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Duality of measures of non-$\mathcal{A}$-compactness

by

JUAN MANUEL DELGADO (Seville) and CÁNDIDO PIÑEIRO (Huelva)

Abstract. Let $\mathcal{A}$ be a Banach operator ideal. Based on the notion of $\mathcal{A}$-compactness in a Banach space due to Carl and Stephani, we deal with the notion of measure of non-$\mathcal{A}$-compactness of an operator. We consider a map $\chi_{\mathcal{A}}$ (respectively, $n_{\mathcal{A}}$) acting on the operators of the surjective (respectively, injective) hull of $\mathcal{A}$ such that $\chi_{\mathcal{A}}(T) = 0$ (respectively, $n_{\mathcal{A}}(T) = 0$) if and only if the operator T is $\mathcal{A}$-compact (respectively, injectively $\mathcal{A}$-compact). Under certain conditions on the ideal $\mathcal{A}$, we prove an equivalence inequality involving $\chi_{\mathcal{A}}(T^*)$ and $n_{\mathcal{A}^d}(T)$. This inequality provides an extension of a previous result stating that an operator is quasi p-nuclear if and only if its adjoint is p-compact in the sense of Sinha and Karn.

1. Introduction. It is well known that if a bounded subset A of a Banach space X is not relatively compact, then there exists $\varepsilon > 0$ such that A cannot be covered by finitely many balls with radii smaller than (or equal to) ε. In this setting, the *Hausdorff measure of noncompactness* (or the *ball measure of noncompactness*), χ, is defined for every bounded set A as follows:

$$\chi(A) = \inf\Big\{\varepsilon > 0\colon A \subset \bigcup_{i=1}^{n} x_i + \varepsilon B_X\Big\},$$

where B_X denotes the closed unit ball of X and the infimum is taken over all possible sets of finitely many vectors $x_1, \ldots, x_n \in X$ [11]. Of course, $\chi(A)$ vanishes if and only if A is relatively compact.

If T is a (bounded) linear operator from the Banach space X to the Banach space Y, the measure of noncompactness of T can be defined in a natural way by setting $\chi(T) = \chi(T(B_X))$. Then χ is a seminorm on $\mathcal{L}(X, Y)$, the space of all bounded linear operators from X to Y, and χ vanishes exactly on $\mathcal{K}(X, Y)$, the subspace of $\mathcal{L}(X, Y)$ consisting of all compact opera-

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tors. According to Schauder's classical theorem, an operator $T \in \mathcal{L}(X, Y)$ is compact if and only if its adjoint operator T^* is. In 1965, Gol'denšteĭn and Markus [12] proved the inequalities

$$\frac{1}{2}\chi(T) \leq \chi(T^*) \leq 2\chi(T),$$

which, in some sense, may be considered as an extension of Schauder's theorem. Another extension is obtained if, for instance, the Kuratowski measure of noncompactness, γ, is considered [15]. The definition of γ is similar to that of χ with "balls with radii" replaced by "bounded subsets with diameter". In this case, Astala [2] showed

$$\gamma(T) = \gamma(T^*) \tag{1.1}$$

for every $T \in \mathcal{L}(X, Y)$.

Based on Grothendieck's characterization of relatively compact sets as those sitting inside the convex hull of the norm null sequences, Sinha and Karn [21] introduced a strengthened form of compactness in Banach spaces. Let $1 \leq p < \infty$ and let p' be the conjugate index of p (i.e., $1/p + 1/p' = 1$). A set $K \subset X$ is said to be *relatively* p-*compact* if there exists a p-summable sequence (x_n) in X such that $A \subset \{\sum_n \alpha_n x_n \colon (\alpha_n) \in B_{\ell_{p'}}\}$ ($(\alpha_n) \in B_{c_0}$ if $p = 1$). The notion of p-compact operator is defined in the obvious way: an operator $T \in \mathcal{L}(X, Y)$ is said to be p-*compact* if $T(B_X)$ is relatively p-compact in Y. Serrano and the present authors have recently proved the following: T (respectively, T^*) is p-compact if and only if T^* (respectively, T) is quasi p-nuclear [9, Corollary 3.4 and Proposition 3.8].

The main purpose of this paper is to obtain an extension of that result using a sort of measures of noncompactness. Indeed, we consider a positive map $\chi_{\Pi_p^d}$ (respectively, n_{Π_p}) acting on Π_p^d, the ideal of operators with p-summing adjoints (respectively, Π_p, the ideal of p-summing operators) vanishing precisely on the class of p-compact operators (respectively, quasi p-nuclear operators). With these maps in hand, an equality like (1.1) relating $\chi_{\Pi_p^d}$ and n_{Π_p} is obtained (Corollary 3.13), which provides the desired generalization.

Our study is carried out in a more general setting. Given an operator ideal $\mathcal{A}$, the notions of surjective (respectively, injective) $\mathcal{A}$-compactness introduced in [4] (respectively, [23]) are basic to this paper. Section 2 is devoted to the study of the map $\chi_{\mathcal{A}}$, defined on a certain class of bounded subsets of a Banach space (the so called $\mathcal{A}$-bounded sets), which gives information about the degree of non-$\mathcal{A}$-compactness of these sets in such a way that $\chi_{\mathcal{A}}$ vanishes precisely on the class of (surjectively) $\mathcal{A}$-compact sets. In Section 3, the notion of measure of non-$\mathcal{A}$-compactness is extended to the operator setting using two different (but related) approaches. Indeed, the map $\chi_{\mathcal{A}}$ (respectively, $n_{\mathcal{A}}$) gives information about the degree of non-$\mathcal{A}$-compactness of an operator, and it vanishes precisely on the class of surjectively (respectively, injectively) $\mathcal{A}$-compact operators. Under certain conditions on the ideal $\mathcal{A}$, we obtain several inequalities involving $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ acting on an operator and its adjoint. We show that this approach is different from that appearing in [2] and [24], where the notion of (outer and inner) $\mathcal{A}$-variation of an operator is defined and studied. Finally, we introduce the notion of $\mathcal{A}$-essential norm $\rho_{\mathcal{A}}$ of an operator in Section 4 and we study the equivalence between $\chi_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ under certain conditions on X or Y.

Our notation is standard. X, Y and Z are always reserved for Banach spaces. A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $i_X\colon X \to X^{**}$. We denote the closed unit ball of X by B_X. The Banach space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $\mathcal{A}$ is an operator ideal, then $\mathcal{A}^d$ denotes its dual operator ideal, i.e., the one with components $\mathcal{A}^d(X,Y) = \{T \in \mathcal{L}(X,Y)\colon T^* \in \mathcal{A}(Y^*,X^*)\}$.

Recall that an operator ideal $\mathcal{A}$ is *surjective* if, given $S \in \mathcal{A}(Z,Y)$ and $T \in \mathcal{L}(X,Y)$, the condition $T(B_X) \subset S(B_Z)$ implies that $T \in \mathcal{A}(X,Y)$. For an arbitrary ideal $\mathcal{A}$, the *surjective hull* $\mathcal{A}^{\mathrm{sur}}$ of $\mathcal{A}$ is the operator ideal whose components are

$$\mathcal{A}^{\mathrm{sur}}(X,Y) = \{T \in \mathcal{L}(X,Y)\colon T(B_X) \subset S(B_Z),\ S \in \mathcal{A}(Z,Y)\},$$

that is, $\mathcal{A}^{\mathrm{sur}}$ is the smallest surjective ideal containing $\mathcal{A}$. If $D \subset X$ is a bounded set and U_D denotes the surjection of $\ell_1(D)$ onto X defined by $U_D(\xi) = \sum_{x\in D} \xi(x)x$, then it is easy to show that an operator T belongs to $\mathcal{A}^{\mathrm{sur}}(X,Y)$ if and only if $T \circ U_{B_X} \in \mathcal{A}(\ell_1(B_X),Y)$. In the case of a Banach ideal $[\mathcal{A},\alpha]$, $\mathcal{A}^{\mathrm{sur}}$ becomes a Banach ideal when equipped with the norm

$$\begin{aligned}\alpha^{\mathrm{sur}}(T) &= \inf\{\alpha(S)\colon T(B_X) \subset S(B_Z),\ S \in \mathcal{A}(Z,Y)\}\\ &= \alpha(T \circ U_{B_X}).\end{aligned}$$

An operator ideal $\mathcal{A}$ is *injective* if, given $S \in \mathcal{A}(X,Z)$ and $T \in \mathcal{L}(X,Y)$, the inequality $\|Tx\| \le \|Sx\|$ for all $x \in X$ implies that $T \in \mathcal{A}(X,Y)$. For an arbitrary ideal $\mathcal{A}$, the *injective hull* $\mathcal{A}^{\mathrm{inj}}$ of $\mathcal{A}$ is the operator ideal with components

$$\mathcal{A}^{\mathrm{inj}}(X,Y) = \{T \in \mathcal{L}(X,Y)\colon \|Tx\| \le \|Sx\| \text{ for all } x \in X,\ S \in \mathcal{A}(X,Z)\},$$

that is, $\mathcal{A}^{\mathrm{inj}}$ is the smallest injective ideal containing $\mathcal{A}$. If J_Y denotes the canonical embedding of Y into $\ell_\infty(B_{Y^*})$, defined by $J_Y(y)(y^*) = \langle y^*, y\rangle$, then it is easy to show that an operator T belongs to $\mathcal{A}^{\mathrm{inj}}(X,Y)$ if and only if $J_Y \circ T \in \mathcal{A}(X, \ell_\infty(B_{Y^*}))$. In the case of a Banach ideal $[\mathcal{A},\alpha]$, $\mathcal{A}^{\mathrm{inj}}$ becomes a Banach ideal when equipped with the norm

$$\begin{aligned}\alpha^{\mathrm{inj}}(T) &= \inf\{\alpha(S)\colon \|Tx\| \le \|Sx\| \text{ for all } x \in X,\ S \in \mathcal{A}(X,Z)\}\\ &= \alpha(J_Y \circ T).\end{aligned}$$

We denote by $\mathcal{L}$, $\mathcal{K}$, $\mathcal{W}$ and $\mathcal{F}$ the operator ideals of bounded, compact, weakly compact and finite rank linear operators, respectively. We also need the following operator ideals: $\mathcal{QN}_p$—quasi p-nuclear operators, $\mathcal{I}_p$—p-integral operators and Π_p—p-summing operators. We refer to Pietsch's book [19] for operator ideals (see also Diestel, Jarchow and Tonge [10] for common operator ideals such as $\mathcal{I}_p$ and Π_p, and Persson and Pietsch [18] for $\mathcal{QN}_p$).

2. A measure of non-$\mathcal{A}$-compactness of a set. Let $\mathcal{A}$ be an operator ideal. A subset A of the Banach space X is said to be $\mathcal{A}$-*bounded* if there exist a Banach space Z and an operator $S \in \mathcal{A}(Z,X)$ with $A \subset S(B_Z)$ [22]. The class of $\mathcal{A}$-bounded subsets of X is denoted by $\mathfrak{M}^{\mathcal{A}}(X)$. Note that an operator belongs to $\mathcal{A}^{\mathrm{sur}}(X,Y)$ if and only if it maps bounded subsets of X to $\mathcal{A}$-bounded subsets of Y. The first examples rely on the following fact.

Proposition 2.1. *A set $A \subset X$ is $\mathcal{A}$-bounded if and only if*

$$U_A \in \mathcal{A}(\ell_1(A), X).$$

Proof. If $A \subset X$ is $\mathcal{A}$-bounded and $S \in \mathcal{A}(Z,X)$ is such that $A \subset S(B_Z)$, then

$$\begin{aligned}U_A(B_{\ell_1(A)}) &= \Big\{\sum_n \alpha_n x_n \colon x_n \in A,\ (\alpha_n) \in B_{\ell_1}\Big\}\\ &\subset \Big\{\sum_n \alpha_n x_n \colon x_n \in S(B_Z),\ (\alpha_n) \in B_{\ell_1}\Big\} = S(B_Z),\end{aligned}$$

and it follows that $U_A(B_{\ell_1(A)})$ is $\mathcal{A}$-bounded. Thus, $U_A \in \mathcal{A}^{\mathrm{sur}}(\ell_1(A), X) = \mathcal{A}(\ell_1(A), X)$ [19, Lemma 4.7.3].

The converse is a direct consequence of the inclusion $A \subset U_A(B_{\ell_1(A)})$. ∎

Example 2.2. (1) The class of all $\mathcal{L}$-bounded sets in X coincides with that of all bounded sets.

(2) The class of all $\mathcal{K}$-bounded sets in X coincides with that of all relatively compact sets.

(3) Let $p \in [1,\infty)$. A bounded set $A \subset X$ is said to be p-*limited* if for every weakly p-summable sequence (x_n^*) in X^* there exists $(\alpha_n) \in \ell_p$ such that $|\langle x_n^*, x\rangle| \le \alpha_n$ for all $x \in A$ and $n \in \mathbb{N}$ [14]. By [8, Proposition 2.1], $A \subset X$ is p-limited if and only if U_A^* is p-summing. So the class of all Π_p^d-bounded sets in X is precisely that of all p-limited sets.

(4) Let $1 \le p < \infty$ and let p' be the conjugate index of p. Denote by $\mathcal{K}_p$ the ideal consisting of all p-compact operators in the sense of Sinha and

Karn. Since $A \subset X$ is relatively p-compact if and only if $U_A \in \mathcal{K}_p(\ell_1(A), X)$ [9, Proposition 3.5], we deduce that the class of all $\mathcal{K}_p$-bounded sets in X is precisely that of all relatively p-compact sets.

In [4], a special type of $\mathcal{A}$-bounded sets was introduced by Carl and Stephani as a refinement of compactness related to a given operator ideal. A set $A \subset X$ is said to be $\mathcal{A}$-*compact* if there exist a Banach space Z, a compact set $K \subset Z$ and an operator $S \in \mathcal{A}(Z, X)$ such that $A \subset S(K)$ (actually, this is the characterization of $\mathcal{A}$-compact sets appearing in [4, Theorem 1.2]). We denote by $\mathfrak{M}_c^{\mathcal{A}}(X)$ the class of $\mathcal{A}$-compact subsets of X.

Relying on the notion of $\mathcal{A}$-compactness, the notion of $\mathcal{A}$-compact operator is defined in the obvious way: $T \in \mathcal{L}(X, Y)$ is said to be $\mathcal{A}$-*compact* if T maps bounded sets in X to relatively $\mathcal{A}$-compact sets in Y. If $\mathcal{K}^{\mathcal{A}}$ denotes the class of $\mathcal{A}$-compact operators, then $\mathcal{K}^{\mathcal{A}}$ is a surjective operator ideal and $\mathcal{K}^{\mathcal{A}} = \mathcal{A}^{\mathrm{sur}} \circ \mathcal{K} = \mathcal{K}^{\mathcal{A}} \circ \mathcal{K}$ [4, Theorem 2.1]. From this, it is easy to deduce that $A \subset X$ is $\mathcal{A}$-compact if and only if $U_A \in \mathcal{K}^{\mathcal{A}}(\ell_1(A), X)$ and that $\mathfrak{M}_c^{\mathcal{A}}(X) = \mathfrak{M}_c^{\mathcal{A}^{\mathrm{sur}}}(X) = \mathfrak{M}_c^{\mathcal{A}\circ\mathcal{K}}(X)$.

Example 2.3. (1) If $\mathcal{A} = \mathcal{L}$ or $\mathcal{A} = \mathcal{K}$, the class of all $\mathcal{A}$-compact sets in X coincides with that of all relatively compact sets.

(2) Having in mind the equality $\mathcal{K}_p = \Pi_p^d \circ \mathcal{K}$ (see, for instance, [1, Corollary 4.9]) and the surjectivity of the ideal Π_p^d (being the dual of an injective ideal), it follows that $\mathcal{K}^{\Pi_p^d} = \mathcal{K}_p$. So $A \subset X$ is Π_p^d-compact if and only if U_A is p-compact. By [9, Proposition 3.5], we deduce that the class of all Π_p^d-compact sets in X is precisely that of all relatively p-compact sets.

(3) Using the above properties, we have

$$\mathfrak{M}_c^{\Pi_p^d}(X) = \mathfrak{M}_c^{\Pi_p^d\circ\mathcal{K}}(X) = \mathfrak{M}_c^{\mathcal{K}_p}(X),$$

that is, the class of all $\mathcal{K}_p$-compact sets in X is precisely that of all relatively p-compact sets.

The notion of $\mathcal{A}$-compactness may be expressed in a similar way to the notion of precompactness in a Banach space.

Theorem ([4, Theorem 3.1]). *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal, X a Banach space and $A \in \mathfrak{M}^{\mathcal{A}}(X)$. The following statements are equivalent:*

(a) *A is $\mathcal{A}$-compact.*
(b) *For every $\varepsilon > 0$, there are finitely many elements $x_1, \ldots, x_n \in X$, a Banach space Z and an operator $S \in \mathcal{A}(Z, X)$ with $\alpha(S) \le \varepsilon$ such that*

$$A \subset \bigcup_{i=1}^{n} x_i + S(B_Z).$$

The above result is a basis for the following definition of measure of noncompactness referring to a given Banach operator ideal $\mathcal{A}$.

Definition 2.4. Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal, X a Banach space and $A \in \mathfrak{M}^{\mathcal{A}}(X)$. The (*outer*) *measure of non-$\mathcal{A}$-compactness* of A is

$$\chi_{\mathcal{A}}(A) = \inf\Big\{\varepsilon > 0\colon A \subset \bigcup_{i=1}^{n} x_i + S(B_Z)\Big\},$$

the infimum taken over all possible $x_1, \dots, x_n \in X$, Banach spaces Z and operators $S \in \mathcal{A}(Z, X)$ with $\alpha(S) \le \varepsilon$.

The condition $A \in \mathfrak{M}^{\mathcal{A}}(X)$ ensures that in the above definition we take the infimum of a nonempty set of positive numbers. Of course, if $\mathcal{A} \subset \mathcal{B}$, then $\chi_{\mathcal{B}}(\cdot) \le \chi_{\mathcal{A}}(\cdot)$ and $\chi_{\mathcal{L}} \equiv \chi$.

In this section, we omit the word "outer" when referring to "outer measures of non-$\mathcal{A}$-compactness".

Remark 2.5. It is clear that

$$\chi_{\mathcal{A}}(A) = \inf\Big\{\alpha(S)\colon A \subset \bigcup_{i=1}^{n} x_i + S(B_Z)\Big\},$$

the infimum taken over all possible $x_1, \dots, x_n \in X$, Banach spaces Z and operators $S \in \mathcal{A}(Z, X)$. From this, it follows that $\chi_{\mathcal{A}}(A) = \lim_n e_n(A, \mathcal{A})$, where $(e_n(A, \mathcal{A}))$ is the sequence of generalized (outer) entropy numbers of the set A with respect to $\mathcal{A}$ introduced in [4, Definition 3]. Theorem 3.2 in [4] may be used to obtain the equality $\chi_{\mathcal{A}}(A) = \chi_{\mathcal{A}^{\mathrm{sur}}}(A)$ for every $A \in \mathfrak{M}^{\mathcal{A}}(X) = \mathfrak{M}^{\mathcal{A}^{\mathrm{sur}}}(X)$.

On the other hand, [7, Proposition 5] shows that

$$\chi_{\mathcal{A}}(A) = \inf\{\alpha(S)\colon A \subset T(B_E) + S(B_Z)\},$$

where the infimum is taken over all Banach spaces E and Z and operators $T \in \mathcal{K}^{\mathcal{A}}(E, X)$ and $S \in \mathcal{A}(Z, X)$.

Remark 2.6. Taking a glance at Proposition 2.1, it is also possible to conclude that

$$\chi_{\mathcal{A}}(A) = \inf\Big\{\varepsilon > 0\colon A \subset \bigcup_{i=1}^{n} x_i + B\Big\},$$

the infimum taken over all possible $x_1, \dots, x_n \in X$ and $\mathcal{A}$-bounded subsets B of X with $\alpha(U_B) \le \varepsilon$.

Remark 2.7. In [16], a way to measure the "size" of $\mathcal{A}$-compact sets is introduced as follows. If $A \subset X$ is $\mathcal{A}$-compact, then one can define $m_{\mathcal{A}}(A) = \inf\{\alpha(S)\colon A \subset S(K),\ S \in \mathcal{A}(Z, X),\ K \subset B_Z \text{ compact}\}$, where the infimum is taken over all Banach spaces Z. It must be pointed out that this notion is different from that in Definition 2.4; in fact, a bounded set is $\mathcal{A}$-compact if and only if its $m_{\mathcal{A}}$-measure is finite.

Most of the proofs of the following properties are routine, so they are omitted.

PROPOSITION 2.8. *Assume $\mathcal{A}$ is a Banach operator ideal and $A, A_1, A_2 \subset X$ are $\mathcal{A}$-bounded. Then:*

(1) $\chi_{\mathcal{A}}(A) = 0$ *if and only if A is $\mathcal{A}$-compact.*
(2) *If $A_1 \subset A_2$, then $\chi_{\mathcal{A}}(A_1) \leq \chi_{\mathcal{A}}(A_2)$. Thus,*
$$\chi_{\mathcal{A}}(A_1 \cap A_2) \leq \min\{\chi_{\mathcal{A}}(A_1), \chi_{\mathcal{A}}(A_2)\}.$$
(3) $\chi_{\mathcal{A}}(A_1 + A_2) \leq \chi_{\mathcal{A}}(A_1) + \chi_{\mathcal{A}}(A_2)$. *As a consequence,*
$$\chi_{\mathcal{A}}(\Delta + A) = \chi_{\mathcal{A}}(A)$$
whenever $\Delta \subset X$ is finite.
(4) $\chi_{\mathcal{A}}(\lambda A) = |\lambda| \chi_{\mathcal{A}}(A)$ *for every $\lambda \in \mathbb{R}$.*
(5) *If $T \in \mathcal{L}(X, Y)$, then $\chi_{\mathcal{A}}(T(A)) \leq \|T\| \chi_{\mathcal{A}}(A)$.*
(6) *If $D \subset X$ is bounded and $T \in \mathcal{A}^{\mathrm{sur}}(X, Y)$, then*
$$\chi_{\mathcal{A}}(T(D)) \leq \alpha^{\mathrm{sur}}(T) \chi(D),$$
where $\chi(D)$ denotes the Hausdorff measure of noncompactness of D.
(7) *If A_2 is $\mathcal{A}$-compact, then $\chi_{\mathcal{A}}(A_1 \cup A_2) = \chi_{\mathcal{A}}(A_1)$.*
(8) $\chi_{\mathcal{A}}(U_A(B_{\ell_1(A)})) = \chi_{\mathcal{A}}(A)$.

Proof. (3) Although the idea of the proof is included in [4, Section 4], we give a sketch for completeness. By [4, p. 89, property A], it can be deduced that
$$e_{2n-1}(A_1 + A_2, \mathcal{A}) \leq e_n(A_1, \mathcal{A}) + e_n(A_2, \mathcal{A});$$
hence
$$\begin{aligned} \chi_{\mathcal{A}}(A_1 + A_2) &= \lim_n e_{2n-1}(A_1 + A_2, \mathcal{A}) \\ &\leq \lim_n (e_n(A_1, \mathcal{A}) + e_n(A_2, \mathcal{A})) = \chi_{\mathcal{A}}(A_1) + \chi_{\mathcal{A}}(A_2). \end{aligned}$$

(6) If $D \subset X$ is bounded and $T \in \mathcal{A}^{\mathrm{sur}}(X, Y)$, it is clear that $T(D)$ is $\mathcal{A}^{\mathrm{sur}}$-bounded. Let $\varepsilon > \chi(D)$ and choose $x_1, \ldots, x_n \in X$ so that $D \subset \bigcup_{i=1}^n x_i + \varepsilon B_X$. Then $T(D) \subset \bigcup_{i=1}^n T(x_i) + \varepsilon T(B_X)$, so in view of Remark 2.5 we have
$$\chi_{\mathcal{A}^{\mathrm{sur}}}(T(D)) \leq \alpha^{\mathrm{sur}}(\varepsilon T) = \alpha^{\mathrm{sur}}(T)\varepsilon.$$
Letting $\varepsilon \searrow \chi(D)$, we obtain $\chi_{\mathcal{A}^{\mathrm{sur}}}(T(D)) \leq \alpha^{\mathrm{sur}}(T)\chi(D)$, and the property follows since $\chi_{\mathcal{A}} \equiv \chi_{\mathcal{A}^{\mathrm{sur}}}$ [4, Theorem 3.2].

(7) By monotonicity, $\chi_{\mathcal{A}}(A_1) \leq \chi_{\mathcal{A}}(A_1 \cup A_2)$. For the converse inequality, fix $\varepsilon > \chi_{\mathcal{A}}(A_1)$ so that $A_1 \subset \bigcup_{i=1}^n x_i + S_1(B_{Z_1})$ with $\alpha(S_1) \leq \varepsilon$. Now, for a given $\delta > 0$, the $\mathcal{A}$-compactness of A_2 ensures the existence of

$u_1, \dots, u_m \in X$ as well as a Banach space Z_2 and $S_2 \in \mathcal{A}(Z_2, X)$ with $\alpha(S_2) \le \delta$ satisfying $A_2 \subset \bigcup_{j=1}^m u_j + S_2(B_{Z_2})$. Setting $\Delta_1 = \{x_1, \dots, x_n\}$ and $\Delta_2 = \{u_1, \dots, u_m\}$, it is clear that $A_1 \cup A_2 \subset (\Delta_1 \cup \Delta_2) + S_1(B_{Z_1}) + S_2(B_{Z_2})$. So, in view of (2), (3) and (6), and having in mind that $\chi(B_E) = 1$ whenever E is infinite-dimensional [3, Theorem 2.5], we conclude that

$$\begin{aligned}\chi_{\mathcal{A}}(A_1 \cup A_2) &\le \chi_{\mathcal{A}}(S_1(B_{Z_1})) + \chi_{\mathcal{A}}(S_2(B_{Z_2}))\\ &\le \alpha(S_1)\chi(B_{Z_1}) + \alpha(S_2)\chi(B_{Z_2})\\ &\le \varepsilon + \delta.\end{aligned}$$

Letting $\delta \searrow 0$ and $\varepsilon \searrow \chi_{\mathcal{A}}(A_1)$ yields the desired inequality. ∎

REMARK 2.9. As a consequence of Proposition 2.8(6), every T in $\mathcal{A}^{\mathrm{sur}}(X, Y)$ maps relatively compact subsets of X to $\mathcal{A}$-compact subsets of Y. For $\mathcal{A} = \Pi_p^d$, this means that every operator with p-summing adjoint maps relatively compact subsets to p-compact subsets (as already proved in [9, Theorem 3.14]).

It is easy to show that the Hausdorff measure of noncompactness is semiadditive, that is, $\chi(D_1 \cup D_2) = \max\{\chi(D_1), \chi(D_2)\}$. Apart from the case stated in Proposition 2.8(7), we have not been able to establish whether this property remains true for measures of non-$\mathcal{A}$-compactness with $\mathcal{A}$ different from $\mathcal{L}$. In this connection, we have the following result.

PROPOSITION 2.10. *Let $p \ge 1$ and let $A_1, A_2 \subset X$ be Π_p^d-bounded sets. Then*

$$\chi_{\Pi_p^d}(A_1 \cup A_2) \le 2^{1/p} \max\{\chi_{\Pi_p^d}(A_1), \chi_{\Pi_p^d}(A_2)\}.$$

Proof. Suppose $\varepsilon > \chi_{\Pi_p^d}(A_1) \ge \chi_{\Pi_p^d}(A_2)$ and consider coverings $A_j \subset \bigcup_{i=1}^{n_j} x_i^j + B_j$ with $\pi_p(U_{B_j}) \le \varepsilon$, $j = 1, 2$ (Remark 2.6). Then

$$A_1 \cup A_2 \subset \bigcup_{x \in \Delta} x + B$$

where $\Delta = \{x_i^j : i = 1, \dots, n_1,\ j = 1, \dots, n_2\}$ and $B = B_1 \cup B_2$. It suffices to see that $\pi_p(U_B^*) \le 2^{1/p}\varepsilon$. For any fixed weakly p-summable sequence (x_n^*) in X^*, it is possible to find a partition of $\mathbb{N}$ into two sets G_1 and G_2 such that

$$\sum_n \|U_B^* x_n^*\|^p \le \sum_{n \in G_1} \|U_{B_1}^* x_n^*\|^p + \sum_{n \in G_2} \|U_{B_2}^* x_n^*\|^p.$$

Hence,

$$\pi_p(U_B^*) \le (\pi_p(U_{B_1}^*)^p + \pi_p(U_{B_1}^*)^p)^{1/p} \le 2^{1/p}\varepsilon. \ \blacksquare$$

REMARK 2.11. If $D \subset X$ is bounded then $\chi(D) = \chi(\overline{D})$. For an arbitrary Banach operator ideal $\mathcal{A}$, we cannot even ensure that $\overline{A}$ is $\mathcal{A}$-bounded whenever $A \subset X$ is. Much more can be said if $\mathcal{A}$ enjoys the following property:

Property (P). There exists a positive constant C such that, for any Banach spaces X and Y and $T \in \mathcal{A}(X,Y)$, we have:

(i) $T^{**}(B_{X^{**}}) \subset Y$ (that is, $\mathcal{A} \subset \mathcal{W}$).
(ii) The operator $\widetilde{T}\colon B_{X^{**}} \ni x^{**} \mapsto T^{**}x^{**} \in Y$ belongs to $\mathcal{A}(X^{**}, Y)$.
(iii) $\alpha(\widetilde{T}) \le C\alpha(T)$.

Proposition 2.12. *Suppose $\mathcal{A}$ is a Banach operator ideal with property* (P) *and X is a Banach space. Then:*

(1) *$A \subset X$ is $\mathcal{A}$-bounded if and only if $\overline{A}$ is.*
(2) *$\chi_{\mathcal{A}}(A) \le \chi_{\mathcal{A}}(\overline{A}) \le C\chi_{\mathcal{A}}(A)$.*

Proof. Let $S \in \mathcal{A}(Z,X)$ with $A \subset S(B_Z)$. Then $A \subset \widetilde{S}(B_{Z^{**}})$. By hypothesis, S is weakly compact, so it factors through a reflexive Banach space. Thus, $\widetilde{S}$ is weak*-weak continuous. From this, $\widetilde{S}(B_{Z^{**}})$ is a weakly compact set in Y and, being absolutely convex, it is norm closed. So we have $\overline{A} \subset \widetilde{S}(B_{Z^{**}})$, and this shows that $\overline{A}$ is $\mathcal{A}$-bounded.

Finally, (2) is obtained using a standard argument. ∎

If a Banach operator ideal $\mathcal{A} \subset \mathcal{W}$ is regular and satisfies $\mathcal{A} = \mathcal{A}^{dd}$, then it enjoys property (P). This is the case of operator ideals $\mathcal{A} \subset \mathcal{W}$ and $\mathcal{A} = \mathcal{A}^{\max}$ [6, pp. 206–207]. Hence, Π_p^d satisfies property (P) (in fact, $\Pi_p^d = \mathcal{K}_p^{\max}$ [20, Theorem 12]).

Corollary 2.13. *If $A \subset X$ is Π_p^d-bounded, then $\chi_{\Pi_p^d}(A) = \chi_{\Pi_p^d}(\overline{A})$.*

3. Measures of non-$\mathcal{A}$-compactness of an operator. If an operator $T\colon X \to Y$ fails to be $\mathcal{A}$-compact, it seems natural to quantify the distance between T and $\mathcal{K}^{\mathcal{A}}(X,Y)$ by evaluating $\chi_{\mathcal{A}}(T(B_X))$ when this expression makes sense.

Definition 3.1. Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal and let T be in $\mathcal{A}^{\mathrm{sur}}(X,Y)$. The (*outer*) *measure of non-$\mathcal{A}$-compactness* of T is

$$\chi_{\mathcal{A}}(T) = \chi_{\mathcal{A}}(T(B_X)).$$

Note that $\chi_{\mathcal{A}}(T) = \lim_n e_n(T, \mathcal{A})$ (see [4, Section 4]). When $\mathcal{A} = \mathcal{L}$, we are dealing with the so called *ball measure of noncompactness*.

Example 3.2. Let $A = \{e_n : n \in \mathbb{N}\} \subset c_0$, where (e_n) is the unit vector basis in c_0. Let us check that $\chi_{\mathcal{A}}(A) = 1$ if $\mathcal{A} = \Pi_p$ or $\mathcal{A} = \Pi_p^d$. If I denotes the embedding map from ℓ_1 into c_0, then $\iota_1(I^*) = 1$ (see, for instance, [19, Proposition 6.4.4]), so $\chi_{\mathcal{I}_1^d}(A) \le 1$. In view of [10, Corollary 5.7],

$$\chi_{\Pi_p^d}(A) \le \chi_{\Pi_1^d}(A) = \chi_{\mathcal{I}_1^d}(A) \le 1.$$

From this and the equality $\chi_{\mathcal{L}}(A) = \chi(A) = 1$ [3, p. 24], it follows that $\chi_{\Pi_p^d}(A) = 1$. On the other hand, I is 1-integral and $\iota_1(I) = 1$ [10, Theorem 5.15], so arguing as above shows that $\chi_{\Pi_p}(A) = 1$. Now, notice that I is precisely the operator U_A, so according to Proposition 2.8(8) we conclude that $\chi_{\Pi_p}(I) = \chi_{\Pi_p^d}(I) = 1$.

Remark 3.3. Given a Banach ideal $\mathcal{A}$, the (*outer*) $\mathcal{A}$-*variation* of an operator $T \in \mathcal{L}(X,Y)$ is defined by

$$\gamma_{\mathcal{A}}(T) = \inf\{\varepsilon > 0\colon T(B_X) \subset \varepsilon B_Y + S(B_Z)\},$$

where the infimum is taken over all Banach spaces Z and operators $S \in \mathcal{A}(Z,X)$ [2, Definition 3.1]. Example 3.2 makes it clear that this is a different notion from that appearing in Definition 3.1; in fact, by [2, Theorem 3.8],

$$\gamma_{\Pi_p}(I) = \inf\{\|I - S\|\colon S \in \Pi_p(\ell_1, c_0)\} = 0.$$

The following result shows an alternative way to describe $\chi_{\mathcal{A}}(T)$:

Proposition 3.4. *Let $\mathcal{A}$ be a Banach operator ideal and $T \in \mathcal{A}^{\mathrm{sur}}(X,Y)$. Then*

$$\chi_{\mathcal{A}}(T) = \inf\{k > 0\colon \chi_{\mathcal{A}}(T(D)) \le k\chi(D) \text{ for all } D \subset X \text{ bounded}\}.$$

If X is infinite-dimensional, then

$$\chi_{\mathcal{A}}(T) = \sup\{\chi_{\mathcal{A}}(T(D))\colon D \subset X \text{ bounded with } \chi(D) = 1\}.$$

Proof. We prove the first equality (the second follows by a standard argument). The assertion is clear if X is finite-dimensional. Suppose X is infinite-dimensional and set

$$G = \{k > 0\colon \chi_{\mathcal{A}}(T(D)) \le k\chi(D) \text{ for all } D \subset X \text{ bounded}\}.$$

Notice that Proposition 2.8(6) ensures that $G \neq \emptyset$. Since $\chi(B_X) = 1$, we have $\chi_{\mathcal{A}}(T(B_X)) \le k$ whenever $k \in G$, and this yields $\chi_{\mathcal{A}}(T) \le \inf G$.

For the opposite inequality, it suffices to show $\chi_{\mathcal{A}}(T(D)) \le \chi_{\mathcal{A}}(T(B_X))$ for every bounded set $D \subset X$ satisfying $\chi(D) = 1$. Fix D with those properties and $\delta > 1$. There exist $x_1, \ldots, x_m \in X$ such that

$$D \subset \bigcup_{i=1}^{m} x_i + \delta B_X. \tag{3.1}$$

On the other hand, if $\varepsilon > \chi_{\mathcal{A}}(T(B_X))$, one can find $y_1, \ldots, y_n \in Y$, a Banach space Z and $S \in \mathcal{A}(Z,Y)$ satisfying $\alpha(S) \le \varepsilon$ so that

$$T(B_X) \subset \bigcup_{j=1}^{n} y_j + S(B_Z). \tag{3.2}$$

From (3.1) and (3.2), we have a covering $T(D) \subset \bigcup_{y\in\Delta} y + \delta S(B_Z)$, $\Delta \subset Y$ being a finite set. Therefore, $\chi_{\mathcal{A}}(T(D)) \le \alpha(\delta S) \le \delta\varepsilon$, and the proof finishes by just taking the infimum over δ and ε. ∎

The next proposition lists some basic properties of the outer measure of non-$\mathcal{A}$-compactness of an operator; they can be easily obtained from the definition and Propositions 2.8 and 3.4.

PROPOSITION 3.5. *Let $\mathcal{A}$ be a Banach operator ideal and $T \in \mathcal{A}^{\mathrm{sur}}(X,Y)$. Then:*

(1) $\chi_{\mathcal{A}}(\cdot)$ *is a seminorm on* $\mathcal{A}^{\mathrm{sur}}(X,Y)$.
(2) $\chi_{\mathcal{A}}(T) = 0$ *if and only if* $T \in \mathcal{K}^{\mathcal{A}}(X,Y)$.
(3) *If* $S \in \mathcal{K}^{\mathcal{A}}(X,Y)$, *then* $\chi_{\mathcal{A}}(T+S) = \chi_{\mathcal{A}}(T)$.
(4) *If* X_0 *and* Y_0 *are Banach spaces,* $R \in \mathcal{L}(Y,Y_0)$ *and* $S \in \mathcal{L}(X_0,X)$, *then* $\chi_{\mathcal{A}}(R \circ T \circ S) \le \|R\|\chi_{\mathcal{A}}(T)\|S\|$.
(5) *If* $D \subset X$ *is bounded, then* $\chi_{\mathcal{A}}(T(D)) \le \chi_{\mathcal{A}}(T)\chi(D)$.
(6) *If* $S \in \mathcal{A}^{\mathrm{sur}}(Y,Z)$, *then* $\chi_{\mathcal{A}}(S \circ T) \le \chi_{\mathcal{A}}(S)\chi_{\mathcal{A}}(T)$.
(7) *If* $\mathrm{Id}_X \in \mathcal{A}^{\mathrm{sur}}(X,X)$, *then* $\chi_{\mathcal{A}}(\mathrm{Id}_X) = 0$ *if and only if X is finite-dimensional* (*otherwise,* $\chi_{\mathcal{A}}(\mathrm{Id}_X) \ge 1$).

Given a Banach operator ideal $\mathcal{A}$, the outer measure of non-$\mathcal{A}$-compactness of an operator may be considered as a tool to evaluate the degree of non-$\mathcal{A}$-compactness of an operator belonging the surjective hull $\mathcal{A}^{\mathrm{sur}}$. To obtain an extension of the equality $\mathcal{QN}_p = \mathcal{K}_p^d$, we are going to consider another type of measure quantifying the degree of noncompactness (with respect to $\mathcal{A}$) of operators belonging to the injective hull $\mathcal{A}^{\mathrm{inj}}$. The following concept was introduced and studied by Stephani [23, Section 1].

DEFINITION 3.6. Let $\mathcal{A}$ be an operator ideal. An operator $T \in \mathcal{L}(X,Y)$ is said to be *injectively $\mathcal{A}$-compact* if there exist a Banach space Z, a sequence $(z_n^*) \in c_0(Z^*)$ and an operator $S \in \mathcal{A}^{\mathrm{inj}}(X,Z)$ such that $\|Tx\| \le \sup_n |\langle z_n^*, Sx\rangle|$ for all $x \in X$.

REMARK 3.7. It is well known that $T \in \mathcal{L}(X,Y)$ is compact if there exists $(x_n^*) \in c_0(X^*)$ such that $\|Tx\| \le \sup_n |\langle x_n^*, x\rangle|$ for all $x \in X$. Thus, for $\mathcal{A} = \mathcal{L}$ the preceding notion coincides with the notion of compact operator.

If $\mathcal{H}^{\mathcal{A}}$ denotes the class of injectively $\mathcal{A}$-compact operators, then $\mathcal{H}^{\mathcal{A}}$ is an injective operator ideal and $\mathcal{H}^{\mathcal{A}} = \mathcal{K} \circ A^{\mathrm{inj}}$ [23, Theorem 1.1]. For example, $\mathcal{H}^{\Pi_p} = \mathcal{K} \circ \Pi_p = \mathcal{QN}_p$ [23, p. 255].

REMARK 3.8. Since $\mathcal{H}^{\mathcal{A}} = \mathcal{K} \circ A^{\mathrm{inj}}$ [23, Theorem 1.1], $\mathcal{A}^{\mathrm{inj}}(X,Z)$ may be replaced with $\mathcal{A}(X,Z)$ in the preceding definition.

When dealing with a Banach operator ideal $[\mathcal{A},\alpha]$, the following characterization of injectively $\mathcal{A}$-compact operators may be deduced from [23, Theorem 1.1].

THEOREM 3.9. *Let $\mathcal{A}$ be a Banach operator ideal and $T \in \mathcal{L}(X,Y)$. The following statements are equivalent:*

(1) *T is injectively $\mathcal{A}$-compact.*
(2) *For every $\varepsilon > 0$, there are finitely many functionals $x_1^*, \dots, x_n^* \in X^*$, a Banach space Z and an operator $S \in \mathcal{A}(X, Z)$ with $\alpha(S) \leq \varepsilon$ such that*
$$\|Tx\| \leq \sup_{1\leq i\leq n} |\langle x_i^*, x\rangle| + \|Sx\|$$
for all $x \in X$.

Definition 3.10. Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal and let T be in $\mathcal{A}^{\mathrm{inj}}(X, Y)$. The (*inner*) *measure of non-$\mathcal{A}$-compactness* of T is
$$n_{\mathcal{A}}(T) = \inf\Big\{\varepsilon > 0\colon \|Tx\| \leq \sup_{1\leq i\leq n} |\langle x_i^*, x\rangle| + \|Sx\| \text{ for all } x \in X\Big\},$$
the infimum taken over all $x_1^*, \dots, x_n^* \in X^*$, Banach spaces Z and operators $S \in \mathcal{A}(X, Z)$ with $\alpha(S) \leq \varepsilon$.

The condition $T \in \mathcal{A}^{\mathrm{inj}}(X, Y)$ ensures that in the above definition we take the infimum of a nonempty set of positive numbers. In fact, $n_{\mathcal{A}}(T) \leq \alpha^{\mathrm{inj}}(T)$. In this case, $n_{\mathcal{A}}$ vanishes precisely on operators belonging to $\mathcal{H}^{\mathcal{A}}$.

Remark 3.11. Given a Banach ideal $\mathcal{A}$, the (*inner*) *$\mathcal{A}$-variation* of an operator $T \in \mathcal{L}(X, Y)$ is defined by
$$\beta_{\mathcal{A}}(T) = \inf\{\varepsilon > 0\colon \|Tx\| \leq \varepsilon\|x\| + \|Sx\| \text{ for all } x \in X\},$$
where the infimum is taken over all Banach spaces Z and operators $S \in \mathcal{A}(X, Z)$ [24]. Since $\beta_{\mathcal{A}}(T) = 0$ if and only if $J_Y \circ T$ is in the uniform closure of $\mathcal{A}(X, \ell_\infty(B_{Y^*}))$ [13, Theorem 20.7.3], the (inner) $\mathcal{A}$-variation is a different notion from that appearing in Definition 3.10.

Theorem 3.12. *Let $\mathcal{A}$ be a Banach operator ideal with property* (P) (*see Remark* 2.11). *Then*
$$\frac{1}{C}\chi_{\mathcal{A}}(T^*) \leq n_{\mathcal{A}^d}(T) \leq C\chi_{\mathcal{A}}(T^*)$$
for every $T \in (\mathcal{A}^d)^{\mathrm{inj}}(X, Y)$.

Proof. Notice that $\chi_{\mathcal{A}}(T^*)$ makes sense if $T \in (\mathcal{A}^d)^{\mathrm{inj}}(X, Y)$ since $(\mathcal{A}^d)^{\mathrm{inj}} \subset (\mathcal{A}^{\mathrm{sur}})^d$ [19, Theorem 8.5.9]. To prove $n_{\mathcal{A}^d}(T) \leq C\chi_{\mathcal{A}}(T^*)$, we fix $\varepsilon > \chi_{\mathcal{A}}(T^*(B_{Y^*}))$ and consider functionals $x_1^*, \dots, x_n^* \in X^*$, a Banach space Z and $S \in \mathcal{A}(Z, X^*)$ satisfying $\alpha(S) \leq \varepsilon$ and
$$T^*(B_{Y^*}) \subset \bigcup_{i=1}^{n} x_i^* + S(B_Z).$$
This covering of $T^*(B_{Y^*})$ yields
$$|\langle T^*y^*, x\rangle| \leq \sup_{1\leq i\leq n} |\langle x_i^*, x\rangle| + \sup_{z\in B_Z} |\langle Sz, x\rangle| \tag{3.3}$$

for all $y^* \in B_{Y^*}$ and $x \in X$. If we set $S_0 := S^* \circ i_X$, it follows that

$$\|Tx\| \le \sup_{1\le i\le n} |\langle x_i^*, x\rangle| + \|S_0 x\|$$

for all $x \in X$. Hence, as $\mathcal{A}$ enjoys property (P), we have $S_0 \in \mathcal{A}^d(X, Z^*)$ and

$$n_{\mathcal{A}^d}(T) \le \alpha^d(S_0) \le \alpha^d(S^*) = \alpha(S^{**}) = \alpha(i_Y \circ \widetilde{S}) \le C\varepsilon,$$

so that $n_{\mathcal{A}^d}(T) \le C\chi_{\mathcal{A}}(T^*)$ by taking the infimum over ε.

For the reverse inequality, fix $\varepsilon > n_{\mathcal{A}^d}(T)$ and consider $x_1^*, \dots, x_n^* \in X^*$, a Banach space Z and $S \in \mathcal{A}^d(X, Z)$ satisfying $\alpha^d(S) \le \varepsilon$ and

$$\|Tx\| \le \sup_{1\le i\le n} |\langle x_i^*, x\rangle| + \|Sx\|$$

for all $x \in X$. Set

$$A := \overline{\mathrm{aco}}\Big(\bigcup_{i=1}^n \pm x_i^* + S^*(B_{Z^*})\Big).$$

We are going to see that $T^*(B_{Y^*}) \subset A$. For contradiction, suppose there exists $x_0^* \in T^*(B_{Y^*})\setminus A$. According to the Hahn–Banach separation theorem, we can separate x_0^* and A in X^* endowed with the weak* topology: there are $r > 0$ and $x_0 \in X$ such that $|\langle x_0^*, x_0\rangle| > r$ and $|\langle \pm x_i^* + S^* z^*, x_0\rangle| < r$ for all $z^* \in B_{Z^*}$ and $i = 1, \dots, n$. In particular, if $z_0^* \in B_{Z^*}$ with $\|Sx_0\| = \langle z_0^*, Sx_0\rangle$, we can select $\bar{x}^* \in \{\pm x_i^* : i = 1, \dots, n\}$ such that

$$\sup_{1\le i\le n} |\langle x_i^*, x_0\rangle| + \|Sx_0\| = |\langle \bar{x}^* + S^* z_0^*, x_0\rangle| < r. \tag{3.4}$$

Now, choose $y_0^* \in B_{Y^*}$ with $T^* y_0^* = x_0^*$; then

$$r < |\langle x_0^*, x_0\rangle| \le \|Tx_0\| \le \sup_{1\le i\le n} |\langle x_i^*, x_0\rangle| + \|Sx_0\| < r,$$

a contradiction that proves $T^*(B_{Y^*}) \subset A$.

According to properties (2), (3) and (8) in Proposition 2.8, and Proposition 2.12, we have

$$\chi_{\mathcal{A}}(T^*(B_{Y^*})) \le C\chi_{\mathcal{A}}(S^*(B_{Z^*})) \le C\alpha(S^*) \le C\varepsilon.$$

Taking the infimum over ε yields $\chi_{\mathcal{A}}(T^*) \le C n_{\mathcal{A}^d}(T)$. ∎

Setting $\mathcal{A} = \Pi_p^d$ in the previous theorem, we obtain the following extension of the equality $\mathcal{QN}_p = \mathcal{K}_p^d$ [9, Corollary 3.4].

Corollary 3.13. *For every $T \in \Pi_p(X, Y)$, $n_{\Pi_p}(T) = \chi_{\Pi_p^d}(T^*)$.*

Remark 3.14. For every Banach operator ideal $\mathcal{A}$, a direct proof yields $n_{\mathcal{A}}(T^*) \le \chi_{\mathcal{A}^d}(T)$ for every $T \in (\mathcal{A}^d)^{\mathrm{sur}}(X, Y)$ (notice that $n_{\mathcal{A}}(T^*)$ makes sense if $T \in (\mathcal{A}^d)^{\mathrm{sur}}(X, Y)$ since $(\mathcal{A}^d)^{\mathrm{sur}} = (\mathcal{A}^{\mathrm{inj}})^d$ [19, Theorem 8.5.9]). Thus, for $\mathcal{A} = \Pi_p$, we have $n_{\Pi_p}(T^*) \le \chi_{\Pi_p^d}(T)$ for every $T \in \Pi_p^d(X, Y)$, which may be considered as an extension of the inclusion $\mathcal{K}_p \subset \mathcal{QN}_p^d$ [9, Corollary 3.4].

From Corollary 3.13, it is clear that $n_{\Pi_p}(T^*) = \chi_{\Pi_p^d}(T^{**})$ for every T in $\Pi_p^d(X,Y)$. Nevertheless, we do not know if there exists a positive constant C satisfying $n_{\Pi_p}(T^*) \geq C\chi_{\Pi_p^d}(T)$ for every $T \in \Pi_p^d(X,Y)$. The main problem is that the measure of non-$\mathcal{A}$-compactness of a set depends on the ambient space. This implies that the equality $\chi_{\mathcal{A}}(T) = \chi_{\mathcal{A}}(T^{**})$ does not hold in general. Indeed, taking a glance at Example 3.2, we have

$$\chi_{\mathcal{L}}(I) = \chi_{\mathcal{L}}(U_A(B_{\ell_1})) = \chi_{\mathcal{L}}(A) = 1.$$

On the other hand, notice that $A \subset \frac{1}{2}e + \frac{1}{2}B_{\ell_\infty}$, where $e = (1,1,\ldots) \in \ell_\infty$. Thus,

$$I^{**}(B_{\ell_1^{**}}) = I^{**}(\overline{B_{\ell_1}}^{w^*}) \subset \overline{I(B_{\ell_1})}^{w} = \overline{I(B_{\ell_1})}^{\|\cdot\|_\infty} \subset \operatorname{aco}\left(\frac{1}{2}e\right) + \frac{1}{2}B_{\ell_\infty}.$$

From this and [3, Theorem 2.5], it follows that

$$\chi_{\mathcal{L}}(I^{**}) \leq \chi_{\mathcal{L}}\left(\operatorname{aco}\left(\frac{1}{2}e\right)\right) + \chi_{\mathcal{L}}\left(\frac{1}{2}B_{\ell_\infty}\right) = \frac{1}{2}\chi_{\mathcal{L}}(B_{\ell_\infty}) = \frac{1}{2}.$$

Thus, if $A \subset \ell_\infty$, then $\chi_{\mathcal{L}}(A) \leq 1/2$.

4. The $\mathcal{A}$-essential norm. Another way to measure the degree of non-compactness of an operator $T \in \mathcal{L}(X,Y)$ is provided by its essential norm, defined by $\|T\|_{\mathcal{K}} = \inf\{\|T-S\| \colon S \in \mathcal{K}(X,Y)\}$. Of course, $\chi_{\mathcal{L}}(\cdot) \leq \|\cdot\|_{\mathcal{K}}$, so it is natural to ask whether those seminorms are or are not equivalent. Several authors have dealt with this problem using different approaches (see for instance [12] and [24]).

Given a Banach ideal $[\mathcal{A},\alpha]$, Theorem 4.1 in [4] states that the ideal $\mathcal{K}^{\mathcal{A}}$ is complete with respect to the ideal norm α^{sur} on $\mathcal{A}^{\mathrm{sur}}$. This allows one to define the $\mathcal{A}$-essential norm of an operator in $\mathcal{A}^{\mathrm{sur}}(X,Y)$ in a similar way to the classical essential norm, namely, the quotient ideal norm in $\mathcal{A}^{\mathrm{sur}}(X,Y)$ modulo the $\mathcal{A}$-compact operators:

$$\rho_{\mathcal{A}}(T) = \inf\{\alpha^{\mathrm{sur}}(T-S) \colon S \in \mathcal{K}^{\mathcal{A}}(X,Y)\}.$$

This is a seminorm on $\mathcal{A}^{\mathrm{sur}}(X,Y)$ that vanishes precisely on $\mathcal{A}$-compact operators. A straightforward argument shows that $\chi_{\mathcal{A}}(T) \leq \rho_{\mathcal{A}}(T)$ for every $T \in \mathcal{A}^{\mathrm{sur}}(X,Y)$.

The aim of this section is to obtain several results showing the equivalence between $\chi_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ under certain conditions on X, Y or $\mathcal{A}$.

Recall that a Banach space X is said to have the *π_λ-approximation property* if there exists a sequence (P_k) of linear projections on X with finite rank satisfying $\lim_k P_k x = x$ for every $x \in X$ and $\sup_k \|P_k\| \leq \lambda$ [5, p. 295]. The arguments in the following proof are an adaptation of a result due to

Gol'denšteĭn and Markus which connects the essential norm $\|\cdot\|_{\mathcal{K}}$ and the ball measure of noncompactness $\chi_{\mathcal{L}}$ of an operator (see [12] or [17]).

THEOREM 4.1. *Let X and Y be Banach spaces and $1 \le p < \infty$. Suppose that Y has the π_λ-approximation property. Then $\rho_{\Pi_p^d}(T) \le (1+\lambda)\chi_{\Pi_p^d}(T)$ for every $T \in \Pi_p^d(X,Y)$.*

Proof. Let $T \in \Pi_p^d(X,Y)$. Given $\varepsilon > 0$, there exist $y_1,\dots,y_n \in Y$, a Banach space Z and $S \in \Pi_p^d(Z,Y)$ with $\Pi_p^d(S) \le \chi_{\Pi_p^d}(T) + \varepsilon/2$ satisfying

$$T(B_X) \subset \bigcup_{i=1}^{n} y_i + S(B_Z). \tag{4.1}$$

Choose $N \in \mathbb{N}$ such that

$$\|P_N y_i - y_i\| \le \frac{\varepsilon}{2n^{1/p}} \tag{4.2}$$

for all $i \in \{1,\dots,n\}$. We are going to show that

$$\pi_p^d(T - P_N \circ T) \le (1+\lambda)(\chi_{\Pi_p^d}(T) + \varepsilon), \tag{4.3}$$

which yields $\rho_{\Pi_p^d}(T) \le (1+\lambda)(\chi_{\Pi_p^d}(T) + \varepsilon)$, so the proof will be concluded by letting $\varepsilon \searrow 0$.

To see (4.3), let $(y_k^*) \in \ell_p^w(Y^*)$. If (x_k) is a sequence in B_X, inclusion (4.1) provides a sequence (z_k) in B_Z such that $Tx_k = y_{i_k} + Sz_k$, where $y_{i_k} \in \{y_1,\dots,y_n\}$. Thus,

$$\begin{aligned}\Big(\sum_k |\langle (T - P_N\circ T)^* y_k^*, x_k\rangle|^p\Big)^{1/p} &= \Big(\sum_k |\langle y_k^*, (T - P_N\circ T)x_k\rangle|^p\Big)^{1/p}\\ \le \Big(\sum_k |\langle y_k^*, (\mathrm{Id}_Y - P_N)(Tx_k - y_{i_k})\rangle|^p\Big)^{1/p} &+ \Big(\sum_k |\langle y_k^*, (\mathrm{Id}_Y - P_N)y_{i_k}\rangle|^p\Big)^{1/p}.\end{aligned}$$

On the one hand,

$$\begin{aligned}\Big(\sum_k |\langle y_k^*, (\mathrm{Id}_Y - P_N)(Tx_k - y_{i_k})\rangle|^p\Big)^{1/p} &= \Big(\sum_k |\langle y_k^*, (\mathrm{Id}_Y - P_N)Sz_k\rangle|^p\Big)^{1/p}\\ &\le \Big(\sum_k |\langle S^*(\mathrm{Id}_Y - P_N)^* y_k^*, z_k\rangle|^p\Big)^{1/p}\\ &\le \pi_p(S^*(\mathrm{Id}_Y - P_N)^*)\|(y_k^*)\|_p^w\\ &\le \Big(\chi_{\Pi_p^d}(T) + \frac{\varepsilon}{2}\Big)(1+\lambda)\|(y_k^*)\|_p^w.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big(\sum_k |\langle y_k^*, (\mathrm{Id}_Y - P_N) y_{i_k}\rangle|^p\Big)^{1/p} \\ &= \Big(\sum_k |\langle (\mathrm{Id}_Y - P_N)^* y_k^*, (\mathrm{Id}_Y - P_N) y_{i_k}\rangle|^p\Big)^{1/p} \\ &\le \Big(\sum_k \Big(\sum_{i=1}^n |\langle (\mathrm{Id}_Y - P_N)^* y_k^*, (\mathrm{Id}_Y - P_N) y_i\rangle|^p\Big)\Big)^{1/p} \\ &\le \Big(\sum_{i=1}^n \frac{\varepsilon^p}{2^p n}\Big)^{1/p} \|((\mathrm{Id}_Y - P_N)^* y_k^*)\|_p^w \\ &\le \frac{\varepsilon}{2}(1+\lambda)\|(y_k^*)\|_p^w.\end{aligned}$$

Summing up, we have

$$\Big(\sum_k |\langle (T - P_N \circ T)^* y_k^*, x_k\rangle|^p\Big)^{1/p} \le (1+\lambda)(\chi_{\Pi_p^d}(T) + \varepsilon)\|(y_k^*)\|_p^w,$$

which leads to (4.3). ∎

With suitable changes in the preceding result, it is possible to obtain an inequality involving $\rho_{\Pi_p}(T)$ and $\chi_{\Pi_p^d}(T^*)$:

THEOREM 4.2. *Let X and Y be Banach spaces and $1 \le p < \infty$. Suppose that X^* has the π_λ-approximation property. Then $\rho_{\Pi_p}(T) \le (1+\lambda) m_{\Pi_p^d}(T^*)$ for every $T \in \Pi_p(X,Y)$.*

We finish with a general version of Theorem 4.1.

THEOREM 4.3. *Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let Y be a Banach space for which there exists a positive constant L such that if $E \subset Y$ is a finite-dimensional space, there exists a finite-dimensional subspace $E \subset F \subset Y$ and a projection $P\colon Y \to F$ with $\|P\| \le L$. Then $\rho_{\mathcal{A}}(T) \le (1+L)\chi_{\mathcal{A}}(T)$ for every $T \in \mathcal{A}^{\mathrm{sur}}(X,Y)$.*

Proof. Starting as in the proof of Theorem 4.1, set $E = \operatorname{span}\{y_i : i = 1, \dots, n\}$ and consider the corresponding subspace F and the projection P given by the hypothesis. Then the conclusion is a consequence of

$$(T - P \circ T)(B_X) \subset (\mathrm{Id}_Y - P)(S(B_Z)). \ ∎$$

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Juan Manuel Delgado
Departamento de Matemática Aplicada I
Escuela Técnica Superior de Arquitectura
Avenida Reina Mercedes, 2
41012 Seville, Spain
E-mail: jmdelga@us.es

Cándido Piñeiro
Departamento de Matemáticas
Facultad de Ciencias Experimentales
Campus Universitario de El Carmen
21071 Huelva, Spain
E-mail: candido@uhu.es