UNIVERSIDAD DE SEVILLA

DOBLE GRADO EN FÍSICA Y MATEMÁTICAS[1]

TRABAJO DE FIN DE GRADO

INSTITUT FÜR THEORETISCHE PHYSIK

WWU MÜNSTER

Derivation of the Metric of Reissner-Nordström and Kerr-Newman Black Holes

Author
Carlos Francisco ROMERO MADRID[2]

Supervisor
Pr. Dr. Jochen HEITGER[3]

July 25, 2018

[1]Facultad de Física, Facultad de Matemáticas.
[2]carrommad@gmail.com
[3]heitger@uni-muenster.de

”El Universo es enorme,
se calcula que es varias veces la Manga del Mar Menor.”
Pepín Tre[4]

[4] https://youtu.be/hcoiez3PenU

Abstract

In this thesis, both the geometrical and action principle approach to Einstein's field equations are developed, providing an intuitive fundamental path also with a more formal development given by an extremal principle. After this solid introduction, derivations of the metric of two distinct theoretical models of black holes are presented. Firstly, the Reissner-Nordström model is studied and its metric is obtained by solving the differential field equations. Secondly, the Kerr-Newman model is approached by the Newman-Jannis algorithm that provides an easy and straightaway procedure to obtain its metric and energy-momentum tensor just by identifying a seed metric and applying a change of variables. Finally, the solutions are studied, horizons and regions of interest of both black holes are commented.

Keywords: general relativity, gravitation, Reissner-Nordström, Kerr-Newman, Schwarzschild, Minskowski, inertial frame, curvilinear coordinates, covariant derivative, general covariance, Einstein's field equations, metric, affine connection, energy-momentum, action principle, Newman-Jannis algorithm, tetrad formalism, horizon.

Contents

1 Introduction

General relativity is one of the most shocking and best experimentally-proved theories of the last century with enormous changes on the development of Physics due to its profound mathematical structure and geometrical results. Nowadays, general relativity is still a matter of difficulties for many students and scientists due to a deep need of interpretation and mathematical background that intuition is not always willing provide our minds.

The geometrical approach of general relativity is, therefore, a matter of need for most of us to relate *curvature* and gravitation. This path is of course much more technical mathematically and will take us some time but, once we have developed our own intuition of this subject, the following derivations will be easily, substantially, derived. As done in Weinberg (1972) [1], Schutz (2009) [2], Dirac (1975) [3], Hamilton (2015) [4] or Chandrasekhar (1983) [5], geometry is developed and linked to gravitation through identification of geodesics and curvature.

A much more elegant way of deriving *Einstein's field equations* is, of course, an extremal principle, which in Physics gets the name of *action principle* as done firstly by David Hilbert in gravitation and reproduced in Weinberg (1972) [1] and Hamilton (2015) [4]. This action principle will provide us with a highly sophisticated mathematical tool developed properly in Gelfand & Fomin (1963) [6], the *Calculus of Variations*, that will help us, once we have defined and obtained all proper quantities and functionals, to obtain the very same equations as we did using only geometrical procedures.

Finally, we will be studying black holes as done by most of the authors, but mainly focusing on the works of Chandrasekhar (1983) [5], Hamilton (2015) [4] and Weinberg (1972) [1], in that order of relevance, with certain parameters that make them have physically special properties. We will be studying firstly the *Reissner-Nordström black hole*, which consists of a static charged mass, no more no less. This charge will lead us to different solutions like the uncharged case, known as *Schwarzschild's solution*. From this watchtower, a way simpler derivation of the *Kerr-Newman black hole* will be possible just by "turning on" the rotation and giving significant electric charge to the mass. A discussion of physically interesting properties of these two solutions will be exposed, which makes the really important part of a physicist investigation.

2 Geometric Approach

Weinberg (1972) [1] will be mostly followed in this section because of its straightaway development, Schutz (2009) [2], which goes backwards from the field equations to its foundations in *Newtonian gravity* and Dirac (1975) [3], whose compact notation is more than recommendable, eventhough we will be using that of [1] to avoid mistakes through the derivations, are the books that can be consulted to verify and follow, more deeply, this section.

2.1 Principle of Equivalence

The *principle of equivalence of gravitation and inertia* gives a subtle but important property of Newtonian transformations of reference systems matching gravitational forces in an arbitrary reference frame with inertial forces in a locally inertial frame. From the preceding sentence we will have to clarify some terms because, what is a *locally inertial frame*?

In order to develop a theory of gravitation, a mathematical structure must be assumed and, in the case of general relativity, its foundations are based on *Riemannian geometry*. For this interpretation, spacetime will be formalized by a four-dimensional differentiable manifold that will locally resemble an Euclidean space. Locally will therefore mean an open subset of the manifold small enough such that it will behave, in our case, as a *Minkowskian spacetime*. This will translate to the vanishing of the first derivatives of the metric in an arbitrary inertial frame that acts as the subject of study.

More visually, if a system of N particles where forces act upon has as equations of motion the following expression

$$m_N \ddot{\boldsymbol{x}}_{\boldsymbol{N}} = m_N \boldsymbol{g} + \sum_M \boldsymbol{F}(\boldsymbol{x}_{\boldsymbol{M}} - \boldsymbol{x}_{\boldsymbol{N}}) \tag{1}$$

where the primes have been neglected in the right side of the equation because an affine transformation of the coordinates doesn't change the distance. By a simple Galilean transformation $\boldsymbol{x}' = \boldsymbol{x} - \frac{1}{2}\boldsymbol{g}t^2$ and $t' = t$, we get a new system in which the gravitational force is suppressed by an inertial force of the same value. This can be done due to the equivalence of inertial mass and gravitational mass, which has been studied up to a high level of accuracy leading to the equality of both values. After the transformation, the equation is invariant in the right side, where the non-gravitational forces are

$$m_N \ddot{\boldsymbol{x}}_{\boldsymbol{N}}' = \sum_M \boldsymbol{F}(\boldsymbol{x}_{\boldsymbol{M}} - \boldsymbol{x}_{\boldsymbol{N}}) \tag{2}$$

Of course, in this case $\boldsymbol{g}$ was a constant, but we can do this transformations also when the gravitational field depends on time. This is done by a differentiable change of coordinates that will lead to derivatives of the transformation $\boldsymbol{x} \to \boldsymbol{\xi}$ as factors $\partial\xi^\mu/\partial x^\nu$ of the transformation.

So, the question here must be, are the laws of nature the same in an gravitationally-accelerating system than in an inertial reference frame absent of gravitation? The answer is yes, and when we extrapolate this to every point in space, due to homogeneity of spacetime, we get the principle of equivalence that we can state as:

> ”at every spacetime point in an arbitrary gravitational field it is possible to choose a *locally inertial* coordinate system such that [...] the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation.” (Weinberg, 1972, p.68) [1].

2.2 Gravitational Forces

Take a test particle in reference frame where non-zero gravitational forces are the only forces acting on the particle. This particle could be observed in another reference frame with such a coordinate system, called *freely falling*, that makes gravitational forces vanish, and we know this is true due to the principle of equivalence that we just stated. So, therefore, in this new system, the particle moves freely, and we know the equations of motion for this case, leading to

$$\frac{\partial^2 \xi^\lambda}{\partial \tau^2} = 0 \tag{3}$$

with the proper time as $d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$. Remember that *proper time* along a timelike world line is defined as the time as measured by a clock following that line; the particle's clock.

If we now take any other arbitrarily chosen coordinate system x^μ, we could obtain the same information of the previous coordinate system ξ^ν by asking that the coordinate transformation between the proper system and the new one are regularly differentiable. This means nothing but that we could locally "see" the old coordinates as a differentiable function of the new, $\boldsymbol{x} = f(\xi)$. With this, applying the chain rule, we will obtain the equation of motion in the new system

$$\frac{\partial^2 x^\lambda}{\partial \tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{4}$$

where we define the following quantity as *Christoffel symbols* or affine connection

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{5}$$

The proper time, as long as it is also a function of the coordinates, will also experience a change. Its expression in the arbitrarily chosen coordinates is

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{6}$$

where $g_{\mu\nu}$ is the metric tensor, given by

$$g_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \tag{7}$$

In the case of a massless particle, as a photon, $d\tau^2 = 0$ as the particle is travelling at the speed of light. If the proper interval is null, then derivatives using proper time as a variable make no sense, see equation (3). As a solution, we can take another parameter $\sigma \equiv \xi^0$ that we do not need to know in order to find the path of the particle, as its only purpose is parametrization, and applying again the chain rule we get the very same equations

$$\frac{\partial^2 x^\lambda}{\partial \sigma^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = 0 \tag{8}$$

$$- g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = 0 \tag{9}$$

Analyzing the results we can see that equation (4), as well as (8), gives the differential equation of motion and equation (6) tells us how to compute the *proper time*, while in (9) sets initial conditions to massless particles.

To argue that both $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ are relevant quantities that comprise the effects of gravitation, we would expect that coordinate transformations from one coordinate system x^μ to another ξ^α

will only depend on the metric, the affine connection and the coordinates, x^μ. As we are working with frames that are only locally inertial, this means that the coordinates' ξ^α *Taylor's series* up to second order around a spacetime point can be expressed not uniquely, but up to an inhomogeneous Lorentz transformation right hand side of equation (12), determined.

In order to do this, take the affine connection and multiply by $\partial \xi^\beta / \partial x^\lambda$ to get

$$\Gamma^\lambda_{\mu\nu} \frac{\partial \xi^\alpha}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{10}$$

with solution, expanded as a Taylor's series

$$\xi^\alpha(\boldsymbol{x}) = a^\alpha + b^\alpha_\mu (x^\mu - X^\mu) + \frac{1}{2} b^\alpha_\lambda \Gamma^\lambda_{\mu\nu} (x^\mu - X^\mu)(x^\nu - X^\nu) + \dots \tag{11}$$

Thus, by identifying coefficients we clearly see

$$a^\alpha = \xi^\alpha(\boldsymbol{X}) \qquad b^\alpha = \left(\frac{\partial \xi^\alpha(\boldsymbol{x})}{\partial x^\lambda} \right)_{\boldsymbol{x}=\boldsymbol{X}} \tag{12}$$

2.3 Relation between $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$

Just as a remark before deriving the relation between metric and affine connection, we need to be completely clear about the mathematical procedure of derivation and fixing coordinate systems. When we fix a locally inertial coordinate system $\xi^\alpha(\boldsymbol{x})$, we need a specific point in spacetime to do so, call it X, and the coordinates should reflect this information by a proper labelling $\xi^\alpha_{\boldsymbol{X}}(\boldsymbol{x})$. Of course, if $\boldsymbol{X}$ is the origin of our locally inertial coordinate system, we should be able to move in a neighborhood small enough with smooth properties, by means of differentiable functions. Derivatives with arguments the components $\boldsymbol{X}^\alpha$, which are not related at all with the metric or affine connection, appear as new terms. The solution is to use the principle of equivalence by means of another interpretation, that *"the locally inertial coordinates $\xi^\alpha_{\boldsymbol{X}}(\boldsymbol{x})$ that we construct at a given point $\boldsymbol{X}$ can be chosen so that the first derivatives of the metric tensor vanish at $\boldsymbol{X}$"* as developed in Weinberg (1972), p.73-77 [1]. These "derivatives" are respect to $\boldsymbol{X}$, giving us the expression we will be giving in the following lines, as there will be no such terms independent of $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$.

By differentiating the $g_{\mu\nu}$ and identifying terms with $\Gamma^\lambda_{\mu\nu}$ we get

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\mu\rho} \tag{13}$$

and it can be easily seen that, with proper summation

$$\Gamma^\sigma_{\mu\lambda} = \frac{1}{2} g^{\sigma\nu} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right) \tag{14}$$

This shows that $g_{\mu\nu}$ acts as a gravitational potential of the gravitational force, which we find by clearing the term $g_{\mu\nu}$ in equation (4) to be proportional to the affine connection $\Gamma^\lambda_{\mu\nu}$.

2.4 The Newtonian Limit

A new theory is considered valid only when it can predict results that previous theories already predicted. In the case of gravity, the previous theory was *Newtonian theory of gravity* and we will be not only confirming but also using its statements to complete our results for a better comprehension.

Before beginning the approximation, it is interesting to point out the interpretation of the metric tensor done in Dirac (1975), p.26 [3]:

> ”At first sight Einstein’s law of gravitation does not look anything like Newton’s. To see a similarity, we must look on the $g_{\mu\nu}$ as *potentials* describing the gravitational field. There are ten of them, instead of just the one potential of the Newtonian theory. They describe not only the gravitational field, but also the system of coordinates. The gravitational field and the system of coordinates are inextricably mixed up in the Einstein’s theory, and one cannot describe the one without the other.”

Let’s start with a particle whose velocity is Newtonian, non-relativistic, in a *weak stationary gravitational field* and therefore we consider that $|d\boldsymbol{x}/d\tau| \ll dt/d\tau$, with τ being the *proper time* as in equation (6). With this assumption, the equation of motion given by (4) is

$$\frac{\partial^2 x^\lambda}{\partial \tau^2} + \Gamma^\lambda_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{15}$$

Stationary tells us that all time derivatives of $g_{\mu\nu}$ are null, giving

$$\Gamma^\lambda_{00} = -\frac{1}{2}g^{\lambda\rho}\frac{\partial g_{00}}{\partial x^\rho} \tag{16}$$

and *weak* gives us the prerequisite to be able to have a coordinate system such that, in an open set G around a point $\boldsymbol{X}$, we have

$$g_{\mu\nu}(\boldsymbol{x}) = \eta_{\mu\nu}(\boldsymbol{x}) + h_{\mu\nu}(\boldsymbol{x}) \qquad |h_{\mu\nu}(\boldsymbol{x})| \ll 1 \qquad \forall \boldsymbol{x} \in G(\boldsymbol{X}) \tag{17}$$

and to first order in $h_{\mu\nu}$, the rest of the terms can be neglected in this approximation,

$$\Gamma^\lambda_{00} = -\frac{1}{2}\eta^{\lambda\rho}\frac{\partial h_{00}}{\partial x^\rho} \tag{18}$$

$$\frac{\partial^2 \boldsymbol{x}}{\partial \tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \nabla h_{00} \tag{19}$$

and the Newtonian analogue, for $\phi = -GM/r$,is

$$\frac{\partial^2 \boldsymbol{x}}{\partial \tau^2} = -\nabla\phi \tag{20}$$

So, by comparison we get, as $h_{00} \to 0$ at great distances,

$$h_{00} = -2\phi \qquad \text{and} \qquad g_{00} = -1 - 2\phi \tag{21}$$

2.5 Principle of General Covariance

An alternative, and more suitable, version of the principle of equivalence is given by the *principle of general covariance*

> ”states that a physical equation holds in a general gravitational field if two conditions are met:
>
> 1. The equation holds in the absence of gravitation; that is, it agrees with the laws of special relativity when the metric tensor $g_{\mu\nu}$ equals the Minkowski tensor $\eta_{\mu\nu}$ and when the affine connection $\Gamma^{\lambda}_{\mu\nu}$ vanishes.
> 2. The equation is generally covariant; that is, it preserves its form under a general coordinate transformation” (Weinberg, 1972, p.91-92) [1].

In order to validate this new *principle* and related to the previous one, we need arguments in favor. First, from (1) we learn that locally inertial coordinate systems, in which gravitational effects can be neglected, exist at every point of spacetime and that if an equation is valid in such coordinate systems then, from (2), we could take an arbitrary differentiable coordinate transformation to any other coordinate system (for example one where gravitation is non-zero), where it will be valid.

To put this idea briefly: coordinates are a mathematical artifice that has no tangible analogue in nature, they are only used to describe nature and, therefore, any arbitrarily chosen coordinate system should be valid for the development of physical laws.

To finish this introduction to the principle of general covariance, we will write a procedure or recipe on how to used the principle: take the equations in their *special relativity* form, replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$, replace all derivatives with *covariant derivatives* and compute how each quantity transforms under a general coordinate transformation, meaning that we have to look for tensor-transformated objects.

2.6 Energy-Momentum Tensor

The *energy-momentum tensor* is defined to be a tensor quantity that describes the density and flux of energy and momentum in spacetime, being the source of the gravitational field in the Einstein’s field equations that we are looking for. More technically, $T^{\alpha\beta}$ gives the flux of the α-th component of the *momentum vector* across a surface with constant x^{β} coordinate, being symmetric in GR and non-symmetric in theories with non-zero spin tensor.

The energy-momentum tensor is the conserved *Noether current* associated with *spacetime translations*, with $T^{\alpha\beta}$ being conserved both in non-gravitational and gravitational interactions. Conservation means that the *covariant derivative* is null; translating to non-gravitational effects, its divergence vanishes.

We will later see how the energy-momentum tensor acts as source of spacetime curvature. Particularly, we will obtain the energy-momentum tensor for the two specific cases of black holes we will be studied as, in order to solve Einstein's field equations (which we will see at the end of the geometric and the action principle's approach), the tensor provides us with the information about the curvature of spacetime.

2.7 Curvature Tensor

As we mentioned in the last part of section 2.2, both the metric tensor and the affine connection comprise the effects of gravitation. More generally, as the affine connection is derived from the "potential" given by the metric, all the information of our gravitational system is enclose in this latter quantity.

We would imagine that the relevant tensors that we are about to obtain, as the title already tells us, are related to the so-called *curvature tensor*, must be related to this $g_{\mu\nu}$ which contains all the information. So, it is logical to think about what tensors can we form from the metric tensor, its derivatives and, particularly, scalars.

Let's take the terms given by the first derivatives of the metric tensor. We know that, due to the principle of covariance, we can make them vanish in a locally inertial frame properly chosen where gravitational effects are negligible. So, these terms must be dismissed as this "new" tensor will be a combination of $g_{\mu\nu}$, and we know that equality between tensors is true in all frames.

If we now think of the second easier possibility, a tensor formed from the metric tensor and its second derivatives, we look at the transformation of the affine connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\tau}}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\Gamma'^{\tau}_{\rho\sigma} + \frac{\partial x^{\lambda}}{\partial x'^{\tau}}\frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\partial x^{\nu}} \tag{22}$$

but what keeps the affine connection from being a tensor in equation (22) is the inhomogeneous term that appears when trying a coordinate change. Let's clear the term $\frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\partial x^{\nu}}$, take $\partial/\partial x^{\kappa}$ and collect terms

$$\begin{aligned}\frac{\partial^3 x'^{\tau}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}} &= \frac{\partial x'^{\tau}}{\partial x^{\lambda}}(\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta}) \\ &- \frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\frac{\partial x'^{\eta}}{\partial x^{\kappa}}(\frac{\partial \Gamma'^{\tau}_{\rho\sigma}}{\partial x'^{\eta}} - \Gamma^{\tau}_{\rho\lambda}\Gamma'^{\lambda}_{\eta\sigma} - \Gamma'^{\tau}_{\lambda\sigma}\Gamma'^{\lambda}_{\eta\rho}) \\ &- \Gamma'^{\tau}_{\rho\sigma}\frac{\partial x'^{\sigma}}{\partial x^{\lambda}}(\Gamma^{\lambda}_{\mu\nu}\frac{\partial x'^{\rho}}{\partial x^{\kappa}} + \Gamma^{\lambda}_{\kappa\nu}\frac{\partial x'^{\rho}}{\partial x^{\mu}} + \Gamma^{\lambda}_{\mu\kappa}\frac{\partial x'^{\rho}}{\partial x^{\nu}})\end{aligned} \tag{23}$$

And if we follow the procedure of Weinberg (1972), p.131-133 [1], by subtracting the same equation interchanging ν and κ, terms involving products of $\Gamma\cdot\Gamma'$ drop out, and by multiplying by the inverse of $\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}\frac{\partial x'^{\eta}}{\partial x^{\kappa}}$, the following is obtained

$$R'^{\tau}_{\rho\sigma\eta} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}}\frac{\partial x^{\mu}}{\partial x'^{\rho}}\frac{\partial x^{\nu}}{\partial x'^{\sigma}}\frac{\partial x^{\kappa}}{\partial x'^{\eta}}R^{\lambda}_{\mu\nu\kappa} \tag{24}$$

which looks like the transformation rule of a 1-contravariant, 3-covariant tensor. And by definition, the *Riemann-Christoffel curvature tensor*, which is the unique tensor that can be built from $g_{\mu\nu}$ and its first and second derivatives, being linear in the last ones, is given by

$$R^{\lambda}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\eta\nu} \tag{25}$$

It is important to notice the sign of the *curvature tensor* presented in Weinberg (1972) [1], as it is opposite in sign to the general definition. This will imply no more than a change of sign to the usual approach in the energy-momentum tensor when written in the Einstein's field equations.

The amount of equations of these mathematical procedures is overwhelming at least, and we should keep in mind their derivations. What is important is their interpretation, and we shall give the main ideas and reminders of the theory done partially in Schutz (2009), p.165-166 [3]:

- We are working on Riemannian manifolds, which are sufficiently smooths spaces where we define our metric
- The conservation of the sign of our metric is based Sylvester's law of inertia for symmetric spaces. We will have three positive eigenvalues and one negative in any given reference frame.
- "The covariant derivative is the analogous in arbitrary frames to the ordinary derivative in locally inertial frames." [3]
- "The Riemann tensor is the characterization of the curvature. Only if it vanishes identically, is the manifold flat. It has 20 independent components (in four dimensions), and satisfies the Bianchi identities, which are differential equations." [3]

2.8 Gravitation and Curvilinear Coordinates

There is another important remark that we would like to present before deriving the field equations of general relativity, which is about the distinguishability of gravitational fields and curvilinear coordinates. Is it possible to tell if space is really under the effects of gravitation or if our $g_{\mu\nu}$ is just $\eta_{\mu\nu}$ expressed in arbitrarily chosen coordinates?

Citing Weinberg (1972), p.138 [1], the answer is in the following theorem (see [1] for a complete proof of *sufficiency*):

> "The necessary and sufficient conditions for a metric $g_{\mu\nu}(\boldsymbol{x})$ to be equivalent to the Minkowski metric $\eta_{\mu\nu}$ [...] are, first, that the curvature tensor [...] must everywhere vanish, $R^{\lambda}_{\mu\nu\kappa}(\boldsymbol{x}) = 0$ and, second, that at some point $\boldsymbol{X}$ the matrix $g^{\mu\nu}(\boldsymbol{X})$ has three positive and one negative eigenvalues."

The first condition gives us a global condition for the entries of the curvature tensor. The second condition makes us remember directly *Sylvester's law of inertia* between matrices from linear algebra. The first is obvious if you think about it for one second: if, thanks to the principle of equivalence, we are able to find a coordinate system such that the metric tensor becomes that of Minkowski in a point $\boldsymbol{X}$ of spacetime, the affine connection must vanish, making the curvature tensor vanish as well. But as $R^{\lambda}_{\mu\nu\kappa}(\boldsymbol{x}) = 0$ is an equality between tensors, and we know that "a tensor has the property that if all the components vanish in one system of coordinates, they vanish in every system of coordinates" (Dirac, 1975, p.8) [3], in particular, in the previous coordinate system.

This leaves in perfect conditions to finally develop and obtain the field equations we have been looking for. Most of the concepts presented were completely necessary for both approaches to the Einstein's field equations, and their understanding will make it easier for the reader to follow the arguments.

2.9 Derivations of the Field Equations

As a cherry to the cake for all the mathematical formalism and generalization of Minkowski spacetime we have developed, in this section we are obtaining the Einstein's field equations through an heuristic process that may not convince the most conventional physicist, which is a coherent process which agrees with the action principle derivation of the field equations.

To make our minds, we will take the path of Weinberg (1972), p.151-155 [1], remember *Maxwell's equations*. This equations describe the relativistic behaviour of electromagnetic waves through spacetime, and they are linear differential equations due to the non-self-interacting nature of the field, As the electromagnetic field does not carry charge. However, gravitational fields carry energy and momentum, which interact with itself causing non-linearity in the field equations that we are about to obtain. As argued in Dirac (1975), p.45 [3]: "In curved space the conservation of energy and momentum is only approximate. The error is to be ascribed to the gravitation field working on the matter and having itself some energy and momentum."

So, first of all in this process, remember the principle of equivalence, choose a point $\boldsymbol{X}$ of spacetime and build a locally inertial coordinate system that behaves as the following

$$g_{\alpha\beta}(\boldsymbol{X}) = \eta_{\alpha\beta} \tag{26}$$

$$\left(\frac{\partial g_{\alpha\beta}(\boldsymbol{x})}{\partial x^{\gamma}}\right)_{\boldsymbol{x}=\boldsymbol{X}} = 0 \tag{27}$$

If expanded the metric tensor around $\boldsymbol{X}$, we will find that it only contains quadratic or higher terms that we will neglect, apart from the Minkowski metric. We plan on taking the Newtonian limit and "undo" the process in order to obtain the field equations. So, for a weak static field produced by a non-relativistic mass density ρ, we got equation (21) for the time-time component of the metric tensor, and by using Poisson's equation for the potential we obtain

$$\nabla^2\phi = 4\pi G\rho \tag{28}$$

$$\nabla^2 g_{00} = -8\pi G T_{00} \tag{29}$$

as the energy density T_{00} for non-relativistic matter is approximately equal to its mass density $T_{00} \simeq \rho$. We are lead to "guess" [1] that for a general distribution of energy-momentum $T_{\alpha\beta}$ the weak-field equations look like

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{30}$$

where $G_{\mu\nu}$, called the *Einstein's tensor*, is a linear combination of the metric, its first and second derivatives. And, from the principle of equivalence, if we were able to find this equations for an arbitrary locally inertial frame of reference, we can generalize them for fields of arbitrarily strong, gravitationally.

Now, in order to derive $G_{\mu\nu}$ we have two possible approaches: the first is considering that this tensor is only formed up to linear terms of the second derivatives and quadratic terms of the first derivatives; the second, which is more general, we can suppose that this tensor depends on elements unrelated to the metric tensor, just by coordinate transformation derivatives, for further development in this direction, see Weinberg (1972), p.155-157 [1].

Our information about this $G_{\mu\nu}$, and all we need to find it, reduces to the following

- $G_{\mu\nu}$ is a tensor, because $T_{\mu\nu}$ is.
- $G_{\mu\nu}$ is symmetric, because $T_{\mu\nu}$ is.
- $G_{\mu\nu}$ is conserved, because $T_{\mu\nu}$ is.
- As assumed in the first possible approach, this tensor is only formed up to linear terms of the second derivatives and quadratic terms of the first derivatives.
- For weak stationary fields, equation (29) must hold.

As explained in section 2.7, the Riemann-Christoffel curvature tensor, and its contractions, are the most general tensor we were able to form from the metric tensor and the affine connection. So, we could write our tensor as a combination of the contractions of the curvature tensor: R is the *curvature scalar*, which is "defined in order to be positive in the surface of a sphere embedded in three dimensions." (Dirac, 1975, p.24) [3], and the *Ricci tensor* $R_{\mu\nu}$

$$G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R \tag{31}$$

and using the so-called *Bianchi identity*

$$R^{\mu}{}_{\nu;\mu} = \frac{1}{2} R_{;\nu} \tag{32}$$

$$G^{\mu}{}_{\nu;\mu} = \left(\frac{C_1}{2} + C_2\right) R_{;\nu} = 0 \tag{33}$$

where the null equality comes from the fact that $G_{\mu\nu}$ is conserved, leaving us with the equality $C_1 = -2C_2$. We have neglected the possibility of $R_{;\nu} = 0$ everywhere because, by equation (30), it would lead us to $\partial T^{\mu}{}_{\mu}/\partial x^{\nu} = 0$, which is false under the action of inhomogeneous non-relativistic matter.

And, finally, to fix the missing constant we take advantage of the limit given by equation (29), and of the fact that our system is non-relativistic by means of

$$|T_{ij}| \ll |T_{00}| \rightarrow |G_{ij}| \ll |G_{00}| \rightarrow R_{ij} \simeq \frac{1}{2} g_{ij} R \tag{34}$$

$$g_{\alpha\beta} \simeq \eta_{\alpha\beta} \rightarrow R \simeq R_{kk} - R_{00} \simeq \frac{3}{2} R - R_{00} \rightarrow R \simeq 2R_{00} \rightarrow G_{00} \simeq 2C_1 R_{00} \tag{35}$$

To obtain R_{00}, we approximate and take only the linear part of the expression of the lowered *Riemann-Christoffel curvature tensor*

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2} \left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right] \tag{36}$$

For an static field, time derivatives must vanish, leaving us with

$$R_{0000} \simeq 0, \quad R_{i0j0} \simeq \frac{1}{2} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j} \rightarrow G_{00} \simeq 2C_1 (R_{i0j0} - R_{0000}) \simeq C_1 \nabla^2 g_{00} \tag{37}$$

We find that $C_1 = 1$ by equations (29) and (30). So we finally find the Einstein's field equations that relates curvature of spacetime to the energy-momentum tensor

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{38}$$

As we said, there is another derivation that does not contemplate the assumption of $G_{\mu\nu}$ being formed only up to linear terms of the second derivatives and quadratic terms of the first derivatives. If we allowed the metric tensor itself to enter the field equations we would get

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{39}$$

being λ the so-called *Einstein's cosmological constant*, Einstein

> ”inserted it many years later in order to obtain static cosmological solutions [...] that he felt at the time were desirable. Observations of the expansion of the universe subsequently made him reject the term [...]. However, recent astronomical observations strongly suggest that it is small but not zero.” (Schutz, 2009, p.188) [2].

3 Action Principle Approach

Dirac (1975) [3] reproduces the approach done originally by David Hilbert of an action principle applied to relativity that we consider in this section. As well as Gelfand (1963) [6], a book on the foundations of Calculus of Variations that might be consulted in case of mathematical need.

This formulation of the dynamic equations has the great advantage of allowing us to establish a connection between symmetries and conservation laws [1].

3.1 The Gravitational Action Principle

Let us define the *action scalar* I as

$$I = \int_{\Omega} R\sqrt{g(\boldsymbol{x})}d^4\boldsymbol{x} \tag{40}$$

where the integration volume can be compactly finite or even infinity, if the *Ricci scalar* R has proper vanishing behaviour. The fact that we have defined this action quantity as the integral of R makes sense because the integral must have a scalar argument, and the only possible scalar formed from the metric tensor and the affine connection, as argued in (2.7), a contraction of the *curvature tensor*, the *Ricci scalar*. It is also worth to mention that the element $\sqrt{g(\boldsymbol{x})}$ can be regarded as the Jacobian of the coordinate transformation.

Take the mathematical theory from Gelfand (1975) [6], and make small variations $\delta g_{\mu\nu}$, while keeping the $g_{\mu\nu}$ and their derivatives of first order constant at the boundaries

$$\delta g_{\mu\nu}(\partial\Omega) = 0 \tag{41}$$

$$\delta\left(\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right)_{\partial\Omega} = 0 \tag{42}$$

We will assume that, for a properly defined action, making $\delta I = 0$ will provide us with a functional equation leading to Einstein's field equations in vacuum. We have

$$R = g^{\mu\nu}R_{\mu\nu} = R^* - L \tag{43}$$

with

$$R^* = g^{\mu\nu}(\frac{\partial\Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}}) \equiv g^{\mu\nu}(\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma}) \tag{44}$$

$$L = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \tag{45}$$

where second derivatives of $g_{\mu\nu}$ are only occurring in R^* and only linearly and can be removed by partial integration. Also, we find perfect differentials, or the product rule, that will contribute nothing to the action I in the first to terms of

$$R^*\sqrt{g} = (g^{\mu\nu}\Gamma^{\sigma}_{\mu\sigma}\sqrt{g})_{,\nu} - (g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu}\sqrt{g})_{,\sigma} - (g^{\mu\nu}\sqrt{g})_{,\nu}\Gamma^{\sigma}_{\mu\sigma} + (g^{\mu\nu}\sqrt{g})_{,\sigma}\Gamma^{\sigma}_{\mu\nu} \tag{46}$$

Expanding and using $g^{\nu\beta}\Gamma^{\mu}_{\nu\beta}\sqrt{g} = (g^{\mu\nu}\sqrt{g})_{,\nu}$, we get that the remaining two terms are $2L\sqrt{g}$, so the action becomes

$$I = \int_{\Omega} L\sqrt{g}d^4\boldsymbol{x} \tag{47}$$

which only contains the $g_{\mu\nu}$ and its first derivatives. If we write $\mathcal{L} = L\sqrt{g}$, which may be considered a *Lagrangian density* in three dimensions or as an action density in spacetime, as

$$I = \int_{\Omega} \mathcal{L}d^4\boldsymbol{x} = \int_{\Omega_0} dx_0 \int_{\Omega'} \mathcal{L}dx^1dx^2dx^3 \tag{48}$$

being $\mathcal{L}$, the argument correspondent to the time variable of integration dx^0, the Lagrangian.

Now, applying the rules of variational calculus, we vary $\mathcal{L}$ to obtain

$$\delta\mathcal{L} = \Gamma^{\alpha}_{\mu\nu}\delta(g^{\mu\nu}\sqrt{g})_{,\alpha} - \Gamma^{\beta}_{\alpha\beta}\delta(g^{\alpha\nu}\sqrt{g})_{,\nu} + (\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu})\delta(g^{\mu\nu}\sqrt{g}) \tag{49}$$

we get, after adding and subtracting terms to make the first two terms of equation (49) a perfect differential,

$$\delta I = \delta\int_{\Omega} \mathcal{L}d^4\boldsymbol{x} = \int_{\Omega} R_{\mu\nu}\delta(g^{\mu\nu}\sqrt{g})d^4\boldsymbol{x} \tag{50}$$

where $R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}$.

And we can also deduce, from the usual partial derivation of the factor, that the variation must take the following form (due to the behaviour of Leibniz's rule)

$$\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\beta\nu}\delta g_{\alpha\beta} \tag{51}$$

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\alpha\beta}\delta g_{\alpha\beta} \tag{52}$$

Providing us with

$$\delta(g^{\mu\nu}\sqrt{g}) = -(g^{\mu\alpha}g^{\beta\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta})\sqrt{g}\delta g_{\alpha\beta} \tag{53}$$

turning equation (50) into

$$\delta I = -\int_{\Omega} R_{\mu\nu}(g^{\mu\alpha}g^{\beta\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta})\sqrt{g}\delta g_{\alpha\beta}d^4\boldsymbol{x} \tag{54}$$

which gives the vanishing functional condition that of Einstein's field equations for vacuum as it must vanish $\forall\delta g_{\alpha\beta}$ arbitrarily chosen:

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 0 \tag{55}$$

3.2 The Comprehensive Action Principle

In this section we will rename our action from $I \to I_g$ in order to denote that this is the gravitational part of the action and, I' will be the action of all the other fields, consisting of a sum of terms corresponding each one to one field in particular.

A really interesting feature of the action principle is linearity in the variational derivative. We only need to obtain the action for each of the fields and add them all together.

Let's start with a renormalized equation (48), where $\mathcal{L} \to (16\pi)^{-1}\mathcal{L}$, so (we will omit the region of integration Ω from now on)

$$I_g = \int \mathcal{L} d^4\boldsymbol{x} \tag{56}$$

and applying a variation and partial integration in the second term

$$\delta I_g = \int \left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} \delta g_{\mu\nu,\alpha} \right) d^4\boldsymbol{x} = \int \left[\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} \right)_{,\alpha} \right] \delta g_{\mu\nu} d^4\boldsymbol{x} \tag{57}$$

where the work on section (3.1) leads to

$$p^{\mu\nu}\sqrt{g} \equiv -(16\pi)^{-1} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \sqrt{g} = \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} \right)_{,\alpha} \tag{58}$$

where we have defined the quantity $p^{\mu\nu}$ so as to reduce notation.

Now, let's calculate the part of the action corresponding to the rest of the fields ϕ_n ($n \in \mathbb{N}$), each of them assumed to be independent but unspecified. The action looks

$$I' = \int \mathcal{L}' d^4\boldsymbol{x} \tag{59}$$

where $\mathcal{L}' = f(\phi_n, \phi_{n,\nu})$, in the form of an integral of a scalar density.

The total variation of the action leads to

$$0 = \delta(I_g + I') = \int \left(p^{\mu\nu} \delta g_{\mu\nu} + N^{\mu\nu} \delta g_{\mu\nu} + \sum_n \chi^n \delta\phi_n \right) \sqrt{g} d^4\boldsymbol{x} \tag{60}$$

where $p^{\mu\nu}\sqrt{g}$ is the argument of integration coming from equation (58) of the I_g derivation (note that in [3] these quantities $p^{\mu\nu}$ are renamed while we keep their first definition and introduce the $N^{\mu\nu}$ as other new terms outside $p^{\mu\nu}$), the second tensor in the argument

$$N^{\mu\nu}\sqrt{g} \equiv \frac{1}{2} \frac{\partial f(\phi_n, \phi_{n,\nu})}{\partial \sqrt{g}} \sqrt{g} g^{\mu\nu} = \frac{\partial \mathcal{L}'}{\partial g_{\mu\nu}} \tag{61}$$

is formed from $\mathcal{L}'_{,\sqrt{g}}$, as what we are really varying is the metric; and, the last terms in the argument of the variation of the action

$$\chi^n \equiv \frac{1}{\sqrt{g}}\left[\frac{\partial f(\phi_n, \phi_{n,\nu})}{\partial \phi_n} + \sum_\nu \left(\frac{\partial f(\phi_n, \phi_{n,\nu})}{\partial \phi_{n,\nu}}\right)_{,\nu}\right] = 0 \tag{62}$$

are the coefficients due to the partial derivation of the scalar fields, which are assumed to be independent and that is the reason why they should vanish to fulfill the variational principle.

To make the argument of equation (60) completely vanish, equation (62) must give us

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - 16\pi N^{\mu\nu} = 0 \tag{63}$$

where both $p^{\mu\nu}$ and $N^{\mu\nu}$ are symmetrical. These are no less than the Einstein's field equations, when a proper normalization of $N^{\mu\nu}$ is taken, derived from an action principle we can define the energy-momentum tensor to be the tensor that

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + 8\pi T^{\mu\nu} = 0 \tag{64}$$

where any tensor produced by matter must fulfill, in order to be consistent,

$$T^{\mu\nu}_{\ ;\nu} = 0 \tag{65}$$

This same procedure is derived, for example, in Weinberg (1972) p. 387-365, [1] by defining the energy-momentum tensor as the functional derivative (in our case, variational derivative) of the gravitational action itself, obtaining therefore the same results as with the normalized $N^{\mu\nu}$.

3.3 Further Reading

For a deeper understanding of the subject, several generalizations have been proposed over the decades after General Relativity came to light.

Among them, both *Brans-Dicke theory* and *Einstein-Cartan theory* are of high interest for those who would like to continue into more general gravitation theories.

Brans-Dicke theory [1] provides a different approach by defending that the gravitational constant G is not so. Instead, gravitational interaction is mediated by a scalar field and that gives G as an average of the interaction of the field with all the matter in the Universe. In fact, general relativity theory can be obtained from Brans-Dicke's as a limit.

Einstein-Cartan theory [7] relaxes the assumption of General Relativity that the affine connection has vanishing antisymmetric part (torsion tensor), so that the torsion can be coupled to the intrinsic angular momentum (spin) of matter, much in the same way in which the curvature is coupled to the energy and momentum of matter. This theory provides a possible different origin of the Universe, the *Big Bounce*, that will make the *Big Bang* an obsolete theory. It also provides a mathematical basis for the theory of *wormholes* that may be interesting for further reading.

All three Einstein's, Brans-Dicke's and Einstein-Cartan's theories are relativistic classical field theories of gravitation, called *metric theories*. For a further insight, also see *quantum theories of gravity* [1].

4 Black Holes

For this interesting part of the paper, we will be following the both deep and conceptual books by Hamilton (2015) [4] and Chandrasekhar (1983) [5], as well as Nordebo (2016) [8] and Newman & Adamo (2004) [11]. This will provide us with an intuition of the particular stellar configurations we are going to work with by learning key concepts of the subject.

4.1 Introduction

A *black hole* is defined [4] to be a region where space itself falls faster than the speed of light, whose main result is that, of course, no light can escape from a black hole from inside. A way to see it is by applying the *cosmological expansion* analogue: imagine that you are a photon inside a black hole, you know you are the fastest entity ever known. You follow a path that hypothetically carries you out of the black hole, but this path will be longer and longer and, somehow, space is being created just in front of you at a rate that you cannot surpass in any possible way.

So, theoretical physics had demonstrated the possibility of existence of black holes in our Universe, but we know that theoretical physics has not always been correct. We need evidence, and there is non-direct observational evidence of their ubiquitous presence by means of the study of their surroundings. To the day, there are two kinds of black holes that can be observed: *stellar-sized black holes in X-ray binary systems*, mostly in our own Milky Way galaxy, and *supermassive black holes in Active Galactic Nuclei* (AGN) found at the centres of our own and other galaxies [1].

Secondary evidences of the presence of a black hole can also be inferred by facts like high luminosity, non-stellar spectrum, rapid variability and relativistic jets. To the day, there is also evidence of gravitational waves measured at Earth by the experiment LIGO, setting the general theory of relativity as a more-than-proved gravitational theory.

The types of black holes that we will be studying are a generalization of the so-called ”ideal black holes” which are stationary, static and electrovac (null-energy-momentum tensor except for the contribution of a stationary electromagnetic field). The Schwarzschild solution of a black hole may be the most known of all of them, we assume, because of their simplicity and importance.

4.2 Reissner-Nordström Black Hole

The first attempt we are dealing with, in the matter of black holes, is the *Reissner-Nordström metric* surrounding a stationary, non-rotating, charged spherically-symmetric mass discovered independently by Hans Reissner, Hermann Weyl and Gunnar Nordström. This metric describes the unique solution in asymptotically flat spacetime.

This types of black holes are of little interest, regarding their charge, as astrophysical objects like stars or clouds tend to neutralize electrically, providing a little net charge for the body. The interest comes from its solution, the physical interpretation and its resemblance to the *Kerr solution*.

4.2.1 Reissner-Nordström Metric

The Reissner-Nordström metric for a black hole of mass M and charge Q, in geometric units $c = G = k = 1$, is given by

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \tag{66}$$

where the so-called *horizon function* is given by

$$\Delta(r) \equiv \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \tag{67}$$

If you have seen or studied *Schwarzschild metric* before, this particular metric will sound familiar to you or, at least, resemble that other metric. Just take $Q \longrightarrow 0$ to obtain the most simple case of a black hole. Another possible interpretation of the mass is to take Schwarzschild's M_S and identify its analogue in Reissner-Nordström's M_{RN}:

$$M_S \longrightarrow M_{RN}(r) = M - \frac{Q^2}{2r} \tag{68}$$

Where the quantity $M_{RN}(r)$ has the coordinate-independent interpretation [4] as the mass interior to radius r, being the mass seen by an observer at infinity M minus the *mass* (or energy, thanks to mass-energy equality in relativity theory) in the electric field $E = Q/r^2$ outside r

$$4\pi \int_r^\infty \frac{E^2}{8\pi r'^2} dr' = \frac{Q}{2r} \tag{69}$$

This may look like a Newtonian calculation, but it turns out to be true as the electromagnetic field is unchanged under Lorentz boosts in the radial direction of movement.

4.2.2 Derivation of the Metric and the Energy-Momentum Tensor

If we consider spacetime to be spherically symmetric, we can write a general metric, as concisely developed in Nordebo (2016), p.25-30 [8],

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{70}$$

where we have assumed dependence on time t, eventhough we will later find out that the dependence is not so, and the unknowns functions of the metric will only depend on r.

Since we are working in vacuum, the energy-momentum tensor we need is given by the electromagnetic energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\gamma}F_{\nu\alpha}g^{\alpha\gamma} - \frac{1}{4}g_{\mu\nu}F_{\gamma\delta}F^{\gamma\delta}\right) \tag{71}$$

and as we know, the trace of the electromagnetic $T_{\mu\nu}$ vanishes, turning Einstein's field equations to be

$$R_{\mu\nu} = -8\pi T_{\mu\nu} \tag{72}$$

Finally we also need the source-free Maxwell's equations

$$F^{\mu\nu}_{\ ;\nu} = 0 \tag{73}$$

$$F_{\mu\nu;\gamma} + F_{\gamma\mu;\nu} + F_{\nu\gamma;\mu} = 0 \tag{74}$$

So, let's compute the affine connection applying equation (14) to the functions $A(t,r)$ and $B(t,r)$; and, thereafter, we will compute the *Ricci tensor* using contracted equation (25) by $\lambda = \nu$. Firstly, all non-vanishing Christoffel symbols are given

$$\begin{array}{lll}
\Gamma^0_{00} = \dfrac{\dot{A}}{2A} & \Gamma^0_{01} = \Gamma^0_{10} = \dfrac{A'}{2A} & \Gamma^0_{11} = \dfrac{\dot{B}}{2A} \\
\Gamma^1_{00} = \dfrac{A'}{2B} & \Gamma^1_{01} = \Gamma^0_{10} = \dfrac{\dot{B}}{2B} & \Gamma^1_{11} = \dfrac{B'}{2B} \\
\Gamma^1_{22} = -\dfrac{r}{B} & \Gamma^1_{33} = -\dfrac{r\sin^2(\theta)}{B} & \\
\Gamma^2_{12} = \Gamma^2_{21} = \dfrac{1}{r} & \Gamma^2_{33} = -\sin(\theta)\cos(\theta) & \\
\Gamma^3_{13} = \Gamma^3_{31} = \dfrac{1}{r} & \Gamma^3_{23} = \Gamma^3_{32} = \cot(\theta) &
\end{array} \tag{75}$$

where a dot represents differentiation with respect to t, and the prime, with respect to r. Analogously, all the non-vanishing component of the *Ricci tensor* are

$$R_{00} = \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A''}{2B} - \frac{A'}{rB} + \frac{\ddot{B}}{2B} - \frac{\dot{B}}{4B}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) \tag{76}$$

$$R_{11} = -\frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A''}{2A} - \frac{B'}{rB} + \frac{\ddot{B}}{2A} + \frac{\dot{B}}{4A}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) \tag{77}$$

$$R_{22} = \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{B} - 1 \tag{78}$$

$$R_{33} = R_{22}\sin^2(\theta) \tag{79}$$

$$R_{01} = R_{10} = -\frac{\dot{B}}{rB} \tag{80}$$

By generalizing the spherically symmetric gravitational field, we can get no further. Now, if we want to determine the functions A and B, what we need is to plug these relations into the Einstein's field equations with a proper energy-momentum tensor, computed with the electromagnetic tensor and the metric tensor.

So, as we have only a radial component of the electric field, we can write

$$E_r(t,r) = E_1(t,r) = F_{01}(t,r) = F_{10}(t,r) \tag{81}$$

and using equation (71) we get

$$T_{00} = \frac{1}{2\mu_0} A F_{01} F^{01} \tag{82}$$

$$T_{11} = -\frac{1}{2\mu_0} B F_{01} F^{01} \tag{83}$$

$$T_{22} = \frac{1}{2\mu_0} r^2 F_{01} F^{01} \tag{84}$$

$$T_{33} = \frac{1}{2\mu_0} g_{33} F_{01} F^{01} = T_{22} sin^2(\theta) \tag{85}$$

All non-diagonal components of the energy-momentum tensor are zero, which means

$$T_{01} = 0 = R_{01} = \frac{\dot{B}}{rB} \to \dot{B} = 0 \to B = B(r) \tag{86}$$

the dependence on time of B has been eliminated. Also note that

$$\frac{T_{00}}{A} + \frac{T_{11}}{B} = 0 \to 0 = \frac{R_{00}}{A} + \frac{R_{11}}{B} = \frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right) \tag{87}$$

which implies

$$\frac{\partial \ln(AB)}{\partial r} = \left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \to AB = f(t) \tag{88}$$

as it must be constant with respect to r.

Solving Maxwell's equations, given in a compact form by equation (74), and applying Gauss' law to determine the arbitrary constant of integration, we get

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} \tag{89}$$

And now, using explicitly Einstein's field equations we can write by identifying $B = A/f$ and $B' = -fA'/A^2$,

$$-8\pi T_{22} = R_{22} = \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{B} - 1 = \frac{1}{f(t)}\frac{\partial(rA(r))}{\partial r} - 1 \tag{90}$$

and by plugging in the expression of T_{22}

$$\frac{1}{f(t)}\frac{\partial(rA(r))}{\partial r} - 1 = -\frac{1}{f(t)}8\pi\frac{1}{2\mu_0}r^2E_r^2 \tag{91}$$

which gives

$$\frac{\partial(rA(r))}{\partial r} = f - \frac{Q^2}{4\pi\epsilon_0 r^2} \tag{92}$$

and integrating and dividing by r, we obtain the result

$$A(t,r) = f(t) + \frac{C(t)}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2} \tag{93}$$

Finally, to get the values of the unknown functions of time f and C, we must take the limit to the Schwarzschild metric $Q \to 0$, and therefore g_{00} must approach

$$g_{00} = 1 - \frac{2M}{r} \tag{94}$$

giving

$$f(t) = 1, \qquad C(t) = -2M \equiv r_S, \qquad r_Q^2 \equiv \frac{Q^2}{4\pi\epsilon_0} \tag{95}$$

$$g_{00}(r) = A(r) = 1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2} \tag{96}$$

So, a nice visualization of the metric in this particular case would be to show the tensor

$$g_{\mu\nu}(r,\theta) = \begin{bmatrix} -\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}$$

and by setting $k = 1/4\pi\epsilon_0 = 1$, we get the result from section 4.2.1.

4.2.3 Horizons

In the Reissner-Nordström metric we find two possible horizons or singularities in the term g_{11}, given by the inverse of the horizon function in equation (67), that make this term tend to infinity. The horizons occur where an object at rest in the geometry, $dr = d\theta = d\phi = 0$, follows a null geodesic, $ds^2 = 0$, which occurs where the horizon function vanishes. This gives us a quadratic equation that we can easily solve

$$R_\pm = M \pm \sqrt{M^2 - Q^2} \tag{97}$$

We can check that the time coordinate has different behaviours depending on the value of r: t is a timelike coordinate when $r > r_+$ and $r < r_-$, so outside the outer horizon and inside the inner horizon; and t is a spacelike coordinate between the inner and outer horizons $r_- < r < r_+$.

We can interpret this differing behaviour from the Schwarzschild solution: outside the outer horizon, spacetime is falling at the speed of light, as it does inside the inner horizon. This is caused by electromagnetic repulsion (negative pressure of the electric field). In between the horizons, space behaves as in the Schwarzschild black hole, spacetime is falling at velocities higher than the speed of light.

If we made $M^2 = Q^2$, with their correspondent units, we would see that both the inner and outer horizon are the same, providing just a 2-dimensional spherical surface where spacetime is falling at exactly the speed of light. Eventhough there is only a "shell" of spacetime with velocity c, light could not get out of the inside of the black hole.

And if we take a vanishing amount of charge we get, as expected from last section, only one horizon that corresponds to that of Schwarzschild's solution, $r = 2M$.

4.3 Kerr-Newman Black Hole

The *Kerr-Newman metric* is the metric corresponding to a stationary, rotating, charged black hole in an asymptotically flat empty space. The original solution was given by Kerr [12] for the uncharged case and completed by Newman for the case with non-vanishing charge. We can also state that is a generalization of the Reissner-Nordström metric developed in the previous section, where the angular momentum was assumed to vanish.

4.3.1 Kerr-Newman Geometry

The metric for the Kerr-Newman geometry is better given by the *Boyer-Lindquist metric* [4], which states

$$ds^2 = -\frac{R^2\Delta}{\rho^2}(dt - asin^2(\theta)d\phi)^2 + \frac{\rho^2}{R^2\Delta}dr^2 + \rho^2 d\theta^2 + \frac{R^4 sin^2(\theta)}{\rho^2}\left(d\phi - \frac{a}{R^2}dt\right)^2 \tag{98}$$

where

$$R \equiv \sqrt{r^2 + a^2} \qquad \rho \equiv \sqrt{r^2 + a^2cos^2(\theta)} \tag{99}$$

and Δ is the horizon function given in equation (67), and a is the term related to the angular momentum of the system.

It is a matter of checking that if we set the constants $M = Q = 0 \rightarrow \Delta = 1$, the Boyer-Lindquist metric gives the Minkowski metric. Also, if we set $a = 0$, we get the Reissner-Nordström metric.

The Boyer-Lindquist line-element of equation (98) defines not only a metric but also a tetrad carefully chosen to exhibit the symmetries of the geometry. In the locally inertial frame defined by the Boyer-Lindquist tetrad, the energy-momentum tensor (which is non-vanishing for charged Kerr-Newman) is diagonal. These assertions only become apparent in the tetrad frame, and are obscure in the coordinate frame.

4.3.2 Tetrad Formalism

The formalism used in the derivation of the Kerr-Newman black hole solution is called the *tetrad formalism*, where the *coordinate basis* choice is transformed to a much more mathematically general concept of *local basis* of the tangent bundle, where we define four linearly independent vector fields or *tetrad* that are able to span the 4-dimensional tangent space at each point in spacetime.

The fact of using a local basis of the tangent will help, not only notation, but also the extension of general relativity to particles with $\frac{1}{2}$-spin. Conceptually, the results obtained by the theory will keep the same but its expressions will be of a much simpler nature due to notation.

For a further reading about the tetrad formalism, we recommend both Weinberg (1972) [1] or De Felice & Clarke [10].

4.3.3 Newman-Janis Algorithm

Following the discovery of the Kerr metric, Newman and Janis developed and *ad hoc* procedure to generalize the static solution given by Schwarzschild to the rotating case of Kerr by using a complex transformation of the coordinates. This procedure has been thought, for many years, to be a lucky path, without mathematical proof of its development.

Lately, as done in [9] by S. P. Drake and Peter Szekeres, proofs of several particular cases have been provided, making this procedure an interesting and conceptually easier way of achieving the so wanted Kerr metric.

For the Newman-Janis approach, we are about to follow a five-step algorithm developed in [9] which will provide the reader with a recipe to generalize static metrics to rotating metrics:

1. Write a static spherically symmetric seed line element in advanced null coordinates $\{u, r, \theta, \phi\}$
$$ds^2 = e^{2\Phi(r)}du^2 + e^{\Phi(r)+\lambda(r)}dudt - r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{100}$$
In the Newman-Janis algorithm the metric seed is Reissner-Nordström's given in equation (66).

2. Express the contravariant form of the metric in terms of a null tetrad
$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu \tag{101}$$
where the null tetrad vectors are denoted by $Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, and
$$l_\mu l^\mu = l m_\mu m^\mu = n_\mu n^\mu = 0, \qquad l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \qquad l_\mu m^\mu = n_\mu m^\mu = 0 \tag{102}$$
and for our spacetime, the null tetrad vectors look like
$$l^\mu = \delta_1^\mu \tag{103}$$

$$n^\mu = \delta_0^\mu - \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\delta_1^\mu \tag{104}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\left(\delta_2^\mu + \frac{i}{\sin(\theta)}\delta_3^\mu\right) \tag{105}$$

3. Extend the coordinates x^ρ to a new set of complex coordinates $\tilde{x}^\rho$

$$x^\rho \longrightarrow \tilde{x}^\rho = x^\rho + iy^\rho(x^\sigma) \tag{106}$$

   where $y^\rho(x^\sigma)$ are analytic functions of the real coordinates x^σ. We also require that the transformation can by recovered when $\tilde{x}^\rho = \bar{\tilde{x}}^\rho$

   In the original Newman-Janis algorithm, the tilde transformation of equations (103-105) is

$$l^\mu \longrightarrow \tilde{l}^\mu = \delta_1^\mu \tag{107}$$

$$n^\mu \longrightarrow \tilde{n}^\mu = \delta_0^\mu - \frac{1}{2}\left(1 - M\left(\frac{1}{\tilde{r}} + \frac{1}{\bar{\tilde{r}}}\right) + \frac{Q^2}{\tilde{r}\bar{\tilde{r}}}\right)\delta_1^\mu \tag{108}$$

$$m^\mu \longrightarrow \tilde{m}^\mu = \frac{1}{\sqrt{2}\tilde{r}}\left(\delta_2^\mu + \frac{i}{\sin(\tilde{\theta})}\delta_3^\mu\right) \tag{109}$$

4. The particular choice of equation (106) we are making is

$$\tilde{x}^\rho = x^\rho + ia\cos\left(x^2\left(\delta_0^\rho - \delta_1^\rho\right)\right) \tag{110}$$

   and using equation (101), we get a new metric, with the correspondent transformation of the null tetrad vectors,

$$g_{\mu\nu}(r,\theta) = \begin{bmatrix} 1 - \frac{2Mr-Q^2}{\rho^2} & 1 & 0 & a\sin^2(\theta)\frac{2Mr-Q^2}{\rho^2} \\ . & 0 & 0 & -a\sin^2(\theta) \\ . & . & -\rho^2 & 0 \\ . & . & . & -sin^2(\theta)\left(r^2 + a^2 - a^2\sin^2(\theta)\frac{2Mr-Q^2}{\rho^2}\right) \end{bmatrix}$$

   where ρ was defined in equation (99) and, because the matrix is symmetric, we have avoided redundancy.

5. Finally, we will transform the metric to Boyer-Lindquist coordinates, a set of coordinates in which the metric has only one off-diagonal term $g_{t\phi}$. The usual change of variables is given by

$$u = t - \int \frac{r^2}{r^2 + Q^2 - 2Mr}dr \tag{111}$$

$$\psi = \phi - \int \frac{r^2 + a^2}{r^2 + a^2 + Q^2 - 2Mr}dr \tag{112}$$

4.3.4 Derivation of the Metric and the Energy-Momentum Tensor

As explained in [9], we will be using both the Reissner-Nordström black hole solution already obtained and the Newman-Janis approach explained in the above section to get to the Kerr-Newman solution of the black hole.

Taking as an starting point the Reissner-Nordström metric for an electrically charged, static and spherically symmetric body, by a complex transformation algorithm whose generalization was done in section 4.3.3, we will obtain our desired results.

We transform to a null coordinate system by applying step 5

$$u = t - \int \frac{r^2}{r^2 + Q^2 - 2Mr} dr \tag{113}$$

and equation (66) becomes, adding the suffix RN to denote where it comes from,

$$ds^2_{RN} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{114}$$

Using now step 2, particularly equation (101), we can deduce the line elements of the null tetrad

$$l_\mu dx^\mu = du \tag{115}$$

$$n_\mu dx^\mu = dr + \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) du \tag{116}$$

$$m_\mu dx^\mu = \frac{r}{\sqrt{2}} (d\theta + i\sin(\theta)d\phi) \tag{117}$$

The inner products and nullity conditions given in step 2 allow us to write down the tetrad of null vectors like

$$l = \frac{\partial}{\partial u} \tag{118}$$

$$n = \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\frac{\partial}{\partial r} \tag{119}$$

$$m = \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + i\csc(\theta)\frac{\partial}{\partial\phi}\right) \tag{120}$$

The contravariant Reissner-Nordström metric in null coordinates is

$$g^{\mu\nu}(r,\theta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2(\theta)} \end{bmatrix}$$

where the electromagnetic energy-momentum tensor is that of Reissner-Nordström, being the potential $A_\mu = (Q/r, 0, 0, 0)$.

Now, making a change of coordinates motivated by the analogy taken from the Schwarzschild to Kerr transformation

$$r' = r + ia\cos(\theta), \qquad u' = u - ia\cos(\theta), \qquad \theta' = \theta, \qquad \phi' = \phi \tag{121}$$

where $a = J/M$ is interpreted as the *Kerr parameter*. The complex tetrad given in equations (107-109) will translate, after applying the above transformation, into

$$l' = \frac{\partial}{\partial r} \tag{122}$$

$$n' = \frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{2Mr - Q^2}{\rho^2}\right)\frac{\partial}{\partial r} \tag{123}$$

$$m' = \frac{1}{\sqrt{2}(r + ia\cos(\theta)}\left(ia\sin(\theta)\frac{\partial}{\partial u} - ia\sin(\theta)\frac{\partial}{\partial r}\frac{\partial}{\partial \theta} + i\csc(\theta)\frac{\partial}{\partial \phi}\right) \tag{124}$$

This allows us to write the covariant form of the metric by using equation (101), see [11] for higher detail,

$$g_{\mu\nu}(r,\theta) = \begin{bmatrix} 1 - \frac{2Mr-Q^2}{\rho^2} & 1 & 0 & a\sin^2(\theta)\frac{2Mr-Q^2}{\rho^2} \\ . & 0 & 0 & -a\sin^2(\theta) \\ . & . & -\rho^2 & 0 \\ . & . & . & \frac{sin^2(\theta)}{\rho^2}\left((r^2 + a^2 + Q^2 - 2Mr)a^2\sin^2(\theta) - (a^2 + r^2)^2\right) \end{bmatrix}$$

As done in [9], finally it is assumed that a proper coordinate transformation of the variables $u = t + F(r)$ and $\phi = \psi + G(r)$ will transform the metric to the well-known Boyer-Lindquist coordinates form as in equations (111-112), which will provide only one off-diagonal term corresponding to $g_{t\phi}$, as in equation (98).

As for the electromagnetic energy-momentum tensor, we would leave its derivation out of the scope of this project, but we will provide the reader with its expression, given in its contravariant form as

$$F_{\mu\nu}(r,\theta) = \begin{bmatrix} 0 & (r^4 + a^2r^2\sin^2(\theta) - a^4\cos^2(\theta)) & -2a^2r\cos(\theta)\sin(\theta) & 0 \\ . & 0 & 0 & a(a^2\cos^2(\theta) - r^2) \\ . & . & 0 & 2ar\cot(\theta) \\ . & . & . & 0 \end{bmatrix}$$

which reduces to the Reissner-Nordström electromagnetic field in the limit of $a \to 0$.

4.3.5 Horizons

If a distant observer looks at the horizon of a Kerr-Newman black hole, it clearly rotates, so trying to solve for the position of the horizon, eventhough it is symmetric, might be incorrect.

If we take a photon at the horizon, fighting against the inflow of space, with fixed coordinates (r, θ), it will evolve in the other coordinates (t, ϕ). Taking a look at the photon's 4 velocity

$v^\mu = (v^t, 0, 0, v^\phi)$, the condition that the photon rests in a null-geodesic is given by

$$0 = v_\mu v^\mu = g_{\mu\nu} v^\mu v^\nu = g_{tt}(v^t)^2 + 2g_{t\phi} v^t v^\phi + g_{\phi\phi}(v^\phi)^2 \tag{125}$$

This equation has solutions provided that the determinant is less or equal to zero (see *Frobenius' theorem* for linear algebra). The determinant is given by

$$g_{tt} g_{\phi\phi} - g_{t\phi}^2 = -R^2 sin^2(\theta)\Delta \tag{126}$$

Thus, if $\Delta \geq 0$, then we have solutions for the null-geodesics such that a photon can be at rest instantaneously both in (r, θ); if $\Delta \leq 0$, these null-geodesics do not exist. So, the horizons will be defined analogously as in Reissner-Nordström by setting $\Delta = 0$.

The solution for the outer and inner horizons that we obtain are given by the solution to the quadratic equation

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2} \tag{127}$$

So, between the horizons, $\Delta < 0$ which means that photons cannot be at rest under this condition of the parameters (M, Q, a), because spacetime is falling faster that light in that region. Analogously, outside these two horizons, spacetime will be falling a less than the speed of light.

Before concluding this section, we would like to mention several important regions of this particular solution of Kerr-Newman black hole. We will provide a list that might be of interest to the reader:

- **Ergospheres:** these are finite regions, outside the outer and inside the inner horizon, where the worldline of an object at rest $dr = d\theta = d\phi = 0$ is spacelike. Here, nothing can remain at rest, but objects can escape. The boundary of the ergosphere is given by

  $$g_{tt} = 0$$

- **Turnaround radius:** this is radius inside the inner horizon at which infallers falling from zero velocity and zero angular momentum at infinity turn around. The radius is given by

  $$g_{t\phi} = 0$$

- **Sisytube:** is a toroidal region inside the inner horizon where both (t, ϕ) are timelike coordinates. Within the toroidal region, there are timelike trajectories that go either forward or backwards in time, which are nothing but *closed timelike curves*, trajectories that connect themselves: their own future with their own past. The boundary of this tube is given by

  $$g_{\phi\phi} = 0$$

Simulations for these special regions, as well as for the visualization of a falling camera into a black hole, can be checked in Hamilton (2015) [1] and in its web page given in the reference. There are simulations not only for this Kerr-Newman black hole, but also for Schwarzschild's and Reissner-Nordström that will clarify your imagination of what black holes are as cosmological objects.

Also, another type that visualization that you may be familiar with because of special relativity are diagrams. In special relativity cone diagrams were mostly used, but when we get to general relativity, horizons and singularities make the visualization of the coordinates a little big more complicated, and the reason for this diagrams is no more than helping the student of the subject imagine what the coordinates, the geodesics, the forces, time-development and other parameters evolve around these objects, black holes.

5 Conclusion and Discussion of the Results

Throughout this thesis we have been able to fully understand what theory general relativity is and why it was considered to be an enlightening theory of gravity since its beginning up to the day of today.

Successfully, Einstein's field equations have been obtained both by geometrical and by means of an action principle, having provided the reader with a wide range of mathematical and physical theory and interpretation of the subject that could also be a basis for further approaches and deeper books on general relativity subjects, as gravitational waves, light deflection and another phenomena worth to study. I personally recommend both Berman (2007) [13] and Fernández Barbón (2005) [14], apart from the literature that has been used and followed through the development of the thesis. The first gives an overview of all the so-called metric theories and, most concisely, black holes classification. The second provides a deeper common link between mathematics, quantum physics and gravity theory with key concepts and interpretations that are more than worth to read.

We also obtained, in a successful way, not only the metrics but also the energy-momentum tensors in both cases. We encourage the reader to think on the importance of the huge massive objects spread through the Universe and their cosmic relevance. It is an interesting exercise to do to imagine yourself around one of these bodies.

Of course, there is no need to say that the rest of the books and articles are more than recommended in order to understand and be able to provide other students and reader with a bibliography that can cover all subjects and doubts that may arise.

I personally find really interesting the last section were we spoke about regions around the black holes. I think it provides a imaginative reader with the visual help that is needed in order to consider these astrophysical and theoretical objects a completely astonishing physical entity.

I would also like to mention to the reader to be careful with the notation when following different books and articles. Do not focus in the notation, which is, of course, of value, but I would recommend to get the underlying idea of what is being studied with that particular language.

Also, I would like to encourage universities, in particular my own, University of Sevilla, to provide students with the opportunity of studying this beautiful subject, that nowadays is not present in the curricular programme of any of the degrees done in the Faculty of Physics.

It is also really important to do calculations with these new objects here defined: differentiate, make your own exercises on the subject, do not trust other people's calculations until you have got the same results. We, scientist, know that this is the only way of doing real science. Study, read, think, do, compare and repeat.

References

[1] WEINBERG, S. (1972). *Gravitation and Cosmology, Principles and Applications of the General Theory of Relativity*. NEW YORK, USA: WILEY.

[2] SCHUTZ, B. (2009). *A First Course in General Relativity* (2ND ED.). NEW YORK, USA: CAMBRIDGE UNIVERSITY PRESS.

[3] DIRAC, P.A.M. (1975). *General Theory of Relativity*. NEW YORK, USA: JOHN WILEY AND SONS.

[4] HAMILTON, A. J. S. (2015). *General Relativity, Black Holes and Cosmology*, http://jila.colorado.edu/ ajsh/home.html. Last visited 16 May 2018.

[5] CHANDRASEKHAR, S. (1983). *The Mathematical Theory of Black Holes*. NEW YORK, USA: OXFORD UNIVERSITY PRESS.

[6] GELFAND, I. M.; FOMIN, S. V. (1963). CALCULUS OF VARIATIONS.(RICHARD A. SILVERMAN, TRANS.) NEW JERSEY, USA: PRENTICE-HALL, INC.

[7] CARTAN, É. (1922). *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion*. C. R. ACAD. SCI. (PARIS) 174, 593–595.

[8] NORDEBO, J. (2016). *The Reissner-Nordström metric*,
http://www.diva-portal.org/smash/get/diva2:912393/FULLTEXT01.pdf. Last visited 01 June 2018.

[9] DRAKE S. P., SZEKERES P. (1998). *An explanation of the Newman-Janis Algorithm*, https://arxiv.org/abs/gr-qc/9807001. Last visited 04 June 2018.

[10] DE FELICE, F.; CLARKE, C.J.S. (1990). *Relativity on Curved Manifolds*. NEW YORK, USA: CAMBRIDGE UNIVERSITZ PRESS.

[11] NEWMAN, T.; ADAMO, T. (2004). *Kerr-Newman metric*, http://www.scholarpedia.org/article/Kerr-Newman_metric. Last visited 04 June 2018.

[12] KERR, ROY. (1963). *Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics*. PHYSICAL REVIEW LETTERS - PHYS REV LETT. 11. 237-238. 10.1103/PhysRevLett.11.237

[13] BERMAN, M. S. (FEB 2007). *Simple Derivation of Schwarzschild, Lense-Thirring, Reissner-Nordtröm, Kerr and Kerr-Newman Metrics*, arXiv:physics/0702014v4.

[14] FERNÁNDEZ BARBÓN, J. L. (MARCH 2005). *Geometría no conmutativa y espaciotiempo cuántico* [Non-conmutative geometry and quantum spacetime]. *Investigación y Ciencia (Madrid) 342*, 60-61.