The p-approximation property in Banach spaces

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Outline

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Notation

- X, Y Banach spaces, $B_X = \{x \in X \colon \|x\| \leq 1\}$
- $\mathcal{L}(Y, X)$ is the space of bounded operators from Y into X
- $\mathcal{F}(Y, X) = \{T \in \mathcal{L}(Y, X) \colon \; T \text{ has finite rank}\}$
- $\mathcal{K}(Y, X) = \{T \in \mathcal{L}(Y, X) \colon \; T \text{ is compact}\}$
- $p \in [1, \infty) \;\Rightarrow\; p' = \dfrac{p}{p-1} \quad \left(\Leftrightarrow \frac{1}{p} + \frac{1}{p'} = 1\right)$

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- $\ell_p(X) = \{(x_n) \subset X\colon\ \sum_n \|x_n\|^p < \infty\}$

$$\|(x_n)\|_p = \left(\sum_n \|x_n\|^p\right)^{1/p}$$

- $\ell_p^w(X) = \{(x_n) \subset X\colon\ \sum_n |\langle x^*, x_n\rangle|^p < \infty,$ for all $x^* \in X^*\}$

$$\|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} \left(\sum_n |\langle x^*, x_n\rangle|^p\right)^{1/p}$$

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- $\mathcal{N}_p(Y, X) = \{T \in \mathcal{L}(Y, X)\colon \ T \text{ is } p\text{-nuclear}\}$
- $T\colon Y \to X$ *p-nuclear* $\Leftrightarrow \begin{matrix} \exists (y_n^*) \in \ell_p(Y^*) \\ \exists (x_n) \in \ell_{p'}^w(X) \end{matrix}\colon T = \sum_n y_n^* \otimes x_n$

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The approximation property

- $$\left.\begin{array}{l}(T_n) \subset \mathcal{F}(Y,X) \\ T_n \xrightarrow{\|\cdot\|} T\end{array}\right\} \Rightarrow T \in \mathcal{K}(Y,X)$$

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- $\left.\begin{array}{l}(T_n) \subset \mathcal{F}(Y,X) \\ T_n \xrightarrow{\|\cdot\|} T\end{array}\right\} \Rightarrow T \in \mathcal{K}(Y,X) \iff \overline{\mathcal{F}(Y,X)}^{\|\cdot\|} \subset \mathcal{K}(Y,X)$
- X has Schauder basis $\Rightarrow$ For every Banach Y, $\overline{\mathcal{F}(Y,X)}^{\|\cdot\|} = \mathcal{K}(Y,X)$

The approximation property

Theorem [Grothendieck, 1955]

X a Banach space. The following statements are equivalent:

1. For every Banach space Y, $\overline{\mathcal{F}(Y,X)}^{\|\cdot\|} = \mathcal{K}(Y,X)$.
2. The identity map I_X belongs to $\overline{\mathcal{F}(X,X)}^{\tau_c}$.

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Definition

A Banach space X has the *approximation property* (AP) if the identity map I_X can be approximated by finite rank operators uniformly on every compact subset of X ($\equiv\ I_X \in \overline{\mathcal{F}(X,X)}^{\tau_c}$).

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- All the classical Banach spaces of sequences and functions has the AP.
- Enflo (1973): $\mathcal{L}(\ell_2, \ell_2)$ does not have the AP.

Approximation property in terms of tensor products

$$\begin{array}{ccc} Y^*\hat{\otimes}_\pi X & \overset{J_1}{\longrightarrow} & \mathcal{N}_1(Y,X) \\ \sum_n y_n^* \otimes x_n & \mapsto & \sum_n \langle y_n^*, \cdot \rangle x_n \\ \sum_n \|y_n^*\| \|x_n\| < \infty & & \end{array}$$

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Approximation property of order p via tensor products

- Chevet–Saphar's tensor norm: $p \in [1, \infty)$

$$g_p(u) = \inf \left\{ \|(y_n^*)\|_p \|(x_n)\|_{\ell_{p'}^w(X)} : \ u = \sum_{n=1}^{m} y_n^* \otimes x_n \in Y^* \otimes X \right\}$$

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- Saphar (1970's): $p \in [1, \infty]$
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- Reinov (1980's): $p \in (0, 1]$
  X hast *approximation property of order p* (AP_p) if, for every Banach space Y, the restriction of J_1 to H_p is injective, where $H_p = \{u = \sum_n y_n^* \otimes x_n : \; \sum_n (\|y_n^*\| \|x_n\|)^p < \infty\} \subset Y^* \hat{\otimes}_\pi X$.

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- AP$_1$ = AP

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Theorem [Grothendieck, 1955]

X a Banach space.

- The following statements are equivalent:
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  2. For every Banach space Y, $Y^* \hat{\otimes}_\pi X \simeq \mathcal{N}_1(Y, X)$.
- A set $K \subset X$ is relatively compact if and only if there exists $(x_n) \in c_0(X)$ such that $K \subset \overline{\text{aco}}\,(x_n) := \{\sum_n a_n x_n : \ (a_n) \in B_{\ell_1}\}$.

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Theorem [Bourgain and Reinov, 1984-85]

A Banach space X has the AP_p ($p \in (0,1)$) if and only if I_X can be approximated by finite rank operators uniformly on subsets $K \subset X$ for which there exists $(x_n) \in \ell_q(X)$ such that $K \subset \{\sum_n a_n x_n : \ (a_n) \in B_{\ell_1}\}$ ($p^{-1} - q^{-1} = 1$).

p-compact sets and the p-approximation property

Definition [Sinha and Karn, 2002]

Let $p \geq 1$.

- $K \subset X$ is *relatively p-compact* if there exists $(x_n) \in \ell_p(X)$ such that $K \subset p\text{-co}\,(x_n) := \left\{ \sum_n a_n x_n \colon \ (a_n) \in B_{\ell_{p'}} \right\}$.

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- ∞-AP=AP.
- All Banach spaces have the p-AP for all $p \in [1, 2]$.
- For every $p > 2$, there exist Banach spaces without the p-AP.
- A necessary condition in terms of the trace is obtained for Banach spaces having the p-AP.

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The ideal $\mathcal{K}_p$ of p-compact operators

Let $p \geq 1$ and $1/p + 1/p' = 1$.

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- $T \in \mathcal{L}(X, Y)$ is *p-compact* if $T(B_X)$ is relatively p-compact.

$$\mathcal{K}_p(X, Y) = \{T \in \mathcal{L}(X, Y) : \ T \text{ is } p\text{-compact}\}$$

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- If $1 \leq p \leq q \leq \infty$, $\mathcal{K}_p(X, Y) \subset \mathcal{K}_q(X, Y)$.
- $\mathcal{K}_p$ is an operator ideal.

p-approximation property and p-compact operators

Theorem [Oja, Piñeiro, Serrano and Delgado, 2009]

X a Banach space, $p \in [1, +\infty]$. The following statements are equivalent:

1. X has the p-AP.
2. For every Banach Y, $\mathcal{F}(Y, X)$ is $\|\cdot\|$-dense in $\mathcal{K}_p(Y, X)$.
3. For every Banach Y, $\mathcal{F}(Y, X)$ is τ_c-dense in $\mathcal{K}_p(Y, X)$.

p-approximation property and p-compact operators

- $\Pi_p^d(Y,X) = \{T \in \mathcal{L}(Y,X) : \; T^* \text{ is } p\text{-summing}\}$
- $T \in \Pi_p^d(Y,X) \Leftrightarrow$ T maps relatively compact sets in Y to relatively p-compact sets in X.

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$R \in \mathcal{F}(X,X)$ satisfying $\sup_{x \in K} \|Rx - x\| < \varepsilon$

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$H := Q \circ D_\alpha(B_{\ell_{p'}})$ compact in Y

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Outline

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A trace characterization of the p-AP

Proposition [Sinha and Karn, 2002]

If X has the p-AP then the following holds:
For every $(x_n) \in \ell_p(X)$ and every $(x_n^*) \in \ell_1(X^*)$ such that $\sum_n \langle x_n^*, x \rangle x_n = 0$ for all $x \in X$, we have $\sum_n \langle x_n^*, x_n \rangle = 0$.

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About the subspace structure of $\mathcal{N}_1(X, Y)^*$

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Corollary

$p \in [1, \infty)$.

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- $\Pi_p(Y, X^{**}), \Pi_p^d(Y, X^{**}) \hookrightarrow \mathcal{N}_1(X, Y)^* \ (p \in [1, 2])$.

Outline

The Banach ideal $\mathcal{K}_p$

– $T \in \mathcal{K}_p(X, Y)$ when $T(B_X)$ is relatively p-compact, i.e., if there exists $(y_n)_n \in \ell_p(Y)$ such that

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- A general definition including most of the mentioned approximation properties of order p.