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Robust MPC for tracking zone regions based on nominal predictions6

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This paper deals with the problem of robust tracking of target sets using a model predictive control (MPC) law. Real industries applications often require a control strategy in which some system outputs are controlled within speciﬁed ranges or zones (zone control), while some others variables – possibly including input variables – are steered to ﬁxed target or setpoint. From a theoretical point of view, the control objective of this kind of problem can be seen as a target set (in the output space) instead of a target point, since inside the zones there are no preferences between one point or another. This problem is particularly interesting in case of additive disturbances which might push the outputs out of the zones. In this work, a stable robust MPC formulation for constrained linear systems, based on nominal predictions is presented. The main features of this controller are the use of nominal predictions, restricted constraints and the concept of distance from a point to a set as offset cost function. The controller ensures both recursive feasibility and local optimality. The properties of the controller are shown in a simulation test, in which we consider a subsystem of an industrial FCC system.

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## Introduction

* 1. *Set-interval control*

In modern processing plants, MPC controllers are usually imple- mented as part of a multilevel hierarchy of control functions [[1,2](https://www.researchgate.net/publication/227975041_Robust_Steady_State_Target_Calculation_for_Model_Predictive_Control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. At the intermediary levels of this control structure, the process unit optimizer computes an optimal economic steady state and provides this information to the MPC in a lower level for implementation. The role of the MPC is then to drive the plant to the most proﬁtable operating condition, fulﬁlling the constraints and minimizing the dynamic error along the path.

In many cases, the optimal economic steady state operating con- dition is not given by a point in the output space (ﬁxed setpoint), but by a region or zone into which the output should lie most of the time. Conceptually, these output zones can be seen as a generaliza- tion of the output targets from a point to a set (i.e., a generalized setpoint) rather than an output constraint, since they are desired

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steady state sets that can be transitorily disregarded, while the con- straints must be fulﬁlled at each time step. In this way, the concept of degrees of freedom is substantially altered. In fact, it is gener- alized in such a way that even systems with more outputs than inputs allow (economic) targets for some inputs. A kind of hierar- chy of objectives arises in the MPC control problem, in which the ﬁrst one is to ﬁnd a feasible solution (i.e. one that fulﬁlls the input and output constraints), the second one is to reach and maintain the outputs inside their corresponding zones and the third one is to steer the inputs as close as possible to the desired economic tar- gets. Only once a higher priority objective is reached, the remainder “*degrees of freedom*” can be used to reach the lower one.

From the point of view of real systems, the zone control may appear in different kind of dynamic systems: (i) process systems with highly correlated outputs to be controlled, in which there are not enough inputs to control all the outputs; (ii) process systems with problems to use the surge capacity of tanks to smooth out the operation of a process unit (in this case, it is desired to let the level of the tank to ﬂoat between limits, as necessary, to buffer disturbances between sections of a plant); (iii) biological systems, such as dia- betes patient, in which tracking a given output setpoint (glycemic level) could demand an excessive and unnecessary control effort (insulin administration) while maintaining the glycemic level in a given safety interval is sufﬁcient to guaranty the control objectives [[3](https://www.researchgate.net/publication/45388429_Zone_Model_Predictive_Control_A_Strategy_to_Minimize_Hyper-_and_Hypoglycemic_Events?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)].

Several approaches have been proposed to account for the MPC set-interval or zone control. [4] shows how some commercial MPC controllers are adapted to account for the zone control problem,

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including the so-called *funnel* strategy. In [[5]](https://www.researchgate.net/publication/245158491_Integrating_real-time_optimization_into_the_model_predictive_controller_of_the_FCC_system?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D) and [[3]](https://www.researchgate.net/publication/45388429_Zone_Model_Predictive_Control_A_Strategy_to_Minimize_Hyper-_and_Hypoglycemic_Events?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D) it can be found simple approaches to tackle the zone control problem: they penal- ize an output into the MPC cost function only if it is inside the zone. Although this kind of switching control has shown to be plausible to be applied in real control systems (as diverse as process sys- tem control and biological system control), stability and recursive feasibility cannot be proved under their formulation framework. A closed-loop stable and recursively feasible MPC controller is pre- sented in [[6](https://www.researchgate.net/publication/244358793_A_stable_MPC_with_zone_control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. In this approach, the authors develop a controller that incorporates steady state economic targets for input and output in the control cost function. Assuming open-loop stable systems, classical stability proofs are extended to the zone control strat- egy by considering artiﬁcial output setpoints as additional decision variables. This controller, however, is formulated only for open- loop stable systems, and since it considers a null controller as local controller, it does not achieve local optimality.

* 1. *Robust set-interval control*

The explicit consideration of model uncertainties or distur- bances is quite different in the context of set-interval control. Since the uncertainty affects the system gains, it also affects the compatibility between the available input set (given by the input constraints) and the desired output set (given by the output zones). Thus, an efﬁcient robust design should take these problems into account in order to avoid unfeasibilities, even unfeasibilities at steady state. An extension to the robust case of the strategy pre- sented in [[6]](https://www.researchgate.net/publication/244358793_A_stable_MPC_with_zone_control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D) (considering multi-model uncertainty) was proposed in [[7](https://www.researchgate.net/publication/221911895_Robust_Model_Predictive_Control_for_Time_Delayed_Systems_with_Optimizing_Targets_and_Zone_Control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. Although these approaches account for the tracking of non- zero targets and the second one considers time delayed systems, they also fail to guarantee local optimality and they are only for- mulated for open-loop stable systems.

* 1. *MPC and the tracking problem*

Most of the rigorous MPC stability, feasibility and optimality results consider the regulation problem, that is steering the sys- tem to a ﬁxed steady state (typically the origin) [8,9]. If, for a given non-zero set point, a suitable choice of the steady state is taken, the

In this paper, the controller presented in [19] is extended to cope with the problem of robust tracking of target sets in pres- ence of additive disturbance. Although here we consider a different uncertainty representation than the one used in [[7](https://www.researchgate.net/publication/221911895_Robust_Model_Predictive_Control_for_Time_Delayed_Systems_with_Optimizing_Targets_and_Zone_Control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)], the proposed controller constitutes an improved robust MPC formulation for the zone control problem (i.e. a robust control suitable for non-stable systems, which preserves local optimality). Based on some of the results presented in [[20](https://www.researchgate.net/publication/220157638_Systems_with_persistent_disturbances_Predictive_control_with_restricted_constraints?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)], we propose here an MPC based on nomi- nal predictions and restricted constraints, which ensures stability, robust satisfaction of the constraints, recursive feasibility and local optimality.

The paper is organized as follows. In Section 2 the control prob- lem is stated. Sections 3 and 4 present the proposed controller and its main properties, respectively. In Section 5 the properties of the controller are shown in a simulation test, in which we consider a subsystem of an industrial FCC system. Finally, in Section 6, some conclusions are drawn.

**Notation:** A positive deﬁnite symmetric matrix *T* is denoted as

*T* > 0 and *T* > *P* denotes that *T P* > 0. For a given symmetric matrix

−

*P* > 0, *x* denotes the weighted Euclidean norm of *x*, i.e.

∗ ∗ ∗*x*∗ =*P P*

√*x*r*Px*. Consider *a* ∈ **R***na* and *b* ∈ **R***nb* , the vector made from stacking

both vectors is deﬁned as (*a*, *b*) ¾ [*a*r, *b*r]r**R***na* +*nb* ; for a set *T* **R***na* +*nb* , the projection of *T* onto *a* is deﬁned as *Proja*(*T* ) = *a* **R***na* : *b* **R***nb* ,

*{* ∈ ∃ ∈

⊂

(*a*, *b*) *T* . A vector **t** in bold denotes a ﬁnite sequence of vec- tors, that is, a vector deﬁned as *t*(0), *t*(1), *. . .*, *t*(*N*) , where *N* is deduced from the context. The norm of a signal **t** is deﬁned as

*{ }*

∈ *}*

∗*t*∗∞ = sup(*t*(*k*)). A matrix **0***n*,*m* ∈ **R***n*×*m* denotes a matrix of zeros

*k*≥0

and *In* ∈ **R***n*×*n* denotes the identity matrix. Given two sets *U* and *V*, such that *U* ⊂ **R** and *V* ⊂ **R***n*, the Minkowski sum is deﬁned by *U* ⊕ *V* ¾ {*u* + *v* : *u* ∈ *U, v* ∈ *V*}, the Pontryagin set difference is: *U* g *V* ¾ {*u* : *u* ⊕ *V* ⊆ *U*}; given a matrix *M* ∈ **R***p*×*n*, the set *MU* ⊂ **R***p* is deﬁned as *MU* ¾ {*Mu* : *u* ∈ *U*}; for a given *h*, *hU* ¾ (*hIn*)*U*.

*n*

## Problem statement

Consider a plant described by the following uncertain discrete- time LTI system

problem can be posed as a regulation problem translating the state and input of the system [[10](https://www.researchgate.net/publication/255925774_Model_predictive_control_with_linear_models?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. The steady state target is usually deter- mined by solving an optimization problem that can be formulated

*x*+ = *Ax* + *Bu* + *w*

*y* = *Cx* + *Du*

(1)

as different mathematical programs for the cases of perfect target tracking or non-square systems [[11](https://www.researchgate.net/publication/3704332_Steady-state_target_optimization_in_linear_model_predictive_control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)], or by solving a unique prob- lem for both situations [[12](https://www.researchgate.net/publication/227656903_Steady_States_and_Constraints_in_Model_Predictive_Control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. However, since the stabilizing choice of the terminal cost and constraints depends on the desired steady state, when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference [[13–](https://www.researchgate.net/publication/3022752_Nonlinear_Control_of_Constrained_Linear_Systems_via_Predictive_Reference_Management?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)16], thus requiring to re-design the MPC at each change of the reference.

In [[17](https://www.researchgate.net/publication/220159753_MPC_for_tracking_piecewise_constant_reference_for_constrained_linear_systems?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)], a rigorous MPC formulation for tracking is proposed, which is able to steer the system to any admissible setpoint in an admissible way, by considering the steady conditions as opti- mization variable of the MPC problem. This controller ensures both, recursive feasibility and convergence to the target (if admissible) for any change of the steady state target. Furthermore, if the target is not admissible, the system is steered to the closest admissible steady state. In [[18](https://www.researchgate.net/publication/220156418_MPC_for_tracking_with_optimal_closed-loop_performance?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)], the MPC for tracking is extended considering a general offset cost function. Under some mild sufﬁcient assump- tions, the new offset cost function ensures the local optimality property, letting the controller achieve optimal closed-loop per- formance. In [19] this controller is extended to the case of tracking target sets (a generalized set-interval control) by using the con- cept of distance of a point to a set. In contrast to the approach presented in [[6](https://www.researchgate.net/publication/244358793_A_stable_MPC_with_zone_control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)], this strategy allows local optimality an it is suitable for non-stable systems.

where *x* R*n* is the state of the system at the current time instant, *x*+ denotes the successor state, that is, the state of the system at next sampling time, *u* ∈ R*m* is the manipulated control input, *y* ∈ R*p* is the controlled variables and *w* R*n* is an unknown but bounded state disturbance. In what follows, *x*(*k*), *u*(*k*), *y*(*k*) and *w*(*k*) denote the state, the manipulable variable, controlled variable and the dis- turbance respectively, at sampling time *k*.

The plant is subject to hard constraints on state and control:

∈

∈

(*x*(*k*)*, u*(*k*)) ∈ *Z* (2)

where is a compact convex polyhedron containing the origin in its interior.

*Z* = *X* × *U*

Deﬁne also the plant nominal model, given by (1) neglecting the disturbance input *w*:

*x*+ = *Ax* + *Bu* (3)

*y* = *Cx* + *Du*

The plant model is assumed to fulﬁl the following assumption:

## Assumption 1.

* The pair (*A*, *B*) is controllable.

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* The uncertainty vector *w* is bounded and lies in a compact convex
1. It is given by the recursion *Rj* ⊕ *Aj W* = *Rj*+1 with *R*1 = *W*.

polyhedron containing the origin in its interior

*W* = {*w* ∈ R : *Aww* ≤ *bw*} (4)

*n*

* The state of the system is measured, and hence *x*(*k*) is known at each sample time.

It is remarkable that no assumption is considered on the num- ber of inputs *m* and outputs *p*, allowing thin plants (*p* > *m*), square plants (*p* = *m*) and ﬂat plants (*p* < *m*). Moreover, it is not assumed

*K*

1. *AK Rj* ⊕ *W* = *Rj*+1 = *Rj* ⊕ *Aj W*

*K*

1. *Rj* ⊆ *Rj*+1
2. The sequence of sets *j* has a limit ∞ as *j* , and ∞ is a robust positive invariant set.

*R R* → ∞ *R*

1. *R*∞ is the minimal RPI set.

Based on this, the sets of restricted constraints on the nominal predictions considered in the optimization problem are given by:

that (*A*, *B*, *C*, *D*) is a minimal realization of the state-space model. This allows us to use state-space models derived from input–output models, that is, using as state a collection of past inputs and out- puts of the plant [8]. The necessity of an observer is also avoided while the global uncertainty and the noise can be posed as additive uncertainties in the state-space model (1).

*Xj* ¾ *X* g *Rj*

*Uj* ¾ *U* g *KRj*

These sets are non-empty if the following assumption holds

**Assumption 2.** *R*∞ ⊂ *X* and *KR*∞ ⊂ *U*.

(5)

The aim of this paper is to ﬁnd a control law *u*(*k*) = *nN*(*x*(*k*), *T t*) such that the system is steered into a (possibly time varying) region *T t*, which deﬁnes the range into which the controlled out- puts should remain fulﬁlling the plant constraints (*x*(*k*)*, u*(*k*)) , despite the uncertainties.

∈ *Z*

## Robust MPC for tracking zone regions based on nominal predictions

In this section the proposed controller is presented. The pro- posed controller is an extension to the robust case of the MPC for tracking zone regions [19], using the concepts presented in [[20](https://www.researchgate.net/publication/220157638_Systems_with_persistent_disturbances_Predictive_control_with_restricted_constraints?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)].

* 1. *Preliminaries: the robust MPC based on nominal predictions and restricted constraints*

The keystone of the robust MPC presented in [[20]](https://www.researchgate.net/publication/220157638_Systems_with_persistent_disturbances_Predictive_control_with_restricted_constraints?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D) is to use predictions based on the nominal system for the MPC cost (i.e., predictions that neglect the disturbance input *w*), and to restrict the constraints set and at any step of the prediction horizon.

*X U*

The controller is based on a pre-stabilization of the plant using a state feedback control gain *K*, such that *AK* = *A* + *BK* has all its eigen-

values in the interior of the unit circle. The controlled system is then given by

**Remark 1.** The calculation of the restricted constraints *j* is not an easy task, due to the complexity of the calculation of the set *j*, based on a series of Minkowsky’s sums. The computational burden can be reduced if the set is given by an interval or an afﬁne map of an hypercube [23]. It is also important to note that the calculation of such sets is made off-line, so it has no practical effects on the MPC problem.

**Remark 2.** An important parameter on the design of this con- troller, is the control gain *K*. This parameter determines the dynamic of the closed-loop system in presence of disturbances and hence, it has to ensure that Assumption 2 holds. In [[24]](https://www.researchgate.net/publication/244358772_Robust_tube-based_MPC_for_tracking_of_constrained_linear_systems_with_additive_disturbances?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D) it is pro- posed an LMI-based method for the calculation of the control gain *K* which ensures that Assumption 2 holds and that the set ∞ is minimized.

*W*

*R*

*R*

*X* g *R*

* 1. *Preliminaries: characterization of the steady state*

Every nominal steady state and input *zs* = (*xs*, *us*) is a solution of the equation

Σ *xs* Σ

= **0***n,*1 (6)

[ *A* − *In B* ]

*us*

*x*(*k* + 1) = *AK x*(*k*) + *Bc*(*k*) + *w*(*k*)

*u*(*k*) = *Kx*(*k*) + *c*(*k*)

Neglecting the disturbances *w*, the nominal prediction model is then given by:

*x*(*k* + 1) = *AK x*(*k*) + *Bc*(*k*)

*u*(*k*) = *Kx*(*k*) + *c*(*k*)

The notion of robust positively invariant (RPI) set [[21,22]](https://www.researchgate.net/publication/26406002_Gilbert_EG_Theory_and_computation_of_disturbance_invariant_sets_for_discrete-time_linear_systems_Mathematical_Problems_in_Egineering_4_317-367?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D) plays an important role in the design of robust controllers for constrained systems. This is deﬁned as follows:

**Deﬁnition 1.** A set *˝* is called a robust positively invariant (RPI) set for the uncertain system *x*(*k* + 1) = *AK x*(*k*) + *w*(*k*) with *w*(*k*) ∈ *W* if *AK ˝* ⊕ *W* ⊆ *˝*.

It is also necessary to deﬁne the so-called reachable sets, that represents the forced response of the system due to the uncertainty.

**Deﬁnition 2.** The reachable set in *j* steps, *Rj*, is given by

*j*−1

Therefore, there exists a matrix *M&* R(*n*+*m*)×*m* such that every nominal steady state and input can be posed as

*zs* = *M&&* (7)

∈

for certain *&* R*m* [[17](https://www.researchgate.net/publication/220159753_MPC_for_tracking_piecewise_constant_reference_for_constrained_linear_systems?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. The nominal steady outputs are then given by

∈

*ys* = *N&&* (8)

where *N&* ¾ [*C D*]*M&* .

*Z X* × *U*

Deﬁning ¾ *N N* , the set of admissible nominal steady states and inputs and the set of admissible nominal controlled vari- ables are given by

*Zs* ¾ {(*x, u*) ∈ *Z* : (*A* − *In*)*x* + *Bu* = **0***n,*1}

*Ys* ¾ {*Cx* + *Du* : (*x, u*) ∈ *hZs*}

where *h* (0, 1) is a given parameter added to avoid those steady states and inputs that provide active constraints.

∈

*Rj* ¾

*K*

 *i*=0

*Ai W*

* 1. *The proposed controller*

As in [19], the proposed controller maintains the main ingredi- ents of the MPC for tracking target sets: the steady state conditions

This is the set of states of the nominal closed-loop systems which

are reachable in *j* steps from the origin, under the disturbance input

*w* [[20](https://www.researchgate.net/publication/220157638_Systems_with_persistent_disturbances_Predictive_control_with_restricted_constraints?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. This set satisﬁes the following properties:

of the system are decision variables in the optimization problem (artiﬁcial reference), the stage cost is a measure of the distance to the artiﬁcial reference, the so-called *offset cost* function is added in

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order to penalize the deviation between the artiﬁcial reference and the target, the terminal constraint is an invariant set for tracking.

The proposed controllers is derived following the results pre- sented in Section 3.1. Therefore, the plant is pre-stabilized by the following control law

where *˝a* is a suitable polyhedral set. Notice that the decision variables are: (i) the sequence of the future actions of the nomi- nal system **c** and (ii) the parameter vector *&* that determines the artiﬁcial target steady state, input and output (*xs*, *us*, *ys*).

Considering the receding horizon policy, the control law is given

*t*

*u*(*k*) = *Kx*(*k*) + *L&* + *c*(*k*) (9)

where *L* = [ *K Im*]*M&* . Then the nominal system can be rewritten as follows:

−

by

*nN* (*x, Tt* ) ¾ *Kx* + *L&*0(*x, Tt* ) + *c*0(0; *x, Tt* )

where *c*0(0 ; *x*, *T t*) is the ﬁrst element of the control sequence **c**0(*x*,

*x*+ = *AK x* + *BL&* + *Bc u* = *Kx* + *L&* + *c*

The cost function to minimize is given by:

*N*−1

Σ

(10)

*T t*) which is the optimal solution of problem *PN*(*x*, *T t*). Notice also that, in the following, the optimal value of the cost function will be denoted as *V* 0 (*x, Tt* ), the optimal value of the other decision vari-

able as *&*0(*x*, *T t*), the nominal optimal state trajectory as **x**0(*x, Tt* ) and the optimal artiﬁcial reference (*x*0(*x, Tt* )*, u*0(*x, Tt* )*, y*0(*x, Tt* )).

*N*

*s*

*s*

*s*

*VN* (*x, Tt* ; **c***, &*) ¾ ∗*c*(*j*)∗2 + *VO*(*ys, Tt* ) (11)

*˙*

*j*=0

Since the set of constraints of *PN*(*x*, *T t*) does not depend on *T t*, its feasibility region does not depend on the target region *T t*. The feasible set of the proposed controller is a polyhedral region *XN* ⊆

where **c** = *{c*(0), *c*(1), *. . .*, *c*(*N* − 1)*}*, *W* = *W* r > 0, the pair (*xs*, *us*) = *M& &*

is the artiﬁcial steady state and input and *ys* = *N& &* the artiﬁcial out- put, all of them parameterized by *&*; *T t* is the zone in which the controlled variables have to be steered. *VO*(*ys*, *T t*) is the so-called offset cost function and it is such that the following assumption is ensured

## Assumption 3.

1. *T t* is a compact convex set.
2. *VO*(*ys*, *T t*) is convex w.r.t. *ys*.
3. If *ys* ∈ *T t*, then *VO*(*ys*, *T t*)≥ 0. Otherwise, *VO*(*ys*, *T t*)>0.

**Remark 3.** Following same arguments as in [[20](https://www.researchgate.net/publication/220157638_Systems_with_persistent_disturbances_Predictive_control_with_restricted_constraints?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D),25], it is possible to prove that, in the case that *K* is the gain of the LQR, minimiz- ing *VN*(*x*, *T t* ; **c**, *&*) is equivalent to minimizing the following cost

R*n* given by the set of initial states that can be steered into *˝t* =

*Projx*(*˝a*) in *N* steps fulﬁlling the constraint (16), for all admissible disturbances.

*t*

* 1. *Stability of the proposed controller*

Consider the following assumption on the controller parame- ters:

## Assumption 4.

1. Deﬁne the extended state *xa* = (*x*, *&*), and

Σ=

*a* Σ

*A A* + *BK BL*

0 *Im*

function

*V*˜*N* (*x, Tt* ; **c***, &*) =

Σ

*N*−1

∗*x*(*j*) − *xs*∗ + ∗*u*(*j*) − *us*∗ + ∗*x*(*N*) − *xs*∗

2 2 2

where *L* = [− *K Im*]*M&* . Deﬁne also

*Xa* = {(*x, &*) : *x* ∈ *Xi, Kx* + *L&* ∈ *Ui, M&&* ∈ *hZs*}

*i*

*Q R*

*j*=0

*P*

and

+ *VO*(*ys, Tt* ) (12)

where *x*(*j*) is the nominal prediction of the model for *u*(*j*) *Kx*(*j*) *L& c*(*j*); *Q* is a real symmetric positive semideﬁnite matrix such that the couple (*Q*1/2, *A*) is detectable; *R* is a real symmetric positive deﬁnite matrix; *P* is the unique solution of the Riccati equation

+

= +

(*A* + *BK* )r*P*(*A* + *BK* ) − *P* = −(*Q* + *K* r*RK* )

In fact, if *W* is chosen as *W* = *R* + *B*r*PB*, the equivalence between cost

1. and (12) holds since

*V*˜*N* (*x, Tt* ; **c***, &*) = *VN* (*x, Tt* ; **c***, &*) + ∗*x*(0) − *xs*∗2

*P*

Then, taking *K* = *KLQR*, minimizing the cost (11) is equivalent to min- imize the cost of the predicted nominal trajectory.

The optimization problem *PN*(*x*, *T t*) is now given by:



*˙t* = {*xa* : *Ai xa* ∈ *Xi ,* for *i* ≥ 0} Then

*˝a* = *˙t* g (*RN* × {0})

*t*

*a a*

In the following theorem, stability and constraints satisfaction of the controlled system are stated.

**Theorem 1** (Stability). *Consider that Assumptions* 1–4 *hold and con- sider a given target operation zone T t. The system controlled by the proposed MPC controller nN*(*x*, *T t*) *is such that:*

* 1. *For all initial condition x*(0) ∈ *XN and for every T t, the evolution of the system is robustly feasible and admissible, that is*, *x*(*j*) ∈ *XN and* (*x*(*j*)*, nN* (*x*(*j*)*, Tt* )) , *w*(*k*) *, k* = 0, 1, , *j* 1*.*

∈ *Z* ∀ ∈ *W* · · · −

* 1. lim *c*(*k*) 0

=

*k*→∞

* 1. *If Tt* ∩ *Ys* =*/* ∅ *then the closed-loop system asymptotically con- verges to a set y*(∞) ⊕ (*C* + *DK* )*R*∞*, such that y*(∞) ∈ *Tt.*
	2. *If Tt* ∩ *Ys* = ∅*, the closed-loop system asymptotically converges to a set ys*∗ (*C DK* ) ∞*, where ys*∗ *is the reachable nominal steady output such that*

⊕ + *R*

*ys*∗ ¾ *arg* min*VO*(*ys, Tt* )

*ys* ∈*Ys*

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## Properties of the proposed controller

The proposed controller is a robust formulation of the MPC for

tracking target sets presented in [19]. As a consequence, it inherits

robustly converge to the target zone *T t*. Since Theorem 1 ensures that the output *y* converges to the set *ys*∗ ⊕ (*C* + *DK* )*R*∞, then

*y*∗ ⊕ (*C* + *DK* )*R* ⊆ *T*

*s*

∞

*t*

all the good properties of that controllers:

* **Steady state optimization**. The offset cost function can be con- sidered as a steady state target optimizer (SSTO) built in the same MPC, since the proposed controller drives the system to a neigh- borhood of the optimal operating point minimizing the offset cost function *VO*(*ys*, *T t*).
* **Feasibility for any reachable target zone**. Since the set of con-

straint of the proposed controller does not depend on the target set *T t*, feasibility is ensured for any *T t* and for any prediction horizon *N*. Therefore, if the initial condition is an admissible equi- librium point, the proposed controller is able to drive the system to any admissible target zone (i.e. *Tt* ∩ *Ys* =*/* ∅) even for *N* = 1.

Moreover, if *T t* varies with the time, the results of Theorem 1

still hold.

* **Input target**. The proposed controller can be formulated consid- ering input targets of the form *umin* ≤ *ut* ≤ *umax*, by deﬁning an offset cost function *VO*(*us*, *T u*,*t*) convex w.r.t. *us*, where *T u*,*t* is a convex polyhedron.
* **Enlargement of the domain of attraction**. The terminal con-

straint of the proposed controller is an invariant set for *any* equilibrium point. In standard MPC, the invariant set is calculated for a ﬁxed equilibrium point. Therefore, the terminal constraint, and as a consequence the domain of attraction of the proposed controller are (potentially) larger than in standard MPC. This property allows to consider small values of the control horizon.

* **Optimization problem posed as a QP**. Since all the ingredients

(functions and sets) of the optimization problem *PN*(*x*, *T t*) are convex, then it derives that *PN*(*x*, *T t*) is a convex mathematical programming problem that can be efﬁciently solved in poly- nomial time by specialized Algorithms [[26](https://www.researchgate.net/publication/224557816_Fast_Model_Predictive_Control_Using_Online_Optimization?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D),27]. As in [19], this problem can be re-casted as a standard QP problem, choosing one of the following formulations of the offset cost function:

* 1. distance from a set as ∞-norm

*VO*(*ys, Tt* ) ¾ min *ys y* (20)

∗ − ∗∞

*y*∈*Tt*

Hence *y* converges to a point *ys*∗ ∈ *T*˜*t* , where

*T*˜*t* = *Tt* g (*C* + *DK* )*R*∞

Due to this fact, the robust convergence of the closed-loop sys- tem to the target zone *T t* is ensured if the proposed controller control law is given by *nN* (*x, T*˜*t* ). In particular

* If *T*˜*t* ∩ *Ys* =*/* ∅ then the closed-loop system asymptotically con- verges to *T t*.
* If *T*˜*t* ∩ *Ys* = ∅, the closed-loop system asymptotically converges to *ys*∗ ⊕ (*C* + *DK* )*R*∞.

**Remark 4.** The calculation of ∞ is not trivial. In [[22,](https://www.researchgate.net/publication/3032065_Invariant_approximations_of_the_minimal_robust_positively_Invariant_set?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)23] approx- imation methods are proposed based on outer estimations.

*R*

In the case of the formulation based on the scaling factor, robust convergence to *T t* is ensured without calculating the set *R*∞ if (*C* + *DK* )*R*∞ ⊆ *÷t* and *yt* ∈ *Ys*. In this case the closed-loop system converges to *yt* ⊕ (*C* + *DK* )*R*∞.

## Simulation results

To test the proposed control strategy, a subsystem of a ﬂuid cat- alytic cracking (FCC) unit, presented in [[28](https://www.researchgate.net/publication/224369731_Robust_model_predictive_control_with_zone_control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)], will be used. The main objective of this simpliﬁed choice is to clearly show the ability of the proposed robust controller to handle both, persistent disturbance rejection and output zone control in systems with more output than inputs. The original system has two manipulated inputs (*u*1 repre- sents the air ﬂow rate to the catalyst regenerator and *u*2 represents the opening of the regenerated catalyst valve) and three controlled outputs (*y*1 represents the riser temperature, *y*2 the regenerator dense phase temperature and *y*3 the regenerator dilute phase tem- perature), while the selected subsystem only consider the second input and the ﬁrst two controlled outputs.

*5.1. Nominal system description*

The nominal linear model of the selected subsystem is given by:

(ii) distance from a set as 1-norm

*O*

*s*

*t*

⎡ 0*.*2033 ⎤

*V* (*y , T* ) ¾ min *y y*

*y*∈*T* ∗ *s* −

∗1

*t*

(21)

*G*(*S*) = ⎢⎣

1*.*7187*s* + 1

⎥⎦ (23)

(iii) distance from a set as a scaling factor: in this implementation, the target region is deﬁned as

 0*.*1886*s* + 3*.*8087

17*.*7347*s*2 + 10*.*8348*s* + 1

For a sample time of *T* = 1, the following discrete state space model is obtained:

*Tt* ¾ *yt* ⊕ *÷t*

where *yt* is a desired target point and *÷t* is a polyhedron that

⎡ 0*.*5589 0 0

⎤ ⎡ 0*.*1895 ⎤

deﬁnes the zone. Then

*A* = ⎢

0 0*.*5240 −0*.*1672

⎥ *, B* = ⎣ 0*.*7413 ⎦ (24)

*V* (*y , T* ) = min *h*

⎣ 0 0*.*1853 0*.*9769 ⎦

0*.*1025

*O s t*

*h,y*

*s.t. h* ≥ 0

*y* − *yt* ∈ *h÷t*

(22)

*t*

then *h* > 1,

and

(25)

*C* =

Σ 0*.*4731 0 0 Σ

*Y*

× *Y*

0 0*.*0106 −0*.*8590

and if *y T t* then *h* [0, [1](https://www.researchgate.net/publication/227975041_Robust_Steady_State_Target_Calculation_for_Model_Predictive_Control?el=1_x_8&enrichId=rgreq-6c3e4580-e451-40e6-9af3-7e73fa24431c&enrichSource=Y292ZXJQYWdlOzI1NzQwNzU5MztBUzoxMDE4MDUxNjU5MDc5NzJAMTQwMTI4MzY1MTE3Mg%3D%3D)]. In particular, if *y* = *yt*, hence *h* = 0.

∈ ∈

Notice that, this measure is such that, if *y* ∈*/ T*

Therefore, *h* has the double role of measuring the distance to a set and to a point.

*4.1. Robust convergence to the target zone*

The objective of the robust MPC for tracking zone regions pro- posed in this paper is to ensure that the output of the system *y* will

To complete the system description, the manipulated input will be constrained to be in *U* = {*u* ∈ **R** : ∗*u*∗∞ ≤ 5}. Notice that for this (2 1) system, the set of admissible nominal steady state output, *s*, is in a subspace of dimension 1. Therefore, from the operation point of view, only the output desired zones with a non-empty inter- section with *s* will be reachable at steady state. The sequence of desired zones proposed for the simulations is given by 4 sets of the form *T t* = *{ymin* ≤ *y* ≤ *ymax}*, which are shown in Table 1. Fig. 1

**Table 1**

Target zones used in the simulation example.

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2.5

Ry



Wy

2

*T t ymin ymax*

*T t*,1 (− 1.6, 12.5) (0, 17.5)

*T t*,2 (0, − 17.5) (1.6, − 13.5)

*T t*,3 (− 1.8, − 13) (− 0.2, − 8.5)

 *T t*,4 (− 0.8, − 2.5) (0.8, 2.5)

shows these desired set together with the set of admissible nomi- nal steady state output, *s*. The intersection of these sets constitutes the nominal reachable desired outputs zones. Notice that the sets *T t*,*i*, for *i* = 1, 2, 3, 4, constitute disjoint sets of the output space. Fur- thermore, as can be seen, the third target set is unreachable for the nominal system.

*Y*

* 1. *Disturbance description*

The set *W* of possible disturbance realizations is given by *W* = *w* **R** : *w* ∞ 0*.*5 . This choice allows a possible disturbance of 2 percent of the maximal state excursion selected for the simula-

3

{ ∈ ∗ ∗ ≤ }

1.5

1

0.5

0

y2

−0.5

−1

−1.5

−2

−2−.50.8

−0.6

−0.4 −0.2 0 0.2 0.4 0.6 0.8

y1

tion, which means that it can be, in many cases, the same order of

*y y*

the current system state. The sets *W* = *CW* and *R* = *CR*∞ (placed

in the output space), which derive from the set ∞ , are shown in

*W W*

Fig. 2. Notice that the set is such that the nominal MPC con- troller, even if it is designed to handle target zones, as the one presented in [19], cannot reject the disturbance realization used in these simulations.

* 1. *Dynamic simulations*

The simulation starts at *x*0 = (0, 0). The parameters of the pro- posed MPC are as follows: *N* = 3, *Q* = 100*I*3 and *R* = *I*2. This particular choice of *Q* and *R* is motivated by the fact that it provides a reason- ably small ∞, thus reducing the conservatism of the controller. The gain matrix *K* of the local controller is given by the LQR and matrix *P* is the solution of the Riccati equation.

*R*

As it was already said, the simulation consists in the four output target (zone) changes shown in Table 1. Furthermore, a persistent disturbance *w* that remain switching between extreme points of

*W*

is injected to the system along the complete simulation. To clearly show that the disturbance *w* is in fact difﬁcult to reject (given that it has not a stationary behavior), we simulate the closed-loop

20

15

10

5

0

y2

−5

−10

−15

−20

−2 −1.5 −1 −0.5 0 0.5 1 1.5 2

y1

**Fig. 1.** Set *Ys* (red-dashed line), the desired output sets *T t*,1 , *T t*,2 , *T t*,3 and *T t*,4 (blue- solid line) and the intersection of these sets (green-solid line). (For interpretation of the references to color in this ﬁgure legend, the reader is referred to the web version of the article.)

**Fig. 2.** The sets *Wy* and *Ry* for the selected disturbance set *W*.

under a nominal controller; i.e., a controller that accounts for the zone control but does not include the disturbance model. As can be seen in Fig. 3 the closed-loop performance is clearly unacceptable for the second output, while the input saturates at different time intervals. In a second simulation stage, we simulate the closed- loop under the proposed robust controller, using an offset cost *VO*(*ys, Tt* ) min *ys y* 1. The system evolution in the output space

∞

= ∗ − ∗

*y*∈*Tt*

is shown in Fig. 4. As can be seen, the nature of the offset cost *VO* is crucial when the target zone is not reachable, as it does occur in the third change. Fig. 4 clearly shows that the controller steers the system to the corresponding output zone, if possible, and to a region around a steady state point which minimizes the 1-norm, if not. The corresponding time evolutions of the input and outputs are shown in Fig. 5.

Finally, the same simulation sequence is repeated for the pro- posed controller, but now using an offset cost given by *VO*(*ys, Tt* ) = min *ys y* . Figs. 6 and 7 show the system evolution in the output

∗ − ∗∞

*y*∈*Tt*

space and the input and outputs time evolution. The main differ- ence between this simulation and the previous one, is that for the third change the distance to the unreachable target is determined by the ∞-norm.

2

0

y1

−20 200 400 600 800 1000 1200

40

0

y2

−400 200 400 600 800 1000 1200

5

0

u

−5

0 200 400 600 800 1000 1200

samples

**Fig. 3.** Input and the outputs time evolutions for the nominal controller.

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20

t,1

Ys

t,4

t,3

t,2

15

10

5

0

y2

−5

−10

−15

−2−02 −1.5 −1 −0.5 0 0.5 1 1.5 2

y1

2

0

y1

−20 200

20

0

y2

−200 200

5

0

u

−5

0 200

400 600 800 1000 1200

400 600 800 1000 1200

400 600 800 1000 1200

samples

**Fig. 4.** System evolution in the output space, for *V* (*y , T* ) = min∗*y* − *y*∗ .

**Fig. 7.** Input and the outputs time evolutions, for *VO*(*ys, Tt* ) = min∗*ys* − *y*∗∞.

*y*∈*Tt*

*O s t s* 1

*y*∈*Tt*

2

0

y1

−20 200 400 600 800 1000 1200

20

0

y2

−200 200 400 600 800 1000 1200

5

0

u

−5

0 200 400 600 800 1000 1200

samples

**Fig. 5.** Input and the outputs time evolutions, for *VO*(*ys, Tt* ) = min∗*ys* − *y*∗1.

*y*∈*Tt*

20

t,1

Ys

t,4

t,3

t,2

15

10

5

0

y2

−5

−10

−15

−20

−2 −1.5 −1 −0.5 0 0.5 1 1.5 2

y1

**Fig. 6.** System evolution in the output space, for *VO*(*ys, Tt* ) = min∗*ys* − *y*∗∞.

*y*∈*Tt*

* 1. *Comment about the nature of the disturbance and the robust controller conservatism*

This subsection is devoted to clearly elucidate the meaning of and the relation with the conservatism of the proposed strat- egy. The set *y* corresponds to the complete dynamic of the system output under the effect of the disturbance, i.e., it includes both, the stationary and the transitory regime of the evolution. Here we assume for the simulation a disturbance realization with a permanent variation along the time (i.e., with no steady state). Most of the disturbance models, however, assume a permanent but constant disturbance, since this assumption has sense in real application as they account for model mismatches usually more signiﬁcant than the disturbance itself. In fact, a more realistic (and less conservative) situation is to consider a constant disturbance (maybe, plus a small variable signal) once the system reach a given target zone. In this case, the system output will be stabi- lized at a ﬁxed point inside the desired zone (if reachable), and mainly, this desired zone could be too much tighter (i.e., the zone will conserve the reachability condition for tighter limits). The reason for that is that the set to be subtracted from the output zone to obtain *T*˜*t* is no longer *y* , but an approximation of *y*, which is clearly smaller (see Fig. 2). This could be an important point since it shows that the conservatism of the proposed strat- egy could be signiﬁcantly reduced for some frequent application

*y*

∞

*R*

*R*∞

∞

*R W*

cases.

## Conclusion

The zone control strategy is implemented in applications where the exact values of the controlled outputs are not important, as long as they remain inside a range with speciﬁed limits. In this paper, a robust extension of the MPC for tracking zone regions control has been presented, based on nominal predictions and restricted con- straints. From a tracking point of view, the controller considers a set, instead of a point, as target. The concept of deviation between two points used in the offset cost function has been generalized to the concept of distance from a point to a set. A characteri- zation of the offset cost function has been given as the minimal distance between the output and some point inside the target set. The controller ensures recursive feasibility and robust satisfaction of the constraints by using nominal predictions and restricted con- straints.

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## Appendix A. Stability proof

In this section, the stability proof of Theorem 1 is presented. Firstly, it is necessary to introduce some lemmas. To this aim, deﬁne as (**c**0(*x*(*k*), *T t*), *&*0(*x*(*k*), *T t*)) the optimal solution of problem *PN*(*x*, *T t*) at the time instant *k*, where

hence

*K x*˜(*j* − 1; *k* + 1) + *c*˜(*j* − 1; *k* + 1) + *L&*˜(*k* + 1) ∈ *Uj* ⊕ *KAj*−1*W*

*K*

and

*Uj* ⊕ *KAj*−1*W* = *U* g *KRj* ⊕ *KAj*−1*W* = *U* g *KRj*−1 = *Uj*−1

*K K*

**c**0(*x*(*k*)*, Tt* ) = {*c*0(0; *x*(*k*)*, Tt* )*, c*0(1; *x*(*k*)*, Tt* )*, . . . , c*0(*N* − 1; *x*(*k*)*, Tt* )} A

Deﬁne the control sequence

**c**˜(*x*(*k* + 1)*, Tt* ) = {*c*0(1; *x*(*k*)*, Tt* )*, . . . , c*0(*N* − 1; *x*(*k*)*, Tt* )*,* 0}

and deﬁne *&*˜(*x*(*k* + 1)*, Tt* ) = *&*0(*x*(*k*)*, Tt* ). Moreover, deﬁne as

**Lemma 4.** *[Recursive feasibility of the terminal constraint]For all k* ≥ 0,

(*x*0(*N*; *k*)*, &*0(*k*)) ∈ *˝a*

*t*

*x*˜(*j*; *x*(*k* + 1)*, Tt* ) the *j*th step prediction, given *x*(*k* + 1). Hence

**Proof.** Consider that at time *k* (*x*0(*N*; *k*)*, &*0(*k*)) ∈ *˝a*. Since *˝a* =

*j*−1

*t t*

*˙t* g (*RN* × 0), hence

*x*˜(*j*; *x*(*k* + 1)*, Tt* ) = *Aj x*(*k* + 1) + Σ*Ai B*[*c*˜(*j* − *i* − 1; *x*(*k* + 1)*, Tt* ) 0 0

+ + = − + +

*K*

*K*

(*x* (*N* − 1; *k* + 1)*, &* (*k* + 1)) ∈ *˙t* g (*RN* × 0) ⊕ (*AK W* × 0)

*N*−1

*i*=0

0 0 0 0

+ *L&*˜(*x*(*k* + 1)*, Tt* )]

In what follows, the dependence from (*x*, *T t*) will be omitted for the sake of clarity, namely, *x*(*j* ; *k*) will denote *x*(*j* ; *x*(*k*), *T t*).

**Lemma 1.** *For all j* = 0, *. . .*, *N* − 1

*x*˜(*j*; *k* + 1) − *x*(*j* + 1; *k*) = *Aj w*(*k*)

*K*

Then, since (*x* (*N*; *k* 1)*, &* (*k* 1)) *Aa*(*x* (*N* 1; *k* 1)*, &* (*k*

1)), hence

(*x*0(*N*; *k* + 1)*, &*0(*k* + 1)) ∈ *Aa*(*˙t* g (*RN* × 0) ⊕ (*AN*−1*W* × 0))

*K*

Taking into account that

*Aa*(*˙t* g (*RN* × 0) ⊕ (*AN*−1*W* × 0))

*K*

= *Aa˙t* g (*AK RN* × 0) ⊕ (*AK W* × 0)

*N*

⎛

**Proof.** Since

*K*

*j*−1

*K*

= *Aa˙t* g ⎝ *A W* × 0⎞

*N*

*j*

⎠ ⊕ (*AN W* × 0)

*x*0(*j* + 1; *k*) = *Aj x*0(1; *k*) +

Σ

*i*=0

*Ai B*[*c*0(*j* − *i*; *k*) + *L&*0(*k*)]

*K K*

*j*=1

⎛ *N*−1 ⎞

and

*K*

*K*

= *A ˙* g ⎝ *Aj W* × 0⎠ g (*AN W* × 0) ⊕ (*AN W* × 0)

*j*−1

Σ

*i*=0

*x*˜(*j*; *k* + 1) = *Aj x*(*k* + 1) + *Ai B*[*c*˜(*j* − *i* − 1; *k* + 1) + *L&*˜(*k* + 1)]

*K*

*a t K K K*

*j*=1

⎛

*N*−1

⎞

*j*−1

Σ

*j*

*K*

⊆ *Aa˙t* g ⎝ *Aj W* × 0⎠

hence

= *AK x*(*k* + 1) +

*i*=0

*Ai B*[*c*0(*j* − *i*; *k*) + *L&*0(*k*)]

*j*=1

⎛ *N*−1 ⎞

*j*

*j* 0 *j*

⊆ (*˙t* g (*W* × 0)) g ⎝

*AK W* × 0⎠

*x*˜(*j*; *k* + 1) − *x*(*j* + 1; *k*) = *AK* [*x*(*k* + 1) − *x* (1; *k*)] = *AK w*(*k*)

A

0

*N.*

⎛ *N*−1

*j*

*j*=1

# ⎞

**Lemma 2.** *If x* (*j*; *k*) ∈ *Xj, then x*˜(*j* − 1; *k* + 1) ∈ *Xj*−1*, for all j* = 0, *. . .*,

*K*

= *˙t* g ⎝

*j*=0

*AK W* × 0⎠

**Proof.** Since *x*˜(*j* − 1; *k* + 1) = *x*0(*j*; *k*) + *Aj*−1*w*(*k*), then

*t*

*N*

= *˙* g (*R*

× 0)

*j*−1

*x*˜(*j* − 1; *k* + 1) ∈ *Xj* ⊕ *Aj*−1*W* = *X* g [

*Ai W*] ⊕ *Aj*−1*W*

where the second from last equality comes from *Aa˙t* ⊕ (*W* × 0) ⊆

*˙t* ⇔ *Aa˙t* ⊆ *˙t* g (*W* × 0).

*K K K*

*i*=0

*j*−2

*i*

Hence,

(*x*0(*N*; *k* + 1)*, &*0(*k* + 1)) ∈ *˙t* g (*RN* × 0) = *˝a*

⊆ *X* g [

⊆ *Xj*−1

A

*i*=0

*t*

*AK W*]

A

*A.1. Proof of Theorem 1*

Before starting with the proof, we introduce the notion of Input-

**Lemma 3.** *If Kx*0(*j*; *k*) + *c*0(*j*; *k*) + *L&*0(*k*) ∈ *Uj, then K x*˜(*j* − 1; *k* +

1) + *c*˜(*j* − 1; *k* + 1) + *L&*˜(*k* + 1) ∈ *Uj*−1*, for all j* = 1, *. . .*, *N* − 1*.*

**Proof.** Taking into account that

*Kx*0(*j*; *k*) + *c*0(*j*; *k*) + *L&*0(*k*) = *K x*˜(*j* − 1; *k* + 1) − *KAj*−1*w*(*k*)

*K*

+ *c*˜(*j* − 1; *k* + 1) + *L&*˜(*k* + 1)

to-State Stability (ISS) [29]. To this aim, consider a closed-loop disturbed system

*x*(*k* + 1) = *fK* (*x*(*k*)*, w*(*k*)) (19)

where *fK* (*x, w*) ¾ *f* (*x, K* (*x*)*, w*). The solution of this equation at sampling time *k*, for the initial state *x*(0) and the sequence of dis- turbances **w**, is denoted as *$K*(*k* ; *x*(0), **w**).

1974 *A. Ferramosca et al. / Journal of Process Control 22 (2012) 1966–1974*

**Deﬁnition 3.** System (19) is ISS for all initial conditions *x*(0) and sequence of disturbances **w** if there exist a function *ˇ* and a function *˝* such that

*KL K*

|*$K* (*k*; *x*(0)*,* **w**)| ≤ *ˇ*(|*x*(0)|*, k*) + *˝*(∗**w**∗)

In what follows, it will be proved that the closed-loop system is ISS for all *x*(0) ∈ *XN* .

**Proof.** From Lemmas 1–4, it is derived that the couple (**c**˜(*k* 1)*, &*˜(*k* 1)) is a feasible solution of problem *PN*(*x*, *T t*).

+

+

Consider now the optimal value of the cost function *V* 0 (*x*(*k*)*, Tt* ),

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due to the optimal solution of problem *P* (*x*(*k*), *T*

*N*

**c**0(*k*),

2004.

*&*0(*k*)). Deﬁne

*N*−1

Σ

*N t*), given by (

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*V*˜*N* (*x*(*k* + 1)*, Tt* ; **c**˜*, &*˜) = ∗*c*˜(*j*; *k* + 1)∗2 + *VO*(*ys, Tt* )

*˙*

*j*=0

Comparing *V*˜*N* (*x*(*k* + 1)*, Tt* ; **c**˜*, &*˜) with *V* 0 (*x*(*k*)*, Tt* ), we have that

*N*

*V*˜*N* (*x*(*k* + 1)*, Tt* ; **c**˜*, &*˜) − *V* 0 (*x*(*k*)*, Tt* ) = −∗*c*0(0; *k*)∗2

*N W*

and hence, by optimality:

*V* 0 (*x*(*k* + 1)*, Tt* ) − *V* 0 (*x*(*k*)*, Tt* ) ≤ −∗*c*0(0; *k*)∗2

*N*

*N*

*W*

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Since *W* > 0, we can state that:

lim *c*0(0; *k*) 0

=

*k*→∞

and (*ii*) is proved.

The fact that *c*0(0 ; *k*) → 0 implies that *u*(*k*) → *K* (*x*(*k*) − *x*0(*k*)) +

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*u*0(*k*), and hence:

*s*

*x*(*k*) → *x*0(*k*) ⊕ *R*

*, u*(*k*) → *u*0(*k*) ⊕ *KR*

*s* [19] A. Ferramosca, D. Limon, A.H. González, D. Odloak, E.F. Camacho, MPC for track-

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*s* ∞ *s* ∞

Using the same arguments as in [19], it can be proved that (*x*0(*k*)*, u*0(*k*)) converges to the optimal equilibrium point (*x*∗*, u*∗) which is the minimizer of the offset cost function *VO*(*ys*, *T t*).

*s s s s*

Now, the stability of the equilibrium point will be proved. If the uncertainty is null, then (following [19]) the system is asymptot-

ically stable in (*xs*∗*, u*∗*s* ). If *w /* 0, the continuity of the control law

=

provides that the closed-loop system is such that the closed-loop prediction *$cl* (*j*; *x, w*) *$*(*j*; *x, kN* (*x, Tt* )*, w*) is continuous in *x* and

=

*w*. Then, resorting to ISS arguments [29], it can be proved that there

exist a *KL* function *ˇ* and a *K* function *˝* such that

|*x*(*k*) − *xs*∗| ≤ *ˇ*(|*x*(0) − *xs*∗|*, k*) + *˝*(∗*w*∗)

for all initial state *x*(0) ∈ *XN* and all disturbances *w*(*k*). A

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