Polynomial graph invariants from homomorphism numbers

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ABSTRACT

We give a new method of generating strongly polynomial sequences of graphs, i.e., sequences $(H_{\mathbf{k}})$ indexed by a tuple $\mathbf{k} = (k_1, \ldots, k_h)$ of positive integers, with the property that, for each fixed graph G, there is a multivariate polynomial $p(G; x_1, \ldots, x_h)$ such that the number of homomorphisms from G to $H_{\mathbf{k}}$ is given by the evaluation $p(G; k_1, \ldots, k_h)$. A classical example is the sequence of complete graphs (K_k), for which $p(G; x)$ is the chromatic polynomial of G. Our construction is based on tree model representations of graphs. It produces a large family of graph polynomials which includes the Tutte polynomial, the Averbouch–Godlin–Makowsky polynomial, and the Tittmann–Averbouch–Makowsky polynomial. We also introduce a new graph parameter, the branching core size of a simple graph, derived from its representation under a particular tree model, and related to how many involutive automorphisms it has. We prove that a countable family of graphs of bounded branching core size is always contained in the union of a finite number of strongly polynomial sequences.

1. Introduction

Let $\mathbb{N}^h$ denote the set of h-tuples of positive integers ($h \geq 1$), and let $\mathcal{H}$ be a countably infinite set of graphs possibly with loops and/or weights on edges. Suppose $\mathcal{H}$ is presented as a sequence $(H_{\mathbf{k}})$ indexed by tuples $\mathbf{k} = (k_1, \ldots, k_h) \in \mathbb{N}^h$. (We extend the usual meaning of "sequence", corresponding to the case $h = 1$, to double sequences ($h = 2$) and more generally to countable sets equipped with a bijection to $\mathbb{N}^h$ for some finite $h \in \mathbb{N}$. Of course, we may present such a set as a conventional sequence by using a bijection from $\mathbb{N}^h$ to $\mathbb{N}$, but the particular indexing of the set by tuples is important in what follows.) Countable families of graphs are often given in the form of such a sequence, for example, the complete graphs (K_k), or the complete bipartite graphs (K_{k_1, k_2}). In these and other concrete examples, the indices $k_1, \ldots, k_h$ correspond to some natural graph parameter (such as number of vertices) or indicate how to construct the graph in that position of the sequence (such as substituting every vertex of a base graph H by k twin copies to obtain H_k).

In this paper, we are interested in the number of homomorphisms of a graph G to $H_{\mathbf{k}}$, denoted by $\hom(G, H_{\mathbf{k}})$, as a function of $\mathbf{k}$ and G. More specifically, when is this function a multivariate polynomial in $\mathbf{k}$ for every graph G? If this is the case for all $\mathbf{k}$, we follow [5] and say that the sequence $(H_{\mathbf{k}})$ is *strongly polynomial*. A well-known example, and one which motivated the "chromatic invariants" of [5], is the sequence of complete graphs (K_k), where $\hom(G, K_k)$ is the value of the chromatic polynomial of G at k for each $k \in \mathbb{N}$.

De la Harpe and Jaeger [5] considered the case $h = 1$ in our setting, i.e. sequences indexed by a single positive integer k. (An exception is their Example B.6, which concerns the bivariate dichromatic polynomial and an anticipation of the two-variable generalization of the chromatic polynomial by Dohmen, Pönitz and Tittmann [6].) They obtained necessary and sufficient criteria that enabled them to verify, for several graph sequences (H_k) with $k \in \mathbb{N}$, that $\hom(G, H_k)$ is strongly polynomial. Further, they provided a general method of generating strongly polynomial sequences of graphs (although by no means all such sequences). On the other hand, in our paper [8] we established precisely for which edge-weighted graphs H homomorphism functions from multigraphs G to H are specializations of the Tutte polynomial $T(G; x, y)$, the Averbouch–Godlin–Makowsky polynomial $\xi_G(x, y, z)$ [1], and the Tittmann–Averbouch–Makowsky polynomial $Q_G(x, y)$ [16]. The edge-weighted graphs H obtained for the three polynomials take the form of a sequence of graphs $(H_\mathbf{k})$ indexed by a tuple $\mathbf{k}$ (shown by cotree representations in Fig. 5).

Here, our main contribution is to introduce a new method to generate strongly polynomial sequences of graphs $(H_\mathbf{k})$, each of which determines a multivariate graph polynomial. These include the chromatic polynomial, the Tutte polynomial, recent generalizations of the chromatic polynomial such as the Dohmen–Pönitz–Tittmann polynomial [6] and the Averbouch–Godlin–Makowsky polynomial [1], and the Tittmann–Averbouch–Makowsky polynomial [16] (a generalization of the independence polynomial). We formulate our results in the general framework of rooted tree models for graphs, but in fact all the above-mentioned polynomials can be generated by just using the cotree representation of cographs.

1.1. Outline

In Section 2 we define homomorphism numbers and strongly polynomial sequences of graphs formally, including in Section 2.1 a useful lemma restating the property of being strongly polynomial in terms of induced subgraph counts rather than homomorphism numbers. In Section 2.2 we define coloured rooted trees and an operation on them ("branching") that will produce all our strongly polynomial sequences of graphs. In Section 2.3 we abstract a definition of tree models for graphs (representation by coloured rooted trees) from the cases of cotrees, clique-width expression trees, embeddings of graphs in closures of rooted trees (used for defining tree-depth) and shrub-depth expression trees. We also isolate the relevant properties of these tree models required for the proof of our main theorem in Section 3.1 to go through.

Section 3 contains in Section 3.1 the statement of our three main results, and Section 3.2 contains their proofs. Theorem 3.1 is the fundamental result for producing strongly polynomial sequences, and Theorem 3.2 is a useful adjunct. Theorem 3.3 gives a sufficient condition to decompose a countable set of graphs into finitely many strongly polynomial (sub)sequences, and introduces a new graph parameter – the branching core size – which according to the tree model used, is related to clique-width, tree-depth and shrub-depth. (In rough terms, the size of the most compact representation of the graph by the given tree model.)

In Section 4 we apply the first two main results of Section 3.1 to obtain strongly polynomial sequences of graphs from each of the tree models related to clique-width, tree-depth and shrub-depth. Even one of the simplest specializations of these two main theorems – to cotree representations of cographs – produces strongly polynomial sequences that determine the chromatic polynomial, the Tutte polynomial, the Averbouch–Godlin–Makowsky polynomial (includes the matching polynomial), and the Tittmann–Averbouch–Makowsky polynomial (includes the independence polynomial). For each tree model we illustrate how it produces sequences in different ways by using the running example of strongly polynomial sequences that start with the cycle C_4 as their initial term.

1.2. Remarks

We conclude this introduction with two remarks.

On the one hand, the "generalized colourings" of [11,12] include only colourings invariant under all permutations of colours, which holds for K_k-colourings (that is, proper k-colourings) and other types of graph colouring such as acyclic colouring that are not defined by counting graph homomorphisms, but not in general for H_k-colourings for other sequences of graphs (H_k). However, in a paper being prepared by A. Goodall, T. Kotek, J.A. Makowsky it will be demonstrated how the model-theoretic framework of [11,12], which gives a general construction of polynomial invariants of relational structures, does in fact include the rooted tree model construction of this paper. In [13] Makowsky also attempts a classification of polynomial graph invariants.

On the other hand, the generalized interpretation model of [10] includes the rooted tree model construction of this paper as a special case. This gives another candidate for the most general class of polynomial graph invariants that includes the chromatic polynomial. Establishing the exact connection between the finite model-theoretic approach of [10] and the countably infinite models used to define graph invariants in [11,12] is a topic of future research.

2. Definitions

2.1. Homomorphism numbers, strongly polynomial sequences of graphs

Let $\hom(G, H)$ denote the number of homomorphisms from a graph G to a simple graph H, i.e., adjacency-preserving maps from $V(G)$ to $V(H)$. This parameter can be extended to weighted graphs as follows.

Let H be a weighted graph given by its adjacency matrix $(a_{i,j})$, where $a_{i,j}$ is the weight of the edge ij. Then, for a multigraph G, the homomorphism function $\mathrm{hom}(G, H)$ is defined by

$$\mathrm{hom}(G, H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)},$$

where the sum is over all functions from $V(G)$ to $V(H)$ and edges of G are taken with multiplicity in the product. When $a_{i,j} \in \{0, 1\}$ this coincides with the number of homomorphisms from G to H as previously defined. When $a_{i,j} \in \mathbb{N}$, the graph H is a multigraph and $\mathrm{hom}(G, H)$ counts the number of homomorphisms from G to H again, where now a homomorphism needs to be defined rather in terms of a pair of maps $f_0 : V(G) \to V(H), f_1 : E(G) \to E(H)$, the defining property being that $f_1(uv)$ has endpoints $f_0(u)$ and $f_0(v)$ for every edge $uv \in E(G)$.

We now introduce the principal object of study in this paper.

Definition 2.1. A sequence $(H_{\mathbf{k}})$ of graphs indexed by $\mathbf{k} = (k_1, \ldots, k_h) \in \mathbb{N}^h$ is a *strongly polynomial sequence in* $\mathbf{k}$ if for every graph G there exists a multivariate polynomial $p(G; x_1, \ldots, x_h)$ such that $\mathrm{hom}(G, H_{\mathbf{k}}) = p(G; k_1, \ldots, k_h)$ for every $\mathbf{k} = (k_1, \ldots, k_h) \in \mathbb{N}^h$.

To simplify notation, henceforth we shall write $p(G; k_1, \ldots, k_h) = p(G; \mathbf{k})$. Also, when no confusion can arise, we shall simply say that $(H_{\mathbf{k}})$ is a strongly polynomial sequence, it being understood that it is strongly polynomial in the given parameter $\mathbf{k}$.

Although a given choice of parametrization of a family $\mathcal{H}$ of graphs may fail to give a strongly polynomial sequence $(H_{\mathbf{k}})$, we are looking from the positive side: we seek instances $\mathcal{H}$ where there exists a parametrization of $\mathcal{H}$ by $\mathbf{k} \in \mathbb{N}^h$ for some $h \geq 1$ which does yield a strongly polynomial sequence. Further, we will be satisfied with such parametrizations over some proper infinite subset $I \subset \mathbb{N}^h$ rather than all tuples in $\mathbb{N}^h$:

Definition 2.2. A sequence $(F_{\mathbf{k}})$ of graphs indexed by tuples $\mathbf{k} \in I \subseteq \mathbb{N}^h$ is a *strongly polynomial subsequence in* $\mathbf{k}$ if it is a subsequence of a strongly polynomial sequence in $\mathbf{k}$, i.e., there is a strongly polynomial sequence $(H_{\mathbf{k}})$ with $\mathbf{k} \in \mathbb{N}^h$, such that $H_{\mathbf{k}} = F_{\mathbf{k}}$ when $\mathbf{k} \in I$.

Clearly, the property of being a strongly polynomial subsequence is unaffected by removing any number of its terms.

For simple graphs G and H, $\mathrm{ind}(G, H)$ counts the number of embeddings of G as an induced subgraph of H, see [2]. A useful equivalent formulation of the property of being strongly polynomial is given by the following:

Lemma 2.3. *A sequence $(H_{\mathbf{k}})$ of simple graphs indexed by a tuple $\mathbf{k} = (k_1, \ldots, k_h) \in \mathbb{N}^h$ is a strongly polynomial sequence in $\mathbf{k}$ if and only if for every graph G there exists a polynomial $q(G; x_1, \ldots, x_h)$ such that $\mathrm{ind}(G, H_{\mathbf{k}}) = q(G; k_1, \ldots, k_h)$ for every $\mathbf{k} = (k_1, \ldots, k_h) \in \mathbb{N}^h$.*

Proof. We first move from counting homomorphisms to injective homomorphisms by the identity (for this and the following identities see for example [2])

$$\mathrm{hom}(G, H_{\mathbf{k}}) = \sum_{\Theta} \mathrm{inj}(G/\Theta, H_{\mathbf{k}}),$$

where the sum is over partitions Θ of $V(G)$, and G/Θ is the graph obtained from G by identifying vertices that lie in the same block of Θ. We then have

$$\mathrm{inj}(G, H_{\mathbf{k}}) = \sum_{\Theta} \mu(\Theta) \mathrm{hom}(G/\Theta, H_{\mathbf{k}}),$$

where

$$\mu(\Theta) = \prod_{I\in\Theta} (-1)^{|I|-1}(|I| - 1)!$$

is the Möbius function of the lattice of partitions of $V(G)$. Thus, it follows that $\mathrm{hom}(G, H_{\mathbf{k}})$ is a polynomial in $\mathbf{k}$ if and only if $\mathrm{inj}(G, H_{\mathbf{k}})$ is a polynomial in $\mathbf{k}$.

Now, to move between counting induced substructures and counting injective homomorphisms, we use the identity

$$\mathrm{inj}(G, H_{\mathbf{k}}) = \sum_{G'\supseteq G} \mathrm{ind}(G', H_{\mathbf{k}}),$$

in which $\mathrm{inj}(G, H)$ denotes the number of injective homomorphisms from G into H. From this identity, by inclusion–exclusion we obtain

$$\mathrm{ind}(G, H_{\mathbf{k}}) = \sum_{G'\supseteq G} (-1)^{|E(G)|-|E(G')|}\mathrm{inj}(G', H_{\mathbf{k}}).$$

Hence, $\mathrm{inj}(G, H_{\mathbf{k}})$ is a polynomial in $\mathbf{k}$ if and only if $\mathrm{ind}(G, H_{\mathbf{k}})$ is a polynomial in $\mathbf{k}$.

Finally, putting these two conclusions together, $\mathrm{hom}(G, H_{\mathbf{k}})$ is a polynomial in $\mathbf{k}$ if and only if $\mathrm{ind}(G, H_{\mathbf{k}})$ is a polynomial in $\mathbf{k}$. $\square$
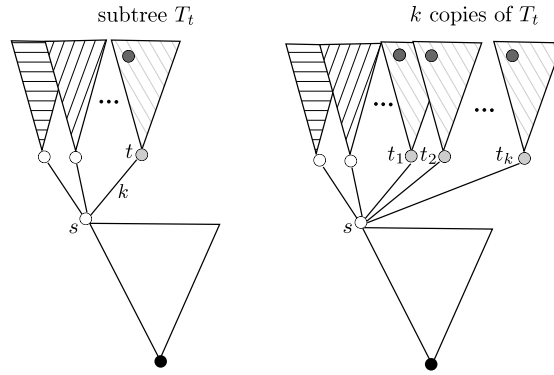
Fig. 1. Branching on edge (s, t) with multiplicity given by its edge-mark k. (See Definition 2.5.)

The property of being a strongly polynomial sequence is preserved under operations such as complementation, disjoint union, and join (as shown by De la Harpe and Jaeger [5]), taking line graphs and lexicographic and Cartesian products, although we do not prove this here as these facts are not needed in this paper. A wide-ranging description of operations that preserve the property of being strongly polynomial is given in [10], in which the model-theoretic notion of an *interpretation scheme* is applied to sequences of relational structures (not just graphs) in order to produce a large class of strongly polynomial sequences of graphs.

2.2. Coloured rooted trees and branching

Let T be a rooted tree with vertex set $V(T)$, edge set $E(T)$, root r, and set of leaves $L(T)$. For $s \in V(T) - r$, let $P(s)$ denote the unique path from root r to vertex s. A vertex $t \neq s$ is an *ancestor* of s if $t \in P(s)$, and a *descendant* of s if $s \in P(t)$. Let $D(T) = \{(s, t) : st \in E(T), s \in P(t)\}$ be the set of edges of T directed away from the root. The *predecessor* of $s \neq r$ is the vertex adjacent to s in $P(s)$, and is denoted by $p(s)$; we have $(p(s), s) \in D(T)$.

The *level* of $s \in V(T)$ is the number of its ancestors, i.e., $|P(s)| - 1$. The root of T is the unique vertex at level 0, and its *height* is the maximum level, i.e., height$(T) = \max\{|P(s)| - 1 : s \in V(T)\}$. The set $\{t \in V(T) : s \in P(t)\}$ is the vertex set of the subtree T_s of T that is rooted at vertex s (tree T_s is the minimal subtree of T containing all descendants of s).

Definition 2.4. Let A be a set. An *A-coloured rooted tree* (T, α_A) is a rooted tree T together with a partial function $\alpha_A : V(T) \rightarrow A$ assigning an element of A to some vertices of T.

A *marked A-coloured rooted tree* (T, α_A, β) is an A-coloured rooted tree (T, α_A) together with a partial function $\beta : D(T) \rightarrow \mathbb{N}$ assigning positive integers to some edges of T.

A vertex not in the domain of α_A is called an *uncoloured* vertex, and an edge not in the domain of β is called *unmarked*. These domains are denoted by dom(α_A) and dom(β), respectively. If all the edges in (T, α_A, β) are unmarked then (T, α_A, β) can be regarded simply as the A-coloured rooted tree (T, α_A). Given a rooted subtree T' of T, we shall write (T', α_A, β) to refer to T' as an A-coloured rooted subtree of (T, α_A, β), where α_A and β are restricted to $V(T')$ and $D(T')$ respectively.

Isomorphism between (marked) A-coloured rooted trees includes preservation of colours on vertices (and integers on edges) as well as isomorphism of the rooted tree structure.

In Section 4, containing applications, A-coloured rooted trees are used for representing graphs (for example, clique-width expression trees). Marked A-coloured rooted trees will be used in this paper together with the branching operation defined below to produce sequences of A-coloured rooted trees that represent strongly polynomial sequences of graphs. Section 4 contains concrete examples.

Definition 2.5. Let (T, α_A, β) be a marked A-coloured rooted tree with edge $(s, t) \in D(T)$ such that $\beta((s, t)) = k$. The *branching* of (T, α_A, β) *on* (s, t) is the marked A-coloured rooted tree which

(i) coincides with (T, α_A, β) on $T \setminus T_t$,
(ii) the edge (s, t) that was marked with k is replaced by k unmarked edges $(s, t_1), \ldots, (s, t_k)$,
(iii) the subtree (T_t, α_A, β) that was pendant from edge (s, t) is replaced by k isomorphic copies $(T_{t_i}, \alpha_A, \beta)$, $1 \leq i \leq k$, each pendant from the corresponding edge (s, t_i).

The value $\beta((s, t)) = k$ is the *branching multiplicity* of edge (s, t).

See Fig. 1 for an illustration of Definition 2.5.

Branching consecutively on a pair of edges is independent of the order in which those edges are taken. Indeed, if $(s, t), (s', t') \in D(T)$ are such that $T_{s'} \cap T_t = \emptyset$ (i.e., t is not a descendant of s', and vice versa) then, clearly, it makes no difference to the resulting marked A-coloured rooted tree if we first branch on (s, t) and then branch on (s', t'), or if we branch in the reverse order. When $s' \in P(t)$ or $t \in P(s')$, we require the following lemma.

Lemma 2.6. *Let* (T, α_A, β) *be a marked A-coloured rooted tree, and* $(s, t), (s', t') \in D(T)$ *such that* $\beta((s, t)) = k$, $\beta((s', t')) = k'$, *and* $T_{s'} \subseteq T_t$. *Then the two marked A-coloured rooted trees obtained from* (T, α_A, β) *by, respectively,*

(i) *first branching on* (s, t) *and then branching on each of the* k *copies of* (s', t'),
(ii) *first branching on* (s', t') *and then branching on* (s, t),

are isomorphic.

Proof. Suppose that we first branch on (s, t), producing k isomorphic copies of (T_t, α_A, β) each pendant from s. In particular, as $T_{t'} \subseteq T_t$, this produces k copies of $(T_{t'}, \alpha_A, \beta)$, each rooted by a copy of vertex t', and each pendant from its own copy of s'. Branching on each copy of edge (s', t') then produces k' isomorphic copies of $(T_{t'}, \alpha_A, \beta)$ pendant from each corresponding copy of s'.

On the other hand, starting by branching on (s', t') first produces k' copies of $(T_{t'}, \alpha_A, \beta)$, each of which are pendant from s'. As $T_{s'} \subseteq T_t$, this gives a marked A-coloured rooted tree which contains the k' copies of $(T_{t'}, \alpha_A, \beta)$ and is rooted at t; branching on (s, t) produces k copies of this tree.

Either way, we obtain kk' isomorphic copies of $(T_{t'}, \alpha_A, \beta)$, with k' copies of $(T_{t'}, \alpha_A, \beta)$ pendant from s' in each of the k copies of (T_t, α_A, β) pendant from s. The remainder of the marked A-coloured rooted tree, $(T, \alpha_A, \beta) \setminus (T_t, \alpha_A, \beta)$, remains unchanged. $\square$

Definition 2.7. The *full branching* of a marked A-coloured rooted tree (T, α_A, β) is the A-coloured rooted tree obtained by branching on (s, t) for each $(s, t) \in \mathrm{dom}(\beta)$.

By Lemma 2.6, the full branching of (T, α_A, β) is a well-defined A-coloured rooted tree.

Definition 2.8. Let (T, α_A) be an A-coloured rooted tree, and let $\mathbf{k} = (k_{s,t} : (s, t) \in M \subseteq D(T))$ be a tuple of positive integers. The $\mathbf{k}$-*branching* of (T, α_A) is the full branching of (T, α_A, β) where $\beta : D(T) \to \mathbb{N}$ is defined by $\beta((s, t)) = k_{s,t}$ for $(s, t) \in M$. We denote this A-coloured rooted tree by $(T^{\mathbf{k}}, \alpha_A)$.

In Definition 2.8 we abuse notation slightly, in that we use the same symbol α_A for the A-colouring of vertices of $T^{\mathbf{k}}$ as for the A-colouring of vertices of T: the former is determined by the latter by definition of branching, copies of a vertex $v \in V(T)$ in $V(T^{\mathbf{k}})$ receiving the same colour as v.

Further, we shall (when possible) omit altogether indicating the functions α_A and β in the notation for (marked) A-coloured rooted trees. Thus, for example, we shall simply say (marked) coloured rooted tree T, branching of T, and write $T^{\mathbf{k}}$ for the $\mathbf{k}$-branching of (T, α_A) when the colouring α_A does not need to be specified.

2.3. Tree models for graphs

We shall use the $\mathbf{k}$-branching of coloured rooted trees of Definition 2.8 in order to produce strongly polynomial sequences of graphs $(H_{\mathbf{k}})$. To do this we take a coloured rooted tree T that represents a graph H (such as a cotree if H is a cograph) and form the $\mathbf{k}$-branching of T, thereby producing a coloured rooted tree that represents the graph $H_{\mathbf{k}}$. Here, we introduce the relevant properties required by such a representation of graphs (by coloured rooted trees) for this construction to work.

Definition 2.9 (*cf.* [7], *Def. 3.1*)**.** Let $\mathcal{H}$ be a class of graphs and A a finite set. A *rooted tree model* for graphs in $\mathcal{H}$ by a class of A-coloured rooted trees $\mathcal{T}$ is given by a surjective function $\gamma : \mathcal{T} \to \mathcal{H}$ with the property that
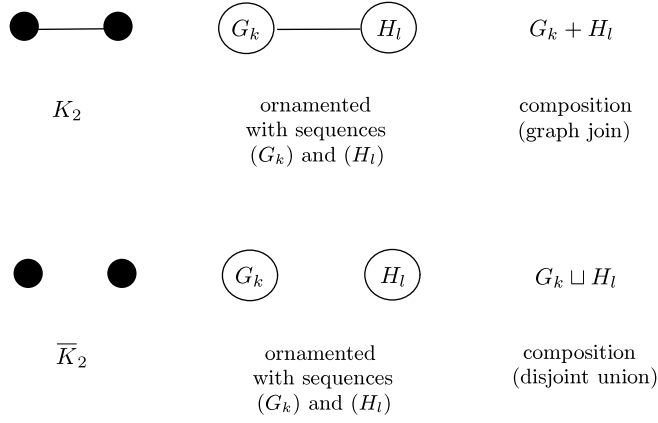
(i) γ is an isomorphism invariant, i.e., $\gamma(T) = \gamma(T')$ when T and T' are isomorphic as coloured rooted trees,
(ii) the vertices of $\gamma(T)$ are in bijective correspondence with a subset of vertices of T containing $L(T)$ (the leaves of T),
(iii) for each $H \in \mathcal{H}$ there are only finitely many coloured rooted trees $T \in \mathcal{T}$ such that $\gamma(T) \cong H$,
(iv) if $H \in \mathcal{H}$, and $T \in \mathcal{T}$ are such that $\gamma(T) \cong H$, then for each connected induced subgraph G of H there is an induced subtree S of T such that $S \in \mathcal{T}$ and $\gamma(S) \cong G$.

Examples of rooted tree models for graphs include:

(a) cotrees for cographs, and more generally clique-width expression trees for graphs [3,4] (where we bound the clique-width so as to restrict to a finite colour set A),
(b) m-partite cotrees, used to define the shrub-depth of a graph [7],
(c) the rooted trees used to define tree-depth [14,15] (where the tree-depth must be bounded so as to have a finite colour set A).

In Section 4 we shall see examples of strongly polynomial sequences of graphs built using the models (a)–(c).

Remark 2.10. Under condition (i) of Definition 2.9 we follow the usual convention, as for graphs, of identifying an isomorphism class of A-coloured rooted trees with a representative of the class.

K_2 G_k —— H_l $G_k + H_l$

ornamented with sequences (G_k) and (H_l)

composition (graph join)

$\overline{K}_2$ G_k H_l $G_k \sqcup H_l$

ornamented with sequences (G_k) and (H_l)

composition (disjoint union)

Fig. 2. Ornamentation of K_2 and $\overline{K}_2$ by graph sequences and the resulting compositions. In the diagram we write G_k instead of, more properly, (G_k), so as to make the diagram more readable.

Condition (ii) allows us to identify $V(\gamma(T))$ with a subset of $V(T)$. Thus, for models (a) and (b) we have $V(\gamma(T)) = L(T)$, and $V(\gamma(T)) = V(T)$ for model (c). The condition that $L(T) \subseteq V(\gamma(T))$ has as a consequence that to each automorphism of T, which can be described by its action on $L(T)$, there is a unique automorphism of $\gamma(T)$ defined by the corresponding action on vertices: this fact will be used in the proof of the key result Theorem 3.1.

Condition (iii) is required for our main Theorems 3.1 and 3.3 in order to have, for given graph H, a bound on the size of A-coloured rooted tree T in $\mathcal{T}$ such that $\gamma(T) \cong H$. This condition is satisfied by the three models (a)–(c) above.

In condition (iv) by *subtree* is meant not necessarily a maximal rooted subtree, just an induced subgraph of T that is a tree and whose root is the vertex at the minimum level. Making the identification of $V(H)$ with a subset of $V(T)$ allowed by (ii), the tree S representing G is the minimal connected subtree of T that spans $V(G)$.

3. Strongly polynomial sequences by branching

3.1. Main results

For the meaning of the notation $T^{\mathbf{k}}$ (the **k**-branching of T) see Definition 2.8.

Theorem 3.1. *Let $\gamma : \mathcal{T} \to \mathcal{H}$ be a rooted tree model, $T \in \mathcal{T}$, $\gamma(T) = H$, and $M \subseteq D(T)$. Let $\mathbf{k} = (k_{s,t} : (s, t) \in M)$ be a tuple of positive integers with the property that branching T on any edge (s, t) yields another tree in $\mathcal{T}$.*
Then the sequence $(\gamma(T^{\mathbf{k}}))$ is a strongly polynomial sequence of graphs in $\mathbf{k}$ with initial term H.

Let $\mathcal{S}$ be the set of all graph sequences (indexed by finite tuples $\mathbf{n} \in \mathbb{N}^h$, for some $h \in \mathbb{N}$). For two graph sequences $(G_{\mathbf{m}})$ and $(H_{\mathbf{n}})$ the *disjoint union* is the graph sequence $(G_{\mathbf{m}} \sqcup H_{\mathbf{n}})$, indexed by the tuple $(\mathbf{m}, \mathbf{n})$, and the *join* is the graph sequence $(G_{\mathbf{m}} + H_{\mathbf{n}})$, also indexed by the tuple $(\mathbf{m}, \mathbf{n})$. (If $\mathbf{m}$ and $\mathbf{n}$ share coordinates, for example if $\mathbf{m} = \mathbf{n}$, then the tuple $(\mathbf{m}, \mathbf{n})$ can be reduced to correspondingly fewer coordinates.)

For a graph H, let $\phi : V(H) \to \mathcal{S}$ assign a graph sequence to each vertex of H. The *ornamented* graph (H, ϕ) is the graph H together with the sequence $\phi(v)$ associated with each vertex $v \in V(H)$. The *composition* of the ornamented graph (H, ϕ) is the graph sequence obtained by taking the disjoint union of the sequences $\{\phi(v) : v \in V(H)\}$ and then joining sequences $\phi(u)$ and $\phi(v)$ whenever $uv \in E(H)$, and otherwise leaving the disjoint union of the sequences. When $H = K_2$ composition is simply the join of the two sequences ornamenting the two vertices, and when $H = \overline{K}_2$ composition is the disjoint union. See Fig. 2 for how graph ornamentation will be depicted later (Figs. 7–11 in Section 4).

We now augment a tree model $\gamma : \mathcal{T} \to \mathcal{H}$ for graphs by A-coloured rooted trees to a tree model for ornamented graphs by $A \times \mathcal{S}$-coloured rooted trees. We make the identification allowed by Definition 2.9(ii) and assume that $L(T) \subseteq V(\gamma(T)) \subseteq V(T)$ for each $T \in \mathcal{T}$.

Let (α_A, ϕ) denote the $A \times \mathcal{S}$-colouring that assigns $(\alpha_A(v), \phi(v))$ to vertex $v \in V(T)$. Then γ interprets the $A \times \mathcal{S}$-coloured rooted tree $(T, (\alpha_A, \phi))$ as the ornamented graph $(\gamma(T), \phi)$. The tree model is defined for *marked* $A \times \mathcal{S}$-coloured rooted trees $(T, (\alpha_A, \phi), \beta)$ by taking the full branching and then interpreting the resulting $A \times \mathcal{S}$-coloured rooted tree as an ornamented graph as before.

Theorem 3.2. *Let $\gamma : \mathcal{T} \to \mathcal{H}$ be a rooted tree model, $T \in \mathcal{T}$, $\gamma(T) = H$, and $M \subseteq D(T)$. Let $\mathbf{k} = (k_{s,t} : (s, t) \in M)$ be a tuple of positive integers with the property that branching T on any edge (s, t) yields another tree in $\mathcal{T}$.*
Then the sequence $(\gamma(T^{\mathbf{k}}), \phi)$ of ornamented graphs yields by composition a strongly polynomial sequence of graphs in $\mathbf{k}$ and the parameters indexing the sequences $\{\phi(v) : v \in V(\gamma(T))\}$.

Our third main result gives a sufficient criterion for a family of graphs to be decomposable into finitely many strongly polynomial subsequences.

Let $\mathcal{H}$ be a class of graphs given by a rooted tree model $\gamma : \mathcal{T} \to \mathcal{H}$. For $H \in \mathcal{H}$, we define $\upsilon_{\mathcal{T}}(H)$ to be the minimum value of $|V(T)|$ over $T \in \mathcal{T}$ such that $\gamma(T^{\mathbf{k}}) = H$ for some $\mathbf{k} = (k_{s,t} : (s, t) \in D(T))$. We call $\upsilon_{\mathcal{T}}(H)$ the *branching core size* of H in the rooted tree model $\gamma : \mathcal{T} \to \mathcal{H}$.

Theorem 3.3. *Let $\mathcal{H}$ be a class of graphs given by a rooted tree model $\gamma : \mathcal{T} \to \mathcal{H}$. Suppose that $\mathcal{F} \subseteq \mathcal{H}$ is such that $\{\upsilon_{\mathcal{T}}(F) : F \in \mathcal{F}\}$ is bounded. Then $\mathcal{F}$ can be partitioned into a finite number of strongly polynomial subsequences of graphs. Moreover, the strongly polynomial subsequences can be produced by branching a finite number of rooted trees in $\mathcal{T}$.*

It is easy to give examples of a family of unbounded branching core size in one tree model that is of bounded branching core size in another: complete graphs $\{K_k : k \in \mathbb{N}\}$ have bounded branching core as cotrees but unbounded tree-depth branching core size. Theorem 3.3 ensures that such a family can be finitely partitioned into strongly polynomial subsequences, even though of unbounded branching core size relative to some tree models.

A partial converse to Theorem 3.3 would require some strengthening of its hypothesis. For example, we believe that although the sequence $(K_k \sqcup K_{k+1} \sqcup \cdots \sqcup K_{k+\ell-1})_{k,\ell \in \mathbb{N}}$ is strongly polynomial, the family of graphs $\{K_k \sqcup K_{k+1} \sqcup \cdots \sqcup K_{k+\ell-1} : k, \ell \in \mathbb{N}\}$ has unbounded branching core size in any tree model.

3.2. Proofs

Proof of Theorem 3.1. By Lemma 2.3 it suffices to prove that, for each connected graph G, the number $\mathrm{ind}(G, \gamma(T^{\mathbf{k}}))$ of induced copies of G in $\gamma(T^{\mathbf{k}})$ is a polynomial in $\mathbf{k}$. Further, by Definition 2.9(ii)–(iv), there are finitely many distinct isomorphism classes of subtrees $S_1, \ldots, S_j$ of $T^{\mathbf{k}}$, where j depends only on G and not on $\mathbf{k}$, and corresponding isomorphic copies $G_1, \ldots, G_j$ of G in $\gamma(T^{\mathbf{k}})$, such that for each $1 \le i \le j$ we have $\gamma(S_i) = G_i \cong G$ and $L(S_i) \subseteq V(G_i) \subseteq V(S_i)$. Note that $S_i \not\cong S_{i'}$ when $i \ne i'$. Conversely, if S is a subtree of $T^{\mathbf{k}}$ such that $\gamma(S) \cong G$ then $S \cong S_i$ for some $1 \le i \le j$. Hence,

$$\mathrm{ind}(G, \gamma(T^{\mathbf{k}})) = \sum_{i=1}^{j} \mathrm{ind}(S_i, T^{\mathbf{k}}). \tag{1}$$

Thus it suffices to prove that $\mathrm{ind}(S_i, T^{\mathbf{k}})$ is a polynomial in $\mathbf{k}$ for each $1 \le i \le j$.

For each subtree T_t rooted at $t \ne r$ in the coloured rooted tree T, there are $k_{p(t),t}$ isomorphic copies of T_t in $T^{\mathbf{k}}$ produced by branching. Consequently, $\mathrm{Aut}(T^{\mathbf{k}})$ contains $k_{p(t),t}!$ elements arising from permutations of these $k_{p(t),t}$ copies of T_t in $T^{\mathbf{k}}$. Let Σ be the wreath product of these permutation groups for each t. Then, $\mathrm{Aut}(T^{\mathbf{k}}) \le \Sigma \times \mathrm{Aut}(T)$ and $|\Sigma| = \mathbf{k}! := \prod_{t \in V(T)-r} k_{p(t),t}!$, while $|\mathrm{Aut}(T)|$ is a constant independent of $\mathbf{k}$. (An automorphism of T corresponds to T having isomorphic subtrees T_s and T_t with $p(s) = p(t)$. In this case there is a corresponding automorphism of $T^{\mathbf{k}}$ precisely when $k_{p(s),s} = k_{p(t),t}$. Thus $\mathrm{Aut}(T^{\mathbf{k}}) \cong \Sigma$ when the components of $\mathbf{k}$ are distinct, and $\mathrm{Aut}(T^{\mathbf{k}}) \cong \Sigma \times \mathrm{Aut}(T)$ when the components of $\mathbf{k}$ are all the same.)

For each $1 \le i \le j$, the set of isomorphic copies of S_i appearing in $T^{\mathbf{k}}$ is partitioned into a finite number r_i of orbits under the action of Σ. Let $S_{i,\ell}$ with $1 \le \ell \le r_i$ be representatives from these orbits, and let $p_{i,\ell}$ denote the size of the orbit containing $S_{i,\ell}$. We next prove that each $p_{i,\ell}$ is a polynomial $p_{i,\ell}(\mathbf{k})$ in $\mathbf{k}$ and so $\mathrm{ind}(S_i, T^{\mathbf{k}}) = \sum_{\ell=1}^{r_i} p_{i,\ell}(\mathbf{k})$ is a polynomial in $\mathbf{k}$ for each $1 \le i \le j$. Let us fix $1 \le i \le j$ and $1 \le \ell \le r_i$.

Observe that there is a minimal $\mathbf{m} = (m_{s,t} : (s, t) \in D(T)) \le \mathbf{k}$, depending on $S_{i,\ell}$ but not on $\mathbf{k}$, such that $S_{i,\ell}$ is an induced subgraph of the induced subtree $T^{\mathbf{m}}$ of $T^{\mathbf{k}}$. (The notation $\mathbf{m} \le \mathbf{k}$ or $\mathbf{k} \ge \mathbf{m}$ indicates that $m_{s,t} \le k_{s,t}$ for each $(s, t) \in D(T)$.)

For $\mathbf{k} \ge \mathbf{m}$, the stabilizer of $S_{i,\ell}$ under the action of Σ contains all permutations of $k_{s,t} - m_{s,t}$ copies of a subtree T_t not containing any vertices of $S_{i,\ell}$. The stabilizer of $S_{i,\ell}$ therefore has size a multiple of $(\mathbf{k} - \mathbf{m})! := \prod_{(s,t) \in D(T)}(k_{s,t} - m_{s,t})!$ and so $p_{i,\ell}$ is a divisor of $\mathbf{k}!/(\mathbf{k} - \mathbf{m})!$, which is polynomial in $\mathbf{k}$. On the other hand, an automorphism in Σ that stabilizes $S_{i,\ell}$ cannot move any of the $m_{s,t}$ branches containing vertices of $S_{i,\ell}$ to any of the $k_{s,t} - m_{s,t}$ branches that do not contain a vertex of $S_{i,\ell}$, but only these $m_{s,t}$ branches among themselves. Hence, the stabilizer of $S_{i,\ell}$ has size a divisor of $(\mathbf{k} - \mathbf{m})!\mathbf{m}!$ which implies that $p_{i,\ell}$ is a multiple of $\frac{\mathbf{k}!}{(\mathbf{k}-\mathbf{m})!\mathbf{m}!}$.

Thus, when $\mathbf{k} \ge \mathbf{m}$, $p_{i,\ell} = p_{i,\ell}(\mathbf{k})$ is a polynomial in $\mathbf{k}$ that is a multiple of $\frac{\mathbf{k}!}{(\mathbf{k}-\mathbf{m})!\mathbf{m}!}$ and divisor of $\frac{\mathbf{k}!}{(\mathbf{k}-\mathbf{m})!}$. In particular, we have $p_{i,\ell}(\mathbf{k}) = 0$ when $k_{s,t} < m_{s,t}$ for some $(s, t) \in D(T)$. By definition of $\mathbf{m}$, if $k_{s,t} < m_{s,t}$ for some $(s, t) \in D(T)$ then $S_{i,\ell}$ does not occur as a subgraph of $T^{\mathbf{k}}$. So the polynomial $p_{i,\ell}(\mathbf{k})$ counts the number of occurrences of $S_{i,\ell}$ in $T^{\mathbf{k}}$ not only for $\mathbf{k} \ge \mathbf{m}$ but also for $\mathbf{k} \le \mathbf{m}$, when it is equal to zero. In other words, $p_{i,\ell} = p_{i,\ell}(\mathbf{k})$ for all $\mathbf{k}$, not just for $\mathbf{k} \ge \mathbf{m}$.

We therefore conclude that $\mathrm{ind}(S_i, T^{\mathbf{k}}) = \sum_{1 \le \ell \le r_i} p_{i,\ell}(\mathbf{k})$ is a polynomial in $\mathbf{k}$ for each $1 \le i \le j$. By Eq. (1) and Lemma 2.3 we have that the sequence $(\gamma(T^{\mathbf{k}}))$ is strongly polynomial in $\mathbf{k}$. $\square$

Proof of Theorem 3.2. For a fixed connected graph G, a copy of G in the composition $(\gamma(T^{\mathbf{k}}), \phi)$ falls into a finite number of types, each of which can be described by a subtree S of $T^{\mathbf{k}}$ and, for each $s \in V(S)$, an isomorphism type G_s of induced subgraph of $\phi(s)$, such that the composition of $\gamma(S)$ ornamented by G_s on vertex $s \in V(S)$ is isomorphic to G. By assumption, the number of copies of G_s in $\phi(s)$ is a polynomial in the parameters indexing the sequence $\phi(s)$. This polynomial multiplied by the number of copies of S in $T^{\mathbf{k}}$, which is a polynomial in $\mathbf{k}$ by the argument given in the proof of Theorem 3.1, is a

polynomial in the parameters indexing the sequence $\phi(s)$ and $\mathbf{k}$. Summing over the finitely many choices of S and induced subgraph isomorphism types of G that occur in $\phi(v)$ we obtain the desired conclusion. $\quad\square$

Proof of Theorem 3.3. By the assumption that $\mathcal{F}$ is of bounded branching core size, there is a bound B such that for each $F \in \mathcal{F}$ we have $\gamma(T^{\mathbf{k}}) = F$ for some asymmetric A-coloured rooted tree T with $|V(T)| = \upsilon_{\mathcal{T}}(F) \leq B$ and some $\mathbf{k} \in \mathbb{N}^{D(T)}$.

The set of A-coloured asymmetric rooted trees T (no colour-preserving automorphisms) of bounded size ($|V(T)| \leq B$) is finite, as A is finite [15]. Hence there is a finite list of asymmetric A-coloured rooted trees $T_1, \ldots, T_\ell$ such that for each $F \in \mathcal{F}$ there is $1 \leq i \leq \ell$ and $\mathbf{k} \in \mathbb{N}^{D(T_i)}$ such that $F = \gamma(T_i^{\mathbf{k}})$.

Hence the terms of the strongly polynomial sequences $(T_i^{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}^{D(T_i)}}$, $1 \leq i \leq \ell$, together cover all of $\mathcal{F}$. $\quad\square$

Proof that $(K_k \sqcup K_{k+1} \sqcup \cdots \sqcup K_{k+\ell-1})$ **is strongly polynomial.** The number of homomorphisms from a graph G to the graph $K_k \sqcup K_{k+1} \sqcup \cdots \sqcup K_{k+\ell-1}$ is given by

$$\sum_{V_1 \sqcup V_2 \sqcup \cdots \sqcup V_\ell = V(G)} \prod_{1 \leq i \leq \ell} P(G[V_i]; k+i-1) \tag{2}$$

where $P(G[V_i])$ is the chromatic polynomial of the subgraph of G induced on V_i, and $\prod P(G[V_i]; k+i-1)$ is a polynomial in $k+i-1$ of degree $|V_i|$.

The expression (2) is therefore a symmetric polynomial in variables $k, k+1, \ldots, k+\ell-1$ of degree bounded by $|V(G)|$, and so is a linear combination of elementary symmetric polynomials in $k, k+1, \ldots, k+\ell-1$ (at most up to degree $|V(G)|$). These elementary symmetric polynomials are coefficients of the falling factorial $(x-k)^{\underline{\ell}}$ (up to degree $|V(G)|$), which are polynomials in k and ℓ (for example, the first elementary symmetric function is $\ell k + \binom{\ell}{2}$) and which can be given explicitly in terms of Stirling numbers of the second kind. Hence expression (2) defines a polynomial in just the two variables k and ℓ. $\quad\square$

4. Examples

We present three examples of rooted tree models for graphs: in Section 4.1 clique-width expression trees (including cotrees for cographs), in Section 4.2 tree-depth embeddings of graphs in the closure of rooted trees, and in Section 4.3 m-partite cotrees (related to shrub-depth). For each we specify the set of A-coloured rooted trees $\mathcal{T}$, the class of graphs $\mathcal{H}$ that they encode (by specifying a rooted tree T together with a colouring $\alpha_A : V(T) \to A$ such that (T, α_A) encodes H), and the edges of a rooted tree in $\mathcal{T}$ on which branching can be applied while staying in the class $\mathcal{T}$. We then illustrate the diversity of strongly polynomial sequences that Theorems 3.1 and 3.2 provide.

We also give representations of the cycle C_4 for each of the three rooted tree models, and apply Theorem 3.1 to obtain a strongly polynomial sequence of graphs to which C_4 belongs as a term.

We begin with cotrees, as a special case of clique-width expression trees, for it is by this tree model for cographs that the strongly polynomial sequences determining the chromatic polynomial, the Tutte polynomial, the Averbouch–Godlin–Makowsky polynomial, and the Tittmann–Averbouch–Makowsky polynomial are most simply described.

4.1. Cotrees and clique-width expression trees

4.1.1. Cotrees and cographs

Let $\mathcal{H}$ be the set of cographs (graphs with no induced path on 4 vertices) and $\mathcal{T}$ the set of A-coloured rooted trees in which $A = \{0, 1\}$ and each $T \in \mathcal{T}$ has a colour from A on each non-leaf vertex, while $L(T)$ is uncoloured. The leaf set $L(T)$ is the vertex set of the graph encoded by the coloured rooted tree T.

The *least common ancestor* $s \wedge t$ of two vertices s and t in a rooted tree T is the vertex at the maximum level in the common subpath $P(s) \cap P(t)$. The encoding $\gamma : \mathcal{T} \to \mathcal{H}$ sets $V(\gamma(T)) = L(T)$ and joins s and t by an edge if and only if $\alpha_A(s \wedge t) = 1$.

Some simple examples of marked cotrees are given in Fig. 3. For each of them, we indicate the strongly polynomial sequence of cographs obtained by Theorem 3.1 (arbitrarily branching the corresponding cotree; marks indicating branching multiplicities). We write $\overline{K}_k$ for the complement of the complete graph K_k.

In Fig. 4 we display a cotree for the cycle C_4, its condensed representation as a marked cotree (again, marks indicating branching multiplicities), and finally the strongly polynomial sequence of cographs obtained by γ (arbitrarily branching this condensed cotree). Here, we use the notation $K_k[\overline{K}_l]$ to indicate the composition of the ornamented graph (K_k, ϕ) where ϕ assigns the graph $\overline{K}_l$ to every vertex of K_k. This composition gives a complete multipartite graph with variable number and size of parts.

Fig. 5 shows the sequences of cographs (defined by branching cotrees ornamented with loops) from which one can obtain, by Theorems 3.1 and 3.2, the chromatic polynomial, the Tutte polynomial, the Averbouch–Godlin–Makowsky polynomial, and the Tittmann–Averbouch–Makowsky polynomial (see [16,8,16] for the homomorphisms functions that are specializations of these polynomials). We use K_k^l to denote the complete graph K_k with a loop of multiplicity l attached at each vertex. The graph $K_{1,k}[K_1^1(\text{centre}); K_j^1(\text{leaves})]$ is the composition of the ornamented graph $(K_{1,k}, \phi)$ where $K_{1,k}$ is the star on $k+1$ vertices, and ϕ assigns the indicated graphs to the centre and the leaves of $K_{1,k}$. For example, the Tittmann–Averbouch–Makowsky polynomial is obtained as follows.
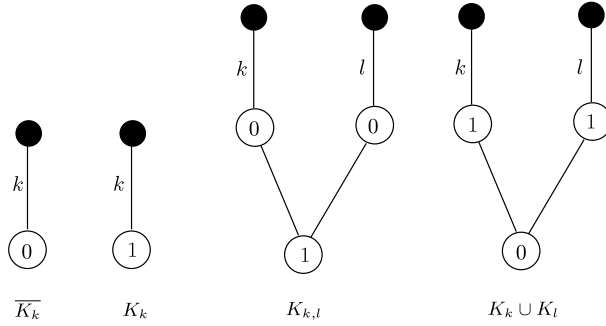
Fig. 3. Examples of strongly polynomial sequences produced by branching of cotrees according to multiplicities given by edge marks. (Unmarked edges in the diagram have mark set equal to 1.) The root is the lowest vertex. The operation of disjoint union is represented by a white circle labelled 0, that of join by a white node labelled 1. Vertices of the graph represented by the cotree are the black circles (taken with multiplicity given by branching).
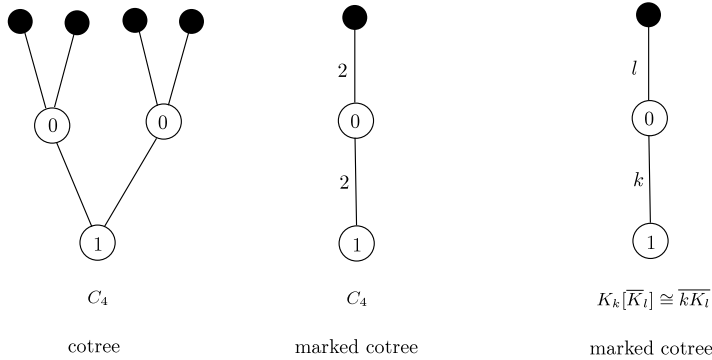
Fig. 4. Cotree for C_4 as a marked cotree, and the result of arbitrary branching on each edge. The right-most marked cotree represents the multipartite graph $K_{k,\dots,k}$ with l parts.

First, branch the corresponding marked cotree in Fig. 5 obtaining an A-coloured rooted tree $T^{(j,k)}$ with $kj+1$ leaves, which are the vertices of $\gamma(T^{(j,k)})$. The encoding $\gamma : \mathcal{T} \to \mathcal{H}$ connects two of those vertices if their least common ancestor in $T^{(j,k)}$ is labelled 1. Thus, the resulting graph $\gamma(T^{(j,k)})$ can be viewed as a star on $k+1$ vertices in which the centre is replaced by K_1^1 and each leaf is replaced by K_j^1. By Theorem 10 in [16], $\mathrm{hom}(G, \gamma(T^{(j,k)}))$ is the Tittmann–Averbouch–Makowsky polynomial $Q(G; x, y)$ evaluated at the point (j, k).

We can clearly see in Fig. 5 how the chromatic polynomial is a specialization of the Tutte polynomial, and the Tutte polynomial a specialization of the Averbouch–Godlin–Makowsky polynomial.

4.1.2. Clique-width expression trees

Let $\mathcal{H}$ be the class of all simple graphs of clique-width at most k and $\mathcal{T}$ the set of A-coloured rooted trees in which $A = [k] \cup \{\eta_{i,j} : i, j \in [k], i < j\} \cup \{\rho_{i \to j} : i, j \in [k], i \neq j\} \cup \{\oplus\}$ and a tree in $\mathcal{T}$ has leaves coloured by $\mathbb{N}$, non-leaves by either $\oplus$ (disjoint union), $\eta_{i,j}$ (join vertices labelled i and j) or $\rho_{i \to j}$ (relabel vertices i with label j). Further, a vertex coloured $\eta_{i,j}$ has only one child, which is coloured $\oplus$, and $\rho_{i \to j}$ has also only one child. Therefore in branching we only mark edges whose vertex nearest the root is coloured $\oplus$.
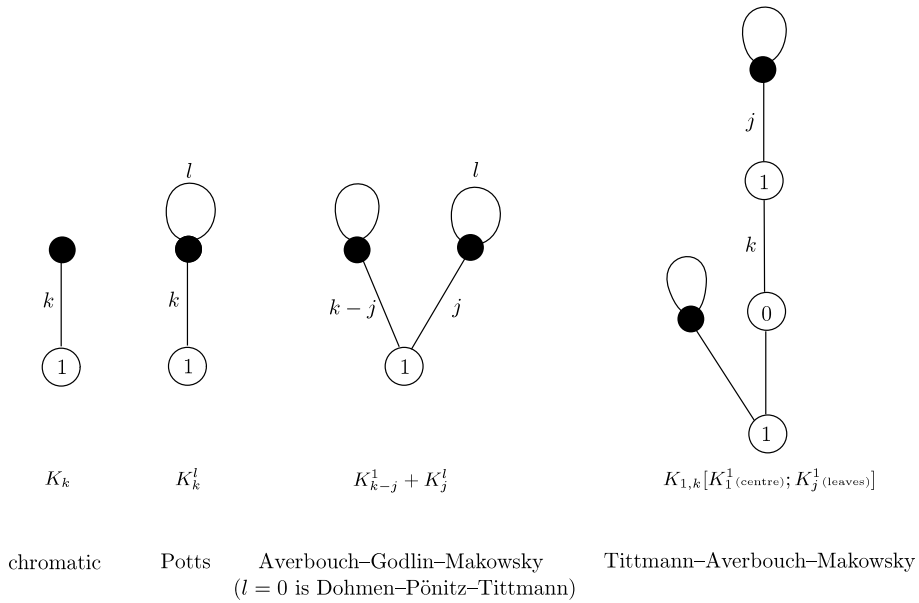
For $k = 2$ clique-width expression trees can be transformed into cotrees as defined above. Indeed, cographs are precisely the graphs of clique-width at most 2.

Fig. 4 can be redrawn as a clique-width 2 expression tree, but we omit this here and proceed in Figs. 6–8 to other representations of C_4 by clique-width expression trees and the associated families of graphs obtained by general branching on edges of the expression tree.

4.2. Tree depth model

The *closure* of a rooted tree T is the graph $\mathrm{clos}(T)$ on vertex set $V(T)$ where st is an edge if $s \in P(t)$ or $t \in P(s)$, and $s \neq t$. A simple connected graph H has *tree-depth d*, denoted by $\mathrm{td}(H) = d$, if H is a subgraph of $\mathrm{clos}(T)$ for some rooted tree T of height $d - 1$, and it is not a subgraph of the closure of a rooted tree of smaller height. For example, the path P_k has tree-depth $\lceil \log_2(k + 1) \rceil$ and the complete graph K_k has tree-depth k. See [14].

Let $\mathcal{H}$ be the class of all simple graphs of tree depth at most d and $\mathcal{T}$ the set of all A-coloured rooted trees in which the colour set A is the set of all finite subsets of $\{0, 1, \dots, d - 1\}$ and in which a vertex at level ℓ in a tree $T \in \mathcal{T}$, $1 \leq \ell \leq d$,

K_k K_k^l $K_{k-j}^1 + K_j^l$ $K_{1,k}[K_1^{1\,(\text{centre})}; K_j^{1\,(\text{leaves})}]$

chromatic   Potts   Averbouch–Godlin–Makowsky   Tittmann–Averbouch–Makowsky
($l = 0$ is Dohmen–Pönitz–Tittmann)

Fig. 5. Strongly polynomial sequences of cographs determining known graph polynomials. A loop attached to a vertex of a cotree remains attached in its branched copies, and is attached with multiplicity l when marked by l.

clique-width
expression tree
for C_4

with general branching
encodes $K_{k,l}$
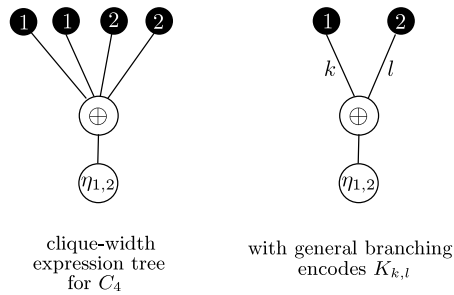
Fig. 6. Simplest clique-width expression tree for C_4, which with general branching situates this graph among complete bipartite graphs $K_{k,l}$.

clique-width
expression tree
for C_4

with general branching

(l times)
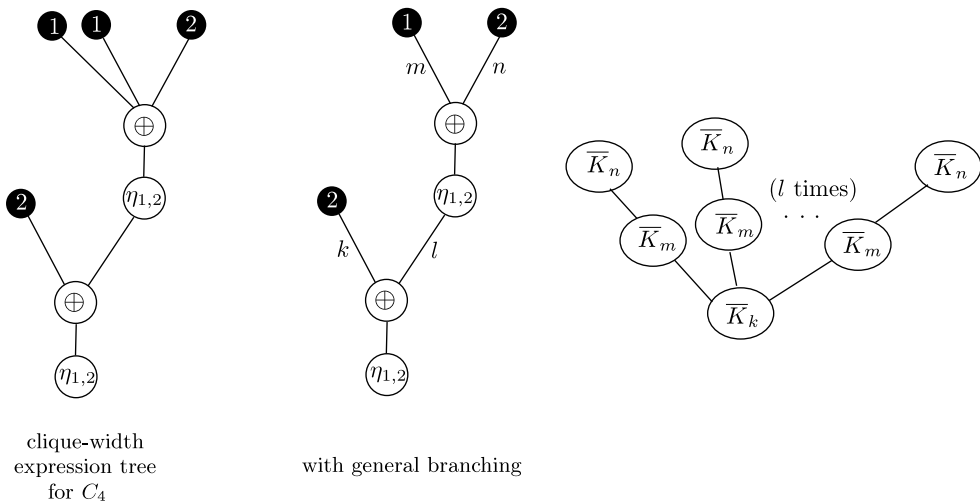
Fig. 7. C_4 encoded by a different clique-width 2 expression tree to that of Fig. 6, and the same tree with general branching multiplicities, giving the composition of the ornamented graph illustrated rightmost. The latter generalizes the graph obtained by general branching of the first tree-depth encoding of C_4 illustrated in Fig. 9.
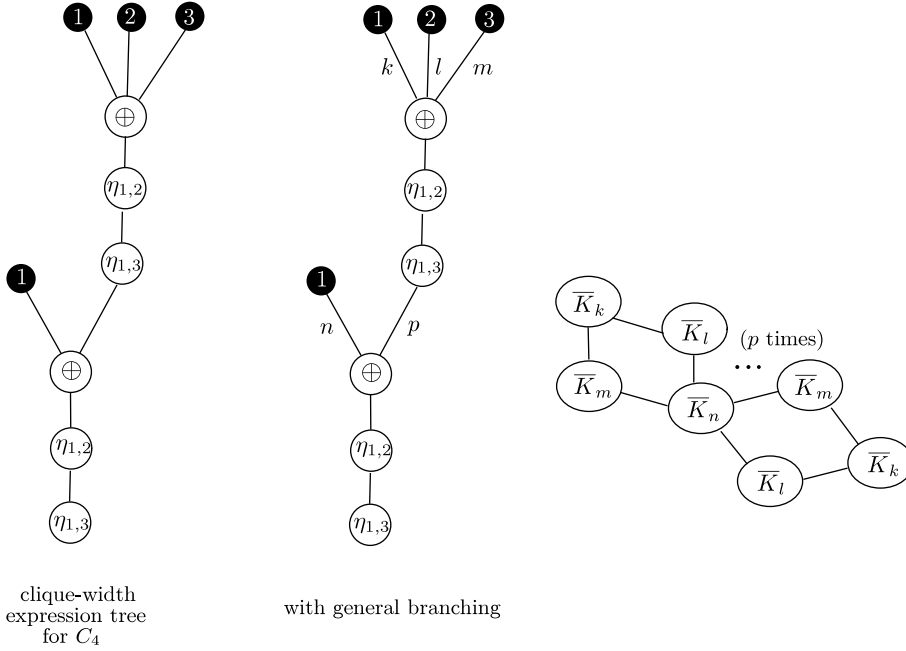
Fig. 8. C_4 encoded by a clique-width 3 expression tree and the same tree with general branching multiplicities, giving the composition of C_4 ornamented by cocliques, illustrated on the right as an ornamented graph.

receives a colour that is a subset of $\{0, 1, \ldots, \ell - 1\}$, and the root always receives the emptyset as colour. For a graph contained in the closure of some rooted tree T of height $\leq d$, the colour set A is thus restricted to subsets of $\{0, 1, \ldots, d-1\}$.

Let $H \in \mathcal{H}$ be contained in the closure of a rooted tree T as a spanning subgraph. For given H there are many choices for T, such as a depth-first search tree for H, or a tree T of minimal height $\text{td}(H) - 1$ whose closure contains H as a subgraph.

The subgraph H of $\text{clos}(T)$, where T has height $d \geq \text{td}(H) - 1$, is encoded by assigning to the vertices of T subsets of $\{0, 1, 2, \ldots, d - 1\}$. The colour of vertex s indicates which of the vertices on the chain $P(s)$ vertex s is adjacent to in H. Specifically, a non-root vertex $s \in V(T)$ is assigned the set $\alpha(s) \subseteq \{0, 1, \ldots, |P(s)| - 2\}$ when in the subgraph H the vertex s is joined to its ancestors (other vertices in the chain $P(s)$) precisely at levels $i \in \alpha(s)$. (The root, the only vertex at level 0, is always assigned the empty set.) In other words,

$$\alpha(s) = \{|P(s)| - 1 - d(s, t) : t \in P(s), st \in E(H)\},$$

where $d(s, t)$ is the distance between s and t in T. For example, s receives the empty set if it is joined to none of its ancestors, and the set $\{0, 1, \ldots, |P(s)| - 2\}$ when it is joined to all its ancestors.

In Fig. 9 we display two strongly polynomial sequences produced by branching different representations of C_4 embedded in the closure of a rooted tree.

4.3. Shrub depth model

The notions of shrub depth and m-partite cographs were introduced in [7]. We just make the definitions of required in order to describe the relevant coloured rooted tree encoding of graphs as m-partite cotrees.

Let $\mathcal{H}$ be the set of m-partite cographs and $\mathcal{T}$ the set of A-coloured rooted trees in which A is the set of $\binom{m+1}{2}$ unordered pairs of elements in $[m]$, and each $T \in \mathcal{T}$ has a colour set equal to a subset of A on each non-leaf vertex, while each $L(T)$ is coloured with an element of $[m]$. The leaf set $L(T)$ is the vertex set of the graph encoded by the coloured tree T.

As for cotrees, in a m-partite cotree adjacencies between leaf vertices $s, t \in L(T)$ are determined by the colour of the least common ancestor $s \wedge t$. The encoding $\gamma : \mathcal{T} \to \mathcal{H}$ sets $V(\gamma(T)) = L(T)$ and joins s and t by an edge if and only if the pair of colours $\alpha_A(s), \alpha_A(t)$ belongs to $\alpha_A(s \wedge t)$. Branching can be performed on any edge of shrub-depth tree representation so as to represent another graph.

The clique-width expression trees of Figs. 7 and 8 have direct translations as 2-partite cotrees. Another pair of 2-partite cotrees representing C_4 is displayed in Fig. 10. By representing C_4 by a 4-partite cotree with 6 edges we see in Fig. 11 a strongly polynomial sequence indexed by 6 parameters k, l, m, n, p, q that contains C_4 as an initial term.

5. Concluding remarks

Theorems 3.1 and 3.2 allow the construction of a large family of strongly polynomial sequences, which in-cludes those determining the chromatic polynomial, the Tutte polynomial, the Averbouch–Godlin–Makowsky, and the
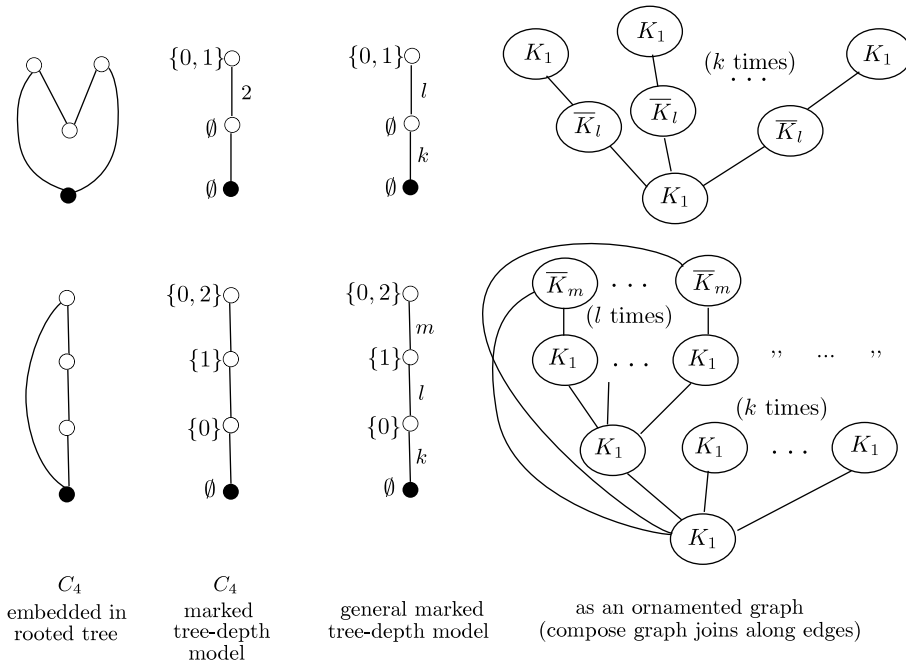
Fig. 9. Two embeddings of C_4 in a rooted tree, and the result of branching more generally. In the rightmost graphs, the lines represent graph joins of the ornament graphs on the vertices. The top-right graph can be obtained as a composition of an ornamented graph (the star $K_{1,k}$ with each edge replaced by a path of length two, ornamented with $\overline{K}_l$ on some vertices). The bottom right graph requires composition of an ornamentation of a more complicated underlying graph.
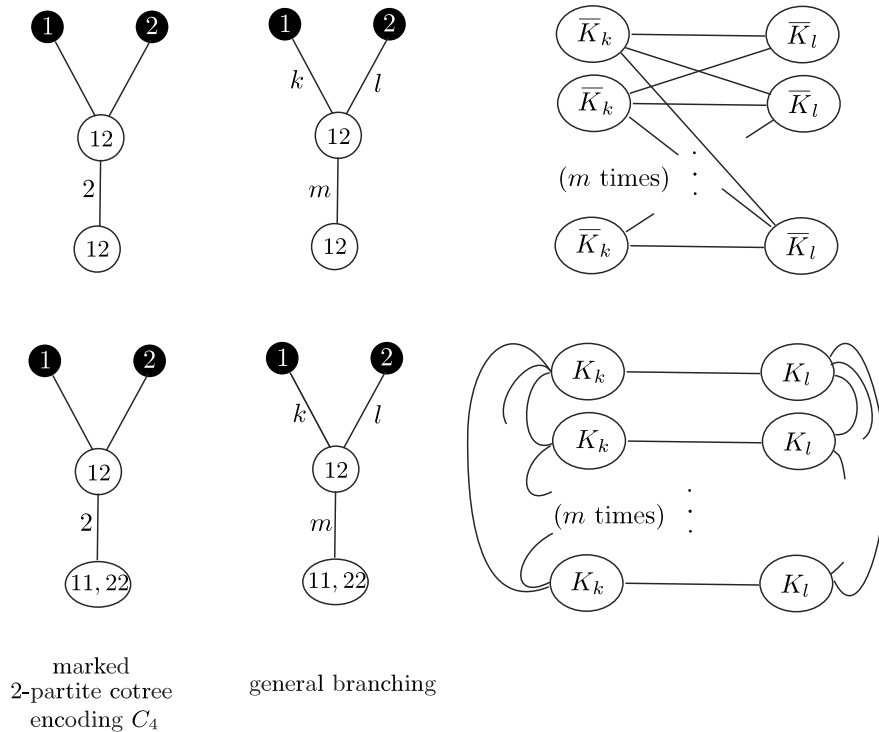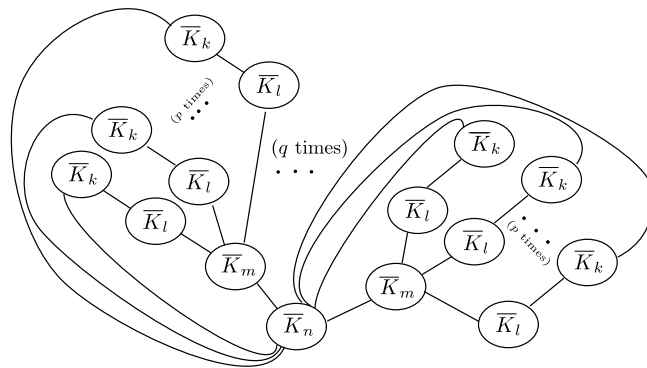

Fig. 10. C_4 encoded in two ways by a 2-partite cotree and the same trees with general branching multiplicities. The top one gives the composition of the complete bipartite graph $K_{m,m}$ ornamented by cocliques, while the bottom one consists of cliques K_{km} and K_{lm} together with edges joining each of m subcliques of size k to m subcliques of size l, as shown.
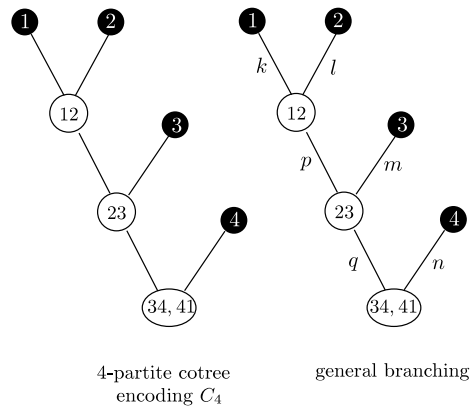
Fig. 11. C_4 encoded by a 4-partite cotree and the same tree with general branching multiplicities along with the graph it represents, illustrated as a graph ornamented by cocliques.

Tittmann–Averbouch–Makowsky polynomial. (See the cotree representations in Fig. 5.) However, not all strongly polynomial sequences are covered by them.

The *generalized Johnson graph* ($J_{k,\ell,D}$), $1 \leq \ell \leq k \in \mathbb{N}, \emptyset \subset D \subseteq \{0, 1, \ldots, \ell\}$, is the graph whose vertices are subsets of $\{1, 2, \ldots, k\}$ of size ℓ, two vertices being adjacent if and only if their intersection has size belonging to D. In particular, when $D = \{0\}$ the graph $J_{k,\ell,\{0\}}$ is the Kneser graph $KG_{k,\ell}$.

For fixed ℓ, D, the sequence ($J_{k,\ell,D}$) is shown in [5, Prop. 3] to have the property that $\hom(G, J_{k,\ell,D})$ is a fixed polynomial in k for sufficiently large k. It is not difficult to develop the proof of this result to show that in fact ($J_{k,\ell,D}$) is a strongly polynomial sequence. However, we cannot find a rooted tree model that can produce this sequence by branching, except when $\ell = 1$, where $J_{k,1,\{0\}} = K_k$, $J_{k,1,\{1\}}$ is $\overline{K}_k$ with a loop on each vertex, and $J_{k,1,\{0,1\}}$ is K_k with a loop on each vertex. A way to build up the sequence of generalized Johnson graphs ($J_{k,\ell,D}$) from more basic strongly polynomial sequences has been found in [10]. Using a model-theoretic approach, our branching of coloured rooted trees becomes a special case of interpretation of graphs in another relational structure: the generalized Johnson graphs find their interpretation in transitive tournaments rather than in coloured rooted trees.

A previous version of this paper [9] contains some extra material that has not been included here. In this earlier work we reviewed and extended some constructions given in [5] for generating new polynomial sequences from old by using different types of operations such as lexicographic product and composition. However, the main contribution of both versions is the same: to introduce a new method of generating strongly polynomial graph sequences. In this regard, in this paper we improve upon [9] since we formulate our results in the general framework of rooted tree models for graphs rather than the special case of the tree depth model (treated in Section 4.2) that is the focus of [9].

The problem of formulating and proving a partial converse to Theorem 3.3, mentioned just after the statement of that theorem in Section 3.1, has not yet been resolved.

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