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Families of symmetric periodic orbits in the three body problem and the figure eight

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Abstract

In this paper we show a technique for the continuation of symmetric periodic orbits in systems with time-reversal symmetries. The geometric idea of this technique allows us to generalize the "cylinder" theorem for this kind of systems. We state the main theoretical result without proof (to be published elsewhere). We focus on the application of this scheme to the three body problem (TBP), taking as starting point the figure eight orbit [3] to find families of symmetric periodic orbits.

Key words and expressions: Hamiltonian and conservative systems, periodic solutions, numerical continuation, boundary value problems, three-body problem, figure-8 orbit.

MSC: 34C25, 34C30, 34C14, 37J15, 65L10.

1 Introduction

Our interest is about finding symmetric periodic solutions in the three body problem. In a previous paper [7] we studied the continuation of periodic orbits in the TBP following the scheme given in [13] for continuation in conservative systems. Unlike other methods, this scheme does not make use of symplectic reduction before numeric calculation. Several families of periodic orbits were shown in [7] and the numerical study of their linear stability (characteristic multipliers). Another applications of this technique appeared in [6] where a lot of families of periodic orbits in the TBP and the restricted three body problem were found. Some of these families are formed by symmetric periodic orbits with respect to time-reversal symmetries. However, while the scheme given in [13] does not take advantage of these symmetries, classical methods do [1, 4]. Our scope in this paper is to adapt the scheme in [13], so that families of symmetric periodic orbits can be computed in a more proper way than in [7, 6] where time-reversal symmetries were not used for the computations.

Chenciner and Montgomery [3] proved recently the existence of a spectacular solution in the TBP called *the figure eight*, since the three bodies follow the same curve (choreography) with this shape. The initial conditions of the figure eight, which were computed numerically by Simó, are shown in [3] too. This kind of solution had been predicted by Moore [11] in 1993. A lot of work has been developed around this new solution for the last two years. Simó also studied the region of stability near the figure eight and found countless examples of choreographies [14, 15]. For other results on continuation, see [9, 2].

The figure eight is symmetric with respect to several time-reversal symmetries; therefore, we think that it is a good starting point to look for symmetric periodic solutions.

In this introduction we give a short revision about the main concepts related to symmetries in the TBP and we describe with respect to which time-reversal symmetries the figure eight is symmetric (for more details, see [3, 9]).

In the second section we show geometrically when a first integral plays a role in the continuation of symmetric periodic orbits. In the third section we are going to state the main theoretical result in this paper about persistence (or continuation) of symmetric periodic orbits. As we have said in the abstract, this result is a generalization of the "cylinder" theorem. We adapt the ideas from reference [13] where continuation of periodic orbits in conservative systems is based on a proper unfolding of these systems. For symmetric periodic orbits continuation it is necessary to unfold systems with first integrals which verify a certain property. In the last section we apply these ideas to the figure eight in the TBP.

1.1 On symmetries in the TBP

Let n be the dimension of the configuration space, so if $n = 2$, the planar TBP is set and if $n = 3$, it is the spatial TBP. For each smooth function $F : \mathbf{R}^{6n} \mapsto \mathbf{R}$ the vector field $X_F(\mathbf{x}) = J\nabla F(\mathbf{x})$ is defined, where J is the matrix of size $6n \times 6n$

$$J = \begin{bmatrix} 0_{3n} & -I_{3n} \\ I_{3n} & 0_{3n} \end{bmatrix}. \tag{1}$$

The TBP (for bodies of masses m_j with $j = 1, 2, 3$) is a system $\dot{\mathbf{x}} = X_H(\mathbf{x})$ whose Hamiltonian H is given by

$$H(\mathbf{x}) = \sum_{j=1}^{3} \frac{1}{2m_j} \|\mathbf{p}_j\|^2 - \sum_{1\le i<j\le 3} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}, \tag{2}$$

with $\mathbf{x} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ where $\mathbf{q}_j$ and $\mathbf{p}_j$ are the position vector and momentum vector of the j-th body (with $j = 1, 2, 3$).

Besides the Hamiltonian, the TBP has other first integrals, as the components of the total linear momentum and the total angular momentum. Moreover, every quadratic first integral can be written as

$$F_{\mathbf{a},A}(\mathbf{x}) = \mathbf{a}^* \left(\sum_{j=1}^{3} \mathbf{p}_j \right) + \sum_{j=1}^{3} \mathbf{q}_j^* A \mathbf{p}_j, \tag{3}$$

where $\mathbf{a} \in \mathbf{R}^n$ and A is a skew symmetric matrix. Moreover, in this paper, the transpose of a vector $\mathbf{x} \in \mathbf{R}^n$ is denoted by $\mathbf{x}^*$. The vectors of the canonical basis in $\mathbf{R}^2$ are denoted as $\mathbf{e}_1$ and $\mathbf{e}_2$. In a similar way, for $\mathbf{R}^3$ are denoted as $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

The Noether's symmetry theorem [10] states that the flux of $\dot{\mathbf{x}} = X_{F_{\mathbf{a},A}}(\mathbf{x})$ is a symplectic continuous group of symmetries, i.e. the system $\dot{\mathbf{x}} = X_H(\mathbf{x})$ remains invariant under the changes of variables induced from the flux of $\dot{\mathbf{x}} = X_{F_{\mathbf{a},A}}(\mathbf{x})$.

We can easily check that $\Psi_{Q,\mathbf{b}}$ are symmetries of the TBP

$$\begin{array}{rcl} \Psi_{Q,\mathbf{b}} : \mathbf{R}^{6n} & \to & \mathbf{R}^n \times \quad \mathbf{R}^n \quad \times \quad \mathbf{R}^n \quad \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \\ \mathbf{x} & \mapsto & (Q\mathbf{q}_1 + \mathbf{b}\,,\, Q\mathbf{q}_2 + \mathbf{b}\,,\, Q\mathbf{q}_3 + \mathbf{b}\,,\, Q\mathbf{p}_1\,,\, Q\mathbf{p}_2\,,\, Q\mathbf{p}_3) \end{array} \tag{4}$$

where Q is an orthogonal matrix of size $n \times n$ and $\mathbf{b}$ is any vector in $\mathbf{R}^n$.

If two bodies have equal masses, then there is another symmetry which exchanges positions and momenta for the bodies. For instance, with $m_2 = m_3$, the TBP has the symmetry

$$C : (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \mapsto (\mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2). \tag{5}$$

We recall that the composition of symmetries is a symmetry.

A matrix R of size $6n \times 6n$ such that $R^2 = I_{6n}$ is said to be a time-reversal symmetry of the TBP iff $X_H(R\,\mathbf{x}) = -R\,X_H(\mathbf{x})$. The fixed point subspace of a time-reversal symmetry is $\mathrm{Fix}(R) = \{\mathbf{x} \in \mathbf{R}^{6n} \,:\, R\mathbf{x} = \mathbf{x}\}$. In the TBP a time-reversal symmetry is

$$R_N = \begin{bmatrix} I_{3n} & 0_{3n} \\ 0_{3n} & -I_{3n} \end{bmatrix}. \tag{6}$$

Any composition of a symmetry with R_N is another time-reversal symmetry. Therefore, the TBP is full of time-reversal symmetries. It is well known (see [8]) that an orbit with two points in $\mathrm{Fix}(R)$ is a symmetric periodic orbit, i.e. the orbit is invariant with respect to R.

In the TBP there exists a rescale Φ_λ (with $\lambda \in \mathbf{R} \setminus \{0\}$) of the positions and the conjugate momenta which maps a solution into another solution with a new time scale. This function is

$$\begin{array}{cccc} \Phi_\lambda : & \mathbf{R}^{6n} & \to & \mathbf{R}^{6n} \\ & (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) & \mapsto & (\lambda^{-2}\mathbf{q}_1, \lambda^{-2}\mathbf{q}_2, \lambda^{-2}\mathbf{q}_3, \lambda\mathbf{p}_1, \lambda\mathbf{p}_2, \lambda\mathbf{p}_3), \end{array} \tag{7}$$

and its time–rescale is λ^{-3}.

1.2 The figure eight as a symmetric periodic orbit

Chenciner and Montgomery [3] looked for a planar curve which is a minimum of the action for the TBP with the bodies of unit mass,

$$\int_0^T \left(\frac{1}{2}\sum_{j=1}^{3} \|\mathbf{p}_j(t)\|^2 + \sum_{1\le i<j\le 3} \frac{1}{\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\|}\right) dt, \tag{8}$$

over the subspace of curves which verify that:

- Initial condition: The first body is in the middle of the other two bodies.
- Final condition: The second body (which was in an extremum in the initial condition) is now equidistant of the first and the third bodies.

They proved that the minimum is reached in a curve which points generate a solution of the TBP where the three bodies follow the same curve (choreography) with a shape of an eight, so this orbit was called the figure eight by them. In Figure 1, we see the curve followed by the bodies and the initial and final conditions for the minimization problem.

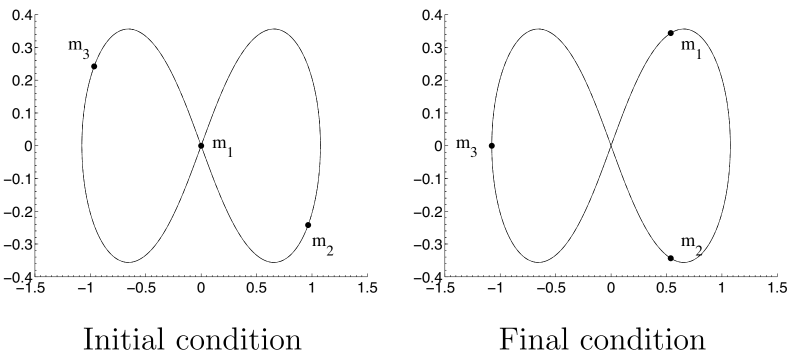


Figure 1: Figure eight with the initial and final conditions of the minimization problem.

The figure eight is symmetric with respect to six time-reversal symmetries in the planar case and nine time-reversal symmetries in the spatial case. These time-reversal symmetries are given by the composition of R_N with an exchange of the bodies (for example, C is one of the three possible exchanges) and with a symmetry $\Psi_{S,\mathbf{0}}$. The matrix S can be chosen from the following list for the planar case:

$$S_{M_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, S_{E_1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9}$$

and for the spatial case:

$$S_{M_1 3D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, S_{\Pi_1 3D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } S_{E_1 3D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{10}$$

In order to simplify (and following [3]), the time-reversal symmetry $R_N \circ C \circ \Psi_{S_{M_1},\mathbf{0}}$ is called M_1 and $R_N \circ C \circ \Psi_{S_{E_1},\mathbf{0}}$ is called E_1 throughout this paper. The fixed points of M_1 have the first body in the axis $\mathbf{e}_1$ and this body forms an isosceles triangle with the other two bodies whose symmetry axis is $\mathbf{e}_1$; the conjugate momentum of the position of the first body, $\mathbf{p}_1$, is in the direction $\mathbf{e}_2$ and the conjugate momenta of the three bodies ($\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$) form an isosceles triangle with symmetry axis in $\mathbf{e}_2$. See Figure 2 for a graphical interpretation.

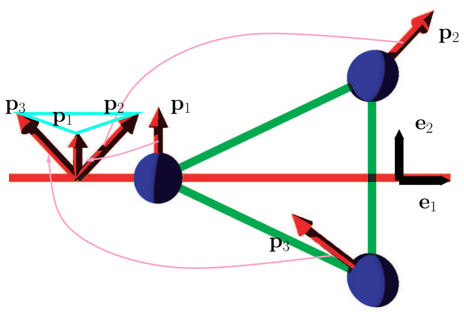


Figure 2: Fixed points of M_1.

The fixed points of E_1 are given by the first body in the origin and in the middle point of the other two bodies; the conjugate momenta of this two bodies are identical, i.e., $\mathbf{q}_1 = \mathbf{0}$, $\mathbf{q}_2 = -\mathbf{q}_3$ and $\mathbf{p}_2 = \mathbf{p}_3$.

2 Geometric Interpretation of the Continuation with Time-Reversal Symmetries

There exists a "cylinder" theorem for systems with a time-reversal symmetry (see [8]). A simple geometric reasoning states that periodic orbits generically arise as a one-parameter family (we are assuming that the dimension of Fix(R) is half the whole dimension).

However, the TBP is a non-generic system, since most of the orbits do not belong to one-parameter families. We have studied why some first integrals avoid a generic behavior for the TBP.

We focus on the time-reversal symmetry M_1 for our explanation. An analogous situation appears for E_1. A translation of a fixed point of M_1 in the direction $\mathbf{e}_1$ produces another fixed point generating a symmetric periodic orbit and the function Φ_λ maps fixed points of M_1 into fixed points of M_1 too. This remark suggests that, under generic conditions, fixed points of M_1 generating symmetric periodic orbits form a two-dimensional submanifold. By theoretical results and numerical computation, we are able to state that the figure eight belongs to a 2-parameter family of symmetric periodic orbits with respect to M_1.

3 Result on the Continuation for Symmetric Orbits

The previous geometric idea still holds in a more general situation: a system with a time-reversal symmetry R and a first integral F such that the flux of the vector field X_F maps fixed points of R into fixed points of R has the same non-generic behavior. If R is anti-symplectic (i.e $R^TJR = -J$), this condition is equivalent to the derivative of the function F restricted to $\mathrm{Fix}(R)$ is zero. Therefore, a first integral constant on $\mathrm{Fix}(R)$ implies that the symmetric periodic orbit does not belong to a one–parameter family any more.

The symplectic framework is not necessary and it is possible to formulate a general result for any kind of system with a time-reversal symmetry. So, we assume in this section that $X : \mathbf{R}^{2m} \to \mathbf{R}^{2m}$ is a vector field with a time-reversal symmetry R and the dimension of $\mathrm{Fix}(R)$ is m. These assumptions are not mandatory, but they simplify the theorem and they hold in most physical applications. We define $\mathcal{R}_{\mathbf{x}_0}$ as the set of derivatives of first integrals which are constant on $\mathrm{Fix}(R)$, i.e.

$$\mathcal{R}_{\mathbf{x}_0} = \left\{ DF(\mathbf{x}_0) \;:\; \begin{array}{l} F \text{ is a first integral of } \dot{\mathbf{x}} = X(\mathbf{x}) \text{ and} \\ F \text{ is constant on } \mathrm{Fix}(R) \end{array} \right\} \tag{11}$$

and let $\varphi_t(\mathbf{x})$ be the flux of the system $\dot{\mathbf{x}} = X(\mathbf{x})$. We denote the subspace spanned by the vector $\mathbf{x} \in \mathbf{R}^{2m}$ with $\mathbf{R}\mathbf{x}$. We state the following theorem for the continuation of symmetric periodic orbits

Theorem 3.1 *Let be $\mathbf{x}_0 \in \mathrm{Fix}(R)$ and $T > 0$. If $\varphi_T(\mathbf{x}_0) \in \mathrm{Fix}(R)$ and*

$$\mathrm{Im}((I - R)D\varphi_T(\mathbf{x}_0)(I + R)) + \mathbf{R}X(\varphi_T(\mathbf{x}_0)) = \mathcal{R}^{\perp}_{\varphi_T(\mathbf{x}_0)} \cap \mathrm{Im}(I - R), \tag{12}$$

then the symmetric periodic orbit generated by $\mathbf{x}_0$ belongs to a $(\dim(\mathcal{R}_{\mathbf{x}_0}) + 1)$-parameter family of symmetric periodic orbits with respect to R.

A proof of this result can be found in [12] and will be published elsewhere. The idea of this proof follows the one given in [13] where "artificial" parameters (or unfolding parameters) are added in the vector field. If F_j (with $1 \leq j \leq k$) are first integrals such that $\{DF_j(\mathbf{x}_0)\}_{1\leq j\leq k}$ is a basis for $\mathcal{R}_{\mathbf{x}_0}$, then we consider the flux $\tilde{\varphi}_t$ of the system

$$\dot{\mathbf{x}} = X(\mathbf{x}) + \sum_{j=1}^{k} \beta_j \nabla F_j(\mathbf{x}) \tag{13}$$

where β_j are the "artificial" parameters and we apply the implicit function theorem to the function

$$(\mathbf{x}, t, \beta_1, \ldots, \beta_k) \in \mathrm{Fix}(R) \times \mathbf{R} \times \mathbf{R}^k \mapsto (I - R)\tilde{\varphi}_t(\mathbf{x}, \beta_1, \ldots, \beta_k), \tag{14}$$

getting a submanifold of zeros for this function and, finally, checking that the artificial parameters are zero along that manifold.

4 Numerical Results about the Figure Eight

The scheme for continuation has been applied to the figure eight as the mass of one of the bodies is allowed to vary. When the masses are identical ($m_1 = m_2 = m_3 = 1$), a consequence of the former theorem is that the figure eight orbit belongs to a two-parameter family given by the translation in the direction $\mathbf{e}_1$ and by mapping Φ_λ. So, for these values of the masses there are no more symmetric periodic orbits with respect to M_1 "near" (with the period close to the original one) the figure eight. If the mass m_1 is varied, the system still has the M_1 time-reversal symmetry; therefore we will try to continue taking m_1 as a continuation parameter, looking for symmetric periodic orbits.

The only quadratic first integral constant on $\mathrm{Fix}(R)$ is the first component of the total linear momentum $F_1(\mathbf{x}) = \mathbf{e}_1^* \left(\sum_{j=1}^{3} \mathbf{p}_j\right)$ where $\mathbf{x} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. Therefore, we set the following boundary value problem

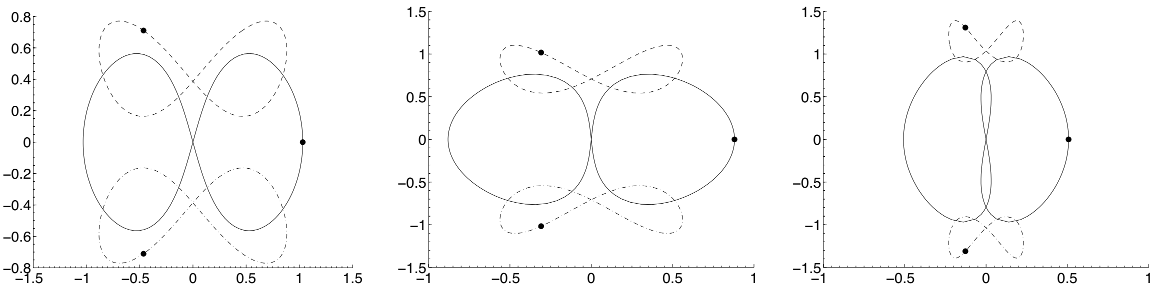
$$\begin{cases} \text{Find } \mathbf{x} \in \mathcal{C}^1([0,1]; \mathbf{R}^{12}) \text{ such that} \\ \dot{\mathbf{x}} = T(J\nabla H(\mathbf{x}) + \beta_1 \nabla F_1(\mathbf{x})), \\ \mathbf{e}_2^*\mathbf{q}_1(0) = 0 \qquad \mathbf{e}_2^*\mathbf{q}_1(1) = 0, \\ \mathbf{q}_3(0) = S_{M_1}\mathbf{q}_2(0) \qquad \mathbf{q}_3(1) = S_{M_1}\mathbf{q}_2(1), \\ \mathbf{e}_1^*\mathbf{p}_1(0) = 0 \qquad \mathbf{e}_1^*\mathbf{p}_1(1) = 0, \\ \mathbf{p}_3(0) = -S_{M_1}\mathbf{p}_2(0) \qquad \mathbf{p}_3(1) = -S_{M_1}\mathbf{p}_2(1), \end{cases} \tag{15}$$

where the half-period T is fixed, S_{M_1} is the matrix defined in (9) and β_1 is the "artificial" parameter needed for the continuation.

The initial conditions for this problem form a two-dimensional submanifold where every translation in the direction $\mathbf{e}_1$ of its points is also included in the submanifold. This situation is avoided with an additional integral condition

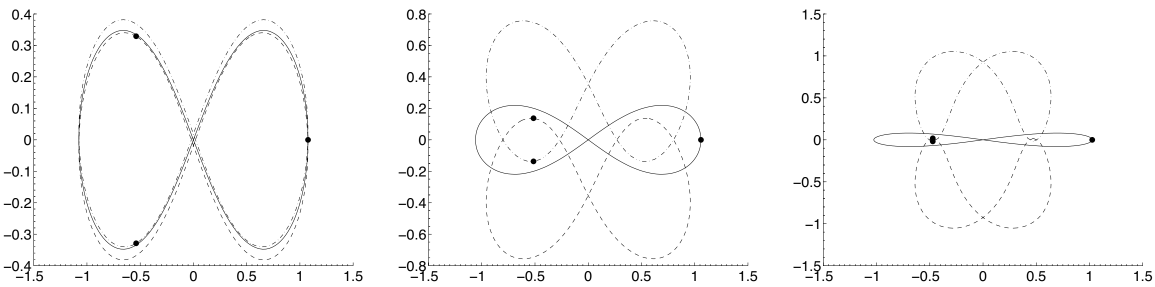
$$\int_0^T DF_1(\mathbf{x}_0(s))J(\mathbf{x}(s) - \mathbf{x}_0(s))\, ds = 0, \tag{16}$$

where $\mathbf{x}_0(t)$ is the previous computed solution. The system (15) together with this integral condition can be numerically continued with a package as AUTO [5]. We get a curve of initial conditions on $\mathrm{Fix}(M_1)$ which generates symmetric periodic orbits. Downwards solutions in the family are shown in Figure 3 and upwards in Figure 4, for this last case there exists a fold in the family for $m_1 = 1.00004$.



$m_1 = 0.9$ $\quad$ $m_1 = 0.7$ $\quad$ $m_1 = 0.5$

Figure 3: We show the family downwards to which the figure eight belongs where the initial conditions in the fixed point subspace of M_1 are marked.

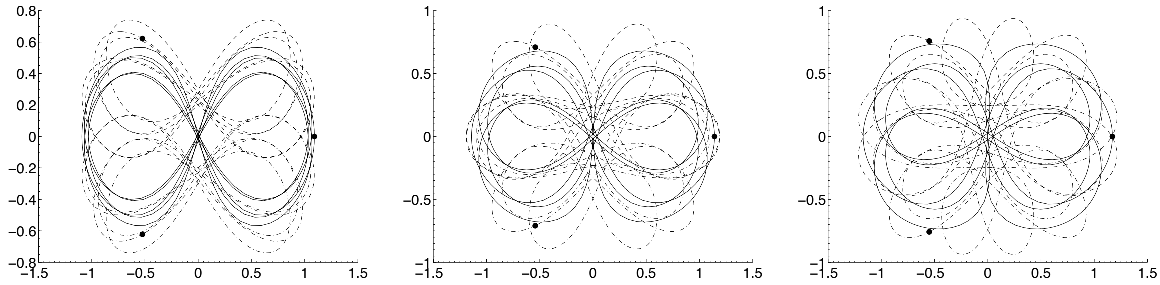


$m_1 = 1.00$ $\quad$ $m_1 = 0.97$ $\quad$ $m_1 = 0.92$

Figure 4: Family upwards to which the figure eight belongs.

We do not detect any point where the condition (12) is not fulfilled along this family, but if T is taken as five times the half period and as starting solution five half windings of the figure eight, then subharmonic bifurcations are detected when the condition (12) is not fulfilled. Using AUTO we get a branch of periodic orbits which start from a 5-subharmonic bifurcation located at $m_1 = 0.9618044$. In Figure 5 some of these solutions are shown.

Using the time–reversal symmetry E_1 the first family can be also continued as symmetric periodic orbits with respect to E_1, but in this case the first integral constant on



$m_1 = 0.96$ $m_1 = 0.95$ $m_1 = 0.94$

Figure 5: We plot some orbits got from a family which starts from a 5-subharmonic bifurcation.

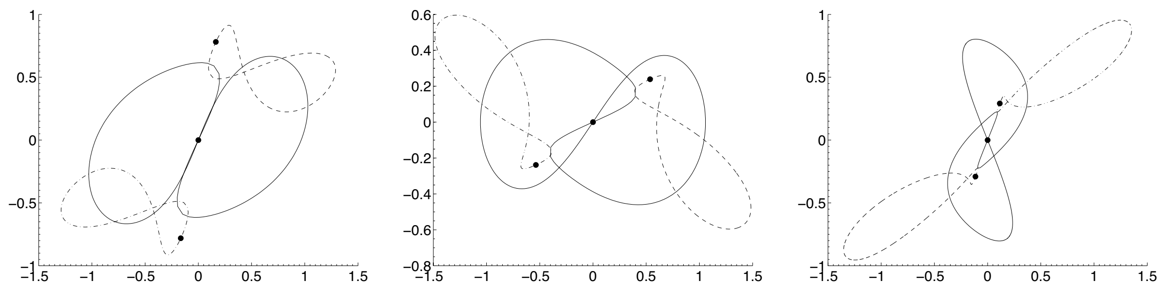
$\mathrm{Fix}(E_1)$ is the angular momentum

$$G_1(\mathbf{x}) = \sum_{i=1}^{3} \mathbf{q}_i^* \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{p}_i.$$

The former theory drives to formulate the following boundary value problem

$$\begin{cases} \text{Find } \mathbf{x} \in \mathcal{C}^1([0,1]; \mathbf{R}^{12}) \text{ such that} \\ \dot{\mathbf{x}} = T(J\nabla H(\mathbf{x}) + \beta_1 \nabla G_1(\mathbf{x})), \\ \mathbf{q}_1(0) \ = \mathbf{0} \qquad \mathbf{q}_1(1) \ = \mathbf{0}, \\ \mathbf{q}_2(0) \ = -\mathbf{q}_3(0) \qquad \mathbf{q}_2(1) \ = -\mathbf{q}_3(1), \\ \mathbf{p}_2(0) \ = \mathbf{p}_3(0) \qquad \mathbf{p}_2(1) \ = \mathbf{p}_3(1), \end{cases} \tag{17}$$

and to add the integral condition $\int_0^T DG_1(\mathbf{x}_0(s))J(\mathbf{x}(s) - \mathbf{x}_0(s))\, ds = 0$. Here β_1 is, again, an "artificial" parameter to allow the continuation. Along the family which was plotted in Figure 3 and 4 an orbit where condition (12) is not fulfilled is detected for the value $m_1 = 0.699779$ and so a new family of symmetric periodic orbits appears from the solution with this value of m_1. Some orbits in this family are shown in Figure 6.



$m_1 = 0.65$ $m_1 = 0.55$ $m_1 = 0.45$

Figure 6: We show symmetric periodic orbits with respect to E_1 (initial conditions in the fixed subspace are marked) which belong to a family appearing from a point where condition (12) is not fulfilled for E_1.

Finally, we want to remark that the same scheme is also useful in the spatial case. For the time-reversal symmetry given by the composition $R_N \circ C \circ S_{M_13D}$ there are two first integrals in a basis of $\mathcal{R}_{\mathbf{x}}$, for instance,

$$F_1(\mathbf{x}) = \mathbf{e}_1^* \sum_{i=1}^{3} \mathbf{p}_i \qquad \text{and} \qquad F_2(\mathbf{x}) = \sum_{i=1}^{3} \mathbf{q}_i^* \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{p}_i. \tag{18}$$

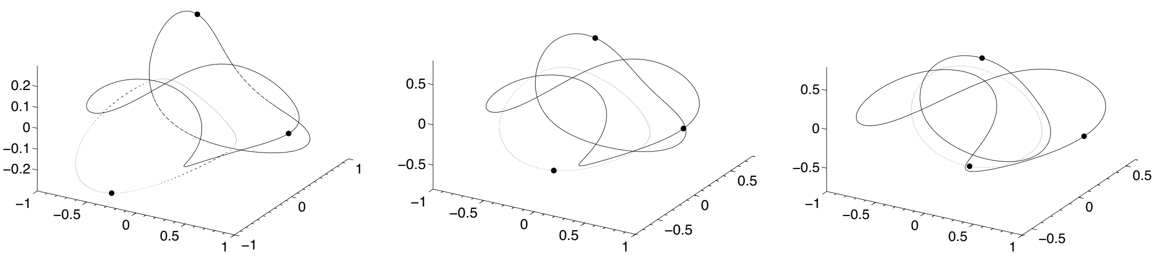
In this case the boundary value problem would be

$$\begin{cases} \text{Find } \mathbf{x} \in \mathcal{C}^1([0,1]; \mathbf{R}^{18}) \text{ such that} \\ \dot{\mathbf{x}} = T(J\nabla H(\mathbf{x}) + \beta_1 \nabla F_1(\mathbf{x}) + \beta_2 \nabla F_2(\mathbf{x})), \\ \mathbf{e}_2^*\mathbf{q}_1(0) \ = 0 \qquad\qquad \mathbf{e}_2^*\mathbf{q}_1(1) \ = 0, \\ \mathbf{e}_3^*\mathbf{q}_1(0) \ = 0 \qquad\qquad \mathbf{e}_3^*\mathbf{q}_1(1) \ = 0, \\ \mathbf{q}_3(0) \ = S_{M_13D}\mathbf{q}_2(0) \qquad \mathbf{q}_3(1) \ = S_{M_13D}\mathbf{q}_2(1), \\ \mathbf{e}_1^*\mathbf{p}_1(0) \ = 0 \qquad\qquad \mathbf{e}_1^*\mathbf{p}_1(1) \ = 0, \\ \mathbf{p}_3(0) \ = -S_{M_13D}\mathbf{p}_2(0) \qquad \mathbf{p}_3(1) \ = -S_{M_13D}\mathbf{p}_2(1), \end{cases} \tag{19}$$

where β_1 and β_2 are "artificial" parameters and S_{M_13D} is the matrix defined in (10) . We add the integral conditions

$$\int_0^T DF_1(\mathbf{x}_0(s))J(\mathbf{x}(s) - \mathbf{x}_0(s))\, ds = 0 \text{ and } \int_0^T DF_2(\mathbf{x}_0(s))J(\mathbf{x}(s) - \mathbf{x}_0(s))\, ds = 0. \tag{20}$$

Its continuation gives the same planar family which was plotted in Figures 4 and 3, but for $m_1 = 0.83883608$ the condition (12) is not true and a branch of non–planar symmetric periodic orbits bifurcates from it (see Figure 7).



$m_1 = 0.8$ $\qquad$ $m_1 = 0.4$ $\qquad$ $m_1 = 0.1$

Figure 7: Branch which appears from a point where condition (12) is not fulfilled for the time-reversal symmetry M_13D.

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