin \ell_1$ there are infinitely many n such that $a_n \neq 0$, so $(a_n u_n)_n \equiv 0$ if and only if $\lambda_i = 0$ for any $i \in I$. As in the above proof, because of $U \subset \mathcal{F} \cup \{0\}$ and by Lemma 6.5, we are done. □

In particular, if in Theorem 6.9 we fix a sequence of functions $(u_n)_n \in \mathcal{F}$, and given a vector space of dimension κ in $(c_0 \setminus \ell_1) \cup \{0\}$ we are able to construct a vector space of the same dimension in the family $\mathcal{AMW} \cup \{0\}$. In the same way, if in Theorem 6.10 we fix a sequence of coefficients $(a_n)_n \in c_0 \setminus \ell_1$, and given a vector space of dimension κ in $\mathcal{F} \cup \{0\}$ we explicitely obtain (again) a vector space of the same dimension in $\mathcal{AMW} \cup \{0\}$. As a consequence of both theorems we obtain the following result about the linear size of the family of Anti M-Weierstrass sequences.

Corollary 6.11. *The family $\mathcal{AMW}$ is maximal lineable.*

Proof. Using Theorems 6.9 and 6.10 we can give two different proofs of the maximal lineability of the family $\mathcal{AMW}$.

(1) Let

$$M := \text{span}\left\{\left(\tfrac{1}{n^c}\right)_n \,:\, c \in (0,1)\right\}.$$

It is clear that $M \subset c_0 \setminus \ell_1$. By Lemma 5.6, it turns out that $\dim(M) = \mathfrak{c}$. Now, taking a sequence of functions $(u_n)_n \in \mathcal{F}$ (for instance, some of the ones previously defined in Examples 6.2 or 6.6), an application of Theorem 6.9 gives us the maximal lineability of the family $\mathcal{AMW}$.

(2) Let $\Lambda := (\alpha_n)_n$ be any sequence of scalars such that

$$a = \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_n < \cdots \to b \qquad (n \to \infty).$$

Consider the vector space $U \subset \mathcal{F} \cup \{0\}$

$$U := \text{span}\left\{\mathcal{J}_\Lambda(x^c) \,:\, c \in (0,+\infty)\right\}.$$

By Proposition 6.8, $\mathcal{J}_\Lambda$ is linear and injective, so $\dim(U) = \mathfrak{c}$.

Now, taking a sequence of coefficients $(a_n)_n \in c_0 \setminus \ell_1$ (for instance, $a_n = \frac{1}{n}$ does the job), an application of Theorem 6.10 gives us the maximal lineability of the family $\mathcal{AMW}$. □

With this last result, we conclude that the family $\mathcal{AMW}$ attains its maximum dimension in terms of Lineability. However, in order to find a dense vector space we need to take a suitable topology to work with. For this, recall that $c_0(C([a,b]))$ denotes the family of all sequences of continuous functions on $[a,b]$ converging to zero in the supremum norm, which becomes a separable Banach space when endowed with the natural norm $\|(f_n)_n\| := \sup_{n\in\mathbb{N}} \|f_n\|_\infty$. Clearly, $\mathcal{AMW} \subset c_0(C([a,b]))$, so we can focus our attention on density properties of the family of Anti M-Weierstrass sequences of functions.

Theorem 6.12. *The family $\mathcal{AMW}$ of Anti-M Weierstrass sequences is maximal dense-lineable in $c_0(C([a,b]))$.*

Proof. From Corollary 6.11, the family $\mathcal{AMW}$ is maximal lineable. Now, the space $c_{00}(C([a,b]))$ of all sequences of continuous functions on $[a,b]$ that are eventually vanishing, that is,

$$c_{00}(C[a,b]) = \big\{(f_n)_n \subset C([a,b]) \,:\, \text{there exists } N \in \mathbb{N} \text{ such that } f_n = 0 \text{ for all } n \geq N\big\}$$

is a dense-lineable subset of $c_0(C([a,b]))$. For a fixed $(f_n)_n \in \mathcal{AMW}$, each sequence $(g_n)_n \subset c_{00}(C([a,b]))$ only modifies a finite number of components of $(f_n)_n$. So $(f_n + g_n)_n \in \mathcal{AMW}$, and $\mathcal{AMW} + c_{00}(C([a,b])) \subset \mathcal{AMW}$. Moreover, it is clear that $c_{00}(C([a,b])) \cap \mathcal{AMW} = \varnothing$, because every sequence in $c_{00}(C([a,b]))$ has trivially a convergent series of supremum norms.

Now, an application of Theorem 2.7, with $A := \mathcal{AMW}$, $B := c_{00}(C([a,b]))$ and $\kappa = \mathfrak{c}$, gives us the maximal dense-lineability of $\mathcal{AMW}$ in $c_0(C([a,b]))$. □

6.3 Algebrability within $\mathcal{AMW}$

In the previous Section we have seen the existence of many linear structures inside the family $\mathcal{AMW}$. Moreover, this could be done focusing our attention in the linear structures of the families of sequences of coefficients in $c_0 \setminus \ell_1$ and of the sequences of functions in $\mathcal{F}$ separately. Now, our interest is to see if this can be translated into algebraic combinations, that is, we look for the existence of (free) algebras in the family $\mathcal{AMW}$. Before we start looking for the existence of such structures, we need to state two technical lemmas that will help us in this search.

Lemma 6.13. *Let $\mathcal{U}$ be a free algebra in $C([a,b])$, generated by $U := \{u_i\}_{i\in I}$. Then, for any family $P = \{p_i\}_{i\in I}$ of polynomials of degree exactly 1, the set $U_P := \{p_i \circ u_i\}_{i\in I}$ is a generator system of a free algebra in $C([a,b])$.*

Proof. Let $\mathcal{U}_\mathcal{P}$ be the algebra generated by U_P. Let us see that $\mathcal{U}_\mathcal{P}$ is free. By hypothesis, $p_i(x) = \alpha_i x + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i \neq 0$, for each $i \in I$. For any $F \in \mathcal{U}_\mathcal{P}$, we have

$$F(x)=\sum_{\mathbf{j}\in J}\lambda_{\mathbf{j}}(p_1(u_1(x)))^{j_1}\cdots(p_N(u_N(x)))^{j_N},\qquad (x\in[a,b])$$

where $J\subset\mathbb{N}_0^N\setminus\{(0,\overset{(N)}{\ldots},0)\}$ is finite, $\lambda_{\mathbf{j}}\in\mathbb{R}\setminus\{0\}$, $\mathbf{j}=(j_1,\ldots,j_N)\in J$. So,

$$\begin{aligned}
F(x)&=\sum_{\mathbf{j}\in J}\lambda_{\mathbf{j}}(\alpha_1u_1(x)+\beta_1)^{j_1}\cdots(\alpha_Nu_N(x)+\beta_N)^{j_N}\\
&=\sum_{\mathbf{j}\in J}\lambda_{\mathbf{j}}\sum_{l_1=0}^{j_1}\binom{j_1}{l_1}\beta_1^{l_1}\cdot(\alpha_1u_1(x))^{j_1-l_1}\cdots\sum_{l_N=0}^{j_N}\binom{j_N}{l_N}\beta_N^{l_N}\cdot(\alpha_Nu_N(x))^{j_N-l_N}\\
&=\sum_{\mathbf{j}\in J}\sum_{l_1=0}^{j_1}\cdots\sum_{l_N=0}^{j_N}\lambda_{\mathbf{j}}\left(\binom{j_1}{l_1}(\beta_1^{l_1}\cdot(\alpha_1u_1(x))^{j_1-l_1})\cdots\binom{j_N}{l_N}(\beta_N^{l_N}\cdot(\alpha_Nu_N(x))^{j_N-l_N})\right)\\
&=\sum_{\mathbf{j}\in J}\sum_{l_1=0}^{j_1}\cdots\sum_{l_N=0}^{j_N}\left(\lambda_{\mathbf{j}}\prod_{\nu=1}^{N}\binom{j_\nu}{l_\nu}\beta_\nu^{l_\nu}\cdot\alpha_\nu^{j_\nu-l_\nu}u_\nu(x)^{j_\nu-l_\nu}\right).
\end{aligned}$$

But $\mathcal{U}$ is free, so if $F\equiv 0$ then, $\lambda_{\mathbf{j}}\prod_{\nu=1}^{N}\binom{j_\nu}{l_\nu}\beta_\nu^{l_\nu}\cdot\alpha_\nu^{j_\nu-l_\nu}=0$ for any $\mathbf{j}=(j_1,\ldots,j_N)\in J$ and $l_k=0,\ldots,j_k$, $k=1,\ldots,N$. In particular, by taking $l_k=0$ for any $k=1,\ldots,N$ we obtain $\lambda_{\mathbf{j}}\cdot\left(\prod_{\nu=1}^{N}\alpha_\nu^{j_\nu}\right)=0$ for any $\mathbf{j}\in J$, and we are done because $\alpha_i\neq 0$ for all $i\in I$. □

Lemma 6.14. *Let $(a_n)_n\in c_0\setminus\bigcup_{p\geq 1}\ell_p$. Let P be a polynomial with real coefficients and without constant term. Then*

$$(P(a_n))_n\in c_0\setminus\ell_1.$$

Proof. Let $P(x)=\sum_{j=0}^{m}p_jx^{t+j}$, where $t\in\mathbb{N}$, $p_j\in\mathbb{R}$ $(0\leq j\leq m)$, $p_0\neq 0$. Then, because $(a_n)_n\in c_0$ we have $(P(a_n))_n\in c_0$. Moreover, $(a_n^t)_n\notin\ell_1$, so there are infinitely many n such that $a_n^t\neq 0$ (without loss of generality we may assume $a_n\neq 0$ for all $n\in\mathbb{N}$), and

$$\lim_{n\to\infty}\frac{|P(a_n)|}{|a_n^t|}=\lim_{n\to\infty}\left|\sum_{j=0}^{m}p_ja_n^j\right|=|p_0|>0.$$

Thus, the result follows from the comparison test. □

Observe that in Lemma 6.13, if we start from a free algebra in $C([a,b])$, then every affine combinations of its elements generates again a free algebra in $C([a,b])$. On the other hand, in Lemma 6.14, if we start from a sequence of coefficients $(a_n)_n \in c_0 \setminus \bigcup_{p\geq 1} \ell_p$ (so, in particular, $(a_n)_n \in c_0 \setminus \ell_1$), any algebraic combination is an element of $c_0 \setminus \ell_1$.

This results allow us to find algebras in $\mathcal{AMW}$.

Theorem 6.15. *Let $(a_n)_n \in c_0 \setminus \bigcup_{p\geq 1} \ell_p$ be a prefixed sequence of scalars. Let $G := \{g_i\}_{i\in I}$ be a (minimal) generator system of a free algebra in $C([a,b])$. Let $\Lambda = (\alpha_n)_n \subset [a,b]$ such that $a = \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} < \cdots \to b$ $(n \to \infty)$. Consider the family*

$$U := \{(a_n u_n^i)_n : (u_n^i)_n = \mathcal{J}_\Lambda(\gamma_i g_i + 1)\},$$

where $\gamma_i := \frac{1}{g_i(a)}$ if $g_i(a) \neq 0$ or $\gamma_i := 1$ if $g_i(a) = 0$. Then U is the (minimal) generator system of a free algebra in $\mathcal{AMW} \cup \{0\}$.

Proof. Let $\mathcal{A}$ be the algebra generated by U, that is, $(F_n)_n \in \mathcal{A}$ if there exist $N \in \mathbb{N}$, mutually different $(u_n^i)_n = \mathcal{J}_\Lambda(\gamma_i g_i + 1)$, $i = 1, \ldots, N$, and a non-zero polynomial P in N real variables without constant term such that

$$F_n = P(a_n u_n^1, a_n u_n^2, \ldots, a_n u_n^N) \qquad (n \in \mathbb{N}).$$

Therefore, there exist a non-empty finite set $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\ldots}, 0)\}$ and scalars $\lambda_{\mathbf{j}} \in \mathbb{R} \setminus \{0\}$ for $\mathbf{j} = (j_1, \ldots, j_N) \in J$ such that for each $n \in \mathbb{N}$ and each $x \in [a,b]$,

$$\begin{aligned} F_n(x) &= \sum_{\mathbf{j}\in J} \lambda_{\mathbf{j}} (a_n u_n^1)(x)^{j_1} \cdots (a_n u_n^N)(x)^{j_N} \\ &= \sum_{\mathbf{j}\in J} \lambda_{\mathbf{j}} a_n^{|\mathbf{j}|} (\mathcal{J}_\Lambda(\alpha_1 p_1 + 1)(x))^{j_1} \cdots (\mathcal{J}_\Lambda(\alpha_N p_N + 1)(x))^{j_N}. \qquad (6.2) \end{aligned}$$

But, by the definition of $\mathcal{J}_\Lambda$ (see Proposition 6.8(3)), for each $n \in \mathbb{N}$ there is a linear affine transformation τ_n such that $\tau_n[\alpha_{3n-1}, \alpha_{3n}] = [a, b]$ and $\mathcal{J}_\Lambda(\gamma_i g_i + 1)(x) = (\gamma_i g_i + 1)(\tau_n(x))$ for any $x \in [\alpha_{3n-1}, \alpha_{3n}]$ and any $i = 1, \ldots, N$. So, for any $n \in \mathbb{N}$,

$$F_n(x) = \sum_{\mathbf{j} \in J} \lambda_{\mathbf{j}} a_n^{|\mathbf{j}|} ((\gamma_1 g_1 + 1)(\tau_n(x)))^{j_1} \cdots (\gamma_N g_N + 1)(\tau_n(x)))^{j_N} \qquad x \in [\alpha_{3n-1}, \alpha_{3n}].$$

Therefore, if $F_n = 0$ in $[a, b]$, and in particular $F_n = 0$ in $[\alpha_{3n-1}, \alpha_{3n}]$,

$$\sum_{\mathbf{j} \in J} \lambda_{\mathbf{j}} a_n^{|\mathbf{j}|} ((\gamma_1 g_1 + 1)(w))^{j_1} \cdots ((\gamma_N g_N + 1)(w))^{j_N} = 0 \qquad w \in [a, b]. \tag{6.3}$$

Now, as $\{g_i\}_{i \in I}$ generates a free algebra in $C([a, b])$, by Lemma 6.13, $\{\gamma_i g_i + 1 \,:\, i \in I\}$ is also a generator system of a free algebra, so from (6.3) and the fact that there are infinitely many $a_n \neq 0$ (this is true because $a_n \notin \ell_1$ for example), we get $\lambda_{\mathbf{j}} = 0$ for $\mathbf{j} \in J$ and the algebra $\mathcal{A}$ is free. Observe that trivially we also obtain that the dimension of $\mathcal{A}$ is the dimension of the algebra generated by $\{g_i\}_{i \in I}$ in $C([a, b])$.

It only rest to show that $\mathcal{A} \subset \mathcal{AMW} \cup \{0\}$, that is, that any non-null sequence $(F_n)_n$ as above is Anti M-Weierstrass. Observe that, again by the definition of the application $\mathcal{J}_\Lambda$ (see Proposition 6.8(2) and (4)), we have:

$$\operatorname{supp}(u_n^i) \subset [\alpha_{3n-2}, \alpha_{3n+1}], \qquad (1 \leq i \leq N, n \in \mathbb{N}), \tag{6.4}$$

and,

$$||u_n^i||_\infty = ||\gamma_i g_i + 1||_\infty \in (0, +\infty) \text{ for each } i = 1, \ldots, N \text{ and } n \in \mathbb{N}. \tag{6.5}$$

Therefore, by (6.4),

$$supp(u_n^1 \cdots u_n^N) \cap supp(u_m^1 \cdots u_m^N) = \varnothing \qquad for\ n \neq m, \tag{6.6}$$

and, by (6.2), we have the absolute convergence of $\sum_{n=1}^{\infty} F_n$. Now, by (6.5),

$$\sup_{n \in \mathbb{N}} ||u_n^1 \cdots u_n^N||_\infty = ||\gamma_1 g_1 + 1||_\infty \cdots ||\gamma_N g_N + 1||_\infty =: C \in (0, \infty), \tag{6.7}$$

and, since $(a_n)_n \in c_0$, and similarly to Lemma 6.5(b), by (6.2), (6.6), and (6.7), the series $\sum_{n=1}^{\infty} F_n(x)$ converges uniformly on $[a,b]$. Finally, again by the definition of $\mathcal{J}_\Lambda$ (see Proposition 6.8(1)),

$$u_n^i(\alpha_{3n-1}) = \gamma_i g_i(a) + 1 =: \delta = 1 \text{ or } 2 \text{ for any } i = 1, \dots, N \text{ and any } n \in \mathbb{N}. \quad (6.8)$$

Now, by (6.2) and (6.8), for each $n \in \mathbb{N}$,

$$|F_n(\alpha_{3n-1})| = \left| \sum_{\mathbf{j} \in J} \lambda_{\mathbf{j}} (a_n u_n^1)^{j_1}(\alpha_{3n-1}) \cdots (a_n u_n^N)^{j_N}(\alpha_{3n-1}) \right| = \left| \sum_{\mathbf{j} \in J} \lambda_{\mathbf{j}} \delta^{|\mathbf{j}|} a_n^{|\mathbf{j}|} \right|.$$

But $(a_n)_n \in c_0 \setminus \bigcup_{p \geq 1} \ell_p$, so by applying Lemma 6.14 to the polynomial $p(x) = \sum_{\mathbf{j} \in J} \lambda_{\mathbf{j}} \delta^{|\mathbf{j}|} x^{|\mathbf{j}|}$, we have

$$\sum_{n=1}^{\infty} ||F_n||_\infty \geq \sum_{n=1}^{\infty} |F_n(\alpha_{3n-1})| = \sum_{n=1}^{\infty} |p(a_n)| = +\infty.$$

Thus $(F_n)_n \in \mathcal{AMW}$, and we are done. □

Theorem 6.16. *Let $\mathcal{L}$ be a free algebra in $c_0 \setminus \ell_1$ generated by $\{(a_n^i)_n\}_{i \in I}$. Let $(u_n)_n \in \mathcal{F}$. Then the algebra $\mathcal{A}$ generated by $\{(a_n^i u_n)_n\}_{i \in I}$ is a free algebra in $\mathcal{AMW} \cup \{0\}$ with the same dimension than $\mathcal{L}$.*

Proof. Let $(F_n)_n \in \mathcal{A}$. Then, for any $x \in [a,b]$ we have,

$$F_n(x) := \sum_{\mathbf{j} \in J} \lambda_{\mathbf{j}} (a_n^1 u_n(x))^{j_1} \cdots (a_n^N u_n(x))^{j_N}$$

$$= \sum_{\mathbf{j} \in J} \lambda_{\mathbf{j}} (a_n^1)^{j_1} \cdots (a_n^N)^{j_N} \cdot (u_n(x))^{|\mathbf{j}|},$$

where $N \in \mathbb{N}$, $J \subset \mathbb{N}_0^N \setminus \{(0, \overset{(N)}{\dots}, 0)\}$ non-empty and finite and $\lambda_{\mathbf{j}} \in \mathbb{R} \setminus \{0\}$ for any $\mathbf{j} = (j_1, \dots, j_N) \in J$.

As $(u_n)_n \in \mathcal{F}$, $L := \inf_{n \in \mathbb{N}} ||u_n||_\infty > 0$; moreover, any u_n ($n \in \mathbb{N}$) takes the value 0 in $[a,b]$. Then, by continuity of u_n's and the intermediate value property, for any $n \in \mathbb{N}$

there is $x_n \in \text{supp}(u_n)$ such that $u_n(x_n) = L$. Hence

$$|F_n(x_n)| = \left|\sum_{\mathbf{j}\in J} \lambda_{\mathbf{j}} (a_n^1)^{j_1} \cdots (a_n^N)^{j_N} \cdot L^{|\mathbf{j}|}\right| \qquad (n \in \mathbb{N}) \tag{6.9}$$

If $(F_n)_n \equiv 0$, by (6.9), for each $n \in \mathbb{N}$

$$\left|\sum_{\mathbf{j}\in J} \lambda_{\mathbf{j}} L^{|\mathbf{j}|} \cdot (a_n^1)^{j_1} \cdots (a_n^N)^{j_N}\right| = 0.$$

But $\mathcal{L}$ is free, so $\lambda_{\mathbf{j}} L^{|\mathbf{j}|} = 0$ for every $\mathbf{j} \in J$, and because $L > 0$, we get that $\lambda_{\mathbf{j}} = 0$ for all $\mathbf{j} \in J$, and $\mathcal{A}$ is also free. As $\mathcal{L} \subset c_0$, then $((a_n^1)^{j_1} \cdots (a_n^N)^{j_N})_n \in c_0$ for any $\mathbf{j} = (j_1, \ldots, j_N) \in J$. Trivially, $(u_n^p)_n \in \mathcal{F}$ for any $p \in \mathbb{N}$. Hence, $(F_n)_n$ is a finite linear combination of products of sequences in $c_0 \setminus \ell_1$ and sequences $\left(u_n^{|\mathbf{j}|}\right)_n \in \mathcal{F}$. Now, by Lemma 6.5, we have the absolute and uniform convergence of the series $\sum_{n=1}^{\infty} F_n(x)$. It is only rest to show that $(||F_n||_\infty)_n \notin \ell_1$ to obtain $(F_n) \in \mathcal{AMW}$ and finish the proof. By (6.9),

$$\sum_{n=1}^{\infty} ||F_n||_\infty \geq \sum_{n=1}^{\infty} |F_n(x_n)| = \sum_{n=1}^{\infty} \left|\sum_{\mathbf{j}\in J} \lambda_{\mathbf{j}} L^{|\mathbf{j}|} \cdot (a_n^1)^{j_1} \cdots (a_n^N)^{j_N}\right| = +\infty,$$

because $\mathcal{L}$ is a free algebra in $c_0 \setminus \ell_1$, and so $\left(\sum_{\mathbf{j}\in J} \lambda_{\mathbf{j}} L^{|\mathbf{j}|} \cdot (a_n^1)^{j_1} \cdots (a_n^N)^{j_N}\right)_n \notin \ell_1$. □

Finally, observe that $(a_n)_n \in \ell_p$ if and only if $(a_n^p)_n \in \ell_1$. In particular, $\mathcal{L}$ is an algebra in $c_0 \setminus \ell_1$ if and only if $\mathcal{L}$ is an algebra in $c_0 \setminus \bigcup_{p\geq 1} \ell_p$. In [10], Bartoszewicz and Glab showed the existence of a free algebra $\mathcal{L}$ in $c_0 \setminus \bigcup_{p\geq 1} \ell_p$ such that the cardinality of any system of generators is the continuum. In fact they consider the algebra generated by $\left\{\left(\frac{1}{\log^c n}\right)_{n\geq 2} : c \in H\right\}$, where $H \subset (0, +\infty)$ is a $\mathbb{Q}$-linearly independent set, and $\text{card}(H) = \mathfrak{c}$. So, we can obtain the next result about the algebrability of the family $\mathcal{AMW}$.

Corollary 6.17. *The family $\mathcal{AMW}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. As in Corollary 6.11, we will give two different approaches.

(1) By Bartoszewicz and Glab result [10, Theorem 2], take the algebra $\mathcal{L}$ generated by the family $\left\{ \left(\frac{1}{\log^c n} \right)_{n \geq 2} : c \in H \right\}$, where $H \subset (0, +\infty)$ is a $\mathbb{Q}$-linearly independent set, and $\operatorname{card}(H) = \mathfrak{c}$, and one of the sequences of functions $(u_n)_n \in \mathcal{F}$ (for instance any sequence of Example 6.2 or Example 6.6), an application of Theorem 6.16 gives us the strong $\mathfrak{c}$-algebrability of the family $\mathcal{AMW}$.

(2) Consider now the free algebras generated by $\{x^c : c \in H\}$ or $\{e^{cx} : c \in H\}$ in $C([a, b])$ (where $H \subset (0, +\infty)$ is a $\mathbb{Q}$-linearly independent set and $\dim(H) = \mathfrak{c}$), and the sequence of coefficients $(a_n)_n \in c_0 \setminus \bigcup_{p \geq 1} \ell_p$ given by $a_n = \frac{1}{\log(n)}$ for $n \geq 2$. Then, an application of Theorem 6.15 gives us (again) the strong $\mathfrak{c}$-algebrability of $\mathcal{AMW}$. $\square$

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