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Enhanced robust NMPC based on nominal predictions ⋆

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Abstract: This paper deals with the design of robust predictive controllers for uncertain constrained nonlinear systems. The considered controller is based on nominal predictions and can be posed as a standard MPC. Robust feasibility is ensured by imposing more conservative constraints in the optimization problem to take into account explicitly the effect of the uncertainty. This is done by means of the calculation of a sequence of sets, which can be easily obtained exploiting the uniform continuity of the model function. Sufficient conditions to prove input-to-state stability of the proposed controller are presented. The proposed approach can be considered an enhanced and updated formulation of a previous approach presented by the authors aimed to the reduction of the conservativeness.

1. INTRODUCTION

Model predictive control is one of the few techniques capable to control a nonlinear plant guaranteeing asymptotic stability to the target operating point fulfilling hard constraints on the state and input. The control law is implicitly derived from the solution of an optimization problem at each sampling time and the receding horizon technique (Mayne et al. [2000]).
In the case that the prediction model differs from the real plant, then the effect of the uncertainty must be considered. Under some mild assumptions, the predictive control law ensure robust stability in the case that the uncertainty is small enough (Grimm et al. [2007], Limon et al. [2002b]). In other case, the uncertainty model must be considered in the controller calculation in order to provide robust stability and robust constraint satisfaction. In this case particularly interesting are those approaches that provide robustness based on the solution of a *nominal* optimization problem. Input-to-state stability appears as a suitable framework for the robust stability analysis while constraint satisfaction can be ensured by means of approximations of the reachable sets. See Limon et al. [2009] and the references there in for a survey on this topic.

In this paper we present an enhanced formulation of the robust NMPC controller based on restricted constraints (Limon et al. [2002a]). In this paper, the uncertainty is modeled as a parametric uncertain signal, not as an additive disturbance. Assuming that the model function is uniformly continuous, enhanced design of the robust controller is achieved: in the calculation of the constraints of the optimization problem and in the stabilizing conditions. The obtained stabilizing design of the controller results particularly interesting to relax the terminal conditions for a certain class of model functions yielding to a less conservative control law.

Notation and basic definitions:

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer and the non-negative integer numbers, respectively. Given two integers $a, b \in \mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{[a,b]} \triangleq \{j \in \mathbb{Z}_{\geq 0} : a \leq j \leq b\}$. Given two vectors $x_1 \in \mathbb{R}^a$ and $x_2 \in \mathbb{R}^b$, $(x_1, x_2) \triangleq [x_1', x_2']' \in \mathbb{R}^{a+b}$. A norm of a vector $x \in \mathbb{R}^a$ is denoted as $|x|$. Given a signal $w \in \mathbb{R}^a$, the signal's sequence is denoted by $\mathbf{w} \triangleq \{w(0), w(1), \cdots\}$ where the cardinality of the sequence is inferred from the context. $\mathbf{0}$ denotes a suitable signal's sequence taking a null value. If a sequence depends on a parameter, as $\mathbf{w}(x)$, $w(j, x)$ denotes its j-th element. The sequence $\mathbf{w}_{[\tau]}$ denotes the truncation of sequence $\mathbf{w}$, i.e. $w_{[\tau]}(j) = w(j)$ if $j \leq \tau$ and $w_{[\tau]}(j) = 0$ if $j > \tau$. For a given sequence, we denote $\|w\| \triangleq \sup_{k \geq 0}\{|w(k)|\}$. The set of sequences $\mathbf{w}$, whose elements $w(j)$ belong to a set $W \subseteq \mathbb{R}^a$ is denoted by $\mathcal{M}_W$. For a compact set A, $A^{sup} \triangleq \sup_{a \in A}\{|a|\}$.
Consider a function $f(x, y) : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$, f is said to be uniformly continuous in x for all $x \in A$ and $y \in B$ if for all $\epsilon > 0$, a $\delta(\epsilon) > 0$ exists such that $|f(x_1, y) - f(x_2, y)| \leq \epsilon$ for all $x_1, x_2 \in A$ with $|x_1 - x_2| \leq \delta(\epsilon)$ and for all $y \in B$. For a given set $A \subset \mathbb{R}^a$, the range of the function is $f(A, y) \triangleq \{f(x, y) : x \in A\} \subset \mathbb{R}^c$.
A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ (or a "$\mathcal{K}$-function") if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$-function and $\gamma(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot, k)$ is of class $\mathcal{K}$, for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing and $\beta(s, k) \to 0$ as $k \to +\infty$. Consider a couple of $\mathcal{K}$-functions σ_1 and σ_2, then $\sigma_1 \circ \sigma_2(s) \triangleq \sigma_1(\sigma_2(s))$, besides $\sigma_1^j(s)$ denotes the j-th composition of σ_1, i.e. $\sigma_1^{j+1}(s) = \sigma_1 \circ \sigma_1^j(s)$

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with $\sigma_1^1(s) \triangleq \sigma_1(s)$. A function $V : \mathbb{R}^a \to \mathbb{R}_{\geq 0}$ is called positive definite if $V(0) = 0$ and there exists a $\mathcal{K}$-function α such that $V(x) \geq \alpha(|x|)$.

2. PROBLEM STATEMENT

In this paper it is considered that the plant to be controlled is described by a discrete-time invariant nonlinear difference equation as follows

$$x(k+1) = f(x(k), u(k), w(k)), \; k \geq 0 \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ is the current controlled variable and $w(k) \in \mathbb{R}^p$ is a signal which models mismatches between the real plant and the model. The origin is an equilibrium point for the plant (i.e. $f(0,0,0) = 0$) which is the control target.

The solution of system (1) at sampling time k for the initial state $x(0)$, a sequence of control inputs $\mathbf{u}$ and uncertainty signal $\mathbf{w}$ is denoted as $\phi(k, x(0), \mathbf{u}, \mathbf{w})$, where $\phi(0, x(0), \mathbf{u}, \mathbf{w}) = x(0)$. It is assumed that there is no trajectory $\phi(k, x(0), \mathbf{u}, \mathbf{w})$ that exhibits finite escape time for any $x(0)$, $\mathbf{u}$ and $\mathbf{w}$. It is also assumed that the state of the plant $x(k)$ can be measured at each sample time.

It is considered that the uncertainty signal $w(k)$ lies in a known ball $\mathcal{W} = \{w : |w| \leq \mu\}$. Furthermore, the control input and state of the plant must fulfill the following hard constraint:

$$(x(k), u(k)) \in \mathcal{Z} \tag{2}$$

where $\mathcal{Z} \subseteq \mathbb{R}^{n+m}$ is closed and contains the origin in its interior.

The model function is assumed to be uniformly continuous in all its arguments in the set $\mathcal{Z} \times \mathcal{W}$. Then, there are three $\mathcal{K}$-functions σ_x, σ_u and σ_w such that

$$\begin{aligned}|f(x_1, u_1, w_1) - f(x_2, u_2, w_2)| \leq \sigma_x(|x_1 - x_2|) \\ +\sigma_u(|u_1 - u_2|) \\ +\sigma_w(|w_1 - w_2|)\end{aligned} \tag{3}$$

for all (x_1, u_1, w_1) and (x_2, u_2, w_2) in $\mathcal{Z} \times \mathcal{W}$.

The nominal model of the plant (1) denotes the system considering zero-disturbance and it is given by

$$\tilde{x}(k+1) = \tilde{f}(\tilde{x}(k), u(k)), \; k \geq 0 \tag{4}$$

where $\tilde{f}(x, u) \triangleq f(x, u, 0)$. The solution to this equation for a given initial state $x(0)$ is denoted as $\tilde{\phi}(k, x(0), \mathbf{u}) \triangleq \phi(k, x(0), \mathbf{u}, \mathbf{0})$.

The aim of the paper is to design a model predictive controller based on nominal predictions such that the controlled plant is robustly stable while satisfying the constraints throughout the evolution. In the following sections, the stability notion used in this paper is briefly introduced: the regional input-to-state stability.

2.1 Regional input-to-state stability (ISS)

The existence of constraints limits the domain where the system can be stabilized. Then, a regional definition of the stability notions must be considered. In this paper, robust stability is studied resorting in the notion of input-to-state stability (Sontag and Wang [1996], Jiang and Wang [2001]). ISS has demonstrated to be a useful framework to analyze robust stability of predictive controllers (see Limon et al. [2009] and the references there in).

Consider that the system (1) is controlled by the law $u = \kappa(x)$ leading the following closed-loop system

$$x^+ = f_\kappa(x, w) \triangleq f(x, \kappa(x), w) \tag{5}$$

$$x \in X_\kappa \triangleq \{x \in \mathbb{R}^n : (x, \kappa(x)) \in \mathcal{Z}\} \tag{6}$$

Now, some definitions and well-known results on regional ISS are summarized.

Definition 1. (Robust positively invariant (RPI) set). A set $\Gamma \subseteq \mathbb{R}^n$ is a robust positively invariant (RPI) set for system (5) if $f_\kappa(x, w) \in \Gamma$, for all $x \in \Gamma$ and all $w \in \mathcal{W}$. Furthermore, if $\Gamma \subseteq X_\kappa$, then Γ is called admissible RPI set. □

Notice that the fact that the RPI set Γ is admissible ensures the robust satisfaction of the constraints since for any initial $x_0 \in \Gamma$, $\Phi(k, x_0, \mathbf{w}) \in \Gamma \subseteq X_\kappa$ for all $k \in \mathbb{Z}_{\geq 0}$ and $\mathbf{w} \in \mathcal{M}_\mathcal{W}$.

Definition 2. (Regional ISS in Γ). Let $\Gamma \subseteq \mathbb{R}^n$ be an admissible RPI for system (5) including the origin as an interior point. The system (5) is input-to-state stable (ISS) in Γ if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function σ such that

$$|\phi_\kappa(j, x(0), \mathbf{w})| \leq \beta(|x(0)|, j) + \sigma(\|\mathbf{w}_{[j-1]}\|) \tag{7}$$

for all $x(0) \in \Gamma$, $\mathbf{w} \in \mathcal{M}_\mathcal{W}$ and $j \in \mathbb{Z}_{\geq 0}$.

ISS can be determined by means of a Lyapunov-like condition (Jiang and Wang [2001], Magni et al. [2006]), as follows.

Definition 3. (ISS-Lyapunov function in Γ) Let Γ be a RPI set containing the origin in its interior. A function V: $\mathbb{R}^n \to \mathbb{R}_{\geq}0$ is called an ISS-Lyapunov function in Γ for system (5) if there exist a compact set $\Omega \subseteq \Gamma$ (including the origin as an interior point), suitable $\mathcal{K}_\infty$-functions α_1, α_2, α_3 and $\mathcal{K}$-function λ such that:

$$V(x) \geq \alpha_1(|x|), \quad \forall x \in \Gamma \tag{8}$$

$$V(x) \leq \alpha_2(|x|), \quad \forall x \in \Omega \tag{9}$$

and for all $x \in \Gamma$ and $w \in \mathcal{W}$, the following condition holds

$$V(f_\kappa(x, w)) - V(x) \leq -\alpha_3(|x|) + \lambda(|w|) \tag{10}$$

□

Based on this Lyapunov-like functions, the following stability theorem can be derived (Jiang and Wang [2001], Magni et al. [2006]):

Theorem 1. If system (5) admits an ISS-Lyapunov function in Γ then it is ISS in Γ.

3. PROPOSED ROBUST MPC

3.1 Semi-feedback approach

The most simple robust MPC formulations derive the control law from the solution of an optimization problem based on open-loop predictions of the uncertain system evolution. This open-loop scheme results to be very conservative from both a performance and domain of attraction points of view (see [Mayne et al., 2000, Section 4]). In order to reduce this conservativeness, a closed-loop (or feedback) formulation of the MPC has been proposed (Scokaert and Mayne [1998]). In this case, control policies instead of control actions are taken as decision variables, yielding to an infinite dimensional optimization problem that is in general very difficult to solve and for which there exists few efficient algorithm in the literature in the case of linear systems (Muñoz de la Peña et al. [2006], Goulart et al. [2006]). A practical formulation between these two approaches is the so-called semi-feedback formulation, where a family of parameterized control laws is used (Chisci et al. [2001], Fontes and Magni [2003]). Thus the decision variables are the sequence of the parameters of the control laws, and hence the optimization problem is a finite-dimensional mathematical programming problem.

Consider that the control actions are derived from a given family of controllers parameterized by $v \in \mathbb{R}^s$,

$$u(k) = \pi(x(k), v(k))$$

which is assumed to be uniformly continuous in its domain. The family of control laws is typically chosen as an affine function of the state (Chisci et al. [2001]). Thus, system (1) is transformed in

$$x(k+1) = f_\pi(x(k), v(k), w(k)),\ k \geq 0 \tag{11}$$

where $f_\pi(x, v, w) \triangleq f(x, \pi(x, v), w)$. Notice that v plays the role of the input of the modified system. The solution of this equation is denoted as $\phi_\pi(k, x, \mathbf{v}, \mathbf{w})$. The nominal model of system (11) is denoted as $\tilde{f}_\pi(x, v) \triangleq f_\pi(x, v, 0)$ and its solution as $\tilde{\phi}_\pi(k, x, \mathbf{v}) \triangleq \phi_\pi(k, x, \mathbf{v}, \mathbf{0})$. Analogously, the constraints can be rewritten as

$$(x(k), v(k)) \in \mathcal{Z}_\pi \tag{12}$$

where $\mathcal{Z}_\pi$ is such that $(x, \pi(x, v)) \in \mathcal{Z}$ for all $(x, v) \in \mathcal{Z}_\pi$.

3.2 Nominal model predictive control

The proposed predictive controller is based on the nominal prediction of the trajectories and follows the standard formulation of the MPC (Mayne et al. [2000]). The control law is derived from the solution of the following mathematical programming problem $P_N(x)$ parameterized in the current state x.

$$\min_{\mathbf{v}} \sum_{j=0}^{N-1} L_\pi(\tilde{x}(j), v(j)) + V_f(\tilde{x}(N)) \tag{13}$$

$$s.t.\ \tilde{x}(j) = \tilde{\phi}_\pi(j, x, \mathbf{v}),\ j \in \mathbb{Z}_{[0,N]} \tag{14}$$

$$(\tilde{x}(j), v(j)) \in \mathcal{Z}_\pi(j),\ j \in \mathbb{Z}_{[0,N-1]} \tag{15}$$

$$\tilde{x}(N) \in \mathcal{X}_f \tag{16}$$

where $L_\pi(x, v) \triangleq L(x, \pi(x, v))$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is the stage cost function, $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the terminal cost function. The sequence of constraint sets $\{\mathcal{Z}_\pi(j)\}$ will be defined later on and $X_f \subseteq \mathbb{R}^n$ is the terminal region. It is assumed that $P_N(x)$ is feasible in a non-empty region denoted $\mathcal{X}_N$. For each $x \in X_N$, the argument of $P_N(x)$ is denoted $\mathbf{v}^*(x)$ and the optimal cost is $V_N^*(x)$. The MPC control law derives from the application of the solution in a receding horizon manner $\kappa_N(x) = v^*(0; x)$ and it is defined for all $x \in \mathcal{X}_N$.

3.3 Robust design of the proposed controller

The proposed controller is based on the availability of two sequence of sets $\{\mathcal{R}(j)\}$ and $\{\mathcal{F}(j)\}$ that are assumed to be calculated off-line (see next section). The sequence $\{\mathcal{F}(j)\}$ is related with the free response of the nominal system and must satisfy the following hypothesis:

Assumption 1. The sequence of sets $\{\mathcal{F}(j)\}$ is such that: For every $(x, \mathbf{v})$, $\tilde{\phi}_\pi(k, \hat{x}, \mathbf{v}) \in \tilde{\phi}_\pi(k, x, \mathbf{v}) \oplus \mathcal{F}(k)$ for all $\hat{x}$ such that $|\hat{x} - x| \leq \sigma_w(\mu)$.

On the other hand, the sequence $\{\mathcal{R}_j\}$ is related to the reachable sets, that is, the sequence of possible trajectories due to the effect of the disturbances. This sequence must satisfy the following conditions

Assumption 2. The sequence of sets $\{\mathcal{R}(j)\}$ is such that:

(1) For every $(x, \mathbf{v})$, $\phi_\pi(k, x, \mathbf{v}, \mathbf{w}) \in \tilde{\phi}_\pi(k, x, \mathbf{v}) \oplus \mathcal{R}(k)$ for all $\mathbf{w} \in \mathcal{M}_W$
(2) $\mathcal{F}(j) \oplus \mathcal{R}(j) \subseteq \mathcal{R}(j+1)$

The first condition states that each set of the sequence is an outer bound of the effect of the uncertainty throughout the trajectory, while the second condition ensures that the sequence is monotone. This fact will be more clearly demonstrated in the proof of lemma 2. Practical methods to calculate the proposed sequences are presented in the following section.

Since sequence of sets $\{\mathcal{R}(j)\}$ provides an estimation of the effect of the disturbance with respect to the nominal predictions, this can be used to counteract the effect of the disturbances in the constraint satisfaction. This is done by using a sequence of tighter constraint sets $\{\mathcal{Z}_\pi(j)\}$ defined as follows:

Definition 4. Let the sequence $\{\mathcal{Z}_\pi(j)\}$ be defined as follows

$$\mathcal{Z}_\pi(j) = \mathcal{Z} \ominus (\mathcal{R}(j) \times \{0\})$$

On the other hand, the terminal constraint set $\mathcal{X}_f$ must satisfy the following assumption

Assumption 3. The input $v_f \in \mathbb{R}^s$ and the sets Ω and $\mathcal{X}_f$ are such that

(i) Ω and $\mathcal{X}_f$ are invariant sets for the system $\tilde{x}^+ = \tilde{f}_\pi(\tilde{x}, v_f)$
(ii) $\Omega \times \{v_f\} \subseteq \mathcal{Z}_\pi(N-1)$
(iii) $\mathcal{X}_f \oplus \mathcal{F}(N-1) \subseteq \Omega$
(iv) $\tilde{f}_\pi(\tilde{x}, v_f) \in \mathcal{X}_f$, for all $\tilde{x} \in \Omega$.

Notice that this assumption requires that Ω is an invariant set for the nominal system and $\mathcal{X}_f$ a set where the system evolves in one step. The main restriction is that Ω must be a contractive invariant set such that $\tilde{f}_\pi(\Omega, v_f) \oplus \mathcal{F}(N-1) \subseteq \Omega$. This implicitly states that Ω is a robust positively invariant set for the system $x^+ = f_\pi(x, v_f) + \omega$ where $\omega \in \mathcal{F}(N-1)$. In the case that the control law π makes the system asymptotically stable in $(x, v) \in \mathcal{Z}_\pi$, which is usual for simple systems such that linear systems (Chisci et al. [2001]) or feedback linearizable system (Raković et al. [2006]), set $\mathcal{F}(N-1)$ can be arbitrarily small for large enough prediction horizon. This relaxes the standard assumption on the terminal constraint, that must be a robust positively invariant set for the whole uncertainty set.

This proposed method to design the constraints of the optimization problem P_N has been chosen in order to ensure the robust feasibility of the controller, as it is demonstrated in the following lemma.

Lemma 2. Consider the system (11) and the sequence of sets $\{\mathcal{Z}_\pi(j)\}$ based on the sequences of sets $\{\mathcal{R}(j)\}$ and $\{\mathcal{F}(j)\}$ which satisfy assumptions 1 and 2. Let the triplet $(v_f, \Omega, \mathcal{X}_f)$ fulfill assumption 3. Consider now a feasible state $x \in \mathcal{X}_N$ and $\mathbf{v}^*$ the argument of $P_N(x)$. Let x^+ be the uncertain successor state and define the sequence of inputs $\mathbf{v}^+ \triangleq \{v^*(1), \cdots, v^*(N-1), v_f\}$. Then the following properties hold.

(1) $(\tilde{\phi}_\pi(j, x^+, \mathbf{v}^+), v^+(j)) \in \mathcal{Z}_\pi(j)$
(2) $\tilde{\phi}_\pi(N, x^+, \mathbf{v}^+) \in \mathcal{X}_f$

The proof can be found in the appendix.

3.4 Calculation of the sequence of sets

The sequence of sets $\{\mathcal{F}(j)\}$ and $\{\mathcal{R}(j)\}$ provides outer bounds on the effect of the uncertainty throughout the prediction, then these can be calculated by methods that provides guaranteed prediction of the uncertain system (Bravo et al. [2006], Raković et al. [2006], Limon et al. [2009]). Among these, it is worth to cite those based on polytopic algorithms, interval arithmetics, zonotopic methos or DC-programming based techniques.

In this paper we provide a simpler, although probably more conservative method, based on the uniform continuity of the model function.

Lemma 3. Let a system be given by model (1) and let define the following sets:

$$\mathcal{F}(j) \triangleq \{x \in \mathbb{R}^n : |x| \leq \sigma_x^j \circ \sigma_w(\mu)\} \tag{17}$$

$$\mathcal{R}(j) \triangleq \{x \in \mathbb{R}^n : |x| \leq c_j(\mu)\} \tag{18}$$

where $c_j(\mu)$ is given by the following recursion

$$c_j(\mu) = max\{\ \sigma_w(\mu) + \sigma_x \circ c_{j-1}(\mu),\ c_{j-1}(\mu) + \sigma_x^{j-1} \circ \sigma_w(\mu)\} \tag{19}$$

with $c_1(\mu) = \sigma_x(\mu)$.

Then the sequence of sets $\{\mathcal{F}(j)\}$ and $\{\mathcal{R}(j)\}$ satisfy the assumptions 1 and 2.

This lemma is proved in the appendix.

As it can be seen, these sets can be easily calculated off-line once provided the bounding functions. In the case that Lipschitz continuity is exploited to derive the bounding functions, the resulting sets are equal to those presented in Limon et al. [2002a]. Notice that if the uniform continuity is exploited, tighter (non-linear) bounding functions can be used, and hence less conservative results will be obtained.

4. INPUT-TO-STATE STABILITY OF THE CONTROLLED SYSTEM

In the previous sections, conditions on the constraints of the optimization problem $P_N(x)$ that suffices to ensure robust feasibility are provided. However these conditions are not sufficient to derive robust stability of the closed-loop system. To this aim, the following additional assumptions are required.

Assumption 4.

(1) Let the stage cost function $L_\pi(x, v)$ be a definite positive function in (x, v) uniformly continuous in $\mathcal{Z}_\pi$ such that
$$L_\pi(x, v) \geq \alpha_L(|x|)$$
$$|L_\pi(x_1, v_1) - L_\pi(x_2, v_2)| \leq \lambda_x(|x_1 - x_2|) + \lambda_v(|v_1 - v_2|)$$
where α_L, λ_x and λ_v are $\mathcal{K}$-functions.
(2) Let the terminal cost function $V_f(x)$ be a definite positive function uniformly continuous in Ω (see assumption 3) such that
$$\alpha_V(|x|) \leq V_f(x) \leq \beta_V(|x|)$$
$$V_f(\tilde{f}_\pi(x, v_f)) - V_f(x) \leq -L_\pi(x, v_f)$$
$$|V_f(x_1) - V_f(x_2)| \leq \delta(|x_1 - x_2|)$$

These assumptions are standard for the stabilizing design of nominal MPC (Mayne et al. [2000]). The only additional requirement is the uniform continuity of the functions. Based on this, stability is stated in the following theorem.

Theorem 4. Consider that assumptions 1 ,2, 3 and 4, hold. Then the system (1) controlled by $\kappa_{MPC}(x) = \pi(x, \kappa_N(x))$ is ISS in $\mathcal{X}_N$ and satisfies the constraints throughout the evolution.

The proof can be found in the appendix.

5. CONCLUSIONS

This paper has demonstrated that outer estimation of the reachable sets can be used to derive robust stabilizing predictive controller based on nominal predictions. This class of controllers are appealing from a practical point of view since can be constructed from standard nominal MPC. On the other hand, the open-loop nature of the problem may yield to the results to be useful only for small uncertainties. In order to reduce this effect, semi-feedback approach is proposed. This is a simple and practical method, but requires an analysis of the system to be controlled in order to find a nice family of control laws.

Based on the uniform continuity of the model function and the defining functions of the MPC, sufficient conditions for input-to-state stability has been proposed. Moreover, uniform continuity can also be exploited to calculate the sequence of sets necessary for the design of the proposed controller.

As future work on this topic, the authors will study how to use these ideas in the context of tube-based robust MPC.

Appendix A. PROOF OF LEMMA 2

First statement: Since $\tilde{\phi}_\pi(j, x^+, \mathbf{v}^+) - \tilde{\phi}_\pi(j+1, x, \mathbf{v}^*) \in \mathcal{F}(j)$ we have that

$$\tilde{\phi}_\pi(j, x^+, \mathbf{v}^+) \in \phi_\pi(j+1, x, \mathbf{v}^*) \oplus \mathcal{F}(j)$$
$$\in \phi_\pi(j+1, x, \mathbf{v}^*) \oplus \mathcal{R}(j+1) \ominus \mathcal{R}(j)$$

Since $(\phi_\pi(j+1, x, \mathbf{v}^*), v^*(j+1)) \in \mathcal{Z}_\pi(j+1)$, we have that

$$(\phi_\pi(j+1, x, \mathbf{v}^*), v^*(j+1)) \in \Big(\mathcal{Z}_\pi(j+1) \oplus(\mathcal{R}(j+1) \times \{0\}) \ominus(\mathcal{R}(j) \times \{0\})\Big) \quad \text{(A.1)}$$

By definition,

$$\mathcal{Z}_\pi(j+1) \oplus (\mathcal{R}(j+1) \times \{0\}) \subseteq \mathcal{Z}_\pi$$

and $v^*(j+1) = v^+(j)$ which implies that

$$(\tilde{\phi}_\pi(j, x^+, \mathbf{v}^+), v^+(j)) \in \mathcal{Z}_\pi \ominus (\mathcal{R}(j) \times \{0\})$$
$$\in \mathcal{Z}_\pi(j).$$

Second statement: Feasibility of the solution $\mathbf{v}^*$ implies that $\tilde{\phi}_\pi(N, x, \mathbf{v}^*) \in \mathcal{X}_f$. On the other hand, from the definition of set $\mathcal{F}(N-1)$ we have that

$$\tilde{\phi}_\pi(N-1, x^+, \mathbf{v}^+) \in \tilde{\phi}_\pi(N, x, \mathbf{v}^*) \oplus \mathcal{F}(N-1)$$

Then

$$\tilde{\phi}_\pi(N-1, x^+, \mathbf{v}^+) \in \mathcal{X}_f \oplus \mathcal{F}(N-1)$$

Since $\mathcal{X}_f \oplus \mathcal{F}(N-1) \subseteq \Omega$, from assumption 3 we infer that the terminal state $\tilde{\phi}_\pi(N-1, x^+, \mathbf{v}^+)$ must be contained in the terminal region $\mathcal{X}_f$.

Appendix B. PROOF OF LEMMA 3

First, it is proved that equation 17 satisfies assumption 1. This is proved by induction. For $j = 1$ we have that

$$|\tilde{\phi}_\pi(1, \hat{x}, \mathbf{v}) - \tilde{\phi}_\pi(1, x, \mathbf{v})| = |\tilde{f}_\pi(\hat{x}, v) - \tilde{f}_\pi(x, v)|$$
$$\le \sigma_x(|\hat{x} - x|)$$
$$\le \sigma_x \circ \sigma_w(\mu)$$

Assume that

$$|\tilde{\phi}_\pi(j, \hat{x}, \mathbf{v}) - \tilde{\phi}_\pi(j, x, \mathbf{v})| \le \sigma_x^j \circ \sigma_w(\mu)$$

then

$$|\tilde{\phi}_\pi(j+1, \hat{x}, \mathbf{v}) - \tilde{\phi}_\pi(j+1, x, \mathbf{v})|$$
$$\le \sigma_x(|\tilde{\phi}_\pi(j, \hat{x}, \mathbf{v}) - \tilde{\phi}_\pi(j, x, \mathbf{v})|)$$
$$\le \sigma_x^{j+1} \circ \sigma_w(\mu)$$

Now it is proved that the set defined in equation (18) satisfies assumption 2. For $j = 1$, we have that

$$|\phi_\pi(1, x, \mathbf{v}, \mathbf{w}) - \tilde{\phi}_\pi(1, x, \mathbf{v})| \le \sigma_w(\mu) = c_1(\mu)$$

Assume that

$$|\phi_\pi(j, x, \mathbf{v}, \mathbf{w}) - \tilde{\phi}_\pi(j, x, \mathbf{v})| \le c_j(\mu)$$

then

$$|\phi_\pi(j+1, x, \mathbf{v}, \mathbf{w}) - \tilde{\phi}_\pi(j+1, x, \mathbf{v})|$$
$$\le |\phi_\pi(j+1, x, \mathbf{v}, \mathbf{w}) - \tilde{f}_\pi(\phi_\pi(j, x, \mathbf{v}, \mathbf{w}), v(j+1))|$$
$$+|\tilde{f}_\pi(\phi_\pi(j, x, \mathbf{v}, \mathbf{w}), v(j+1)) - \tilde{\phi}_\pi(j+1, x, \mathbf{v})|$$
$$\le \sigma_w(\mu) + \sigma_x(|\phi_\pi(j, x, \mathbf{v}, \mathbf{w}) - \tilde{\phi}_\pi(j, x, \mathbf{v})|)$$
$$\le \sigma_w(\mu) + \sigma_x(c_j(\mu)) \le c_{j+1}(\mu)$$

Condition $\mathcal{F}(j) \oplus \mathcal{R}(j) \subseteq \mathcal{R}(j+1)$ is fulfilled iff

$$c_j(\mu) + \sigma_x^j \circ \sigma_w(\mu) \le c_{j+1}(\mu)$$

Notice that this is directly inferred from equation (19).

Appendix C. PROOF OF THEOREM 4

Robust feasibility of the problem has been stated in lemma 2, which implies the robust constraint satisfaction. Then, it suffices to prove the input-to-state stability. This is achieved by demonstrating that the optimal cost function $V_N^*(x)$ is a ISS-Lyapunov function.

Consider that $x \in \mathcal{X}_N$ and the optimal solution of $P_N(x)$ is $\mathbf{v}^*$. Let define $x^*(j) = \tilde{\phi}_\pi(j, x, \mathbf{v}^*)$. Let x^+ the successor state and let $\mathbf{v}^+$ the feasible solution proposed in lemma 2. Let define $x^+(j) = \tilde{\phi}_\pi(j, x^+, \mathbf{v}^+)$. Then the cost $V_N(x^+, \mathbf{v}^+)$ is given by

$$V_N(x^+, \mathbf{v}^+) = \sum_{j=1}^{N-1} L_\pi(x^+(j), v^+(j)) + V_f(x^+(N))$$

From the uniform continuity of the model we have that $|x^+(j) - x^*(j+1)| \le \sigma_x^j \circ \sigma_w(|w|)$ (this is immediate from the proof of lemma 2). Then considering the uniform continuity of L_π and V_f, we have that $\Delta V = V_N(x^+, \mathbf{v}^+) - V_N^*(x) + L_\pi(x, \kappa_N(x))$ is such that

$$\Delta V = \sum_{j=1}^{N-1} (L_\pi(x^+(j), v^+(j)) - L_\pi(x^*(j+1), v^*(j+1))$$
$$+L_\pi(x^+(N-1), v_f) + V_f(x^+(N)) - V_f(x^*(N))$$

Since

$$L_\pi(x^+(j), v^+(j)) - L_\pi(x^*(j+1), v^*(j+1))$$
$$\le |L_\pi(x^+(j), v^+(j)) - L_\pi(x^*(j+1), v^*(j+1))|$$
$$\le \lambda_x \circ \sigma_x^j \circ \sigma_w(|w|)$$

and

$$V_f(x^+(N-1)) - V_f(x^*(N))$$
$$\le |V_f(x^+(N-1)) - V_f(x^*(N))|$$
$$\le \delta(|x^+(N-1) - x^*(N)|)$$
$$\le \delta \circ \sigma_x^{N-1} \circ \sigma_w(|w|)$$

we have that

$$\Delta V \le \sum_{j=1}^{N-1} \lambda_x \circ \sigma_x^j \circ \sigma_w(|w|) + \delta \circ \sigma_x^{N-1} \circ \sigma_w(|w|)$$
$$+L_\pi(x^+(N-1), v_f) + V_f(x^+(N)) - V_f(x^+(N-1))$$

From assumption 4 we have that

$$L_\pi(x^+(N-1), v_f) + V_f(x^+(N)) - V_f(x^+(N-1)) \le 0$$

and hence, there exists a $\mathcal{K}$ function θ such that $\Delta V \le \theta(|w|)$. In virtue of the optimality, we have that

$$V_N^*(x^+) - V_N^*(x) \le -L_\pi(x, \kappa_N(x)) + \theta(|w|)$$

On the other hand, for all $x \in \mathcal{X}_N$,

$$V_N^*(x) \ge L_\pi(x, \kappa_N(x)) \ge \alpha_L(|x|)$$

Besides, from the optimality of the solution we have that for all $x \in \mathcal{X}_f$,

$$V_N^*(x) \le V_f(x) \le \beta_V(|x|)$$

Then $V_N^*(x)$ is a ISS-Lyapunov function in $\mathcal{X}_N$, and from theorem 1 we derive that the closed-loop system is ISS in $\mathcal{X}_N$.

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